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DECAY OF ENERGY FOR VISCOELASTIC WAVE EQUATIONS WITH BALAKRISHNAN-TAYLOR DAMPING AND MEMORIES

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Abstract. In this article, we consider a viscoelastic wave equation with Balakrishnan-Taylor damping, and finite and infinite memory terms in a bounded domain. Under suitable assumptions on relaxation functions and with certain initial data, by adopting the perturbed energy method, we establish a decay of energy which depends on the behavior of the relaxation functions.

1. Introduction

In this article, we study the following viscoelastic problem with Balakrishnan-Taylor damping, a nonlinear source term and finite and infinite memories:

$$\begin{gathered}
|u_t|^\rho u_{tt} - M(t)\Delta u - \Delta u_{tt} - \Delta u_t + \int_0^t g_1(t-s)\operatorname{div}(a_1(x)\nabla u(s))ds \\
+ \int_0^\infty g_2(s)\operatorname{div}(a_2(x)\nabla u(t-s))ds + \gamma(t)h(u_t) \\
= |u|^{p-1}u, \quad \text{in } \Omega\times(0,\infty), \\
u(x,t) = 0, \quad \text{on } \partial\Omega\times(0,\infty), \\
u(x,-t) = u_0(x,t), \quad \text{in } \Omega\times(0,\infty), \\
u_t(x,0) = u_1(x), \quad \text{in } \Omega,
\end{gathered} \tag{1.1}$$

where $M(t) = a + b\|\nabla u\|_2^2 + \sigma\int_\Omega \nabla u\cdot\nabla u_t\,dx$, a, b, σ are positive constants, Ω is a bounded domain of R^n $(n\geq 1)$ with smooth boundary $\partial\Omega$, g_1 and g_2 are positive non-increasing functions defined on R^+, $a_1(x)$ and $a_2(x)$ are essentially bounded non-negative functions, h is a non-decreasing function, p and ρ satisfy

$$\begin{gathered}
1 < p \leq \frac{n}{n-2}, \quad \text{for } n\geq 3, \\
1 \leq p < \infty, \quad \text{for } n = 1,2, \\
0 < \rho \leq \frac{2}{n-2}, \quad \text{for } n\geq 3, \\
0 \leq \rho < \infty, \quad \text{for } n = 1,2.
\end{gathered} \tag{1.2}$$

From the physical point of view, equation (1.1) is related to the panel flutter equation and spillover problem with memories and damping. The case of $\sigma = 0$,

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in the absence of the Balakrishnan-Taylor damping, equation (1.1) can be used to describe the motion of viscoelastic materials. It is well known that viscoelastic materials have a wide application in science and engineering because they have the capacity of storage and dissipation of mechanical energy, which is modeled by the convolution terms (as in (1.1)). Many authors have considered the behavior of the partial differential equations (PDEs) with convolution term, see for example [5, 14, 7, 16, 8, 4, 18] and references therein. Guesmia and Messaoudi [7] discussed the problem

$$\begin{gathered}
u_{tt} - \Delta u + \int_0^t g_1(t-s)\operatorname{div}(a_1(x)\nabla u(s))ds \\
+ \int_0^\infty g_2(s)\operatorname{div}(a_2(x)\nabla u(t-s))ds = 0, \quad \text{in } \Omega\times(0,\infty), \\
u(x,t) = 0, \quad \text{on } \partial\Omega\times(0,\infty), \\
u(x,-t) = u_0(x,t), \quad \text{in } \Omega\times(0,\infty), \\
u_t(x,0) = u_1(x), \quad \text{in } \Omega.
\end{gathered} \tag{1.3}$$

Under suitable conditions on a_1, a_2 and for a wide class of relaxation functions g_1 and g_2, they established a general decay result, from which the usual exponential and polynomial decay rates are only special cases. Guesmia and Messaoudi [8] were concerned with the long-time behavior of the solution of the Timoshenko system

$$\begin{gathered}
\rho_1\varphi_{tt} - k_1(\varphi_x+\psi)_x + b(x)h(\varphi_t) \\
+ \int_0^\infty g(s)(a(x)\varphi_x(t-s))_x ds = 0, \quad \text{in } (0,L)\times(0,\infty), \\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi) = 0, \quad \text{in } (0,L)\times(0,\infty), \\
\varphi(0,t) = \psi_x(0,t) = \varphi(L,t) = \psi_x(L,t) = 0, \quad \text{in } (0,\infty), \\
\varphi(x,-t) = \varphi_0(t), \varphi_t(x,0) = \varphi_1(x), \quad \text{in } (0,L)\times(0,\infty), \\
\psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), \quad \text{in } (0,L).
\end{gathered} \tag{1.4}$$

They showed that the dissipation generated by these two complementary controls guarantees the stability of the system in case of the equal-speed propagation as well as in the opposite case. Mustafa [18] studied the following equation with the Dirichlet boundary condition

$$u_{tt} - \Delta u + \int_0^t \operatorname{div}[a(x)g(t-\tau)\nabla u(\tau)]d\tau + \eta(t)b(x)h(u_t) = 0, \quad \text{in } \Omega\times(0,\infty), \tag{1.5}$$

and established an explicit and general decay rate result, using some properties of convex functionals.

