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EXISTENCE OF SOLUTIONS FOR SEMILINEAR PROBLEMS ON EXTERIOR DOMAINS

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Abstract. In this article we prove the existence of an infinite number of radial solutions to $\Delta u + K(r)f(u) = 0$ on $\mathbb{R}^N$ such that $\lim_{r\to\infty} u(r) = 0$ with prescribed number of zeros on the exterior of the ball of radius $R > 0$ where f is odd with $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞) with f superlinear for large u, and $K(r) \sim r^{-\alpha}$ with $\alpha > 2(N-1)$.

1. Introduction

In this article we study radial solutions of

$$\Delta u + K(|x|)f(u) = 0 \quad \text{for } R < |x| < \infty, \tag{1.1}$$

$$u(x) = 0 \text{ when } |x| = R, \quad \lim_{|x|\to\infty} u(x) = 0, \tag{1.2}$$

where $u : \mathbb{R}^N \to \mathbb{R}$ with $N > 2$, $R > 0$, $f : \mathbb{R} \to \mathbb{R}$ is odd and locally Lipschitz with

- (H1) $f'(0) < 0$, there exists $\beta > 0$ such that $f(u) < 0$ on $(0, \beta)$, $f(u) > 0$ on (β, ∞).
- (H2) $f(u) = |u|^{p-1}u + g(u)$ where $p > 1$ and
$$\lim_{u\to\infty} \frac{|g(u)|}{|u|^p} = 0.$$
- (H3) Denoting $F(u) \equiv \int_0^u f(t)\,dt$ we also assume that thee exists γ with $0 < \beta < \gamma$ such that $F < 0$ on $(0, \gamma)$ and $F > 0$ on (γ, ∞).
- (H4) Further we assume K and K' are continuous on $[R, \infty)$ and $K(r) > 0$, there exists $\alpha > 2(N-1)$ such that $\lim_{r\to\infty} rK'/K = -\alpha$.
- (H5) There exist positive constants d_1, d_2 such that
$$2(N-1) + \frac{rK'}{K} < 0, \quad d_1 r^{-\alpha} \le K(r) \le d_2 r^{-\alpha} \quad \text{for } r \ge R.$$

Our main result read as follows.

Theorem 1.1. *Assume* (H1)–(H5) *and $N > 2$. Then for each nonnegative integer n there exists a radial solution, u_n, of* (1.1)–(1.2) *such that u_n has exactly n zeros on (R, ∞).*

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The radial solutions of (1.1)–(1.2) on $\mathbb{R}^N$ with $K(r) \equiv 1$ have been well-studied. These include [2, 3, 8, 9, 10]. Recently there has been an interest in studying these problems on $\mathbb{R}^N \backslash B_R(0)$. These include [1, 5, 6, 7]. In these papers $0 < \alpha < 2(N-1)$. In this paper we consider $\alpha > 2(N-1)$. Here we use a scaling argument as in [9] to prove existence of solutions.

A key difference between the $0 < \alpha < 2(N-1)$ case and the $\alpha > 2(N-1)$ case is that the function $E(r) = \frac{1}{2}\frac{u'^2}{K(r)} + F(u)$ is non-increasing for $0 < \alpha < 2(N-1)$ and nondecreasing for $\alpha > 2(N-1)$. For $0 < \alpha < 2(N-1)$ this allows us to obtain important estimates on the growth of solutions. For $\alpha > 2(N-1)$ we are unable to do this so instead we make the change of variables $u(r) = u_1(r^{2-N})$ and investigate the differential equation for u_1 on $[0, R^{2-N}]$. For this equation it turns out there is a function $E_1 = \frac{1}{2}\frac{u_1'^2}{h(t)} + F(u_1)$ that is nondecreasing and so we can apply some similar analysis as we did in the $0 < \alpha < 2(N-1)$ case.

The outline of this paper is as follows: in section two we establish existence of a radial solutions of (1.1)–(1.2) with $u(R) = 0$ and $u'(R) > 0$ on $[R, \infty)$. We then make the change of variables $u_1(r) = u(r^{2-N})$ and transform our problem to the compact set $[0, R^{2-N}]$ with $u_1(R^{2-N}) = 0$ and $u_1'(R^{2-N}) = -b^* < 0$. The rest of section two is devoted to showing that $u_1(r)$ stays positive if $b^* > 0$ stays sufficiently small and that $u_1(r)$ has more and more zeros as $b^* \to \infty$. In section 3 we prove the main theorem by choosing appropriate values of the parameter b^*, say b_n^*, such that $u_{1,n}$ is a solution with exactly n zeros on $(0, R^{2-N})$ for each nonnegative integer n and hence converting back to the original notation we get a solution of our original equation with exactly n zeros on (R, ∞) and $u(r) \to 0$ as $r \to \infty$.

