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TRANSITION FRONTS OF TWO SPECIES COMPETITION LATTICE SYSTEMS IN RANDOM MEDIA

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Abstract. This article studies the existence and non-existence of transition fronts for a two species competition lattice system in random media, and explores the influence of randomness of the media on the wave profiles and wave speeds of such transition fronts. We first establish comparison principle for sub-solutions and super-solutions of the related cooperative system. Next, under some proper assumptions, we construct appropriate sub-solutions and super-solutions for the cooperative system. Finally, we show that random transition fronts exist if their least mean speed is greater than an explicit threshold and there is no random transition front with least mean speed less than the threshold.

1. Introduction

This article studies the existence of transition fronts of the two species competition lattice random system

$$\begin{aligned}\dot{u}_i(t) &= u_{i+1}(t) - 2u_i(t) + u_{i-1}(t) + u_i(t)(a_1(\theta_t\omega) \\ &\quad - b_1(\theta_t\omega)u_i(t) - c_1(\theta_t\omega)v_i(t)), \\ \dot{v}_i(t) &= v_{i+1}(t) - 2v_i(t) + v_{i-1}(t) + v_i(t)(a_2(\theta_t\omega) \\ &\quad - b_2(\theta_t\omega)u_i(t) - c_2(\theta_t\omega)v_i(t)),\end{aligned} \tag{1.1}$$

where $i \in \mathbb{Z}$, $t \in \mathbb{R}$, $\omega \in \Omega$, $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space, θ_t is an ergodic metric dynamical system on Ω, $a_i(\cdot) : \Omega \to \mathbb{R}$, $b_i(\cdot) : \Omega \to (0, \infty)$, $c_i(\cdot) : \Omega \to (0, \infty)$ $(i = 1, 2)$ are measurable, and for every $\omega \in \Omega$, $a_i^\omega(t) := a_i(\theta_t\omega)$, $b_i^\omega(t) := b_i(\theta_t\omega)$, $c_i^\omega(t) := c_i(\theta_t\omega)$ $(i = 1, 2)$ are locally Hölder continuous in $t \in \mathbb{R}$. Moreover, we assume $b_i(\theta_t\omega) > 0$, $c_i(\theta_t\omega) > 0$ $(i = 1, 2)$ for every $\omega \in \Omega$ and $t \in \mathbb{R}$.

System (1.1) is a spatial-discrete counterpart of the following two species competition system with random dispersal,

$$\begin{aligned}\partial_t u &= u_{xx} + u(a_1(\theta_t\omega) - b_1(\theta_t\omega)u - c_1(\theta_t\omega)v), \\ \partial_t v &= v_{xx} + v(a_2(\theta_t\omega) - b_2(\theta_t\omega)u - c_2(\theta_t\omega)v),\end{aligned} \tag{1.2}$$

Systems (1.1) and (1.2) are widely used to model the population dynamics of competitive species when the movement or internal dispersal of the organisms occurs between non-adjacent and adjacent locations, respectively (see, for example,

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[6, 21, 25, 26]). Note that system (1.2) often models the evolution of population densities of competitive species in which the internal interaction or movement of the organisms occurs randomly between adjacent spatial locations and is described by the differential operator, referred to as the *random dispersal operator*. System (1.1) arises in modeling the evolution of population densities of competitive species in which the internal interaction or movement of the organisms occurs between non-adjacent spatial locations and is described by the difference operator, referred to as the *discrete dispersal operator*.

In (1.1) and (1.2), the functions a_1, a_2 represent the respective growth rates of the two species, b_1, c_2 account for self-regulation of the respective species, and c_1, b_2 account for competition between the two species. Two of the central dynamical issues about (1.1) and (1.2) are spatial spreading speeds and traveling wave solutions. A huge amount of research has been carried out toward the spatial spreading speeds and traveling wave solutions of system (1.2) in spatially and temporally homogeneous media (see, for example, [7, 8, 13, 14, 15, 16, 17, 19, 20, 28]) or spatially and/or temporally periodic media (see, for example, [9, 18, 29]). Recently, Bao, Li, Shen and Wang in [2] studied the spatial spreading speeds and linear determinacy of diffusive cooperative/competitive system in time recurrent environments. Bao in [1] studied the spatial spreading speeds and generalized traveling waves of competition system in general time heterogeneous media.

As for the lattice system arising in competition models, to the best of our knowledge, there are only a few works on the related topics. The reader is referred to [11, 12, 27] for the study on the spatial spreading speeds and traveling wave solutions for competition lattice system in time independent habitats. We note that Cao and Gao in [3] studied the existence and stability of random transition fronts for KPP-type one species lattice random equations. The reader is referred to [4, 5, 10, 24, 30] for the study on the spatial spreading speeds and traveling wave solutions for KPP-type one species lattice equations in homogeneous or periodic or time heterogeneous media.

In this article we study the traveling wave solutions of two species competition lattice system with general time dependence. Since in nature, many systems are subject to irregular influences arisen from various kind of noise, it is of great importance to take the randomness of the environment into account and study the existence and non-existence of random transition fronts of competition lattice system in random media. Due to the lack of space regularity, we need finding new approach to get the existence of transition fronts when dealing with spatial-discrete system (1.1). We point out that the method used here can also be used to get the existence and non-existence of transition fronts for two species competition lattice system in general time dependent habitats. Besides, we will study the stability of random transition fronts of competition lattice system elsewhere.

Let

$$l^\infty(\mathbb{Z}) = \{u = \{u_i\}_{i\in\mathbb{Z}} : \sup_{i\in\mathbb{Z}} |u_i| < \infty\}$$

with norm $\|u\| = \|u\|_\infty = \sup_{i\in\mathbb{Z}} |u_i|$, and

$$l^{\infty,+}(\mathbb{Z}) = \{u \in l^\infty(\mathbb{Z}) : \inf_{i\in\mathbb{Z}} u_i \geq 0\}.$$

For $u, v \in l^\infty(\mathbb{Z})$, we define

$$u \geq v \quad \text{if} \quad u - v \in l^{\infty,+}(\mathbb{Z}).$$

Then for any given $(u_0, v_0) \in l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$, (1.1) has a unique (local) solution $(u(t; u_0, v_0, \omega), v(t; u_0, v_0, \omega)) = \{(u_i(t; u_0, v_0, \omega), v_i(t; u_0, v_0, \omega))\}_{i\in\mathbb{Z}}$ with initial datum $(u(0; u_0, v_0, \omega), v(0; u_0, v_0, \omega)) = (u_0, v_0)$. Note that, if $u_0 \in l^{\infty,+}(\mathbb{Z})$, $v_0 \in l^{\infty,+}(\mathbb{Z})$, then $(u(t; u_0, v_0, \omega), v(t; u_0, v_0, \omega))$ exists for all $t \geq 0$ and $u(t; u_0, v_0, \omega) \in l^{\infty,+}(\mathbb{Z})$, $v(t; u_0, v_0, \omega) \in l^{\infty,+}(\mathbb{Z})$ for all $t \geq 0$. A solution $(u(t; \omega), v(t; \omega)) = \{u_i(t; \omega), v_i(t; \omega)\}_{i\in\mathbb{Z}}$ of (1.1) is called *spatially homogeneous* if $u_i(t) = u_j(t)$ and $v_i(t) = v_j(t)$ for all $i, j \in \mathbb{Z}$.

Note that (1.1) contains the following two sub-systems,

$$\dot{u}_i(t) = u_{i+1}(t) - 2u_i(t) + u_{i-1}(t) + u_i(t)(a_1(\theta_t\omega) - b_1(\theta_t\omega)u_i(t)), \tag{1.3}$$

and

$$\dot{v}_i(t) = v_{i+1}(t) - 2v_i(t) + v_{i-1}(t) + v_i(t)(a_2(\theta_t\omega) - c_2(\theta_t\omega)v_i(t)). \tag{1.4}$$

First we give some notation and assumptions related to (1.1). Let

$$\underline{a}(\omega) = \liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau := \lim_{r\to\infty}\inf_{t-s\geq r}\frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau,$$
$$\overline{a}(\omega) = \limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau := \lim_{r\to\infty}\sup_{t-s\geq r}\frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau,$$

where $a(\omega)$ could be $a_i(\omega)$, $b_i(\omega)$, $c_i(\omega)$ ($i = 1$ or 2) or any similar function. We call $\underline{a}(\cdot)$ and $\overline{a}(\cdot)$ the least mean and the greatest mean of $a(\cdot)$, respectively. It is easy to obtain

$$\underline{a}(\theta_t\omega) = \underline{a}(\omega), \quad \overline{a}(\theta_t\omega) = \overline{a}(\omega) \quad \text{for all } t \in \mathbb{R},$$

and

$$\underline{a}(\omega) = \liminf_{t,s\in\mathbb{Q},t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau, \quad \overline{a}(\omega) = \limsup_{t,s\in\mathbb{Q},t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau.$$

Then $\underline{a}(\omega)$ and $\overline{a}(\omega)$ are measurable in ω. The ergodicity of the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ implies that, there are $\underline{a}, \overline{a} \in \mathbb{R}$ and a measurable subset $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$\theta_t\Omega_0 = \Omega_0 \quad \forall t \in \mathbb{R}$$
$$\liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau = \underline{a} \quad \forall \omega \in \Omega_0$$
$$\limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t a(\theta_\tau\omega)d\tau = \overline{a} \quad \forall \omega \in \Omega_0,$$

That is, $\underline{a}(\omega)$ and $\overline{a}(\omega)$ are independent of ω in a subset of Ω of full measure (see Lemma 2.3).

Throughout this paper, we assume that the trivial solution $(0, 0)$ of (1.1) is unstable with respect to perturbation in $l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$, i.e.

(H1) $\underline{a_i}(\omega) = \liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t a_i(\theta_\tau\omega)d\tau > 0$ ($i = 1, 2$) for a.a. $\omega \in \Omega$.

Note that (H1) implies that (1.1) has two semi-trivial spatially homogeneous positive solutions $(u^*(t; \omega), 0) = (\phi^*(\theta_t\omega), 0) \in \operatorname{Int} l^{\infty,+}(\mathbb{Z}) \times l^{\infty,+}(\mathbb{Z})$ and $(0, v^*(t; \omega)) = (0, \psi^*(\theta_t\omega)) \in l^{\infty,+}(\mathbb{Z}) \times \operatorname{Int} l^{\infty,+}(\mathbb{Z})$ for some random equilibria ϕ^* and ψ^*, where $u^*(t; \omega) = \phi^*(\theta_t\omega)$ is the unique spatially homogeneous positive solution of (1.3), and $v^*(t; \omega) = \psi^*(\theta_t\omega)$ is the unique spatially homogeneous positive solution of (1.4) (see [4, Theorem 1.1] and [22, Theorem A]).

We also assume that

(H2) $(0, v^*(t;\omega))$ is linearly unstable in $l^{\infty,+}(\mathbb{Z}) \times l^{\infty,+}(\mathbb{Z})$, that is,
$$\underline{a_1(\omega) - c_1(\omega)v^*(\cdot;\omega)} > 0.$$

Note that $(u^*(t;\omega), 0)$ is linearly and globally stable in $l^{\infty,+}(\mathbb{Z}) \times l^{\infty,+}(\mathbb{Z})$, that is,
$$\overline{a_2(\omega) - b_2(\omega)u^*(\cdot;\omega)} < 0,$$
and for any $(u_0, v_0) \in l^{\infty,+}(\mathbb{Z}) \times l^{\infty,+}(\mathbb{Z})$ with $u_0 \neq 0$ and a.a. $\omega \in \Omega$, $u_i(t; u_0, v_0, \theta_{t_0}\omega) - u^*(t + t_0; \omega) \to 0$ and $v_i(t; u_0, v_0, \theta_{t_0}\omega) \to 0$ as $t \to \infty$ uniformly in $i \in \mathbb{Z}$ and $t_0 \in \mathbb{R}$.

We remark that if $\underline{a_1(\omega) - c_1(\omega)v^*(\cdot;\omega)} > 0$, then $(0, v^*(t;\omega))$ is unstable in $l^{\infty,+}(\mathbb{Z}) \times l^{\infty,+}(\mathbb{Z})$, and if $\overline{a_2(\omega) - b_2(\omega)u^*(\cdot;\omega)} < 0$, then $(u^*(t;\omega), 0)$ is locally stable in $l^{\infty,+}(\mathbb{Z}) \times l^{\infty,+}(\mathbb{Z})$, and if $\underline{a_i}(\omega) > 0$ $(i = 1, 2)$, $a^\omega_{1L} > \frac{c^\omega_{1M} a^\omega_{2M}}{c^\omega_{2L}}$ and $a^\omega_{2M} \leq \frac{a^\omega_{1L} b^\omega_{2L}}{b^\omega_{1M}}$ for any $\omega \in \Omega$, then $(u^*(t;\omega), 0)$ is globally stable and $(0, v^*(t;\omega))$ is unstable in $l^{\infty,+}(\mathbb{Z}) \times l^{\infty,+}(\mathbb{Z})$, where $a^\omega_{iL} = \inf_{t\in\mathbb{R}} a_i(\theta_t\omega)$, $a^\omega_{iM} = \sup_{t\in\mathbb{R}} a_i(\theta_t\omega)$ and b^ω_{iL}, b^ω_{iM}, c^ω_{iL}, c^ω_{iM} are defined similarly (This can be proved similarly as [1, Proposition 2.4]).

Now we present the third standing hypothesis.

(H3) For any $\omega \in \Omega$, $\inf_{t\in\mathbb{R}} b_2(\theta_t\omega) > 0$, $b_i(\theta_t\omega) \geq c_i(\theta_t\omega)$ $(i = 1, 2)$ and
$$a_1(\theta_t\omega) - c_1(\theta_t\omega)v^*(t;\omega) \geq a_2(\theta_t\omega) - 2c_2(\theta_t\omega)v^*(t;\omega) + b_2(\theta_t\omega)v^*(t;\omega)$$
for $t \in \mathbb{R}$.