The model in hand, with the Balakrishnan-Taylor damping ($\sigma \neq 0$) and in the absence of Δu_{tt}, the strong damping Δu_t and $\rho = g_1 = g_2 = h = 0$, was proposed by Balakrishnan and Taylor [1], and Bass and Zes [2]. The model is used to solve the overflow problem, that is, to set up an appropriate feedback control function, which consists of a limited number of modes, to achieve a high performance of the closed-loop systems. So far, there are many stability results for the problem having the Balakrishnan-Taylor damping and memory term see for example [3, 9, 11, 17, 23, 24]. Mu and Ma [17] considered the wave equations with

Balakrishnan-Taylor memory terms and source terms:

$$
\begin{gathered}
u_{tt} - (a + b\|\nabla u\|^2 + \sigma \int_\Omega \nabla u \cdot \nabla u_t \, dx) + \int_0^t g_1(t-s)\Delta u(s)ds \\
= f_1(u,v),\ t > 0, \quad x \in \Omega, \\
v_{tt} - (a + b\|\nabla v\|^2 + \sigma \int_\Omega \nabla v \cdot \nabla v_t \, dx) + \int_0^t g_2(t-s)\Delta v(s)ds \\
= f_2(u,v),\ t > 0,\ x \in \Omega, \\
u(x,t) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\
v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega, \\
u(x,t) = v(x,t) = 0, \quad (x,t) \in \Gamma \times [0,\infty),
\end{gathered} \tag{1.6}
$$

and proved that for a certain class of relaxation functions and certain initial data, the decay rate of the solution energy is similar to that of relaxation functionals which is not necessarily of exponential or polynomial type. In addition, they considered problem (1.6) with the added terms $\Delta^2 u + \Delta^2 u_t$ and $\Delta^2 v + \Delta^2 v_t$, namely,

$$
\begin{gathered}
u_{tt} - (a + b\|\nabla u\|^2 + \sigma \int_\Omega \nabla u \nabla u_t \, dx) + \Delta^2 u + \Delta^2 u_t + \int_0^t g_1(t-s)\Delta u(s)ds \\
= f_1(u,v), \quad t > 0,\ x \in \Omega, \\
v_{tt} - (a + b\|\nabla v\|^2 + \sigma \int_\Omega \nabla v \nabla v_t \, dx) + \Delta^2 v + \Delta^2 v_t + \int_0^t g_2(t-s)\Delta v(s)ds \\
= f_2(u,v), \quad t > 0,\ x \in \Omega, \\
u(x,t) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\
v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega, \\
u(x,t) = v(x,t) = 0, \quad (x,t) \in \Gamma \times [0,\infty)\,.
\end{gathered} \tag{1.7}
$$

They established the blow-up of the solution for (1.7) when relaxation functionals and initial data satisfy some conditions even in presence of strong damping.

There are some methods developed to analyze the stability of the PDEs such as Lyapunov's energy method (see [10, 19]), Riesz basis approach (see [21, 13]), frequency multiplier technique (see [15, 20]), Carleman estimates (see [6]) and so on. Motivated by [7, 8, 18], we consider (1.1). Our major contributions in this article are the following:

(1) We put together several useful models of viscoelasticity: dispersion term $|u_t|^\rho$, the Balakrishna-Taylor damping $\int_\Omega \nabla u \nabla u_t \, dx$, strong damping term Δu_t, infinite and finite time history memories $\int_0^\infty g_2(s) \operatorname{div}(a_2(x)\nabla u(t-s))ds$, $\int_0^t g_1(t-s) \operatorname{div}(a_1(x)\nabla u(s))ds$, and a nonlinear source term $|u|^{p-1}u$.

(2) Give the model in (1.1) which is a unified treatment.

(3) Our technical treatment offers, as far as we know, the sharpest assumptions/conditions for the study of the viscoelastic wave equation.

The rest of this article is organized as follows. In section 2, we present preliminary material needed for our work. In section 3, we prove the global existence and the uniform decay of energy.

2. Preliminaries

In this section, we present some materials needed for our main results. Throughout this article, we use the following assumptions and notation. We shall write $\|\cdot\|$ and $\|\cdot\|_p$ to denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm respectively, $(\cdot,\cdot)$ denotes the usual inner product in $L^2(\Omega)$. We denote by c and c_i $(i\in N^+)$ various positive constants, which may be different at different occurrences.

Weuse the following hypotheses.

(A1) $\gamma(t)$: $R_+\to R^+$ is a non-increasing continuous function.

(A2) $g_i(s)$: $R_+\to R^+$ $(i=1,2)$ are differentiable non-increasing functions such that

$$g_i(0)>0,\quad a-\|a_1\|_\infty\int_0^\infty g_1(s)ds-\|a_2\|_\infty\int_0^\infty g_2(s)ds:=l>0.$$

(A3) There exists a positive differentiable non-increasing function $\xi:R_+\to R^+$ satisfying

$$g_1'(t)\le-\xi(t)g_1(t),\quad t\ge0.$$

(A4) There exists a positive constant κ and an increasing strictly convex function $G:R_+\to R_+$ of class $C^1(R_+)\cap C^2(R^+)$ satisfying

$$G(0)=G'(0)=0,\quad \lim_{t\to+\infty}G'(t)=+\infty,$$

such that

$$g_2'(t)\le-\kappa g_2(t),\quad t\ge0,\tag{2.1}$$

or

$$\int_0^\infty\frac{g_2(t)}{G^{-1}(-g_2'(t))}dt+\sup_{t\in R_+}\frac{g_2(t)}{G^{-1}(-g_2'(t))}<+\infty.\tag{2.2}$$

(A5) $h(s)$: $R\to R$ is a non-decreasing function with $sh(s)\ge0$ for all $s\in R$ and there exists a convex and increasing function $H:R_+\to R_+$ of class $C^1(R_+)\cap C^2(R^+)$ satisfying $H(0)=0$, and H is linear on $[0,\epsilon_1]$ or $H'(0)=0$ and $H''>0$ on $(0,\epsilon_1]$ such that

$$m_1|s|\le|h(s)|\le M_1|s|,\quad \text{if } |s|\ge\epsilon_1,\tag{2.3}$$

$$h^2(s)\le H^{-1}(sh(s)),\quad \text{if } |s|<\epsilon_1,\tag{2.4}$$

where ϵ_1, m_1 and M_1 are positive constants.