2. Preliminaries

Since we are interested in radial solutions of (1.1)–(1.2), we denote $r = |x|$ and write $u(x) = u(|x|)$ where u satisfies

$$u'' + \frac{N-1}{r}u' + K(r)f(u) = 0 \quad \text{for } R < r < \infty, \tag{2.1}$$

$$u(R) = 0, u'(R) = b > 0. \tag{2.2}$$

We will occasionally write $u(r, b)$ to emphasize the dependence of the solution on b. By the standard existence-uniqueness theorem [4] there is a unique solution of (2.1)–(2.2) on $[R, R+\epsilon)$ for some $\epsilon > 0$.

We next we consider

$$E(r) = \frac{1}{2}\frac{u'^2}{K(r)} + F(u). \tag{2.3}$$

It is straightforward using (2.1) and (H5) to show that

$$E'(r) = -\frac{u'^2}{2rK}[2(N-1) + \frac{rK'}{K}] \geq 0. \tag{2.4}$$

Thus E is non-decreasing. Therefore,

$$\frac{1}{2}\frac{u'^2}{K(r)} + F(u) = E(r) \geq E(R) = \frac{1}{2}\frac{b^2}{K(R)} \quad \text{for } r \geq R. \tag{2.5}$$

Next we let

$$u(r) = u_1(r^{2-N}) \tag{2.6}$$

where we denote

$$R^* = R^{2-N},\ b^* = \frac{bR^{N-1}}{N-2}. \tag{2.7}$$

This transforms our equation (2.1)–(2.2) into

$$u_1''(t) + h(t)f(u_1(t)) = 0 \quad \text{for } 0 < t < R_1, \tag{2.8}$$

where

$$u_1(R^*) = 0, \quad u_1'(R^*) = -b^* < 0, \tag{2.9}$$

and

$$h(t) = \frac{1}{(N-2)^2} t^{\frac{2(N-1)}{2-N}} K(t^{1/(2-N)}).$$

Since $(r^{2(N-1)}K)' < 0$ (by (H5)) and $t = r^{\frac{1}{2-N}}$ with $N > 2$ it follows that

$$h'(t) > 0 \quad \text{for } 0 < t \le R^*. \tag{2.10}$$

In addition, from (H5) we see that

$$0 < \frac{d_1}{(N-2)^2} \le \frac{h(t)}{t^q} \le \frac{d_2}{(N-2)^2} \quad \text{for } 0 < t \le R^* \tag{2.11}$$

where $q = \frac{\alpha - 2(N-1)}{N-2} > 0$ (by (H4)).

Now let

$$E_1 = \frac{1}{2}\frac{u_1'^2}{h(t)} + F(u_1). \tag{2.12}$$

Then using (2.8) and (2.10) we see that

$$E_1' = -\frac{u_1'^2 h'}{2h^2} \le 0.$$

Therefore,

$$\frac{1}{2}\frac{u_1'^2}{h(t)} + F(u_1) \ge \frac{1}{2}\frac{(b^*)^2}{h(R^*)} \quad \text{on } (t, R^*). \tag{2.13}$$

Also we consider

$$E_2 = \frac{1}{2}u_1'^2 + h(t)F(u_1). \tag{2.14}$$

Using (2.8) this gives

$$E_2' = h'(t)F(u_1).$$

Integrating this on (t, R^*) gives

$$\frac{1}{2}u_1'^2 + h(t)F(u_1) + \int_t^{R^*} h'(s)F(u_1)\,ds = \frac{1}{2}(b^*)^2. \tag{2.15}$$