Under the assumptions (H1)–(H3), one of the most interesting dynamical problems is to study the existence of random transition front (generalized traveling wave) solutions connecting $(u^*(t;\omega), 0)$ and $(0, v^*(t;\omega))$ for (1.1). To do so, we first transform (1.1) to a cooperative system via the standard change of variables,
$$\tilde{u}_i = u_i, \quad \tilde{v}_i = v^*(t;\omega) - v_i.$$
Dropping the tilde, (1.1) is transformed into
$$\begin{aligned} \dot{u}_i &= Hu_i + u_i(a_1(\theta_t\omega) - b_1(\theta_t\omega)u_i - c_1(\theta_t\omega)(v^*(t;\omega) - v_i)),\\ \dot{v}_i &= Hv_i + b_2(\theta_t\omega)(v^*(t;\omega) - v_i)u_i + v_i(a_2(\theta_t\omega)\\ &\quad - 2c_2(\theta_t\omega)v^*(t;\omega) + c_2(\theta_t\omega)v_i), \end{aligned} \tag{1.5}$$
where
$$Hu_i(t) := u_{i+1}(t) - 2u_i(t) + u_{i-1}(t), \quad i \in \mathbb{Z},\ t \in \mathbb{R}.$$

It is clear that (1.5) is cooperative in the region $u_i(t) \geq 0$ and $0 \leq v_i(t) \leq v^*(t;\omega)$, and the trivial solution $(0, 0)$ of (1.1) becomes $(0, v^*(t;\omega))$, the semitrivial solutions $(0, v^*(t;\omega))$ and $(u^*(t;\omega), 0)$ of (1.1) becomes $(0, 0)$ and $(u^*(t;\omega), v^*(t;\omega))$, respectively. To study the random transition front solutions of (1.1) connecting $(u^*(t;\omega), 0)$ and $(0, v^*(t;\omega))$ is then equivalent to study the random transition front solutions of (1.5) connecting $(u^*(t;\omega), v^*(t;\omega))$ and $(0, 0)$.

We denote $(u(t; u^0, v^0, \omega), v(t; u^0, v^0, \omega)) = \{(u_i(t; u^0, v^0, \omega), v_i(t; u^0, v^0, \omega))\}_{i\in\mathbb{Z}}$ as the solution of (1.5) with $(u(0; u^0, v^0, \omega), v(0; u^0, v^0, \omega)) = (u^0, v^0) \in l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$. For any $(u^1, u^2) \in l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$ and $(v^1, v^2) \in l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$, the relation $(u^1, u^2) < (v^1, v^2)$ ($(u^1, u^2) \leq (v^1, v^2)$ resp.) is to be understood componentwise: $u^i < v^i$ ($u^i \leq v^i$) for each i. Other relations like "max", "min", "sup", "inf" can be similarly understood. Then it is clear that, if $(u^0, v^0) \geq (0, 0)$, then

$(u(t;u^0,v^0,\omega),v(t;u^0,v^0,\omega))$ exists for all $t\geq 0$ and $(u(t;u^0,v^0,\omega),v(t;u^0,v^0,\omega))\geq (0,0)$ for all $t\geq 0$ (see Proposition 2.1). A solution $(u(t;\omega),v(t;\omega))$ of (1.5) is called an *entire solution* if it is a solution of (1.5) for $t\in\mathbb{R}$.

Definition 1.1 (Random transition front)**.** An entire solution $(u(t;\omega),v(t;\omega))$ is called a *random transition front* or a *random generalized traveling wave* of (1.5) connecting $(0,0)$ and $(u^*(t;\omega),v^*(t;\omega))$ if for a.a. $\omega\in\Omega$,

$$(u_i(t;\omega),v_i(t;\omega))=(\Phi(i-\int_0^t c(s;\omega)ds,\theta_t\omega),\Psi(i-\int_0^t c(s;\omega)ds,\theta_t\omega))$$

for some $\Phi(x,\omega)$, $\Psi(x,\omega)$ $(x\in\mathbb{R})$ and $c(t;\omega)$, where $\Phi(x,\omega)$, $\Psi(x,\omega)$ and $c(t;\omega)$ are measurable in ω, and for a.a. $\omega\in\Omega$,

$$\begin{gathered}(0,0)<(\Phi(x,\omega),\Psi(x,\omega))<(u^*(t;\omega),v^*(t;\omega)),\\ \lim_{x\to-\infty}(\Phi(x,\theta_t\omega),\Psi(x,\theta_t\omega))=(u^*(t;\omega),v^*(t;\omega)),\\ \lim_{x\to\infty}(\Phi(x,\theta_t\omega),\Psi(x,\theta_t\omega))=(0,0)\quad\text{uniformly in } t\in\mathbb{R}.\end{gathered}$$

Suppose that $(u(t;\omega),v(t;\omega))=\{(u_i(t;\omega),v_i(t;\omega))\}_{i\in\mathbb{Z}}$ with $(u_i(t;\omega),v_i(t;\omega))=(\Phi(i-\int_0^t c(s;\omega)ds,\theta_t\omega),\Psi(i-\int_0^t c(s;\omega)ds,\theta_t\omega))$ is a *random transition front* of (1.5). If $\Phi(x,\omega)$ and $\Psi(x,\omega)$ are non-increasing with respect to x for a.a. $\omega\in\Omega$ and all $x\in\mathbb{R}$, then $(u(t;\omega),v(t;\omega))$ is said to be a *monotone random transition front.* If there is $\bar{c}_{\rm inf}\in\mathbb{R}$ such that for a.a. $\omega\in\Omega$,

$$\liminf_{t-s\to\infty}\frac{1}{t-s}\int_s^t c(\tau;\omega)d\tau=\bar{c}_{\rm inf},$$

then $\bar{c}_{\rm inf}$ is called its *least mean speed.*

Note that the ergodicity of the metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ implies $\underline{a_1(\omega)-c_1(\omega)v^*(\cdot;\omega)}=\liminf_{t-s\to\infty}\frac{1}{t-s}\int_s^t(a_1(\theta_\tau\omega)-c_1(\theta_\tau\omega)v^*(\tau;\omega))d\tau$ is independent of ω in a subset $\Omega_0\subset\Omega$ of full measure. We denote

$$\underline{\lambda}=\underline{a_1(\omega)-c_1(\omega)v^*(\cdot;\omega)}$$

for $\omega\in\Omega_0$. For given $\mu>0$, let

$$c_0:=\inf_{\mu>0}\frac{e^\mu+e^{-\mu}-2+\underline{\lambda}}{\mu},$$

By [4, Lemma 5.1], there is a unique $\mu^*>0$ such that

$$c_0=\frac{e^{\mu^*}+e^{-\mu^*}-2+\underline{\lambda}}{\mu^*}$$

and for any $\gamma>c_0$, the equation $\gamma=\frac{e^\mu+e^{-\mu}-2+\underline{\lambda}}{\mu}$ has exactly two positive solutions for μ.

Now we are in a position to state the main results on the existence and non-existence of random transition fronts of two species cooperative lattice systems in random media.

Theorem 1.2. *Assume* (H1)–(H3) *hold. Then we have:*

(i) For any given $\gamma>c_0$, there is a monotone random transition front of (1.5) *with least mean speed $\bar{c}_{\rm inf}=\gamma$. More precisely, for any given $\gamma>c_0$, let $0<\mu<\mu^*$ be such that $\frac{e^\mu+e^{-\mu}-2+\underline{\lambda}}{\mu}=\gamma$. Then* (1.5) *has a monotone random transition front $(u(t;\omega),v(t;\omega))=\{(u_i(t;\omega),v_i(t;\omega))\}_{i\in\mathbb{Z}}$ with $u_i(t;\omega)=\Phi(i-\int_0^t c(s;\omega)ds,\theta_t\omega)$ and*

$v_i(t;\omega) = \Psi(i - \int_0^t c(s;\omega)ds, \theta_t\omega)$, where $c(t;\omega) = \frac{e^{\mu}+e^{-\mu}-2+a_1(\theta_t\omega)-c_1(\theta_t\omega)v^*(t;\omega)}{\mu}$ and hence $\overline{c}_{\inf} = \frac{e^{\mu}+e^{-\mu}-2+\underline{\lambda}}{\mu} = \gamma$. Moreover, for any $\omega \in \Omega_0$,

$$\lim_{x\to-\infty}(\Phi(x,\theta_t\omega),\Psi(x,\theta_t\omega)) = (u^*(t;\omega), v^*(t;\omega)),$$
$$\lim_{x\to\infty}(\Phi(x,\theta_t\omega),\Psi(x,\theta_t\omega)) = (0,0)$$

uniformly in $t \in \mathbb{R}$.

(ii) There is no random transition front of (1.5) *with least mean speed less than c_0.*

Remark 1.3. (i) When a_i, b_i, c_i $(i = 1,2)$ are constants, our existence result of the transition front is consistent with the result obtained in [12, Theorems 1, 4]. Also, we obtain the non-existence result of the transition front.

(ii) We leave as an open problem the case $\overline{c}_{\inf} = c_0$, that is, the existence of random transition front of (1.5) with least mean speed $\overline{c}_{\inf} = c_0$.

The rest of this article is organized as follows. In Section 2, we establish the comparison principle for sub-solutions and super-solutions of (1.5) and prove some basic properties and fundamental lemmas to be used in later section. We prove the existence and non-existence of random transition fronts after constructing appropriate sub-solutions and super-solutions of (1.5) in Section 3.

2. Preliminaries

In this section, we present some preliminary material to be used in later sections. We first present a comparison principle for sub-solutions and super-solutions of (1.5) and prove the convergence of solutions on compact subsets. Next, we present some useful lemmas including a technical lemma.

Consider first the following space continuous version of (1.5),

$$\begin{aligned} \partial_t u &= Hu + u(a_1(\theta_t\omega) - b_1(\theta_t\omega)u - c_1(\theta_t\omega)(v^*(t;\omega) - v)),\\ \partial_t v &= Hv + b_2(\theta_t\omega)(v^*(t;\omega) - v)u + v(a_2(\theta_t\omega)\\ &\quad - 2c_2(\theta_t\omega)v^*(t;\omega) + c_2(\theta_t\omega)v), \end{aligned} \tag{2.1}$$

where

$$u = u(x,t), \quad v = v(x,t),$$
$$Hu(x,t) := u(x+1,t) + u(x-1,t) - 2u(x,t), \quad x \in \mathbb{R},\ t \in \mathbb{R}.$$

Let

$$l^{\infty}(\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R} : \sup_{x\in\mathbb{R}} |u(x)| < \infty\}$$

with the norm $\|u\| = \sup_{x\in\mathbb{R}} |u(x)|$, and

$$l^{\infty,+}(\mathbb{R}) = \{u \in l^{\infty}(\mathbb{R}) : \inf_{x\in\mathbb{R}} u(x) \geq 0\}.$$

For $u, v \in l^{\infty}(\mathbb{R})$, we define

$$u \geq v \quad \text{if} \quad u - v \in l^{\infty,+}(\mathbb{R}).$$

Recall that for any $(u^0, v^0) \in l^{\infty}(\mathbb{Z}) \times l^{\infty}(\mathbb{Z})$,

$$(u(t;u^0,v^0,\omega), v(t;u^0,v^0,\omega)) = \{(u_i(t;u^0,v^0,\omega), v_i(t;u^0,v^0,\omega))\}_{i\in\mathbb{Z}}$$

is the solution of (1.5) with $(u_i(0;u^0,v^0,\omega), v_i(0;u^0,v^0,\omega)) = (u_i^0, v_i^0)$ for $i \in \mathbb{Z}$. For any $(u_0, v_0) \in l^{\infty}(\mathbb{R}) \times l^{\infty}(\mathbb{R})$, let $(u(x,t;u_0,v_0,\omega), v(x,t;u_0,v_0,\omega))$ be the solution of (2.1) with $(u(x,0;u_0,v_0,\omega),v(x,0;u_0,v_0,\omega))=(u_0(x),v_0(x))$. For any (u^1,u^2), $(v^1,v^2)\in l^\infty(\mathbb{R})\times l^\infty(\mathbb{R})$, the relation $(u^1,u^2)<(v^1,v^2)$ ($(u^1,u^2)\le(v^1,v^2)$ resp.) is also to be understood componentwise: $u^i<v^i$ ($u^i\le v^i$) for each i.

Let

$$\begin{gathered}f(t,u,v,\omega)=u(a_1(\theta_t\omega)-b_1(\theta_t\omega)u-c_1(\theta_t\omega)(v^*(t;\omega)-v)),\\ g(t,u,v,\omega)=b_2(\theta_t\omega)(v^*(t;\omega)-v)u+v(a_2(\theta_t\omega)-2c_2(\theta_t\omega)v^*(t;\omega)+c_2(\theta_t\omega)v).\end{gathered}$$

A pair of function $(u(x,t;\omega),v(x,t;\omega))$ on $\mathbb{R}\times[0,T)$ which is continuous in t is called a *super-solution* or *sub-solution* of (2.1) (resp. (1.5)) if for a.a. $\omega\in\Omega$ and any given $x\in\mathbb{R}$ (resp. $x\in\mathbb{Z}$), $u(x,t;\omega)$ and $v(x,t;\omega)$ are absolutely continuous in $t\in[0,T)$, and

$$\begin{aligned}u_t(x,t;\omega)&\ge Hu(x,t;\omega)+f(t,u,v,\omega)\\ v_t(x,t;\omega)&\ge Hv(x,t;\omega)+g(t,u,v,\omega)\end{aligned}$$

for a.a. $t\in[0,T)$, or

$$\begin{aligned}u_t(x,t;\omega)&\le Hu(x,t;\omega)+f(t,u,v,\omega)\\ v_t(x,t;\omega)&\le Hv(x,t;\omega)+g(t,u,v,\omega)\end{aligned}$$

for a.a. $t\in[0,T)$.

A pair of function is said to be a generalized super-solution (resp. sub-solution) if it is the infimum (resp. supremum) of a finite number of super-solutions (resp. sub-solutions).

Now we are in a position to present a comparison principle for solutions of (2.1), the comparison principle for solutions of (1.5) can be proved similarly.