(A6) There exists a positive constant m_0, such that

$$\|\nabla u_0(.,s)\|_2\le m_0,\quad s\in R_+.\tag{2.5}$$

(A7) $a_i(x):\overline{\Omega}\to R^+$ $(i=1,2)$ are in $C^1(\overline{\Omega})$ such that, for positive constants ϵ_2 and ϵ_3 and for Γ_1, $\Gamma_2\subset\partial\Omega$ with meas $(\Gamma_i>0)$,

$$\inf_{x\in\overline{\Omega}}(a_1(x)+a_2(x))\ge\epsilon_2,$$

$$a_i=0\quad\text{or}\quad\inf_{\Gamma_i}a_i(x)\ge2\epsilon_3,\quad i=1,2.$$

As in [7], let $d=\min\{\epsilon_2,\epsilon_3\}$ and let $\alpha_i\in C^1(\overline{\Omega})$ $(i=1,2)$, be such that

$$\begin{gathered}0\le\alpha_i(x)\le a_i(x),\\ \alpha_i(x)=0,\quad\text{if } a_i(x)\le\frac{d}{4},\\ \alpha_i(x)=a_i(x),\quad\text{if } a_i(x)\ge\frac{d}{2}.\end{gathered}\tag{2.6}$$

Assumption (2.4) was introduced for the first time in [12].

Lemma 2.1 (Sobolev-Poincare inequality)**.** *Let q be a number with $2 \le q < \infty$ for $n = 1, 2$, and $2 \le q \le \frac{2n}{n-2}$ for $n \ge 3$. Then there exists a constant $c_* = c_*(\Omega, q)$ such that*

$$\|u\|_q \le c_* \|\nabla u\|_2, \quad u \in H_0^1(\Omega).$$

From this lemma and (1.2) we know that there exists some positive constant η such that for any $u \in H_0^1(\Omega)$ one has

$$\|u\|_{p+1}^{p+1} \le \eta (l \|\nabla u\|_2^2)^{\frac{p+1}{2}}. \tag{2.7}$$

Using the Faedo-Galerkin method, we can obtain the following local solution. We omit the proof.

Theorem 2.2. *Suppose that* (A1)–(A7) *hold, and let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Then there exists a unique weak solution u of* (1.1) *such that*

$$u \in C([0,T), H_0^1(\Omega)) \cap L^{p+1}(\Omega), \quad u_t \in C([0,T); H_0^1(\Omega)) \cap L^{\rho+2}(\Omega),$$

for some $T > 0$.

Now, for (1.1), we consider the functionals

$$\begin{aligned} I(t) :=& \int_\Omega \Big[a - a_1(x) \int_0^t g_1(s) ds - a_2(x) \int_0^\infty g_2(s) ds\Big] |\nabla u|^2 dx + b \|\nabla u\|_2^4 \\ &+ (g_1 \circ \nabla u)(t) + (g_2 \circledcirc \nabla u)(t) - \|u\|_{p+1}^{p+1}, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} J(t) :=& \frac{1}{2} \int_\Omega \Big[a - a_1(x) \int_0^t g_1(s) ds - a_2(x) \int_0^\infty g_2(s) ds\Big] |\nabla u|^2 dx + \frac{b}{4} \|\nabla u\|_2^4 \\ &+ \frac{1}{2} (g_1 \circ \nabla u)(t) + \frac{1}{2} (g_2 \circledcirc \nabla u)(t) - \frac{1}{p+1} \|u\|_{p+1}^{p+1}. \end{aligned} \tag{2.9}$$

We define the energy functional of problem (1.1) as

$$E(t) := \frac{1}{\rho+2} \|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2} \|\nabla u_t\|_2^2 + J(t), \tag{2.10}$$

where

$$\begin{aligned} (g_1 \circ \nabla u)(t) &= \int_\Omega a_1(x) \int_0^t g_1(t-s) |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx, \\ (g_2 \circledcirc \nabla u)(t) &= \int_\Omega a_2(x) \int_0^\infty g_2(s) |\nabla u(t) - \nabla u(t-s)|^2 \, ds \, dx. \end{aligned}$$

Lemma 2.3. *$E(t)$ is a non-increasing function for $t \ge 0$, and*

$$\begin{aligned} E'(t) =& -\sigma \Big(\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2\Big)^2 - \|\nabla u_t\|_2^2 + \frac{1}{2} (g_1' \circ \nabla u)(t) + \frac{1}{2} (g_2' \circledcirc \nabla u)(t) \\ &- \frac{1}{2} g_1(t) \int_\Omega a_1(x) |\nabla u|^2 dx - \int_\Omega \gamma(t) u_t h(u_t) dx \le 0, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} (g_1' \circ \nabla u)(t) &= \int_\Omega a_1(x) \int_0^t g_1'(t-s) |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx, \\ (g_2' \circledcirc \nabla u)(t) &= \int_\Omega a_2(x) \int_0^\infty g_2'(s) |\nabla u(t) - \nabla u(t-s)|^2 \, ds \, dx. \end{aligned}$$

Proof. Multiplying the first equation in (1.1) by u_t, integrating over Ω and using integration by parts and hypotheses (A1)–(A7), we obtain (2.10). $\square$

3. Global solution and energy decay results

Lemma 3.1. *Suppose that* (A1)–(A7) *hold. Let* $(u_0, u_1) \in H_0^1(\Omega)\times L^2(\Omega)$, $I(0) > 0$ *and*

$$\eta\Big(\frac{2(p+1)}{p-1}E(0)\Big)^{\frac{p-1}{2}} < 1.$$

Then $I(t) > 0$ *for all* $t \ge 0$.