It follows from (H3) that F is bounded from below so there exists $F_0 > 0$ such that $F(u_1) \ge -F_0$ for all $u_1 \in \mathbb{R}$. Also since $h'(t) > 0$ by (2.10) we see that

$$\int_t^{R^*} h'(s)F(u_1)\,ds \ge -F_0\left[h(R^*) - h(t)\right]. \tag{2.16}$$

Therefore, since $h(t) > 0$ and $h(t)$ is bounded on $[0, R^*]$ by (2.11) we see from (2.15)-(2.16) that

$$\frac{1}{2}u_1'^2 + h(t)F(u_1) \le \frac{1}{2}(b^*)^2 + F_0[h(R^*) - h(t)] \le \frac{1}{2}(b^*)^2 + F_0 h(R^*). \tag{2.17}$$

It follows from (2.17) that for fixed b^*, then u_1 and u_1' are uniformly bounded on $[0, R^*]$ and therefore the solution u_1 exists on $[0, R^*]$. Therefore, the solution u of (2.1)–(2.2) exists on $[R, \infty)$.

Lemma 2.1. *If $b^* > 0$ is sufficiently small, then $0 < u_1 < \beta$ on $(0, R^*)$.*

Proof. We first note that if u_1 has a local maximum then there exists M_{b^*} with $u_1' < 0$ on (M_{b^*}, R^*), $u_1'(M_{b^*}) = 0$, and with $u_1''(M_{b^*}) \leq 0$. Thus $f(u_1(M_{b^*})) \geq 0$ from (2.8) and therefore $u_1(M_{b^*}) \geq \beta$. Thus while $0 < u_1 < \beta$ we see that u_1 is monotone.

So suppose now that the lemma is false. Then for every $b > 0$ with b sufficiently small there exists an s_{b^*} with $0 < s_{b^*} < R^*$ such that $u_1(s_{b^*}) = \beta$ and $u_1' < 0$ on (s_{b^*}, R^*). Now integrating (2.8) on (t, R^*) and using (2.9) gives

$$u_1' = -b^* + \int_t^{R^*} h(s) f(u_1)\, ds.$$

Integrating again on (t, R^*) gives

$$u_1(t) = b^*(R^* - t) - \int_t^{R^*} \int_s^{R^*} h(x) f(u_1(x))\, dx\, ds.$$

Observe from (H1) that there exists $c_1 > 0$ such that

$$f(u_1) \geq -c_1 u_1 \text{ when } u_1 \geq 0. \tag{2.18}$$

Then using (2.18) and the fact that u_1 is decreasing on (s_{b^*}, R^*) we obtain

$$u_1(t) \leq b^*(R^* - t) + \int_t^{R^*} c_1 d(s) u_1(s)\, ds \tag{2.19}$$

where

$$d(s) = \int_s^{R^*} h(x)\, dx > 0. \tag{2.20}$$

Then we let

$$W(t) = \int_t^{R^*} d(s) u_1(s)\, ds \tag{2.21}$$

and from (2.21) we observe $W'(t) = -d(t)u_1(t)$. Next, multiplying (2.19) by $d(t)$ we obtain

$$-W' \leq b^*(R^* - t)d(t) + c_1 d(t) W.$$

Thus

$$-b^*(R^* - t)d(t) \leq W' + c_1 d(t) W.$$

Denoting $D(t) = e^{\int_0^t c_1 d(s)\, ds} > 0$ and multiplying the previous inequality by $D(t)$ gives

$$-b^*(R^* - t)d(t)D(t) \leq (D(t)W(t))'.$$

Integrating on (t, R^*) gives

$$D(t)W(t) \leq b^* \int_t^{R^*} (R^* - s) d(s) D(s)\, ds$$

thus from (2.21) and the definition of $D(t)$ we see that

$$\int_t^{R^*} d(s) u_1(s)\, ds = W(t) \leq b^* e^{-\int_0^t c_1 d(s)\, ds} \int_t^{R^*} (R^* - s) d(s) e^{\int_0^s c_1 d(x)\, dx}\, ds.$$

Then from (2.19) we see that

$$u_1(t) \le b^* \Big((R^* - t) + c_1 e^{-\int_0^t c_1 d(s)\, ds} \int_t^{R^*} (R^* - s) d(s) e^{\int_0^s c_1 d(x)\, dx}\, ds \Big). \tag{2.22}$$

Since $h(t)$ is bounded on $[0, R^*]$, it follows from (2.20) that $d(t)$ is bounded on $[0, R^*]$ and thus the term in the large parentheses in (2.22) is bounded on $[0, R^*]$. Therefore, from (2.22) we see there exists a $c_2 > 0$ which is independent of b^* such that

$$u_1(t) \le c_2 b^* \quad \text{on } [s_{b^*}, R^*].$$

Evaluating this at s_{b^*} give $0 < \beta \le c_2 b^* \to 0$ as $b^* \to 0$ which is a contradiction. Thus we see that if $b^* > 0$ is sufficiently small then $0 < u_1 < \beta$ on $(0, R^*)$. □