Proposition 2.1 (Comparison principle)**.** *(1) Suppose that* $(u_1(x,t;\omega),v_1(x,t;\omega))$ *is a bounded sub-solution of* (2.1) *on* $[0,T)$ *and that* $(u_2(x,t;\omega),v_2(x,t;\omega))$ *is a bounded super-solution of* (2.1) *on* $[0,T)$ *and* $(u_i(x,t;\omega),v_i(x,t;\omega))\in[0,u^*(t;\omega)]\times[0,v^*(t;\omega)]$ $(i=1,2)$ *for* $x\in\mathbb{R}$ *and* $t\in[0,T)$. *If*

$$(u_1(\cdot,0;\omega),v_1(\cdot,0;\omega))\le(u_2(\cdot,0;\omega),v_2(\cdot,0;\omega)),$$

then

$$(u_1(\cdot,t;\omega),v_1(\cdot,t;\omega))\le(u_2(\cdot,t;\omega),v_2(\cdot,t;\omega))\quad \text{for }\ t\in[0,T).$$

(2) Suppose that $(u_i(x,t;\omega),v_i(x,t;\omega))\in[0,u^*(t;\omega)]\times[0,v^*(t;\omega)]$ $(i=1,2)$ *are bounded and satisfy that for any given* $x\in\mathbb{R}$, $u_i(x,t;\omega)$, $v_i(x,t;\omega)$ $(i=1,2)$ *are absolutely continuous in* $t\in[0,\infty)$*, and*

$$\begin{aligned}&\partial_tu_2(x,t;\omega)-(Hu_2(x,t;\omega)+f(t,u_2,v_2,\omega))\\ &>\partial_tu_1(x,t;\omega)-(Hu_1(x,t;\omega)+f(t,u_1,v_1,\omega)),\\ &\partial_tv_2(x,t;\omega)-(Hv_2(x,t;\omega)+g(t,u_2,v_2,\omega))\\ &>\partial_tv_1(x,t;\omega)-(Hv_1(x,t;\omega)+g(t,u_1,v_1,\omega))\end{aligned}$$

for $t>0$*. Moreover, suppose that*

$$(u_2(\cdot,0;\omega),v_2(\cdot,0;\omega))\ge(u_1(\cdot,0;\omega),v_1(\cdot,0;\omega)).$$

Then $(u_2(\cdot,t;\omega),v_2(\cdot,t;\omega))>(u_1(\cdot,t;\omega),v_1(\cdot,t;\omega))$ *for* $t>0$.

Proof. (1) Let

$$Q_1(x,t;\omega) = e^{ct}(u_2(x,t;\omega) - u_1(x,t;\omega)), \quad Q_2(x,t;\omega) = e^{ct}(v_2(x,t;\omega) - v_1(x,t;\omega)),$$

where $c := c(\omega)$ is to be determined later. Then there is a measurable subset $\bar{\Omega}$ of Ω with $\mathbb{P}(\bar{\Omega}) = 0$ such that for any $\omega \in \Omega \setminus \bar{\Omega}$, $Q_1(x,t;\omega)$ and $Q_2(x,t;\omega)$ satisfy

$$\begin{aligned} \partial_t Q_1 &\geq Q_1(x+1,t;\omega) + Q_1(x-1,t;\omega) + a_1(x,t;\omega)Q_1 + b_1(x,t;\omega)Q_2, \\ \partial_t Q_2 &\geq Q_2(x+1,t;\omega) + Q_2(x-1,t;\omega) + a_2(x,t;\omega)Q_1 + b_2(x,t;\omega)Q_2, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} a_1(x,t;\omega) &= c - 2 + f_u(t,u_1^*,v_1^*,\omega), \quad b_1(x,t;\omega) = f_v(t,u_1^*,v_1^*,\omega), \\ a_2(x,t;\omega) &= g_u(t,u_2^*,v_2^*,\omega), \quad b_2(x,t;\omega) = c - 2 + g_v(t,u_2^*,v_2^*,\omega) \end{aligned}$$

for some $u_i^* = u_i^*(x,t;\omega)$ $(i = 1,2)$ between $u_1(x,t;\omega)$ and $u_2(x,t;\omega)$ and some $v_i^* = v_i^*(x,t;\omega)$ $(i = 1,2)$ between $v_1(x,t;\omega)$ and $v_2(x,t;\omega)$.

Since (2.1) is cooperative in $[0,u^*(t;\omega)] \times [0,v^*(t;\omega)]$, we have $b_1(x,t;\omega) \geq 0$ and $a_2(x,t;\omega) \geq 0$. By the boundedness of $u_i(x,t;\omega)$ and $v_i(x,t;\omega)$ $(i = 1,2)$, we can choose $c = c(\omega) > 0$ such that $b_2(x,t;\omega) \geq 0$ and $a_1(x,t;\omega) \geq 0$.

We claim that $Q_i(x,t;\omega) \geq 0$ $(i = 1,2)$ for $x \in \mathbb{R}$ and $t \in [0,T]$. Let $p_0(\omega) := \max_{i=1,2} \max_{(x,t)\in\mathbb{R}\times[0,T]}\{a_i(x,t;\omega), b_i(x,t;\omega)\}$. It suffices to prove the claim for $x \in \mathbb{R}$ and $t \in (0,T_0]$ with $T_0 = \min\{T, \frac{1}{2(1+p_0(\omega))}\}$. Assume that there are some $\tilde{x} \in \mathbb{R}$ and $\tilde{t} \in (0,T_0]$ such that $Q_1(\tilde{x},\tilde{t};\omega) < 0$ or $Q_2(\tilde{x},\tilde{t};\omega) < 0$. Then there is $t^0 \in (0,T_0)$ such that

$$Q_1^{\inf}(\omega) := \inf_{(x,t)\in\mathbb{R}\times[0,t^0]} Q_1(x,t;\omega) < 0 \quad \text{or} \quad Q_2^{\inf}(\omega) := \inf_{(x,t)\in\mathbb{R}\times[0,t^0]} Q_2(x,t;\omega) < 0.$$

Without loss of generality, we assume that $Q_1^{\inf}(\omega) \leq Q_2^{\inf}(\omega)$. Observe that there are $x_n \in \mathbb{R}$ and $t_n \in (0,t^0]$ such that

$$Q_1(x_n,t_n;\omega) \to Q_1^{\inf}(\omega) \quad \text{as } n \to \infty.$$

By (2.2) and the fundamental theorem of calculus for Lebesgue integrals, we obtain

$$\begin{aligned} &Q_1(x_n,t_n;\omega) - Q_1(x_n,0;\omega) \\ &\geq \int_0^{t_n} [Q_1(x_n+1,t;\omega) + Q_1(x_n-1,t;\omega) + a_1(x_n,t;\omega)Q_1(x_n,t;\omega) \\ &\quad + b_1(x_n,t;\omega)Q_2(x_n,t;\omega)]dt \\ &\geq \int_0^{t_n} [2Q_1^{\inf}(\omega) + a_1(x_n,t;\omega)Q_1^{\inf}(\omega) + b_1(x_n,t;\omega)Q_2^{\inf}(\omega)]dt \\ &\geq \int_0^{t_n} [2Q_1^{\inf}(\omega) + 2p_0(\omega)Q_1^{\inf}(\omega)]dt \\ &\geq 2(1+p_0(\omega))t^0 Q_1^{\inf}(\omega) \qquad \text{for } n \geq 1. \end{aligned}$$

Note that $Q_1(x_n,0;\omega) \geq 0$, we then have

$$Q_1(x_n,t_n;\omega) \geq 2(1+p_0(\omega))t^0 Q_1^{\inf}(\omega) \qquad \text{for } n \geq 1.$$

Letting $n \to \infty$, we obtain

$$Q_1^{\inf}(\omega) \geq 2(1+p_0(\omega))t^0 Q_1^{\inf}(\omega) > Q_1^{\inf}(\omega).$$

A contradiction. Hence $Q_i(x,t;\omega) \geq 0$ $(i = 1,2)$ for $x \in \mathbb{R}$ and $t \in [0,T]$, which implies that $(u_1(x,t;\omega), v_1(x,t;\omega)) \leq (u_2(x,t;\omega), v_2(x,t;\omega))$ for $\omega \in \Omega \setminus \bar{\Omega}$, $x \in \mathbb{R}$ and $t \in [0,T]$.

(2) Since (2.1) is cooperative in $[0, u^*(t;\omega)] \times [0, v^*(t;\omega)]$, then for $\omega \in \Omega \setminus \bar{\Omega}$, by the similar arguments as getting (2.2), we can find $c(\omega), \mu(\omega) > 0$ such that for any given $x \in \mathbb{R}$,

$$\partial_t w(x,t;\omega) > w(x+1,t;\omega) + w(x-1,t;\omega) + \mu(\omega)w(x,t;\omega) \quad \text{for } t > 0,$$

where $w(x,t;\omega) = e^{c(\omega)t}(u_2(x,t,\omega) - u_1(x,t,\omega))$. Thus we have that for any given $x \in \mathbb{R}$,

$$w(x,t;\omega) > w(x,0;\omega) + \int_0^t [w(x+1,s;\omega) + w(x-1,s;\omega) + \mu(\omega)w(x,s;\omega)]ds.$$

By the arguments in (1), $w(x,t;\omega) \geq 0$ for all $x \in \mathbb{R}$ and $t \geq 0$. It then follows that $w(x,t;\omega) > w(x,0;\omega) \geq 0$ and and hence $u_2(x,t;\omega) > u_1(x,t;\omega)$ for $\omega \in \Omega \setminus \bar{\Omega}$, $x \in \mathbb{R}$ and $t > 0$. Similarly, we can get that $v_2(x,t;\omega) > v_1(x,t;\omega)$ for $\omega \in \Omega \setminus \bar{\Omega}$, $x \in \mathbb{R}$ and $t > 0$. □

Proposition 2.2. *Suppose that* $(u_n, v_n) \in l^{\infty,+}(\mathbb{R}) \times l^{\infty,+}(\mathbb{R})$ $(n = 1,2,\dots)$ *and* $(u_0, v_0) \in l^{\infty,+}(\mathbb{R}) \times l^{\infty,+}(\mathbb{R})$ *with* $\{\|u_n\|\}$, $\{\|v_n\|\}$ *bounded. If* $(u_n(x), v_n(x)) \to (u_0(x), v_0(x))$ *as* $n \to \infty$ *uniformly in* x *on bounded sets, then for each* $t > 0$, $(u(x,t;u_n,v_n,\theta_{t_0}\omega), v(x,t;u_n,v_n,\theta_{t_0}\omega)) \to (u(x,t;u_0,v_0,\theta_{t_0}\omega), v(x,t;u_0,v_0,\theta_{t_0}\omega))$ *as* $n \to \infty$ *uniformly in* x *on bounded sets and* $t_0 \in \mathbb{R}$.

Proof. Fix any $\omega \in \Omega$, and let

$$\begin{aligned}
u^n(x,t;\theta_{t_0}\omega) &= u(x,t;u_n,v_n,\theta_{t_0}\omega) - u(x,t;u_0,v_0,\theta_{t_0}\omega), \\
v^n(x,t;\theta_{t_0}\omega) &= v(x,t;u_n,v_n,\theta_{t_0}\omega) - v(x,t;u_0,v_0,\theta_{t_0}\omega).
\end{aligned}$$

Then

$$\begin{aligned}
\partial_t u^n &= Hu^n + a_1^n(x,t;\theta_{t_0}\omega)u^n + b_1^n(x,t;\theta_{t_0}\omega)v^n, \\
\partial_t v^n &= Hv^n + a_2^n(x,t;\theta_{t_0}\omega)u^n + b_2^n(x,t;\theta_{t_0}\omega)v^n,
\end{aligned}$$

where

$$\begin{aligned}
a_1^n(x,t;\theta_{t_0}\omega) &= f_u(t, u_1^n(x,t;\theta_{t_0}\omega), v_1^n(x,t;\theta_{t_0}\omega), \theta_{t_0}\omega), \\
b_1^n(x,t;\theta_{t_0}\omega) &= f_v(t, u_1^n(x,t;\theta_{t_0}\omega), v_1^n(x,t;\theta_{t_0}\omega), \theta_{t_0}\omega), \\
a_2^n(x,t;\theta_{t_0}\omega) &= g_u(t, u_2^n(x,t;\theta_{t_0}\omega), v_2^n(x,t;\theta_{t_0}\omega), \theta_{t_0}\omega), \\
b_2^n(x,t;\theta_{t_0}\omega) &= g_v(t, u_2^n(x,t;\theta_{t_0}\omega), v_2^n(x,t;\theta_{t_0}\omega), \theta_{t_0}\omega),
\end{aligned}$$

for $u_1^n(x,t;\theta_{t_0}\omega)$, $u_2^n(x,t;\theta_{t_0}\omega)$ between $u(x,t;u_n,v_n,\theta_{t_0}\omega)$ and $u(x,t;u_0,v_0,\theta_{t_0}\omega)$, and $v_1^n(x,t;\theta_{t_0}\omega)$, $v_2^n(x,t;\theta_{t_0}\omega)$ between $v(x,t;u_n,v_n,\theta_{t_0}\omega)$ and $v(x,t;u_0,v_0,\theta_{t_0}\omega)$.