Proof. Since $I(0) > 0$, by continuity, there exists $T_* \le T$ such that $I(t) \ge 0$ for all $t \in [0, T_*)$. Using that $a - \|a_1\|_\infty \int_0^\infty g_1(s)ds - \|a_2\|_\infty \int_0^\infty g_2(s)ds = l > 0$, for any $t \in [0, T_*)$, we have

$$\begin{aligned} J(t) =& \frac{p-1}{2(p+1)}\int_\Omega \Big(a - a_1(x)\int_0^t g_1(s)ds - a_2(x)\int_0^\infty g_2(s)ds\Big)|\nabla u|^2dx \\ &+ \frac{p-3}{4(p+1)}b\|\nabla u\|_2^4 + \frac{p-1}{2(p+1)}\big((g_1\circ\nabla u)(t) + (g_2\circledcirc\nabla u)(t)\big) + \frac{1}{p+1}I(t) \\ \ge& \frac{p-1}{2(p+1)}l\|\nabla u(t)\|_2^2. \end{aligned}$$

From the above inequality, and (2.7)–(2.11), we have

$$l\|\nabla u\|_2^2 \le \frac{2(p+1)}{p-1}J(t) \le \frac{2(p+1)}{p-1}E(t) \le \frac{2(p+1)}{p-1}E(0), \tag{3.1}$$

and

$$\begin{aligned} \|u\|_{p+1}^{p+1} &\le \eta(l\|\nabla u\|_2^2)^{\frac{p+1}{2}} \\ &\le \eta\Big(\frac{2(p+1)}{p-1}E(0)\Big)^{\frac{p-1}{2}} l\|\nabla u\|_2^2 < l\|\nabla u\|_2^2 \\ &< \int_\Omega \Big(a - a_1(x)\int_0^t g_1(s)ds - a_2(x)\int_0^\infty g_2(s)ds\Big)|\nabla u|^2dx. \end{aligned} \tag{3.2}$$

This shows that $I(t) > 0$ for all $t \in [0, T_*)$. By repeating this procedure, we can extended T_* to T. This completes the proof of Lemma 3.1. $\square$

Theorem 3.2. *Suppose that* (A1)–(A7) *hold. If* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *then the solution of* (1.1) *is global and bounded.*

Proof. Using (2.11) and Lemma 3.1, we have

$$\begin{aligned} E(0) \ge E(t) &= \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_t\|_2^2 + J(t) \\ &\ge \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{p-1}{2(p+1)}l\|\nabla u\|_2^2. \end{aligned}$$

Therefore,

$$\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 + \|\nabla u\|_2^2 \le cE(0),$$

where c is a positive constant that depends on l, ρ and p. This completes the proof. $\square$

By using the Hölder inequality and the properties of the functions α_1 and α_2, we easily obtained the following Lemma. We omit the proof.

Lemma 3.3. *The following inequalities hold,*

$$\int_\Omega \alpha_1(x)\Big(\int_0^t g_1(t-s)(u(t)-u(s))ds\Big)^2 dx \le c(g_1\circ\nabla u)(t), \tag{3.3}$$

$$\int_\Omega \alpha_1(x)\Big(\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))ds\Big)^2 dx \le c(g_1\circ\nabla u)(t), \tag{3.4}$$

$$\int_\Omega |\nabla\alpha_1(x)|\Big(\int_0^t g_1(t-s)(u(t)-u(s))ds\Big)^2 dx \le c(g_1\circ\nabla u)(t), \tag{3.5}$$

$$\int_\Omega |\nabla\alpha_1(x)|\Big(\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))ds\Big)^2 dx \le c(g_1\circ\nabla u)(t), \tag{3.6}$$

$$\int_\Omega \alpha_2(x)\Big(\int_0^\infty g_2(s)(u(t)-u(t-s))ds\Big)^2 dx \le c(g_2\circledcirc\nabla u)(t), \tag{3.7}$$

$$\int_\Omega \alpha_2(x)\Big(\int_0^\infty g_2(s)(\nabla u(t)-\nabla u(t-s))ds\Big)^2 dx \le c(g_2\circledcirc\nabla u)(t), \tag{3.8}$$

$$\int_\Omega |\nabla\alpha_2(x)|\Big(\int_0^\infty g_2(s)(u(t)-u(t-s))ds\Big)^2 dx \le c(g_2\circledcirc\nabla u)(t), \tag{3.9}$$

$$\int_\Omega |\nabla\alpha_2(x)|\Big(\int_0^\infty g_2(s)(\nabla u(t)-\nabla u(t-s))ds\Big)^2 dx \le c(g_2\circledcirc\nabla u)(t). \tag{3.10}$$

We define the perturbed energy functional

$$L(t) = ME(t) + \varepsilon\psi(t) + \chi_1(t) + \chi_2(t), \tag{3.11}$$

where M and ε are positive constants that will be specified later, and

$$\psi(t) = \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u\,dx + \frac{\sigma}{4}\|\nabla u\|_2^4 + \int_\Omega \nabla u\cdot\nabla u_t\,dx + \frac{1}{2}\|\nabla u\|_2^2,$$
$$\chi_1(t) = \int_\Omega \alpha_1(x)(\Delta u + \Delta u_t - \frac{1}{\rho+1}|u_t|^\rho u_t)\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx,$$
$$\chi_2(t) = \int_\Omega \alpha_2(x)(\Delta u + \Delta u_t - \frac{1}{\rho+1}|u_t|^\rho u_t)\int_0^\infty g_2(s)(u(t)-u(t-s))\,ds\,dx.$$

Lemma 3.4. *There exist two positive constants β_1 and β_2 such that the relation*

$$\beta_1 L(t) \le E(t) \le \beta_2 L(t),$$

holds for ε small enough while M is large enough.