Lemma 2.2. *If b^* is sufficiently large then u_1 has a local maximum, M_{b^*}, and $M_{b^*} \to R^*$ as $b^* \to \infty$.*

Proof. Using (2.13) we see that if

$$F(u_1) \le \frac{1}{4} \frac{(b^*)^2}{h(R^*)}, \quad \text{then } \frac{u_1'^2}{h(t)} \ge \frac{1}{2} \frac{(b^*)^2}{h(R^*)}. \tag{2.23}$$

In particular, in a neighborhood of $t = R^*$ we have $F(u_1) \le \frac{1}{4} \frac{(b^*)^2}{h(R^*)}$ since $F(u_1(R^*)) = 0$. Also since $u_1' < 0$ near $t = R^*$ then from (2.23):

$$-u_1' \ge \frac{b^* \sqrt{h(t)}}{\sqrt{2h(R^*)}} \quad \text{on } (t, R^*) \text{ with } t \text{ near } R^*.$$

Integrating this on (t, R^*) gives

$$u_1(t) \ge \frac{b^*}{\sqrt{2h(R^*)}} \int_t^{R^*} \sqrt{h(s)}\, ds \quad \text{when } F(u_1) \le \frac{1}{4} \frac{(b^*)^2}{h(R^*)}. \tag{2.24}$$

Now from (H2)-(H3) it follows that there is a $c_3 > 0$ such that $F(u_1) \ge \frac{1}{2(p+1)} |u_1|^{p+1} - c_3$ for all $u_1 \in \mathbb{R}$. From this and (2.23)-(2.24) we see that

$$\frac{1}{2(p+1)} \Big(\frac{b^*}{\sqrt{2h(R^*)}} \int_t^{R^*} \sqrt{h(s)}\, ds \Big)^{p+1} - c_3 \le F(u_1) \le \frac{(b^*)^2}{4h(R^*)}.$$

Rewriting this gives

$$\int_t^{R^*} \sqrt{h(s)}\, ds \le \Big[2(p+1) \Big(\frac{c_3}{(b^*)^{p+1}} + \frac{1}{4h(R^*)(b^*)^{p-1}} \Big) \Big]^{\frac{1}{p+1}} \sqrt{2h(R^*)}. \tag{2.25}$$

Since $p > 1$, the right-hand side of (2.25) approaches 0 as $b^* \to \infty$. Since $\int_0^{R^*} \sqrt{h(s)}\, ds > 0$ we see that $F(u_1(t))$ cannot be bounded by $\frac{1}{4}(b^*)^2 h(R^*)$ for all $t \in [0, R^*]$ and for all sufficiently large b^*. Thus for sufficiently large b^* there exists $t_{b^*} \in (0, R^*)$ such that

$$F(u_1(t_{b^*})) = \frac{(b^*)^2}{4h(R^*)} \tag{2.26}$$

where $0 < u_1 < u_1(t_{b^*})$ on (t_{b^*}, R^*).

Now evaluating (2.25) at $t = t_{b^*}$ and noticing the right-hand side of (2.25) goes to 0 as $b^* \to \infty$ it follows that

$$t_{b^*} \to R^* \text{ as } b^* \to \infty. \tag{2.27}$$

We also note that from (H2) and (H3), there is a $c_4 \geq 1$ such that $F(u_1) \leq \frac{c_4}{p+1}|u_1|^{p+1}$ for all $u_1 \in \mathbb{R}$. From this and (2.26) we see that

$$\frac{c_4}{p+1} u_1^{p+1}(t_{b^*}) \geq F(u_1(t_{b^*})) = \frac{(b^*)^2}{4h(R^*)} \tag{2.28}$$

and so

$$u_1(t_{b^*}) \geq c_5 (b^*)^{\frac{2}{p+1}} \quad \text{where } c_5 = \Big(\frac{(p+1)}{4h(R^*)c_4}\Big)^{\frac{1}{p+1}} > 0. \tag{2.29}$$

Suppose now that u_1 does not have a local maximum for b^* sufficiently large so that $u_1' < 0$ on $(0, R^*)$ for large b^*.