Take $\rho > 0$, and let

$$X(\rho) = \{(u,v) : \mathbb{R} \to \mathbb{R}^2 : (u(\cdot)e^{-\rho|\cdot|}, v(\cdot)e^{-\rho|\cdot|}) \in l^\infty(\mathbb{R}) \times l^\infty(\mathbb{R})\}$$

with the norm $\|(u,v)\|_{X(\rho)} = \sup_{x\in\mathbb{R}}(|u(x)| + |v(x)|)e^{-\rho|x|}$. Observe that $(H,H) : X(\rho) \to X(\rho)$, given by

$$(H,H)(u,v) = (Hu, Hv),$$

is a bounded linear operator. Note also that $a_i^n(x,t;\theta_{t_0}\omega)$ and $b_i^n(x,t;\theta_{t_0}\omega)$ are uniformly bounded $(i = 1,2)$. Then there are $M > 0$ and $\alpha > 0$ such that

$$\|e^{(H,H)t}\|_{X(\rho)} \leq Me^{\alpha t}$$

and $|a_i^n(x,t;\theta_{t_0}\omega)| \le M$, $|b_i^n(x,t;\theta_{t_0}\omega)| \le M$. Hence,

$$\begin{aligned}
&(u^n(\cdot,t;\theta_{t_0}\omega), v^n(\cdot,t;\theta_{t_0}\omega))\\
&= e^{(H,H)t}(u^n(\cdot,0;\theta_{t_0}\omega), v^n(\cdot,0;\theta_{t_0}\omega))\\
&\quad + \int_0^t e^{(H,H)(t-\tau)}[a_1^n(\cdot,\tau;\theta_{t_0}\omega)u^n(\cdot,\tau;\theta_{t_0}\omega) + b_1^n(\cdot,\tau;\theta_{t_0}\omega)v^n(\cdot,\tau;\theta_{t_0}\omega),\\
&\quad a_2^n(\cdot,\tau;\theta_{t_0}\omega)u^n(\cdot,\tau;\theta_{t_0}\omega) + b_2^n(\cdot,\tau;\theta_{t_0}\omega)v^n(\cdot,\tau;\theta_{t_0}\omega)]d\tau
\end{aligned}$$

and then

$$\begin{aligned}
&\|(u^n(\cdot,t;\theta_{t_0}\omega), v^n(\cdot,t;\theta_{t_0}\omega))\|_{X(\rho)}\\
&\le Me^{\alpha t}\|(u^n(\cdot,0;\theta_{t_0}\omega), v^n(\cdot,0;\theta_{t_0}\omega))\|_{X(\rho)}\\
&\quad + M^2\int_0^t e^{\alpha(t-\tau)}\|(u^n(\cdot,\tau;\theta_{t_0}\omega), v^n(\cdot,\tau;\theta_{t_0}\omega))\|_{X(\rho)}d\tau.
\end{aligned}$$

By Gronwall's inequality,

$$\|(u^n(\cdot,t;\theta_{t_0}\omega), v^n(\cdot,t;\theta_{t_0}\omega))\|_{X(\rho)} \le e^{(\alpha+M^2)t}M\|(u^n(\cdot,0;\theta_{t_0}\omega), v^n(\cdot,0;\theta_{t_0}\omega))\|_{X(\rho)}.$$

Note that $\|(u^n(\cdot,0;\theta_{t_0}\omega), v^n(\cdot,0;\theta_{t_0}\omega))\|_{X(\rho)} \to 0$ as $n\to\infty$ uniformly in $t_0\in\mathbb{R}$. It then follows that

$$(u^n(x,t;\theta_{t_0}\omega), v^n(x,t;\theta_{t_0}\omega)) \to (0,0) \quad \text{as } n\to\infty$$

uniformly in x on bounded sets and $t_0\in\mathbb{R}$. The proof is complete. □

Now we present some lemmas including the technical results.

Lemma 2.3. $\underline{a}(\cdot), a(\cdot), \overline{a}(\cdot) \in L^1(\Omega,\mathcal{F},\mathbb{P})$. *Also* $\underline{a}(\omega)$ *and* $\overline{a}(\omega)$ *are independent of* ω *for a.a.* $\omega\in\Omega$.

The proof of the above lemma follows from [23, Lemma 2.1].

Lemma 2.4. *Suppose that for* $\omega\in\Omega$, $a^\omega(t) = a(\theta_t\omega) \in C(\mathbb{R},(0,\infty))$. *Then for a.a.* $\omega\in\Omega$,

$$\underline{a} = \sup_{A\in W^{1,\infty}_{\rm loc}(\mathbb{R})\cap L^\infty(\mathbb{R})} \operatorname{ess\,inf}_{t\in\mathbb{R}}(A' + a^\omega)(t).$$

The proof of the above lemma follows from [23, Lemma 2.2] and Lemma 2.3. Note that by (H3) there is a strictly positive solution $h(t;\omega)$ of

$$\frac{dv}{dt} - (a_2(\theta_t\omega) - 2c_2(\theta_t\omega)v^*(t;\omega))v - b_2(\theta_t\omega)v^*(t;\omega) = -(a_1(\theta_t\omega) - c_1(\theta_t\omega)v^*(t;\omega))v.$$

Denote

$$c(t;\omega,\mu) = \frac{e^\mu + e^{-\mu} - 2 + a_1(\theta_t\omega) - c_1(\theta_t\omega)v^*(t;\omega)}{\mu}.$$

Lemma 2.5. *Let* $\omega\in\Omega_0$ *and* $0<\sigma\ll 1$. *Then for any* μ, $\tilde\mu$ *with* $0<\mu<\tilde\mu<\min\{2\mu,\mu^*\}$, *there exist* $\{t_k\}_{k\in\mathbb{Z}}$ *with* $t_k<t_{k+1}$ *and* $\lim_{k\to\pm\infty} t_k = \pm\infty$, $A_\omega\in W^{1,\infty}_{\rm loc}(\mathbb{R})\cap L^\infty(\mathbb{R})$ *with* $A_\omega(\cdot)\in C^1((t_k,t_{k+1}))$ *for* $k\in\mathbb{Z}$, *and* $d_\omega>0$ *such that for any* $d\ge d_\omega$ *the functions*

$$\tilde u(x,t,\omega) := e^{-\mu(x-\int_0^t c(s;\omega,\mu)ds)} - de^{(\frac{\tilde\mu}{\mu}-1)A_\omega(t)-\tilde\mu(x-\int_0^t c(s;\omega,\mu)ds)},$$

$$\tilde v(x,t,\omega) := \sigma e^{-\mu(x-\int_0^t c(s;\omega,\mu)ds)}h(t;\omega) - \sigma de^{(\frac{\tilde\mu}{\mu}-1)A_\omega(t)-\tilde\mu(x-\int_0^t c(s;\omega,\mu)ds)}h(t;\omega)$$

satisfy

$$\begin{aligned} \partial_t \tilde{u} &\le H\tilde{u} + \tilde{u}(a_1(\theta_t\omega) - b_1(\theta_t\omega)\tilde{u} - c_1(\theta_t\omega)(v^*(t;\omega) - \tilde{v})), \\ \partial_t \tilde{v} &\le H\tilde{v} + b_2(\theta_t\omega)(v^*(t;\omega) - \tilde{v})\tilde{u} + \tilde{v}(a_2(\theta_t\omega) - 2c_2(\theta_t\omega)v^*(t;\omega) + c_2(\theta_t\omega)\tilde{v}), \end{aligned}$$

for $t \in (t_k, t_{k+1})$, $x \ge \int_0^t c(s;\omega,\mu)ds + \frac{\ln d}{\tilde{\mu}-\mu} + \frac{A_\omega(t)}{\mu}$, $k \in \mathbb{Z}$.

Proof. For a given $\omega \in \Omega_0$ and $0 < \mu < \tilde{\mu} < \min\{2\mu, \mu^*\}$, by the arguments in the proof of [4, Lemma 5.1] we can get that $\frac{e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2+\underline{\lambda}}{\tilde{\mu}} < \frac{e^{\mu}+e^{-\mu}-2+\underline{\lambda}}{\mu}$, and hence $\underline{\lambda} > \frac{\mu(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)-\tilde{\mu}(e^{\mu}+e^{-\mu}-2)}{\tilde{\mu}-\mu}$. Let $0 < \delta \ll 1$ be such that

$$(1-\delta)\underline{\lambda} > \frac{\mu(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)-\tilde{\mu}(e^{\mu}+e^{-\mu}-2)}{\tilde{\mu}-\mu}.$$

It follows from Lemma 2.4 that there exist $T > 0$ and $A_\omega \in W^{1,\infty}_{\rm loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $A_\omega(\cdot) \in C^1((t_k, t_{k+1}))$ with $t_k = kT$ for $k \in \mathbb{Z}$, and

$$(1-\delta)(a_1(\theta_t\omega) - c_1(\theta_t\omega)v^*(t;\omega)) + A'_\omega(t) \ge \frac{\mu(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)-\tilde{\mu}(e^{\mu}+e^{-\mu}-2)}{\tilde{\mu}-\mu} \tag{2.3}$$

for all $t \in (t_k, t_{k+1})$, $k \in \mathbb{Z}$.

Now we fix the above $\delta > 0$ and $A_\omega(t)$. Let

$$\begin{gathered} \xi(x,t;\omega) = x - \int_0^t c(s;\omega,\mu)ds, \\ \tilde{u}(x,t,\omega) = e^{-\mu\xi(x,t;\omega)} - de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}, \\ \tilde{v}(x,t,\omega) = \sigma e^{-\mu\xi(x,t;\omega)}h(t;\omega) - \sigma de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}h(t;\omega) \end{gathered}$$

with $d > 1$ to be determined later. Then we have

$$\begin{aligned} &\partial_t\tilde{u} - [H\tilde{u} + \tilde{u}(a_1(\theta_t\omega) - b_1(\theta_t\omega)\tilde{u} - c_1(\theta_t\omega)(v^*(t;\omega) - \tilde{v}))] \\ &= \mu c(t;\omega,\mu)e^{-\mu\xi(x,t;\omega)} + d[-(\frac{\tilde{\mu}}{\mu}-1)A'_\omega(t) - \tilde{\mu}c(t;\omega,\mu)]e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)} \\ &\quad - [(e^{\mu}+e^{-\mu}-2)e^{-\mu\xi(x,t;\omega)} - d(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}] \\ &\quad - \tilde{u}[a_1(\theta_t\omega) - b_1(\theta_t\omega)\tilde{u} - c_1(\theta_t\omega)(v^*(t;\omega) - \tilde{v})] \\ &= d[-(\frac{\tilde{\mu}}{\mu}-1)A'_\omega(t) - \tilde{\mu}c(t;\omega,\mu) + e^{\tilde{\mu}} + e^{-\tilde{\mu}} - 2 + a_1(\theta_t\omega) - c_1(\theta_t\omega)v^*(t;\omega)] \\ &\times e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)} + \tilde{u}[b_1(\theta_t\omega)\tilde{u} - c_1(\theta_t\omega)\tilde{v}] \end{aligned} \tag{2.4}$$

Recall that

$$c(t;\omega,\mu) = \frac{e^{\mu}+e^{-\mu}-2+a_1(\theta_t\omega)-c_1(\theta_t\omega)v^*(t;\omega)}{\mu}.$$

Then by (2.4) we obtain

$$
\begin{aligned}
&\partial_t\tilde{u}-[H\tilde{u}+\tilde{u}(a_1(\theta_t\omega)-b_1(\theta_t\omega)\tilde{u}-c_1(\theta_t\omega)(v^*(t;\omega)-\tilde{v}))]\\
&=d(\frac{\tilde{\mu}}{\mu}-1)[\frac{\mu(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)-\tilde{\mu}(e^{\mu}+e^{-\mu}-2)}{\tilde{\mu}-\mu}\\
&\quad-(1-\delta)(a_1(\theta_t\omega)-c_1(\theta_t\omega)v^*(t;\omega))-A'_\omega(t)]e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}\\
&\quad+\tilde{u}[b_1(\theta_t\omega)\tilde{u}-c_1(\theta_t\omega)\sigma h(t;\omega)\tilde{u}]\\
&-\delta d(\frac{\tilde{\mu}}{\mu}-1)(a_1(\theta_t\omega)-c_1(\theta_t\omega)v^*(t;\omega))e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}\\
&\le d(\frac{\tilde{\mu}}{\mu}-1)[\frac{\mu(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)-\tilde{\mu}(e^{\mu}+e^{-\mu}-2)}{\tilde{\mu}-\mu}\\
&\quad-(1-\delta)(a_1(\theta_t\omega)-c_1(\theta_t\omega)v^*(t;\omega))\\
&\quad-A'_\omega(t)]e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}+b_1(\theta_t\omega)\tilde{u}^2\\
&\quad-\delta d(\frac{\tilde{\mu}}{\mu}-1)(a_1(\theta_t\omega)-c_1(\theta_t\omega)v^*(t;\omega))e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}\\
&=d(\frac{\tilde{\mu}}{\mu}-1)[\frac{\mu(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)-\tilde{\mu}(e^{\mu}+e^{-\mu}-2)}{\tilde{\mu}-\mu}\\
&\quad-(1-\delta)(a_1(\theta_t\omega)-c_1(\theta_t\omega)v^*(t;\omega))-A'_\omega(t)]e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}\\
&\quad-[d\delta(\frac{\tilde{\mu}}{\mu}-1)e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)}(a_1(\theta_t\omega)-c_1(\theta_t\omega)v^*(t;\omega))\\
&\quad-b_1(\theta_t\omega)e^{-(2\mu-\tilde{\mu})\xi(x,t;\omega)}]e^{-\tilde{\mu}\xi(x,t;\omega)}+d[-2e^{-\mu\xi(x,t;\omega)}\\
&\quad+de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}]e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}b_1(\theta_t\omega)
\end{aligned}
\tag{2.5}
$$

for $t\in(t_k,t_{k+1})$. Note that

$$
\begin{aligned}
&\partial_t\tilde{v}-[H\tilde{v}+b_2(\theta_t\omega)(v^*(t;\omega)-\tilde{v})\tilde{u}+\tilde{v}(a_2(\theta_t\omega)-2c_2(\theta_t\omega)v^*(t;\omega)+c_2(\theta_t\omega)\tilde{v})]\\
&=\sigma[(a_2(\theta_t\omega)-2c_2(\theta_t\omega)v^*(t;\omega))h(t;\omega)+b_2(\theta_t\omega)v^*(t;\omega)\\
&\quad-(a_1(\theta_t\omega)-c_1(\theta_t\omega)v^*(t;\omega))h(t;\omega)]\tilde{u}+\sigma h(t;\omega)\partial_t\tilde{u}-\sigma h(t;\omega)H\tilde{u}\\
&\quad-b_2(\theta_t\omega)v^*(t;\omega)\tilde{u}+\sigma h(t;\omega)\tilde{u}^2b_2(\theta_t\omega)\\
&\quad-\sigma h(t;\omega)\tilde{u}(a_2(\theta_t\omega)-2c_2(\theta_t\omega)v^*(t;\omega)+c_2(\theta_t\omega)\tilde{v})\\
&=\sigma h(t;\omega)\{\partial_t\tilde{u}-[H\tilde{u}+\tilde{u}(a_1(\theta_t\omega)-b_2(\theta_t\omega)\tilde{u}-c_1(\theta_t\omega)v^*(t;\omega)+c_2(\theta_t\omega)\tilde{v})]\}\\
&\quad+b_2(\theta_t\omega)v^*(t;\omega)\tilde{u}(\sigma-1).
\end{aligned}
$$