Proof. By using Young's inequality, Hölder inequality and Lemma 3.3, we obtain

$$\begin{aligned}
&\Big|\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u\,dx\Big| \\
&\le \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+1)(\rho+2)}\|u\|_{\rho+2}^{\rho+2} \\
&\le \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{c_*^{\rho+2}}{(\rho+1)(\rho+2)}\|\nabla u\|_2^{\rho+2} \\
&\le \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{c_*^{\rho+2}}{(\rho+1)(\rho+2)}\Big(\frac{2(p+1)}{(p-1)l}E(0)\Big)^{\rho/2}\|\nabla u\|_2^2,
\end{aligned}$$

$$
\begin{aligned}
&\big|\int_\Omega \alpha_1(x)\Delta u(t)\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\big| \\
&\le \big|\int_\Omega \nabla\alpha_1(x)\nabla u(t)\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\big| \\
&\quad + \big|\int_\Omega \alpha_1(x)\nabla u(t)\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\big| \\
&\le \delta\|\nabla u\|_2^2 + \frac{c}{\delta}(g_1\circ\nabla u)(t),
\end{aligned}
$$

$$
\big|\int_\Omega \alpha_1(x)\Delta u_t(t)\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\big| \le \delta\|\nabla u_t\|_2^2 + \frac{c}{\delta}(g_1\circ\nabla u)(t),
$$

$$
\begin{aligned}
&\big|\frac{1}{\rho+1}\int_\Omega \alpha_1(x)|u_t|^\rho u_t\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\big| \\
&\le \frac{1}{(\rho+1)(\rho+2)}\int_\Omega \alpha_1(x)\Big(\int_0^t g_1(t-s)(u(t)-u(s))ds\Big)^{\rho+2}dx \\
&\quad + \frac{1}{\rho+2}\int_\Omega \alpha_1(x)|u_t|^{\rho+2}dx \\
&\le \frac{c}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{cc_*^{\rho+2}}{(\rho+1)(\rho+2)}\Big(\frac{4(p+1)}{(p-1)l}E(0)\Big)^{\rho/2}(g_1\circ\nabla u)(t),
\end{aligned}
$$

$$
\big|\int_\Omega \alpha_2(x)\Delta u(t)\int_0^\infty g_2(s)(u(t)-u(t-s))\,ds\,dx\big| \le \delta\|\nabla u\|_2^2 + \frac{c}{\delta}(g_2\circledcirc\nabla u)(t),
$$

$$
\big|\int_\Omega \alpha_2(x)\Delta u_t\int_0^\infty g_2(s)(u(t)-u(t-s))\,ds\,dx\big| \le \delta\|\nabla u_t\|_2^2 + \frac{c}{\delta}(g_2\circledcirc\nabla u)(t),
$$

$$
\begin{aligned}
&\big|\frac{1}{\rho+1}\int_\Omega \alpha_2(x)|u_t|^\rho u_t\int_0^\infty g_2(s)(u(t)-u(t-s))\,ds\,dx\big| \\
&\le \frac{1}{\rho+2}\Big(\frac{1}{\rho+1}\int_\Omega \alpha_2(x)\Big(\int_0^\infty g_2(s)|u(t)-u(t-s)|ds\Big)^{\rho+2}dx \\
&\quad + \int_\Omega \alpha_2(x)|u_t|^{\rho+2}dx\Big) \\
&\le \frac{c}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{c_*^{\rho+2}c}{(\rho+1)(\rho+2)}\int_0^\infty g_2(s)\|\nabla u(t)-\nabla u(t-s)\|_2^{\rho+2}ds.
\end{aligned}
$$

Now, using (3.1) and (A6) we obtain

$$
\begin{aligned}
\|\nabla u(t)-\nabla u(t-s)\|_2^2 &\le 2\|\nabla u(t)\|_2^2 + 2\|\nabla u(t-s)\|_2^2 \\
&\le 4\sup_{s>0}\|\nabla u(s)\|_2^2 + 2\sup_{\tau<0}\|\nabla u(\tau)\|_2^2 \\
&\le 4\sup_{s>0}\|\nabla u(s)\|_2^2 + 2\sup_{\tau>0}\|\nabla u_0(\tau)\|_2^2 \\
&\le \frac{8(p+1)}{(p-1)l}E(0) + 2m_0^2 := N_1,
\end{aligned}
$$

$$
\begin{aligned}
&\big|\frac{1}{\rho+1}\int_\Omega \alpha_2(x)|u_t|^\rho u_t\int_0^\infty g_2(s)(u(t)-u(t-s))\,ds\,dx\big| \\
&\le \frac{c}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{c_*^{\rho+2}N_1^{\rho/2}c}{(\rho+1)(\rho+2)}(g_2\circledcirc\nabla u)(t).
\end{aligned}
$$