We then define

$$Q(b^*) = \frac{1}{2} \inf_{[\frac{1}{2}t_{b^*}, t_{b^*}]} h(t) \frac{f(u_1)}{u_1}.$$

Since $t_{b^*} \to R^*$ as $b^* \to \infty$ by (2.27) it follows that the interval $[\frac{1}{2}t_{b^*}, t_{b^*}]$ is bounded from below by a positive constant as $b^* \to \infty$ and so $h(t)$ is bounded from below on $[\frac{1}{2}t_{b^*}, t_{b^*}]$ by a positive constant for large values of b^*. In addition, since u_1 is decreasing on $[\frac{1}{2}t_{b^*}, t_{b^*}]$ then by (2.29),

$$u_1(t) \geq u_1(t_{b^*}) \geq c_5 (b^*)^{\frac{2}{p+1}} \quad \text{on } [\frac{1}{2}t_{b^*}, t_{b^*}] \tag{2.30}$$

and since $\frac{f(u_1)}{u_1} \to \infty$ as $u_1 \to \infty$ by (H2) it follows that

$$Q(b^*) \to \infty \text{ as } b^* \to \infty. \tag{2.31}$$

We now compare the solution of (2.8), i.e.,

$$u_1'' + \big[h(t)\frac{f(u_1)}{u_1}\big]u_1 = 0, \tag{2.32}$$

with the solution of

$$v_1'' + Q(b^*)v_1 = 0, \tag{2.33}$$

where $v_1(t_{b^*}) = u_1(t_{b^*}) > 0$ and $v_1'(t_{b^*}) = u_1'(t_{b^*}) < 0$. Since the general solution of (2.33) is $v_1 = c_6 \sin(\sqrt{Q(b^*)}(t - c_7))$ for some constants $c_6 \neq 0$ and c_7 we see that any interval of length $\frac{\pi}{\sqrt{Q(b^*)}}$ has a zero of v_1. And since $t_{b^*} \to R^*$ as $b^* \to \infty$ by (2.27), it follows from (2.31) that v_1 is zero somewhere on $[\frac{1}{2}t_{b^*}, t_{b^*}]$ since $\frac{\pi}{\sqrt{Q(b^*)}} < \frac{1}{2}t_{b^*}$ for b^* sufficiently large.

In particular, v_1 must have a local maximum, m_{b^*}, with $m_{b^*} \geq \frac{1}{2}t_{b^*}$, $v_1' < 0$ on $(m_{b^*}, t_{b^*}]$, and $v_1 > 0$ on $[m_{b^*}, t_{b^*}]$. We claim now that u_1 also has a local maximum on $(m_{b^*}, t_{b^*}]$ for b^* sufficiently large. So suppose not then $u_1' < 0$ and $u_1 > 0$ on $(m_{b^*}, t_{b^*}]$. Multiplying (2.32) by v_1, multiplying (2.33) by u_1, and subtracting we obtain

$$(v_1 u_1' - u_1 v_1')' + \Big(h(t)\frac{f(u_1)}{u_1} - Q(b^*)\Big)u_1 v_1 = 0.$$

Integrating this on $[m_{b^*}, t_{b^*}]$ gives

$$-v_1(m_{b^*})u_1'(m_{b^*}) + \int_{m_{b^*}}^{t_{b^*}} \Big(h(t)\frac{f(u_1)}{u_1} - Q(b^*)\Big)u_1 v_1 \, dt = 0. \tag{2.34}$$

We note $v_1(m_{b^*}) > 0$ and that both u_1 and v_1 are positive on $[m_{b^*}, t_{b^*}]$. Since $h(t)\frac{f(u_1)}{u_1} - Q(b^*) > 0$ on $[m_{b^*}, t_{b^*}]$, it follows from (2.34) that $u_1'(m_{b^*}) > 0$ which contradicts that $u_1' < 0$ on $[m_{b^*}, t_{b^*}]$. So we see that u_1 must also have a local maximum, M_{b^*}, with $M_{b^*} > m_{b^*}$ and $u_1' < 0$ on $(M_{b^*}, R^*]$. This completes the first part of the proof.