Then by similar arguments as for proving (2.5), we obtain

$$
\begin{aligned}
&\partial_t\tilde{v}-[H\tilde{v}+b_2(\theta_t\omega)(v^*(t;\omega)-\tilde{v})\tilde{u}+\tilde{v}(a_2(\theta_t\omega)-2c_2(\theta_t\omega)v^*(t;\omega)\\
&+c_2(\theta_t\omega)\tilde{v})]\\
&\le\{d(\frac{\tilde{\mu}}{\mu}-1)[\frac{\mu(e^{\tilde{\mu}}+e^{-\tilde{\mu}}-2)-\tilde{\mu}(e^{\mu}+e^{-\mu}-2)}{\tilde{\mu}-\mu}\\
&\quad-(1-\delta)(a_1(\theta_t\omega)-c_1(\theta_t\omega)v^*(t;\omega))-A'_\omega(t)]e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}\\
&\quad+[b_2(\theta_t\omega)e^{-(2\mu-\tilde{\mu})\xi(x,t;\omega)}\\
&\quad-d\delta(\frac{\tilde{\mu}}{\mu}-1)e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)}(a_1(\theta_t\omega)-c_1(\theta_t\omega)v^*(t;\omega))]e^{-\tilde{\mu}\xi(x,t;\omega)}
\end{aligned}
$$

$$\begin{aligned}&+ d[-2e^{-\mu\xi(x,t;\omega)} + de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}]e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)}\\&\times b_2(\theta_t\omega)\}\sigma h(t;\omega) + b_2(\theta_t\omega)v^*(t;\omega)\tilde{u}(\sigma-1)\end{aligned} \tag{2.6}$$

for $t \in (t_k, t_{k+1})$. Let

$$\begin{aligned}d \ge d_\omega = \max\{&\max_{i\in\{1,2\},t\in\mathbb{R}}\{\frac{b_i(\theta_t\omega)}{a_1(\theta_t\omega) - c_1(\theta_t\omega)v^*(t;\omega)}\}\frac{\mu e^{-(\frac{\tilde{\mu}}{\mu}-1)\|A_\omega\|_\infty}}{\delta(\tilde{\mu}-\mu)},\\&e^{(\frac{\tilde{\mu}}{\mu}-1)\|A_\omega\|_\infty}\}.\end{aligned}$$

Then we have

$$d\delta(\frac{\tilde{\mu}}{\mu}-1)e^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)}(a_1(\theta_t\omega) - c_1(\theta_t\omega)v^*(t;\omega)) \ge b_i(\theta_t\omega) \quad (i = 1, 2).$$

For this choice of d, if $\xi(x,t;\omega) = x - \int_0^t c(s;\omega,\mu)ds \ge \frac{\ln d}{\tilde{\mu}-\mu} + \frac{A_\omega(t)}{\mu}$, which is equivalent to $\tilde{u}(x,t,\omega) \ge 0$ and $\tilde{v}(x,t,\omega) \ge 0$, then $\xi(x,t;\omega) \ge 0$ and

$$de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}\xi(x,t;\omega)} \le e^{-\mu\xi(x,t;\omega)}.$$

From this and (2.3), we obtain that each term the right hand side of (2.5) and (2.6) is less than or equal to zero. The lemma thus follows. □

For a given function $t \mapsto u(t) \in l^\infty(\mathbb{Z})$ and $c \in \mathbb{R}$, we define

$$\limsup_{|i|\le ct, t\to\infty} u_i(t) = \limsup_{t\to\infty} \sup_{i\in\mathbb{Z},|i|\le ct} u_i(t).$$

Lemma 2.6. *Let $(u^0, v^0) \in l^{\infty,+}(\mathbb{Z}) \times l^{\infty,+}(\mathbb{Z})$. If there is a positive constant $c(\omega) > 0$ such that*

$$\liminf_{s\in\mathbb{R},|i|\le c(\omega)t,t\to\infty} u_i(t;u^0,v^0,\theta_s\omega) = \liminf_{t\to\infty} \inf_{s\in\mathbb{R},i\in\mathbb{Z},|i|\le c(\omega)t} u_i(t;u^0,v^0,\theta_s\omega) > 0, \tag{2.7}$$

then for any $0 < c < c(\omega)$,

$$\limsup_{|i|\le ct,t\to\infty} [|u_i(t;u^0,v^0,\theta_s\omega)-u^*(t+s;\omega)|+|v_i(t;u^0,v^0,\theta_s\omega)-v^*(t+s;\omega)|] = 0 \tag{2.8}$$

uniformly in $s \in \mathbb{R}$.

Proof. Let $\omega \in \Omega_0$ and $c(\omega)$ satisfy (2.7). We denote

$$\delta_0 = \liminf_{s\in\mathbb{R},|i|\le c(\omega)t,t\to\infty} u_i(t;u^0,v^0,\theta_s\omega).$$

Then there is $T \gg 1$ such that

$$\inf_{|i|\le c(\omega)t} u_i(t;u^0,v^0,\theta_s\omega) \ge \frac{\delta_0}{2}, \quad \forall s \in \mathbb{R},\ t \ge T. \tag{2.9}$$

Suppose by contradiction that there is $0 < c_0 < c(\omega)$ such that (2.8) does not hold. Then there are $\epsilon_0 > 0$, $s_n \in \mathbb{R}$, $i_n \in \mathbb{Z}$, $t_n > 0$ such that $|i_n| \le c_0 t_n$, $t_n \to \infty$, and

$$\begin{aligned}&|u_{i_n}(t_n;u^0,v^0,\theta_{s_n}\omega) - u^*(t_n+s_n;\omega)|\\&+ |v_{i_n}(t_n;u^0,v^0,\theta_{s_n}\omega) - v^*(t_n+s_n;\omega)| \ge \epsilon_0.\end{aligned} \tag{2.10}$$

Let $(\tilde{u}^0,\tilde{v}^0) = \{(\tilde{u}_i^0,\tilde{v}_i^0)\}$ and $(\hat{u}^0,\hat{v}^0) = \{(\hat{u}_i^0,\hat{v}_i^0)\}$, where $\tilde{u}_i^0 = \frac{\delta_0}{2}$, $\tilde{v}_i^0 = 0$, $\hat{u}_i^0 = \|u^0\|$ and $\hat{v}_i^0 = \|v^0\|$ for all $i \in \mathbb{Z}$. By the global stability of $(u^*(t;\omega), v^*(t;\omega))$, there is $\tilde{T} \ge T$ such that

$$|u_i(t;\tilde{u}^0,\tilde{v}^0,\theta_{s_n}\omega) - u^*(t+s;\omega)| + |v_i(t;\tilde{u}^0,\tilde{v}^0,\theta_{s_n}\omega) - v^*(t+s;\omega)| < \frac{\epsilon_0}{4} \tag{2.11}$$

for all $i \in \mathbb{Z}, s \in \mathbb{R}, t \geq \tilde{T}$, and

$$\begin{aligned} u_i(t; u^0, v^0, \theta_s\omega) &\leq u_i(t; \hat{u}^0, \hat{v}^0, \theta_s\omega) < u^*(t+s;\omega) + \frac{\epsilon_0}{2}, \\ v_i(t; u^0, v^0, \theta_s\omega) &\leq v_i(t; \hat{u}^0, \hat{v}^0, \theta_s\omega) < v^*(t+s;\omega) + \frac{\epsilon_0}{2} \end{aligned} \tag{2.12}$$

for all $i \in \mathbb{Z}$, $s \in \mathbb{R}$, $t \geq \tilde{T}$. Observe that $(c(\omega)-c_0)(t_n-\tilde{T})-2c_0\tilde{T} \to \infty$ as $n \to \infty$. Hence there is N such that

$$(c(\omega)-c_0)(t_n-\tilde{T})-2c_0\tilde{T} \geq T, \quad \forall n \geq N.$$

For every $n \geq N$, let $\tilde{u}^n = \{\tilde{u}_i^n\} \in l^\infty(\mathbb{Z})$ with $\|\tilde{u}^n\| \leq \frac{\delta_0}{2}$ and

$$\begin{gathered} \tilde{u}_i^n = \begin{cases} \frac{\delta_0}{2}, & |i| \leq (c(\omega)-c_0)(t_n-\tilde{T})-2c_0\tilde{T}, \\ 0, & |i| \geq (c(\omega)-c_0)(t_n-\tilde{T})-c_0\tilde{T}, \end{cases} \\ \tilde{v}^n \equiv 0. \end{gathered} \tag{2.13}$$

Since $|i| \leq (c(\omega)-c_0)(t_n-\tilde{T})-c_0\tilde{T}$ implies that $|i+i_n| \leq c(\omega)(t_n-\tilde{T})$ for every $n \geq N$, it follows from (2.9) and (2.13) that

$$\tilde{u}_i^n \leq u_{i+i_n}(t_n-\tilde{T}; u^0, v^0, \theta_{s_n}\omega), \quad \forall i \in \mathbb{Z},\ \forall n \geq N.$$

Note that

$$\tilde{v}_i^n = 0 \leq v_{i+i_n}(t_n-\tilde{T}; u^0, v^0, \theta_{s_n}\omega), \quad \forall i \in \mathbb{Z},\ \forall n \geq N.$$

Then by the comparison principle, we have

$$u_i(t; \tilde{u}^n, \tilde{v}^n, \theta_{\tilde{s}_n}\omega) \leq u_{i+i_n}(t+t_n-\tilde{T}; u^0, v^0, \theta_{s_n}\omega), \tag{2.14}$$

$$v_i(t; \tilde{u}^n, \tilde{v}^n, \theta_{\tilde{s}_n}\omega) \leq v_{i+i_n}(t+t_n-\tilde{T}; u^0, v^0, \theta_{s_n}\omega), \tag{2.15}$$

for all $i \in \mathbb{Z}$, $t > 0$, and $n \geq N$, where $\tilde{s}_n = s_n + t_n - \tilde{T}$. It follows from the definition of $(\tilde{u}^n, \tilde{v}^n)$ that

$$\lim_{n\to\infty} (\tilde{u}^n, \tilde{v}^n) = (\tilde{u}^0, \tilde{v}^0) \quad \text{locally uniformly in } i \in \mathbb{Z}.$$

Therefore, from Proposition 2.2 we have that for every $t > 0$,

$$\begin{aligned} &\lim_{n\to\infty} [|u_i(t; \tilde{u}^n, \tilde{v}^n, \theta_{\tilde{s}_n}\omega) - u_i(t; \tilde{u}^0, \tilde{v}^0, \theta_{\tilde{s}_n}\omega)| \\ &+ |v_i(t; \tilde{u}^n, \tilde{v}^n, \theta_{\tilde{s}_n}\omega) - v_i(t; \tilde{u}^0, \tilde{v}^0, \theta_{\tilde{s}_n}\omega)|] = 0 \end{aligned} \tag{2.16}$$

locally uniformly in $i \in \mathbb{Z}$. It then follows from (2.11), (2.14), (2.15) and (2.16) that

$$\begin{gathered} u^*(s_n+t_n;\omega) - \frac{\epsilon_0}{2} < u_0(\tilde{T}; \tilde{u}^n, \tilde{v}^n, \theta_{\tilde{s}_n}\omega) \leq u_{i_n}(t_n; u^0, v^0, \theta_{s_n}\omega), \\ v^*(s_n+t_n;\omega) - \frac{\epsilon_0}{2} < v_0(\tilde{T}; \tilde{u}^n, \tilde{v}^n, \theta_{\tilde{s}_n}\omega) \leq v_{i_n}(t_n; u^0, v^0, \theta_{s_n}\omega) \quad \text{for } n \gg 1. \end{gathered}$$

Note that by (2.12) we have

$$\begin{aligned} u_{i_n}(t_n; u^0, v^0, \theta_{s_n}\omega) &< u^*(s_n+t_n;\omega) + \frac{\epsilon_0}{2}, \\ v_{i_n}(t_n; u^0, v^0, \theta_{s_n}\omega) &< v^*(s_n+t_n;\omega) + \frac{\epsilon_0}{2} \end{aligned}$$

for $n \gg 1$. Then

$$|u_{i_n}(t_n; u^0, v^0, \theta_{s_n}\omega) - u^*(s_n+t_n;\omega)| + |v_{i_n}(t_n; u^0, v^0, \theta_{s_n}\omega) - v^*(s_n+t_n;\omega)| < \epsilon_0$$

for $n \gg 1$, which contradicts (2.10). Hence (2.8) holds. □

3. Random transition fronts

In this section, we study the existence and non-existence of random transition fronts, and prove Theorem 1.2.