Therefore, $|L(t) - ME(t)| \le cE(t)$. The prof is complete. $\square$

Lemma 3.5. *Suppose that* (1.2) *and* (A1)–(A7) *hold. Then*

$$\psi(t) = \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u \, dx + \frac{\sigma}{4}\|\nabla u\|_2^4 + \int_\Omega \nabla u \cdot \nabla u_t \, dx + \frac{1}{2}\|\nabla u\|_2^2$$

along the solution of (1.1), *and for any* $\varepsilon_1 > 0$,

$$\begin{aligned}
\psi'(t) \le & \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 - b\|\nabla u\|_2^4 + \frac{c}{\varepsilon_1}\int_\Omega h^2(u_t)dx \\
& + \frac{c}{2\varepsilon_1}(g_1 \circ \nabla u)(t) + \frac{c}{2\varepsilon_1}(g_2 \circledcirc \nabla u)(t) + \|u\|_{p+1}^{p+1} \\
& - \int_\Omega [a - a_1(x)\int_0^t g_1(s)ds - a_2(x)\int_0^\infty g_2(s)ds - \varepsilon_1]|\nabla u|^2 dx.
\end{aligned} \tag{3.12}$$

Proof. By taking the time derivative of $\psi(t)$ and using problem (1.1), we obtain

$$\begin{aligned}
\psi'(t) = & \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u_t\|_2^2 - a\|\nabla u\|_2^2 - b\|\nabla u(t)\|_2^4 \\
& - \int_\Omega u(t)\int_0^t g_1(t-s)\operatorname{div}(a_1(x)\nabla u(s))\, ds\, dx - \gamma(t)\int_\Omega u(t)h(u_t)dx \\
& - \int_\Omega u(t)\int_0^\infty g_2(s)\operatorname{div}(a_2(x)\nabla u(t-s)ds)dx + \|u(t)\|_{p+1}^{p+1}.
\end{aligned} \tag{3.13}$$

For the fifth term, by using Young's inequality and Hölder inequality, we obtain

$$\begin{aligned}
& - \int_\Omega u(t)\int_0^t g_1(t-s)\operatorname{div}(a_1(x)\nabla u(s))\, ds\, dx \\
& = \int_\Omega \nabla u(t)\int_0^t g_1(t-s)a_1(x)\nabla u(s)\, ds\, dx \\
& = \int_\Omega a_1(x)\int_0^t g_1(t-s)(\nabla u(s) - \nabla u(t))ds\nabla u(t)dx \\
& \quad + \int_0^t g_1(s)ds\int_\Omega a_1(x)|\nabla u(t)|^2 dx \\
& \le \frac{\varepsilon_1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2\varepsilon_1}\int_\Omega a_1(x)(\int_0^t g_1(t-s)(\nabla u(s) - \nabla u(t))ds)^2 dx \\
& \quad + \int_0^t g_1(s)ds\int_\Omega a_1(x)|\nabla u(t)|^2 dx \\
& \le \frac{\varepsilon_1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2\varepsilon_1}\int_0^t g_1(s)ds(g_1 \circ \nabla u)(t) + \int_0^t g_1(s)ds\int_\Omega a_1(x)|\nabla u(t)|^2 dx \\
& \le \frac{\varepsilon_1}{2}\|\nabla u(t)\|_2^2 + \frac{c}{2\varepsilon_1}(g_1 \circ \nabla u)(t) + \int_0^t g_1(s)ds\int_\Omega a_1(x)|\nabla u(t)|^2 dx.
\end{aligned} \tag{3.14}$$

Similarity, for the sixth term we obtain

$$-\int_\Omega \gamma(t)u(t)h(u_t)dx \le c\varepsilon_1\|\nabla u(t)\|_2^2 + \frac{c}{\varepsilon_1}\int_\Omega h^2(u_t)dx. \tag{3.15}$$

For the seventh term, we have

$$
\begin{aligned}
& -\int_\Omega u(t)\int_0^\infty g_2(s)\operatorname{div}(a_2(x)\nabla u(t-s))\,ds\,dx \\
& \le \frac{\varepsilon_1}{2}\|\nabla u(t)\|_2^2 + \frac{c}{2\varepsilon_1}(g_2\circledcirc\nabla u)(t) + \int_0^\infty g_2(s)ds\int_\Omega a_2(x)|\nabla u(t)|^2dx.
\end{aligned}\tag{3.16}
$$

By using (3.14)-(3.16) in (3.13), estimate (3.12) follows. □

Lemma 3.6. *Suppose that* (1.2) *and* (A1)–(A7) *hold. Then*

$$
\chi_1(t) = \int_\Omega \alpha_1(x)(\Delta u + \Delta u_t - \frac{1}{\rho+1}|u_t|^\rho u_t)\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx, \tag{3.17}
$$

along the solution of (1.1), *and for any* $\varepsilon_2, \varepsilon_3 > 0$,

$$
\begin{aligned}
\chi_1'(t) \le & -\Big[\int_0^t g_1(s)ds - c\varepsilon_2\Big]\int_\Omega \alpha_1(x)|\nabla u_t|^2dx + c(\varepsilon_2+\varepsilon_3)\int_\Omega|\nabla u|^2dx \\
& - \frac{1}{\rho+1}\int_0^t g_1(s)ds\int_\Omega \alpha_1(x)|u_t|^{\rho+2}dx + \frac{c}{\varepsilon_3}(g_1\circ\nabla u)(t) + \varepsilon_3(g_2\circledcirc\nabla u)(t) \\
& - \frac{c}{\varepsilon_2}(g_1'\circ\nabla u)(t) - \sigma\frac{4(p+1)}{p-1}E(0)E'(t) + \varepsilon_3\int_\Omega h^2(u_t)dx.
\end{aligned}
$$