Next we show $M_{b^*} \to R^*$ as $b^* \to \infty$. Integrating (2.8) on (M_{b^*}, t) gives

$$-u_1'(t) = \int_{M_{b^*}}^{t} h(s) f(u_1)\, ds. \tag{2.35}$$

Now since $f(u_1) \ge \frac{1}{2} u_1^p$ when $u_1 > 0$ is large (by (H2)) and since u_1 is decreasing on (M_{b^*}, R^*) then when b^* is sufficiently large and when $M_{b^*} < t < t_{b^*}$ then $u_1(t) \ge u_1(t_{b^*}) \to \infty$ as $b^* \to \infty$ by (2.29) so we obtain from (2.35):

$$-u_1'(t) \ge \frac{1}{2} u_1^p(t) \int_{M_{b^*}}^{t} h(s)\, ds.$$

Dividing by u_1^p, integrating on (M_{b^*}, t_{b^*}), and estimating gives

$$\frac{1}{(p-1) u_1^{p-1}(t_{b^*})} \ge \frac{1}{2} \int_{M_{b^*}}^{t_{b^*}} \int_{M_{b^*}}^{s} h(x)\, dx\, ds. \tag{2.36}$$

Now the left-hand side of (2.36) goes to 0 as $b^* \to \infty$ by (2.30) thus we see from (2.36) that $t_{b^*} - M_{b^*} \to 0$ as $b^* \to \infty$. Also from (2.27) we know that $t_{b^*} \to R^*$ as $b^* \to \infty$. Therefore, combining these two statements we see $M_{b^*} \to R^*$ as $b^* \to \infty$. This completes the proof. □

Lemma 2.3. *If b^* is sufficiently large then u_1 has an arbitrarily large number of zeros on $(0, R^*)$.*

Proof. From Lemma 2.2 we know u_1 has a local maximum, M_{b^*}, with $M_{b^*} \to R^*$ as $b^* \to \infty$. Recalling (2.6) it follows that $u(r) = u_1(r^{2-N})$ has a local maximum, M_b, and

$$M_b \to R \quad \text{as } b \to \infty. \tag{2.37}$$

Now we let

$$w_\lambda(r) = \lambda^{-\frac{2}{p-1}} u(M_b + \frac{r}{\lambda})$$

where $\lambda^{\frac{2}{p-1}} = u(M_b)$. Then

$$\begin{gathered} w_\lambda'' + \frac{N-1}{\lambda M_b + r} w_\lambda' + K(M_b + \frac{r}{\lambda}) \lambda^{\frac{-2p}{p-1}} f(\lambda^{\frac{2}{p-1}} w_\lambda) = 0, \\ w_\lambda(0) = 1, w_\lambda'(0) = 0. \end{gathered} \tag{2.38}$$

Since $K'(r) < 0$ and $F(u) \ge -F_0$ for some $F_0 > 0$ (by (H3)), we see that

$$\begin{aligned} &\Big(\frac{1}{2} w_\lambda'^2 + K(M_b + \frac{r}{\lambda}) \lambda^{\frac{-2(p+1)}{p-1}} F(\lambda^{\frac{2}{p-1}} w_\lambda)\Big)' \\ &= -\Big(\frac{N-1}{\lambda M_b + r}\Big) w_\lambda'^2 + \lambda^{\frac{-2(p+1)}{p-1} - 1} K'(M_b + \frac{r}{\lambda}) F(\lambda^{\frac{2}{p-1}} w_\lambda) \\ &\le -\lambda^{\frac{-2(p+1)}{p-1} - 1} K'(M_b + \frac{r}{\lambda}) F_0. \end{aligned}$$

Integrating this on $(0, r)$ gives

$$\begin{aligned} &\frac{1}{2} w_\lambda'^2 + K(M_b + \frac{r}{\lambda}) \lambda^{\frac{-2(p+1)}{p-1}} F(\lambda^{\frac{2}{p-1}} w_\lambda) \\ &\le K(M_b) \lambda^{\frac{-2(p+1)}{p-1}} F(\lambda^{\frac{2}{p-1}}) - \lambda^{\frac{-2(p+1)}{p-1}} F_0 \big[K(M_b + \frac{r}{\lambda}) - K(M_b)\big]. \end{aligned} \tag{2.39}$$

Since K is bounded on $[R,\infty)$ it follows that

$$\lambda^{\frac{-2(p+1)}{p-1}} F_0\big[K(M_b+\frac{r}{\lambda}) - K(M_b)\big] \to 0 \quad \text{as } \lambda\to\infty.$$