For any $\gamma > c_0$, let $0 < \mu < \mu^*$ be such that $\frac{e^{\mu}+e^{-\mu}-2+\underline{\lambda}}{\mu} = \gamma$, where $\underline{\lambda} = \underline{a_1(\omega) - c_1(\omega)v^*(\cdot;\omega)}$ for $\omega \in \Omega_0$. For every $\omega \in \Omega$, denote

$$c(t;\omega,\mu) = \frac{e^{\mu} + e^{-\mu} - 2 + (a_1(\theta_t\omega) - c_1(\theta_t\omega)v^*(t;\omega))}{\mu}$$

and $\hat{u}^{\mu}(x,t;\omega) = e^{-\mu(x-\int_0^t c(s;\omega,\mu)ds)}$. Then $\hat{u}^{\mu}(x,t;\omega)$ satisfies

$$\begin{aligned}
&\partial_t\hat{u}^{\mu}(x,t;\omega) - H\hat{u}^{\mu}(x,t;\omega) - (a_1(\theta_t\omega) - c_1(\theta_t\omega)v^*(t;\omega))\hat{u}^{\mu}(x,t;\omega)\\
&= \hat{u}^{\mu}(x,t;\omega)[\mu c(t;\omega,\mu) - (e^{\mu} + e^{-\mu} - 2) + (a_1(\theta_t\omega) - c_1(\theta_t\omega)v^*(t;\omega))] = 0
\end{aligned}$$

for $x \in \mathbb{R}$, $t \in \mathbb{R}$. Then we have

$$\begin{aligned}
&\partial_t\hat{u}^{\mu} - H\hat{u}^{\mu} - \hat{u}^{\mu}(a_1(\theta_t\omega) - b_1(\theta_t\omega)\hat{u}^{\mu} - c_1(\theta_t\omega)(v^*(t;\omega) - \hat{u}^{\mu}))\\
&= \hat{u}^{\mu}[\mu c(t;\omega,\mu) - (e^{\mu} + e^{-\mu} - 2) - (a_1(\theta_t\omega) - c_1(\theta_t\omega)v^*(t;\omega))]\\
&\quad + \hat{u}^{\mu}(b_1(\theta_t\omega) - c_1(\theta_t\omega))\hat{u}^{\mu}\\
&= \hat{u}^{\mu}(b_1(\theta_t\omega) - c_1(\theta_t\omega))\hat{u}^{\mu} \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t\hat{u}^{\mu} - H\hat{u}^{\mu} - b_2(\theta_t\omega)(v^*(t;\omega) - \hat{u}^{\mu})\hat{u}^{\mu}\\
&- \hat{u}^{\mu}(a_2(\theta_t\omega) - 2c_2(\theta_t\omega)v^*(t;\omega) + c_2(\theta_t\omega)\hat{u}^{\mu})\\
&= \mu c(t;\omega,\mu)\hat{u}^{\mu} - (e^{\mu} + e^{-\mu} - 2)\hat{u}^{\mu} - b_2(\theta_t\omega)v^*(t;\omega)\hat{u}^{\mu} + \hat{u}^{\mu}b_2(\theta_t\omega)\hat{u}^{\mu}\\
&\quad - (a_2(\theta_t\omega) - 2c_2(\theta_t\omega)v^*(t;\omega))\hat{u}^{\mu} - \hat{u}^{\mu}c_2(\theta_t\omega)\hat{u}^{\mu}\\
&= [a_1(\theta_t\omega) - c_1(\theta_t\omega)v^*(t;\omega) - (a_2(\theta_t\omega) - 2c_2(\theta_t\omega)v^*(t;\omega) + b_2(\theta_t\omega)v^*(t;\omega))]\hat{u}^{\mu}\\
&\quad + \hat{u}^{\mu}(b_2(\theta_t\omega) - c_2(\theta_t\omega))\hat{u}^{\mu} \geq 0 \quad \text{for } x \in \mathbb{R},\ t \in \mathbb{R}.
\end{aligned}$$

Hence, $(\hat{u}^{\mu}(x,t;\omega), \hat{u}^{\mu}(x,t;\omega)) = (e^{-\mu(x-\int_0^t c(s;\omega,\mu)ds)}, e^{-\mu(x-\int_0^t c(s;\omega,\mu)ds)})$ is a super-solution of (2.1). Denote

$$(\overline{u}^{\mu}(x,t;\omega), \overline{v}^{\mu}(x,t;\omega)) = \min\{(u^*(t;\omega), v^*(t;\omega)), (\hat{u}^{\mu}(x,t;\omega), \hat{u}^{\mu}(x,t;\omega))\}.$$

Then $(\overline{u}^{\mu}(x,t;\omega), \overline{v}^{\mu}(x,t;\omega))$ is a generalized super-solution of (2.1).

Lemma 3.1. *For $\omega \in \Omega_0$, we have*

$$\begin{aligned}
u(x,t-t_0;\overline{u}^{\mu}(\cdot,t_0;\omega), \overline{v}^{\mu}(\cdot,t_0;\omega), \theta_{t_0}\omega) \leq \overline{u}^{\mu}(x,t;\omega),\\
v(x,t-t_0;\overline{u}^{\mu}(\cdot,t_0;\omega), \overline{v}^{\mu}(\cdot,t_0;\omega), \theta_{t_0}\omega) \leq \overline{v}^{\mu}(x,t;\omega),
\end{aligned}$$

for all $x \in \mathbb{R}$, $t \geq t_0$, $t_0 \in \mathbb{R}$.

Proof. For any constant C, $(\hat{U}(x,t;\omega), \hat{V}(x,t;\omega)) := (e^{Ct}\hat{u}^{\mu}(x,t;\omega), e^{Ct}\hat{u}^{\mu}(x,t;\omega))$ satisfies

$$\begin{aligned}
\partial_t\hat{U}(x,t;\omega) &= (\partial_t\hat{u}^{\mu}(x,t;\omega) + C\hat{u}^{\mu}(x,t;\omega))e^{Ct}\\
&\geq H\hat{U}(x,t;\omega) + C\hat{U}(x,t;\omega) + e^{Ct}f(t,\hat{u},\hat{u},\omega),
\end{aligned}$$

and

$$\begin{aligned}
\partial_t\hat{V}(x,t;\omega) &= (\partial_t\hat{u}^{\mu}(x,t;\omega) + C\hat{u}^{\mu}(x,t;\omega))e^{Ct}\\
&\geq H\hat{V}(x,t;\omega) + C\hat{V}(x,t;\omega) + e^{Ct}g(t,\hat{u},\hat{u},\omega).
\end{aligned}$$

Hence,

$$\hat{U}(x,t;\omega) \ge \hat{U}(x,t_0;\omega) + \int_{t_0}^{t} (H\hat{U}(x,\tau;\omega) + C\hat{U}(x,\tau;\omega) + e^{C\tau} f(\tau,\hat{u},\hat{u},\omega))d\tau,$$

$$\hat{V}(x,t;\omega) \ge \hat{V}(x,t_0;\omega) + \int_{t_0}^{t} (H\hat{V}(x,\tau;\omega) + C\hat{V}(x,\tau;\omega) + e^{C\tau} g(\tau,\hat{u},\hat{u},\omega))d\tau.$$

Denote $(\overline{U}(x,t;\omega),\overline{V}(x,t;\omega)) := (e^{Ct}\overline{u}^{\mu}(x,t;\omega), e^{Ct}\overline{v}^{\mu}(x,t;\omega))$. Then we have

$$\overline{U}(x,t;\omega) \ge \overline{U}(x,t_0;\omega) + \int_{t_0}^{t} (H\overline{U}(x,\tau;\omega) + C\overline{U}(x,\tau;\omega) + e^{C\tau} f(\tau,\overline{u},\overline{v},\omega))d\tau,$$

$$\overline{V}(x,t;\omega) \ge \overline{V}(x,t_0;\omega) + \int_{t_0}^{t} (H\overline{V}(x,\tau;\omega) + C\overline{V}(x,\tau;\omega) + e^{C\tau} g(\tau,\overline{u},\overline{v},\omega))d\tau.$$

Let $Q_1(x,t;\omega) = e^{Ct}(\overline{u}^{\mu}(x,t;\omega) - u(x,t-t_0;\overline{u}^{\mu}(\cdot,t_0;\omega),\overline{v}^{\mu}(\cdot,t_0;\omega),\theta_{t_0}\omega))$ and $Q_2(x,t;\omega) = e^{Ct}(\overline{v}^{\mu}(x,t;\omega) - v(x,t-t_0;\overline{u}^{\mu}(\cdot,t_0;\omega),\overline{v}^{\mu}(\cdot,t_0;\omega),\theta_{t_0}\omega))$. Then

$$\begin{aligned}
&Q_1(x,t;\omega) - Q_1(x,t_0;\omega)\\
&\ge \int_{t_0}^{t} (HQ_1(x,\tau;\omega) + a_1(x,\tau;\omega)Q_1(x,\tau;\omega) + b_1(x,\tau;\omega)Q_2(x,\tau;\omega))d\tau,
\end{aligned}$$

and

$$\begin{aligned}
&Q_2(x,t;\omega) - Q_2(x,t_0;\omega)\\
&\ge \int_{t_0}^{t} (HQ_2(x,\tau;\omega) + a_2(x,\tau;\omega)Q_1(x,\tau;\omega) + b_2(x,\tau;\omega)Q_2(x,\tau;\omega))d\tau,
\end{aligned}$$

where

$$\begin{gathered}
a_1(x,t;\omega) = C + f_u(t,u_1^*,v_1^*,\omega), \quad b_1(x,t;\omega) = f_v(t,u_1^*,v_1^*,\omega),\\
a_2(x,t;\omega) = g_u(t,u_2^*,v_2^*,\omega), \quad b_2(x,t;\omega) = C + g_v(t,u_2^*,v_2^*,\omega).
\end{gathered}$$

Since (2.1) is cooperative, we know that $b_1(x,t;\omega) \ge 0$ and $a_2(x,t;\omega) \ge 0$. By the boundedness of $\bar{u}^{\mu}(x,t;\omega)$, $\bar{v}^{\mu}(x,t;\omega)$, $u(x,t-t_0;\overline{u}^{\mu}(\cdot,t_0;\omega),\overline{v}^{\mu}(\cdot,t_0;\omega),\theta_{t_0}\omega)$ and $v(x,t-t_0;\overline{u}^{\mu}(\cdot,t_0;\omega), \overline{v}^{\mu}(\cdot,t_0;\omega),\theta_{t_0}\omega)$, we can choose $C > 0$ such that $b_2(x,t;\omega) \ge 0$ and $a_1(x,t;\omega) \ge 0$ for all $t \ge t_0$, $x \in \mathbb{R}$ and a.a. $\omega \in \Omega$. By the arguments of Proposition 2.1, we have that

$$Q_i(x,t;\omega) \ge Q_i(x,t_0;\omega) = 0, \quad i = 1,2,$$

and hence for $\omega \in \Omega_0$, we have that $u(x,t-t_0;\overline{u}^{\mu}(\cdot,t_0;\omega),\overline{v}^{\mu}(\cdot,t_0;\omega),\theta_{t_0}\omega) \le \overline{u}^{\mu}(x,t;\omega)$ and $v(x,t-t_0;\overline{u}^{\mu}(\cdot,t_0;\omega),\overline{v}^{\mu}(\cdot,t_0;\omega),\theta_{t_0}\omega) \le \overline{v}^{\mu}(x,t;\omega)$ for all $x \in \mathbb{R}$, $t \ge t_0$, $t_0 \in \mathbb{R}$. □

Next, we construct a sub-solution of (2.1). Let $\tilde{\mu} > 0$ be such that $\mu < \tilde{\mu} < \min\{2\mu,\mu^*\}$ and $\omega \in \Omega_0$. Let A_ω and d_ω be given by Lemma 2.5, and let

$$x_\omega(t) = \int_0^t c(s;\omega,\mu)ds + \frac{\ln d_\omega + \ln\tilde{\mu} - \ln\mu}{\tilde{\mu} - \mu} + \frac{A_\omega(t)}{\mu}.$$

Recall that

$$\begin{gathered}
\tilde{u}(x,t,\omega) = e^{-\mu(x-\int_0^t c(s;\omega,\mu)ds)} - de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}(x-\int_0^t c(s;\omega,\mu)ds)},\\
\tilde{v}(x,t,\omega) = \sigma e^{-\mu(x-\int_0^t c(s;\omega,\mu)ds)}h(t;\omega) - \sigma de^{(\frac{\tilde{\mu}}{\mu}-1)A_\omega(t)-\tilde{\mu}(x-\int_0^t c(s;\omega,\mu)ds)}h(t;\omega)
\end{gathered}$$

By calculations we have that for any given $t \in \mathbb{R}$,

$$\begin{aligned}(\tilde{u}(x_\omega(t),t,\omega),\tilde{v}(x_\omega(t),t,\omega)) &= (\sup_{x\in\mathbb{R}}\tilde{u}(x,t,\omega),\sup_{x\in\mathbb{R}}\tilde{v}(x,t,\omega))\\ &= \Big(e^{-\mu(\frac{\ln d_\omega}{\tilde{\mu}-\mu}+\frac{A_\omega(t)}{\mu})}e^{-\mu\frac{\ln\tilde{\mu}-\ln\mu}{\tilde{\mu}-\mu}}(1-\frac{\mu}{\tilde{\mu}}),\\ &\quad \sigma h(t;\omega)e^{-\mu(\frac{\ln d_\omega}{\tilde{\mu}-\mu}+\frac{A_\omega(t)}{\mu})}e^{-\mu\frac{\ln\tilde{\mu}-\ln\mu}{\tilde{\mu}-\mu}}(1-\frac{\mu}{\tilde{\mu}})\Big).\end{aligned} \tag{3.1}$$

Define

$$\begin{aligned}&(\underline{u}^\mu(x,t;\theta_{t_0}\omega),\underline{v}^\mu(x,t;\theta_{t_0}\omega))\\ &= \begin{cases}(\tilde{u}(x,t+t_0,\omega),\tilde{v}(x,t+t_0,\omega)), & \text{if } x\ge x_\omega(t+t_0),\\ (\tilde{u}(x_\omega(t+t_0),t+t_0,\omega),\tilde{v}(x_\omega(t+t_0),t+t_0,\omega)), & \text{if } x\le x_\omega(t+t_0).\end{cases}\end{aligned}$$

Then $(\underline{u}^\mu(x,t;\omega),\underline{v}^\mu(x,t;\omega))$ is a generalized sub-solution of (2.1). It is clear that

$$\begin{aligned}(0,0) &< (\underline{u}^\mu(\cdot,t;\theta_{t_0}\omega),\underline{v}^\mu(\cdot,t;\theta_{t_0}\omega))\\ &< (\overline{u}^\mu(\cdot,t;\theta_{t_0}\omega),\overline{v}^\mu(\cdot,t;\theta_{t_0}\omega))\\ &\le (u^*(t+t_0;\omega),v^*(t+t_0;\omega))\end{aligned}$$

for all $t,t_0\in\mathbb{R}$, and there exists $\tilde{\sigma}>0$ such that

$$\lim_{x\to\infty}\sup_{t\in\mathbb{R},t_0\in\mathbb{R}}\frac{\underline{u}^\mu(x,t;\theta_{t_0}\omega)}{\overline{u}^\mu(x,t;\theta_{t_0}\omega)}=1,\quad \lim_{x\to\infty}\sup_{t\in\mathbb{R},t_0\in\mathbb{R}}\frac{\underline{v}^\mu(x,t;\theta_{t_0}\omega)}{\overline{v}^\mu(x,t;\theta_{t_0}\omega)}=\tilde{\sigma}. \tag{3.2}$$

Note that by the similar arguments as in Lemma 3.1, we can prove that

$$\begin{aligned}u(x,t-t_0;\underline{u}^\mu(\cdot,t_0;\omega),\underline{v}^\mu(\cdot,t_0;\omega),\theta_{t_0}\omega)\ge\underline{u}^\mu(x,t;\omega),\\ v(x,t-t_0;\underline{u}^\mu(\cdot,t_0;\omega),\underline{v}^\mu(\cdot,t_0;\omega),\theta_{t_0}\omega)\ge\underline{v}^\mu(x,t;\omega)\end{aligned}$$

for $x\in\mathbb{R}$, $t\ge t_0$ and a.a. $\omega\in\Omega$. Now we are in a position to prove the main Theorem.