Proof. Taking the derivative of χ_1 and using (1.1), we obtain

$$
\begin{aligned}
& \chi_1'(t) \\
& = -\int_\Omega \alpha_1(x)M(t)\Delta u\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx \\
& \quad + \int_\Omega \alpha_1(x)\int_0^t g_1(t-s)\operatorname{div}(a_1(x)\nabla u(s))ds\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx \\
& \quad + \int_\Omega \alpha_1(x)\int_0^\infty g_2(s)\operatorname{div}(a_2(x)\nabla u(t-s))ds\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx \\
& \quad + \int_\Omega \alpha_1(x)\gamma(t)h(u_t)\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx \\
& \quad - \int_\Omega \alpha_1(x)|u|^{p-1}u\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx \\
& \quad + \int_\Omega \alpha_1(x)(\Delta u + \Delta u_t - \frac{1}{\rho+1}|u_t|^\rho u_t)(\int_0^t g_1'(t-s)(u(t)-u(s))\,ds\,dx \\
& \quad + \int_0^t g_1(s)ds\int_\Omega \alpha_1(x)(\Delta u + \Delta u_t - \frac{1}{\rho+1}|u_t|^\rho u_t)u_t\,dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\chi_1'(t) = & \int_\Omega b\|\nabla u\|_2^2\nabla\alpha_1\nabla u(t)\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx \\
& - \int_0^t g_1(s)ds\int_\Omega \alpha_1|\nabla u_t|^2dx - \frac{1}{\rho+1}\int_0^t g_1(s)ds\int_\Omega \alpha_1|u_t|^{\rho+2}dx \\
& + \int_\Omega \alpha_1 b\|\nabla u\|_2^2\nabla u(t)\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_\Omega \nabla\alpha_1\Big(a-a_1\int_0^t g_1(s)ds-a_2\int_0^\infty g_2(s)\Big)\nabla u(t)\\
&\times\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\\
&+\int_\Omega \alpha_1\Big(a-a_1\int_0^t g_1(s)ds-a_2\int_0^\infty g_2(s)\Big)\nabla u(t)\\
&\times\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&+\int_\Omega(\sigma\int_\Omega \nabla u\nabla u_t\,dx)\nabla\alpha_1\nabla u(t)\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\\
&+\int_\Omega(\sigma\int_\Omega \nabla u\nabla u_t\,dx)\alpha_1\nabla u(t)\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&+\int_\Omega \nabla\alpha_1 a_1(x)\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))ds\\
&\times\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\\
&+\int_\Omega \alpha_1 a_1(x)(\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))ds)^2dx\\
&+\int_\Omega a_2\nabla\alpha_1\Big(\int_0^\infty g_2(s)(\nabla u(t)-\nabla u(t-s))ds\Big)\\
&\times\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\\
&+\int_\Omega a_2\alpha_1\Big(\int_0^\infty g_2(s)(\nabla u(t)-\nabla u(t-s))ds\Big)\\
&\times\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&+\int_\Omega \alpha_1\gamma h(u_t)\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\\
&-\int_\Omega \alpha_1|u|^{p-1}u\int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\\
&-\int_\Omega \nabla\alpha_1\nabla u\int_0^t g_1'(t-s)(u(t)-u(s))\,ds\,dx\\
&-\int_\Omega \alpha_1\nabla u\int_0^t g_1'(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&-\int_\Omega \nabla\alpha_1\nabla u_t\int_0^t g_1'(t-s)(u(t)-u(s))\,ds\,dx\\
&-\int_\Omega \alpha_1\nabla u_t\int_0^t g_1'(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&-\frac{1}{\rho+1}\int_\Omega \alpha_1|u_t|^\rho u_t\int_0^t g_1'(t-s)(u(t)-u(s))\,ds\,dx
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^t g_1(s)ds \int_\Omega u_t \nabla\alpha_1 \nabla u\, dx \\
&- \int_0^t g_1(s)ds \int_\Omega \alpha_1 \nabla u_t \nabla u\, dx - \int_0^t g_1(s)ds \int_\Omega u_t \nabla\alpha_1 \nabla u_t\, dx.
\end{aligned}
$$

Using Young's inequality, Poincare inequality and that $|\nabla\alpha_1| \le c a_1(x)$, we obtain (3.17). □

With a similar proof as that of Lemma 3.6 we can obtain the following Lemma.

Lemma 3.7. *Suppose that* (1.2) *and (A1)-(A7) hold. Then*

$$
\chi_2(t) = \int_\Omega \alpha_2(x)(\Delta u + \Delta u_t - \frac{1}{\rho+1}|u_t|^\rho u_t) \int_0^\infty g_2(s)(u(t) - u(t-s))\, ds\, dx
$$

along the solution of (1.1)*, and for any* $\varepsilon_2, \varepsilon_3 > 0$,

$$
\begin{aligned}
\chi_2'(t) \le &- \Big[\int_0^\infty g_2(s)ds - c\varepsilon_2 \Big] \int_\Omega \alpha_2(x)|\nabla u_t|^2 dx + \varepsilon_3 (g_1 \circ \nabla u)(t) \\
&- \frac{1}{\rho+1} \int_0^\infty g_2(s)ds \int_\Omega \alpha_2(x)|u_t|^{\rho+2} dx + c(\varepsilon_2 + \varepsilon_3) \int_\Omega |\nabla u|^2 dx \\
&+ \frac{c}{\varepsilon_3}(g_2 \circledcirc \nabla u)(t) - \frac{c}{\varepsilon_2}(g_2' \circledcirc \nabla u)(t) - \sigma \frac{4(p+1)}{p-1} E(0)E'(t) \\
&+ \varepsilon_3 \int_\Omega h^2(u_t) dx.
\end{aligned} \tag{3.18}
$$