Also from (H2) and (H3) it follows that $F(\lambda^{\frac{2}{p-1}}) = \frac{1}{p+1}\lambda^{\frac{2(p+1)}{p-1}} + G(\lambda^{\frac{2}{p-1}})$ where $G(u) = \int_0^u g(s)\,ds$ and thus by (H2) and L'Hôpital's rule $|\frac{G(u)}{u^{p+1}}| \to 0$ as $u\to\infty$. Therefore

$$\lambda^{\frac{-2(p+1)}{p-1}} F(\lambda^{\frac{2}{p-1}}) = \frac{1}{p+1} + \lambda^{\frac{-2(p+1)}{p-1}} G(\lambda^{\frac{2}{p-1}}) \to \frac{1}{p+1} \quad \text{as } \lambda\to\infty.$$

Also by (H2) and(H3) we see that

$$\lambda^{\frac{-2(p+1)}{p-1}} F(\lambda^{\frac{2}{p-1}} w_\lambda) = \frac{1}{p+1} w_\lambda^{p+1} + \lambda^{\frac{-2(p+1)}{p-1}} G(\lambda^{\frac{2}{p-1}} w_\lambda).$$

Then by (2.39) for sufficiently large λ,

$$\frac12 w_\lambda'^2 + K(M_b+\frac{r}{\lambda})\frac{1}{p+1}|w_\lambda|^{p+1} \le \frac{K(R)}{p+1} + 1 - \lambda^{-\frac{2(p+1)}{p-1}} G(\lambda^{\frac{2}{p-1}} w_\lambda). \tag{2.40}$$

Since $|\frac{G(u)}{u^{p+1}}| \to 0$ as $u\to\infty$ it follows that $|G(u)| \le \frac{1}{2(p+1)}|u|^{p+1}$ for $|u|\ge A$ where A is some positive constant and $|G(u)| \le G_0$ for $|u|\le A$ since G is continuous. Thus $|G(u)| \le \frac{1}{2(p+1)}|u|^{p+1} + G_0$ for all u and therefore from (2.40):

$$\frac12 w_\lambda'^2 + K(M_b+\frac{r}{\lambda})\frac{|w_\lambda|^{p+1}}{p+1} \le \frac{K(R)}{p+1} + 1 + K(M_b+\frac{r}{\lambda})\Big(\frac{|w_\lambda|^{p+1}}{2(p+1)} + \lambda^{-\frac{2(p+1)}{p-1}} G_0\Big).$$

Therefore, for sufficiently large λ and since K is bounded we have

$$\frac12 w_\lambda'^2 + K(M_b+\frac{r}{\lambda})\frac{|w_\lambda|^{p+1}}{2(p+1)} \le \frac{K(R)}{p+1} + 2.$$

Thus we see that $|w_\lambda|$ and $|w_\lambda'|$ are uniformly bounded on $[R,\infty)$ for large λ. So by the Arzela-Ascoli theorem a there is a subsequence (still labeled w_λ) such that $w_\lambda \to w$ uniformly on compact sets. Also, since w_λ' is uniformly bounded it follows that $\frac{w_\lambda'}{\lambda M_b + r} \to 0$ as $\lambda\to\infty$. In addition, from (H2) we have

$$K(M_b+\frac{r}{\lambda})\lambda^{\frac{-2p}{p-1}} f(\lambda^{\frac{2}{p-1}} w_\lambda) = K(M_b+\frac{r}{\lambda})[w_\lambda^p + \lambda^{\frac{-2p}{p-1}} g(\lambda^{\frac{2}{p-1}} w_\lambda)].$$

Since $M_b \to R$ by Lemma 2.2 then $K(M_b+\frac{r}{\lambda})w_\lambda^p \to K(R)w^p$ uniformly on compact sets. And since $\frac{g(u)}{u^p} \to 0$ as $u\to\infty$ by (H2) it follows that $K(M_b + \frac{r}{\lambda})\lambda^{\frac{-2p}{p-1}} g(\lambda^{\frac{2}{p-1}} w_\lambda) \to 0$ uniformly on compact sets as $\lambda\to\infty$. It follows then from (2.38) that $|w_\lambda''|$ is uniformly bounded. Then by the Arzela-Ascoli theorem we see for some subsequence (still labeled w_λ) that $w_\lambda \to w$ and $w_\lambda' \to w'$ uniformly on compact sets as $\lambda\to\infty$ and then from (2.38) we see that w satisfies

$$w'' + K(R)|w|^{p-1}w = 0,$$
$$w(0) = 1, w'(0) = 0.$$

Now it is straightforward to show that this has infinitely many zeros on $[0,\infty)$ and therefore w_λ and hence u has an arbitrarily large number of zeros on (R,∞) provided b is chosen sufficiently large. Also it follows that u_1 has an arbitrarily large number of zeros provided b^* is chosen sufficiently large. This completes the proof. $\square$