Proof of Theorem 1.2. (i) By Lemma 3.1 we have

$$u(x,t-t_0;\overline{u}^\mu(\cdot,t_0;\omega),\overline{v}^\mu(\cdot,t_0;\omega),\theta_{t_0}\omega)\le\overline{u}^\mu(x,t;\omega)$$

It then follows that

$$u(x,\tau_2-\tau_1;\overline{u}^\mu(\cdot,-\tau_2;\omega),\overline{v}^\mu(\cdot,-\tau_2;\omega),\theta_{-\tau_2}\omega)\le\overline{u}^\mu(x,-\tau_1;\omega)$$

for $x\in\mathbb{R}$ and $\tau_2>\tau_1$. Then we obtain

$$\begin{aligned}&u\Big(x,t+\tau_1;u(\cdot,\tau_2-\tau_1;\overline{u}^\mu(\cdot,-\tau_2;\omega),\overline{v}^\mu(\cdot,-\tau_2;\omega),\theta_{-\tau_2}\omega),\\ &v(\cdot,\tau_2-\tau_1;\overline{u}^\mu(\cdot,-\tau_2;\omega),\overline{v}^\mu(\cdot,-\tau_2;\omega),\theta_{-\tau_2}\omega),\theta_{-\tau_1}\omega\Big)\\ &\le u(x,t+\tau_1;\overline{u}^\mu(\cdot,-\tau_1;\omega),\overline{v}^\mu(\cdot,-\tau_1;\omega),\theta_{-\tau_1}\omega)\end{aligned}$$

for $x\in\mathbb{R}$, $t\ge-\tau_1$, $\tau_2>\tau_1$, and hence

$$\begin{aligned}&u\big(x,t+\tau_2;\overline{u}^\mu(\cdot,-\tau_2;\omega),\overline{v}^\mu(\cdot,-\tau_2;\omega),\theta_{-\tau_2}\omega\big)\\ &\le u\big(x,t+\tau_1;\overline{u}^\mu(\cdot,-\tau_1;\omega),\overline{v}^\mu(\cdot,-\tau_1;\omega),\theta_{-\tau_1}\omega\big)\end{aligned}$$

for $x\in\mathbb{R}$, $t\ge-\tau_1$, $\tau_2>\tau_1$.

Therefore $\lim_{\tau\to\infty} u(x,t+\tau;\overline{u}^\mu(\cdot,-\tau;\omega),\overline{v}^\mu(\cdot,-\tau;\omega),\theta_{-\tau}\omega)$ exists. Similarly, we can get that $\lim_{\tau\to\infty} v(x,t+\tau;\overline{u}^\mu(\cdot,-\tau;\omega),\overline{v}^\mu(\cdot,-\tau;\omega),\theta_{-\tau}\omega)$ exists. Define

$$\begin{aligned} U(x,t;\omega) &:= \lim_{\tau\to\infty} u(x,t+\tau;\overline{u}^\mu(\cdot,-\tau;\omega),\overline{v}^\mu(\cdot,-\tau;\omega),\theta_{-\tau}\omega), \\ V(x,t;\omega) &:= \lim_{\tau\to\infty} v(x,t+\tau;\overline{u}^\mu(\cdot,-\tau;\omega),\overline{v}^\mu(\cdot,-\tau;\omega),\theta_{-\tau}\omega) \end{aligned}$$

for $x\in\mathbb{R}$, $t\in\mathbb{R}$, $\omega\in\Omega_0$. Then $(U(x,t;\omega),V(x,t;\omega))$ is non-increasing in $x\in\mathbb{R}$ and by dominated convergence theorem we know that $(U(x,t;\omega),V(x,t;\omega))$ is a solution of (2.1).

We claim that, for every $\omega\in\Omega_0$,

$$\begin{aligned} &\lim_{x\to-\infty}\Big(U(x+\int_0^t c(s;\omega,\mu)ds,t;\omega),V(x+\int_0^t c(s;\omega,\mu)ds,t;\omega)\Big) \\ &=(u^*(t;\omega),v^*(t;\omega)) \quad \text{uniformly in } t\in\mathbb{R}. \end{aligned} \tag{3.3}$$

In fact, fixing any $\omega\in\Omega_0$, and letting $\hat{x}_\omega=\frac{\ln d_\omega+\ln\tilde{\mu}-\ln\mu}{\tilde{\mu}-\mu}-\frac{\|A_\omega\|_\infty}{\mu}$, from (3.1), $\inf_{t\in\mathbb{R}} h(t;\omega)>0$ and $(\underline{u}^\mu(x,t;\omega),\underline{v}^\mu(x,t;\omega))\le(U(x,t;\omega),V(x,t;\omega))$ it follows that

$$0<(1-\frac{\mu}{\tilde{\mu}})e^{-\mu(\frac{\ln d_\omega+\ln\tilde{\mu}-\ln\mu}{\tilde{\mu}-\mu}+\frac{\|A_\omega\|_\infty}{\mu})}\le\inf_{t\in\mathbb{R}} U(\hat{x}_\omega+\int_0^t c(s;\omega,\mu)ds,t;\omega),$$

and

$$0<\sigma\inf_{t\in\mathbb{R}} h(t;\omega)(1-\frac{\mu}{\tilde{\mu}})e^{-\mu(\frac{\ln d_\omega+\ln\tilde{\mu}-\ln\mu}{\tilde{\mu}-\mu}+\frac{\|A_\omega\|_\infty}{\mu})}\le\inf_{t\in\mathbb{R}} V(\hat{x}_\omega+\int_0^t c(s;\omega,\mu)ds,t;\omega).$$

Let $(u_0(x),v_0(x))\equiv(u_0,v_0)$, where

$$(u_0,v_0):=(\inf_{t\in\mathbb{R}} U(\hat{x}_\omega+\int_0^t c(s;\omega,\mu)ds,t;\omega),\inf_{t\in\mathbb{R}} V(\hat{x}_\omega+\int_0^t c(s;\omega,\mu)ds,t;\omega)),$$

and $(\tilde{u}_0(x),\tilde{v}_0(x))$ be uniformly continuous such that $(\tilde{u}_0(x),\tilde{v}_0(x))=(u_0(x),v_0(x))$ for $x<\hat{x}_\omega-1$ and $(\tilde{u}_0(x),\tilde{v}_0(x))=(0,0)$ for $x\ge\hat{x}_\omega$. Then $\lim_{n\to\infty}(\tilde{u}_0(x-n),\tilde{v}_0(x-n))=(u_0(x),v_0(x))$ locally uniformly in $x\in\mathbb{R}$. Note that by (H2), we have

$$\lim_{t\to\infty}(u(x,t;u_0,v_0,\theta_{t_0}\omega)-u^*(t+t_0;\omega),v(x,t;u_0,v_0,\theta_{t_0}\omega)-v^*(t+t_0;\omega))=(0,0)$$

uniformly in $t_0\in\mathbb{R}$ and $x\in\mathbb{R}$. Then for any $\epsilon>0$, there is $T:=T(\epsilon)>0$ such that

$$u^*(t_0+T;\omega)>u(x,T;u_0,v_0,\theta_{t_0}\omega)>u^*(t_0+T;\omega)-\epsilon,\quad \forall t_0\in\mathbb{R},\ x\in\mathbb{R}.$$

Therefore, from the definition of $c(t,\omega,\mu)$ we know that,

$$u^*(t_0+T;\omega)>u(x+\int_0^T c(s;\theta_{t_0}\omega,\mu)ds,T;u_0,v_0,\theta_{t_0}\omega)>u^*(t_0+T;\omega)-\epsilon$$

for all $t_0\in\mathbb{R}$ and $x\in\mathbb{R}$. By Proposition 2.2, there is $N:=N(\epsilon)>1$ such that

$$\begin{aligned} u^*(t_0+T;\omega)&>u\Big(\int_0^T c(s;\theta_{t_0}\omega,\mu)ds,T;\tilde{u}_0(\cdot-N),\tilde{v}_0(\cdot-N),\theta_{t_0}\omega\Big) \\ &>u^*(t_0+T;\omega)-2\varepsilon,\quad \forall t_0\in\mathbb{R}. \end{aligned}$$

That is,

$$\begin{aligned} u^*(t_0+T;\omega)&>u(\int_0^T c(s;\theta_{t_0}\omega,\mu)ds-N,T;\tilde{u}_0(\cdot),\tilde{v}_0(\cdot),\theta_{t_0}\omega) \\ &>u^*(t_0+T;\omega)-2\varepsilon,\quad \forall t_0\in\mathbb{R}. \end{aligned}$$

Note that

$$\begin{gathered}
U(x+\int_0^{t-T} c(s;\omega,\mu)ds, t-T;\omega) \geq \tilde{u}_0(x), \quad \forall t\in\mathbb{R},\ x\in\mathbb{R},\\
V(x+\int_0^{t-T} c(s;\omega,\mu)ds, t-T;\omega) \geq \tilde{v}_0(x), \quad \forall t\in\mathbb{R},\ x\in\mathbb{R},\\
\int_0^t c(s;\omega,\mu)ds = \int_0^T c(s;\theta_{t-T}\omega,\mu)ds + \int_0^{t-T} c(s;\omega,\mu)ds.
\end{gathered}$$

Then we obtain

$$\begin{aligned}
u^*(t;\omega) &> U(x+\int_0^t c(s;\omega,\mu)ds, t;\omega)\\
&= u\Big(x+\int_0^T c(s;\theta_{t-T}\omega,\mu)ds, T; U(\cdot+\int_0^{t-T} c(s;\omega,\mu)ds, t-T;\omega),\\
&\quad V(\cdot+\int_0^{t-T} c(s;\omega,\mu)ds, t-T;\omega), \theta_{t-T}\omega\Big)\\
&> u^*(t;\omega)-2\epsilon, \quad \forall t\in\mathbb{R},\ x\leq -N,
\end{aligned}$$

and hence $\lim_{x\to-\infty} U(x+\int_0^t c(s;\omega,\mu)ds, t;\omega) = u^*(t;\omega)$ uniformly in $t\in\mathbb{R}$. Similarly, we can derive $\lim_{x\to-\infty} V(x+\int_0^t c(s;\omega,\mu)ds, t;\omega) = v^*(t;\omega)$ uniformly in $t\in\mathbb{R}$. Thus (3.3) follows.

Note that by (3.2) we have that for every $\omega\in\Omega_0$,

$$\begin{gathered}
\lim_{x\to\infty}\sup_{t\in\mathbb{R}} U(x+\int_0^t c(s;\omega,\mu)ds, t;\omega) = 0,\\
\lim_{x\to\infty}\sup_{t\in\mathbb{R}} V(x+\int_0^t c(s;\omega,\mu)ds, t;\omega) = 0.
\end{gathered}$$

Set

$$\begin{gathered}
(\tilde{\Phi}(x,t;\omega), \tilde{\Psi}(x,t;\omega)) = \Big(U(x+\int_0^t c(s;\omega,\mu)ds, t;\omega), V(x+\int_0^t c(s;\omega,\mu)ds, t;\omega)\Big),\\
(\Phi(x,\omega), \Psi(x,\omega)) = (\tilde{\Phi}(x,0;\omega), \tilde{\Psi}(x,0;\omega)).
\end{gathered}$$

We now claim that $(\tilde{\Phi}(x,t;\omega), \tilde{\Psi}(x,t;\omega))$ is stationary ergodic in t, that is, for a.a. $\omega\in\Omega$,

$$(\tilde{\Phi}(x,t;\omega), \tilde{\Psi}(x,t;\omega)) = (\tilde{\Phi}(x,0;\theta_t\omega), \tilde{\Psi}(x,0;\theta_t\omega)).$$

In fact, note that for $\omega\in\Omega$,

$$\begin{aligned}
\int_{-\tau}^t c(s;\omega,\mu)ds &= \int_{-\tau}^t \frac{e^\mu+e^{-\mu}-2+a_1(\theta_s\omega)-c_1(\theta_s\omega)v^*(s;\omega)}{\mu}ds\\
&= \frac{e^\mu+e^{-\mu}-2}{\mu}(t+\tau) + \int_{-\tau}^t \frac{a_1(\theta_s\omega)-c_1(\theta_s\omega)v^*(s;\omega)}{\mu}ds
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\int_{-(t+\tau)}^{0} c(s;\theta_t\omega,\mu)ds \\ &= \int_{-(t+\tau)}^{0} \frac{e^{\mu}+e^{-\mu}-2+a_1(\theta_s\circ\theta_t\omega)-c_1(\theta_s\circ\theta_t\omega)v^*(s;\theta_t\omega)}{\mu}ds \\ &= \frac{e^{\mu}+e^{-\mu}-2}{\mu}(t+\tau)+\int_{-(t+\tau)}^{0}\frac{a_1(\theta_{s+t}\omega)-c_1(\theta_{s+t}\omega)v^*(s+t;\omega)}{\mu}ds \\ &= \frac{e^{\mu}+e^{-\mu}-2}{\mu}(t+\tau)+\int_{-\tau}^{t}\frac{a_1(\theta_s\omega)-c_1(\theta_s\omega)v^*(s;\omega)}{\mu}ds. \end{aligned} \tag{3.5}$$

Combining (3.4) with (3.5), we derive $\int_{-\tau}^{t} c(s;\omega,\mu)ds = \int_{-(t+\tau)}^{0} c(s;\theta_t\omega,\mu)ds$ for $\tau \geq 0$ and $t \in \mathbb{R}$. Recall that

$$\begin{gathered} (\overline{u}^{\mu}(x,t;\omega),\overline{v}^{\mu}(x,t;\omega)) = \min\left\{(u^*(t;\omega),v^*(t;\omega)),(\hat{u}(x,t;\omega),\hat{u}(x,t;\omega))\right\}, \\ (\hat{u}(x,t;\omega),\hat{u}(x,t;\omega)) = (e^{-\mu(x-\int_0^t c(s;\omega,\mu)ds)}, e^{-\mu(x-\int_0^t c(s;\omega,\mu)ds)}). \end{gathered}$$

Then we have

$$\begin{aligned} &\tilde{\Phi}(x,t;\omega) \\ &= \lim_{\tau\to\infty} u\Big(x+\int_0^t c(s;\omega,\mu)ds, t+\tau; \overline{u}^{\mu}(\cdot,-\tau;\omega),\overline{v}^{\mu}(\cdot,-\tau;\omega),\theta_{-\tau}\omega\Big) \\ &= \lim_{\tau\to\infty} u\Big(x,t+\tau;\overline{u}^{\mu}(\cdot+\int_0^t c(s;\omega,\mu)ds,-\tau;\omega),\overline{v}^{\mu}(\cdot+\int_0^t c(s;\omega,\mu)ds, \\ &\quad -\tau;\omega),\theta_{-\tau}\omega\Big) \\ &= \lim_{\tau\to\infty} u\Big(x,t+\tau;\overline{u}^{\mu}(\cdot,-(t+\tau);\theta_t\omega),\overline{v}^{\mu}(\cdot,-(t+\tau);\theta_t\omega),\theta_{-\tau}\omega\Big) \\ &= \lim_{\tau\to\infty} u\Big(x,t+\tau;\overline{u}^{\mu}(\cdot,-(t+\tau);\theta_t\omega),\overline{v}^{\mu}(\cdot,-(t+\tau);\theta_t\omega),\theta_{t-(t+\tau)}\omega\Big) \\ &= \lim_{\tau\to\infty} u\Big(x,\tau;\overline{u}^{\mu}(\cdot,-\tau;\theta_t\omega),\overline{v}^{\mu}(\cdot,-\tau;\theta_t\omega),\theta_{t-\tau}\omega\Big) \\ &= \tilde{\Phi}(x,0;\theta_t\omega). \end{aligned}$$

Similarly, we can get $\tilde{\Psi}(x,t;\omega) = \tilde{\Psi}(x,0;\theta_t\omega)$, and hence $(\tilde{\Phi}(x,t;\omega),\tilde{\Psi}(x,t;\omega)) = (\tilde{\Phi}(x,0;\theta_t\omega),\tilde{\Psi}(x,0;\theta_t\omega))$. The claim thus follows and we obtain the desired random profile $(\Phi(x,\omega),\Psi(x,\omega))$.

(ii) Let

$$\begin{aligned} c_*(\omega) = \sup\big\{c : \limsup_{|i|\leq ct, t\to\infty} &[|u_i(t;u^0,v^0,\theta_s\omega)-u^*(t+s;\omega)| \\ &+|v_i(t;u^0,v^0,\theta_s\omega)-v^*(t+s;\omega)|]=0 \\ &\text{uniformly in } s\in\mathbb{R} \text{ for all } (u^0,v^0)\in l_0^{\infty}(\mathbb{Z})\times l_0^{\infty}(\mathbb{Z})\big\}, \end{aligned}$$

where

$$l_0^{\infty}(\mathbb{Z}) = \{u=\{u_i\}_{i\in\mathbb{Z}}\in l^{\infty}(\mathbb{Z}) : u_i\geq 0 \text{ for all } i\in\mathbb{Z},\ u_i=0 \text{ for } |i|\gg 1,\ \{u_i\}\neq 0\}.$$

Recall that

$$\underline{\lambda} = \liminf_{t-s\to\infty}\frac{1}{t-s}\int_s^t (a_1(\theta_\tau\omega)-c_1(\theta_\tau\omega)v^*(\tau;\omega))d\tau,$$

$$c_0 := \inf_{\mu>0} \frac{e^{\mu}+e^{-\mu}-2+\underline{\lambda}}{\mu}.$$

We claim that $c_*(\omega)=c_0$ for $\omega\in\Omega_0$. In fact, we consider

$$\dot u_i(t) = u_{i+1}(t)-2u_i(t)+u_{i-1}(t)+u_i(t)(a_1(\theta_t\omega)-c_1(\theta_t\omega)v^*(t;\omega)-b_1(\theta_t\omega)u_i(t)) \tag{3.6}$$

For any $u^0\in l^{\infty,+}(\mathbb{Z})$, let $u^-(t;u^0,\omega)$ be the solution of (3.6) with $u^-(0;u^0,\omega)=u^0$. By comparison principle, for any $(u^0,v^0)\in l^{\infty,+}(\mathbb{Z})\times l^{\infty,+}(\mathbb{Z})$, we have

$$u_i(t;u^0,v^0,\omega)\ge u_i^-(t;u^0,\omega),\quad \forall t\ge 0. \tag{3.7}$$

By [3, Remark 1.1 (1)], for any $c(\omega)$ with $0<c(\omega)<c_0$,

$$\liminf_{s\in\mathbb{R},|i|\le c(\omega)t,t\to\infty} u_i^-(t;u^0,\theta_s\omega)>0.$$

With (3.7), we then have

$$\liminf_{s\in\mathbb{R},|i|\le c(\omega)t,t\to\infty} u_i(t;u^0,v^0,\theta_s\omega)>0.$$

Then by Lemma 2.6, for any $0<c<c(\omega)$,

$$\limsup_{|i|\le ct,t\to\infty}[|u_i(t;u^0,v^0,\theta_s\omega)-u^*(t+s;\omega)|+|v_i(t;u^0,v^0,\theta_s\omega)-v^*(t+s;\omega)|]=0$$

uniformly in $s\in\mathbb{R}$. which implies that $c_*(\omega)\ge c_0$.

Assume that $c_*(\omega)>c_0$ for some $\omega\in\Omega_0$. Fix γ, c' and c'' such that

$$c_0<\gamma<c'<c''<c_*(\omega).$$

Observe that $c_0>0$. For any $(u^0,v^0)\in l_0^{\infty}(\mathbb{Z})\times l_0^{\infty}(\mathbb{Z})$,

$$\begin{aligned}\limsup_{|i|\le c''t,t\to\infty}&[|u_i(t;u^0,v^0,\theta_s\omega)-u^*(t+s;\omega)|\\&+|v_i(t;u^0,v^0,\theta_s\omega)-v^*(t+s;\omega)|]=0\end{aligned} \tag{3.8}$$

uniformly in $s\in\mathbb{R}$.

Let $(u_i(t;\omega),v_i(t;\omega))=(\Phi(i-\int_0^t c(s;\omega)ds,\theta_t\omega),\Psi(i-\int_0^t c(s;\omega)ds,\theta_t\omega))$ be as in (i) with $\bar c_{\rm inf}=\gamma$. Let

$$u_i^s=\Phi(i-\big[\int_0^s c(\tau;\omega)d\tau\big],\theta_s\omega),\quad v_i^s=\Psi(i-\big[\int_0^s c(\tau;\omega)d\tau\big],\theta_s\omega),\quad \forall s\in\mathbb{R}.$$

By (i), there is $(u^0,v^0)\in l_0^{\infty}(\mathbb{Z})\times l_0^{\infty}(\mathbb{Z})$ such that

$$(u^0,v^0)\le(u^s,v^s),\quad \forall s\in\mathbb{R}.$$

Hence

$$\begin{gathered}u_i(t;u^0,v^0,\theta_s\omega)\le u_i(t;u^s,v^s,\theta_s\omega),\\ v_i(t;u^0,v^0,\theta_s\omega)\le v_i(t;u^s,v^s,\theta_s\omega)\end{gathered}$$

for $i\in\mathbb{Z}$, $s\in\mathbb{R}$ and $t\ge 0$. This together with (3.8) implies that

$$\begin{aligned}\limsup_{|i|\le c''t,t\to\infty}&[|u_i(t;u^s,v^s,\theta_s\omega)-u^*(t+s;\omega)|\\&+|v_i(t;u^s,v^s,\theta_s\omega)-v^*(t+s;\omega)|]=0\end{aligned} \tag{3.9}$$

uniformly in $s \in \mathbb{R}$. Note that $\int_0^{t+s} c(\tau;\omega)d\tau = \int_0^s c(\tau;\omega)d\tau + \int_0^t c(\tau;\theta_s\omega)d\tau$. By (i) again, we have

$$\begin{aligned} u_i(t;u^s,v^s,\theta_s\omega) &= \Phi\Big(i - \int_0^t c(\tau;\theta_s\omega)d\tau - \big[\int_0^s c(\tau;\omega)d\tau\big], \theta_{t+s}\omega\Big) \\ &\le \Phi\Big(i - \int_0^{t+s} c(\tau;\omega)d\tau, \theta_{t+s}\omega\Big), \end{aligned}$$

and

$$\begin{aligned} v_i(t;u^s,v^s,\theta_s\omega) &= \Psi\Big(i - \int_0^t c(\tau;\theta_s\omega)d\tau - \big[\int_0^s c(\tau;\omega)d\tau\big], \theta_{t+s}\omega\Big) \\ &\le \Psi\Big(i - \int_0^{t+s} c(\tau;\omega)d\tau, \theta_{t+s}\omega\Big). \end{aligned}$$

Then

$$\limsup_{i \ge (c''-c')(t+s) + \int_0^{t+s} c(\tau;\omega)d\tau, t\to\infty} [u_i(t;u^s,v^s,\theta_s\omega) + v_i(t;u^s,v^s,\theta_s\omega)] = 0 \tag{3.10}$$

uniformly in $s \in \mathbb{R}$. It follows from (3.9) and (3.10) that

$$\bar{c}_{\rm inf} \ge c' > \gamma,$$

which is a contradiction. Therefore, $c_*(\omega) = c_0$.

Suppose that $(u(t;\omega), v(t;\omega)) = \{(u_i(t;\omega), v_i(t;\omega))\}_{i\in\mathbb{Z}}$ with $(u_i(t;\omega), v_i(t;\omega)) = (\Phi(i - \int_0^t c(s;\omega)ds, \theta_t\omega),\ \Psi(i - \int_0^t c(s;\omega)ds, \theta_t\omega))$ is a random transition front of (1.5) connecting $(u^*(t;\omega), v^*(t;\omega))$ and $(0,0)$. We prove that its least mean speed $\bar{c}_{\rm inf} \ge c_0$. Observe that $\inf_{x\le z}\inf_{s\in\mathbb{R}} \Phi(x,\theta_s\omega) > 0$ and $\inf_{x\le z}\inf_{s\in\mathbb{R}} \Psi(x,\theta_s\omega) > 0$ for all $z \in \mathbb{R}$. Therefore, we can choose $(u^0_\omega, v^0_\omega) \in l_0^\infty(\mathbb{Z}) \times l_0^\infty(\mathbb{Z})$ such that $(u^0_\omega, v^0_\omega) \le (\Phi(x,\theta_s\omega), \Psi(x,\theta_s\omega))$ for all $s \in \mathbb{R}$. Let $0 < \epsilon \ll 1$. Then by $c_*(\omega) = c_0$ and the comparison principle, we have

$$\begin{aligned} &\limsup_{t\to\infty}\sup_{s\in\mathbb{R}}[|u_{[(c_0-\epsilon)t]}(t;u^0_\omega,v^0_\omega,\theta_s\omega) - u^*(t+s;\omega)| \\ &+ |v_{[(c_0-\epsilon)t]}(t;u^0_\omega,v^0_\omega,\theta_s\omega) - v^*(t+s;\omega)|] = 0, \end{aligned}$$

and

$$\begin{aligned} &\liminf_{t\to\infty}\inf_{s\in\mathbb{R}}\{u_{[(c_0-\epsilon)t]}(t;u^0_\omega,v^0_\omega,\theta_s\omega) + v_{[(c_0-\epsilon)t]}(t;u^0_\omega,v^0_\omega,\theta_s\omega)\} \\ &\le \liminf_{t\to\infty}\inf_{s\in\mathbb{R}}\{u_{[(c_0-\epsilon)t]}(t;\Phi(\cdot,\theta_s\omega),\Psi(\cdot,\theta_s\omega),\theta_s\omega) \\ &\quad + v_{[(c_0-\epsilon)t]}(t;\Phi(\cdot,\theta_s\omega),\Psi(\cdot,\theta_s\omega),\theta_s\omega)\} \\ &= \liminf_{t\to\infty}\inf_{s\in\mathbb{R}}\{\Phi([(c_0-\epsilon)t] - \int_0^t c(\tau;\theta_s\omega)d\tau, \theta_{t+s}\omega) \\ &\quad + \Psi([(c_0-\epsilon)t] - \int_0^t c(\tau;\theta_s\omega)d\tau, \theta_{t+s}\omega)\}. \end{aligned}$$

From this and $\int_0^{t+s} c(\tau;\omega)d\tau = \int_0^s c(\tau;\omega)d\tau + \int_0^t c(\tau;\theta_s\omega)d\tau$, we know that there is a $M(\omega)$ such that $(c_0-\epsilon)t \le \int_0^{t+s} c(\tau;\omega)d\tau - \int_0^s c(\tau;\omega)d\tau + M(\omega)$ for all $t > 0$, $s \in \mathbb{R}$. Hence,

$$\bar{c}_{\rm inf} = \liminf_{t\to\infty}\inf_{s\in\mathbb{R}} \frac{\int_0^{t+s} c(\tau;\omega)d\tau - \int_0^s c(\tau;\omega)d\tau}{t} \ge c_0 - \epsilon.$$

By the arbitrariness of $\epsilon > 0$, we obtain $\bar{c}_{\rm inf} \ge c_0$. $\square$

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