3. Proof of the main theorem

From Lemma 2.3 we see that the set

$$\{b^* : u_1(r, b^*) \text{ has at least one zero on } (0, R^*)\}$$

is nonempty. And since $0 < u_1(r, b^*) < \beta$ on $(0, R^*)$ for $b^* > 0$ sufficiently small by Lemma 2.2 then we see that this set is bounded from below by a positive constant. So we let

$$b_0^* = \inf\{b^* : u_1(r, b^*) \text{ has at least one zero on } 0 < t < R^*\}$$

and note that $b_0^* > 0$. In addition, it follows by continuity with respect to initial conditions that $u_1(r, b_0^*) \geq 0$ on $(0, R^*)$. We claim next that $u_1(r, b_0^*) > 0$ for $0 < t < R^*$. If not then there is a z with $0 < z < R^*$ such that $u_1(z, b_0^*) = 0$. Since $u_1(r, b_0^*) \geq 0$ it follows that $u_1'(z, b_0^*) = 0$. This however implies $u_1 \equiv 0$ contradicting $u_1'(R^*, b_0^*) = -b_0^* < 0$. Thus it must be that $u_1(t, b_0^*) > 0$ for $0 < t < R^*$. Also, for $b^* > b_0^*$ then by definition of b_0 there is a z_{b^*} such that $u_1(z_{b^*}, b_0^*) = 0$. It follows that $z_{b^*} \to 0$ as $b^* \to (b_0^*)^+$ otherwise a subsequence of these would converge to a z_0 with $0 < z_0 \leq R^*$ such that $u_1(z_0, b_0^*) = 0$. Since $b_0^* > 0$ it follows that $u_1'(R^*, b_0^*) = -b_0^* < 0$ and so $z_0 < R^*$ but then this contradicts that $u_1(r, b_0^*) > 0$ for $0 < t < R^*$. Thus $z_{b^*} \to 0$ as $b^* \to (b_0^*)^+$. Then $0 = u_1(z_{b^*}, b^*) \to u_1(0, b_0^*)$ as $b^* \to (b_0^*)^+$ thus we see that $u_1(0, b_0^*) = 0$. Thus $u_1(t, b_0^*)$ is a positive solution of (2.8)-(2.9). Now if we let $b_0 = \frac{(N-2)b_0^*}{R^{N-1}}$ then it follows that $u(r, b_0)$ is a positive solution of (2.1)–(2.2) and $\lim_{r\to\infty} u(r, b_0) = 0$.

Next by Lemma 2.3 we see that the set

$$\{b^* : u_1(t, b^*) \text{ has at least two zeros on } 0 < t < R^*\}$$

is nonempty and from Lemma 2.1 this set is bounded from below. And so we let

$$b_1^* = \inf\{b^* : u_1(r, b^*) \text{ has at least two zeros on } 0 < t < R^*\}.$$

By [7, Lemma 2.7] it follows that if b is close to b_0 then $u(r, b)$ has at most one zero on (R, ∞) and consequently $u_1(t, b^*)$ has at most zero on $(0, R^*)$ if b^* is close to b_0^*. Therefore $b_0^* < b_1^*$. It can then be shown that $u_1(t, b_1^*)$ has exactly one zero on $(0, R^*)$ and $u_1(0, b_1^*) = 0$. So if we let $b_1 = \frac{(N-2)b_1^*}{R^{N-1}}$ then $u(r, b_1)$ is a solution of (2.1)–(2.2) with $\lim_{r\to\infty} u(r, b_1) = 0$ with exactly one zero on (R, ∞).

Similarly it can be shown that there is a solution, u_n, of (2.1)–(2.2) such that $\lim_{r\to\infty} u(r, b_n) = 0$ and with n interior zeros on (R, ∞) where n is any nonnegative integer. This completes the proof.

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