Assumption (A2) guarantees that for any $t_0 > 0$, we have

$$
g_0 := \min\big\{ \int_0^{t_0} g_1(s)ds, \int_0^\infty g_2(s)ds \big\}.
$$

A differentiation of L, together with Lemmas 3.5, 3.5 and 3.7, give

$$
\begin{aligned}
L'(t) \le &- \frac{1}{\rho+1} \int_\Omega [g_0(\alpha_1 + \alpha_2) - \varepsilon] |u_t|^{\rho+2} dx + (\frac{M}{2} - \frac{c}{\varepsilon_2})(g_1' \circ \nabla u + g_2' \circledcirc \nabla u) \\
&- \int_\Omega [(g_0 - c\varepsilon_2)(\alpha_1 + \alpha_2) - \varepsilon + M] |\nabla u_t|^2 dx + \varepsilon \|u\|_{p+1}^{p+1} \\
&+ (\frac{c\varepsilon}{\varepsilon_1} + \frac{c}{\varepsilon_3} + \varepsilon_3)(g_1 \circ \nabla u + g_2 \circledcirc \nabla u) - \sigma \frac{8(p+1)}{p-1} E(0)E'(t) \\
&+ (2\varepsilon_3 + \frac{c\varepsilon}{\varepsilon_1}) \int_\Omega h^2(u_t)dx - b\varepsilon \|\nabla u\|_2^4 - [(l - \varepsilon_1)\varepsilon - c\varepsilon_3] \int_\Omega |\nabla u|^2 dx.
\end{aligned}
$$

We choose ε small enough and M large enough such that

$$
\frac{c\varepsilon_3}{l - \varepsilon_1} < \varepsilon < (g_0 - c\varepsilon_2)(\alpha_1 + \alpha_2), \quad M > \frac{2c}{\varepsilon_2}.
$$

Therefore, there exist positive constants κ_1, κ_2, κ_3, and κ_4 such that for all $t \ge t_0$ we have

$$
L'(t) \le -\kappa_1 E(t) + \kappa_2 (g_1 \circ \nabla u + g_2 \circledcirc \nabla u) + \kappa_3 \int_\Omega h^2(u_t)dx - \kappa_4 E'(t).
$$

We define $F_1(t) = L(t) + \kappa_4 E(t)$. Thus, we have

$$
F_1'(t) \le -\kappa_1 E(t) + \kappa_2 (g_1 \circ \nabla u + g_2 \circledcirc \nabla u) + \kappa_3 \int_\Omega h^2(u_t)dx. \tag{3.19}
$$

Now, we are ready to prove our main results by adopting and modifying the arguments in [22]. Firstly we give the following Lemma.

Lemma 3.8. *If condition* (2.2) *holds, then for any* $\varepsilon_0 > 0$*, we have*

$$G'(\varepsilon_0 E(t))(g_2 \circledcirc \nabla u) \leq -cE'(t) + c\varepsilon_0 E(t) G'(\varepsilon_0 E(t)). \tag{3.20}$$

Proof. Since $E(t)$ is non-increasing and the assumption (A6), we have

$$\begin{aligned}
&\int_\Omega a_2(x)(\nabla u(t) - \nabla u(t-s))^2 dx \\
&\leq 2\|a_2\|_\infty \int_\Omega |\nabla u(t)|^2 dx + 2\|a_2\|_\infty \int_\Omega |\nabla u(t-s)|^2 dx \\
&\leq \begin{cases} cE(0), & \text{if } 0 \leq s < t, \\ cE(0) + c\int_\Omega |\nabla u_0(s-t)|^2 dx, & \text{if } s \geq t, \end{cases} \leq A,
\end{aligned}$$

in which A is a positive constant.

Let ε_0, δ_1, $\delta_2 > 0$. Then

$$\begin{aligned}
&(g_2 \circledcirc \nabla u)(t) \\
&= \int_\Omega a_2(x) \int_0^\infty g_2(s)(\nabla u(t) - \nabla u(t-s))^2 \, ds \, dx \\
&= \frac{1}{\delta_1 G'(\varepsilon_0 E(t))} \int_0^\infty G^{-1}\Big(- \delta_2 g_2'(s) \int_\Omega a_2(x)(\nabla u(t) - \nabla u(t-s))^2 dx \Big) \\
&\quad \times \frac{\delta_1 G'(\varepsilon_0 E(t)) g_2(s) \int_\Omega a_2(x)(\nabla u(t) - \nabla u(t-s))^2 dx}{G^{-1}\left(-\delta_2 g_2'(s) \int_\Omega a_2(x)(\nabla u(t) - \nabla u(t-s))^2 dx\right)} ds \\
&\leq \frac{1}{\delta_1 G'(\varepsilon_0 E(t))} \int_0^\infty G^{-1}\Big(- \delta_2 g_2'(s) \int_\Omega a_2(x)(\nabla u(t) - \nabla u(t-s))^2 dx \Big) \\
&\quad \times \frac{A\delta_1 G'(\varepsilon_0 E(t)) g_2(s)}{G^{-1}(-A\delta_2 g_2'(s))} ds.
\end{aligned}$$

Let G^* be the dual function of the convex function G defined by

$$G^*(t) = \sup_{s \geq 0} \{ts - G(s)\}, \quad t \in R_+.$$

Obviously, G' is increasing and defines a bijection from R_+ to R_+, and then, for any $t \in R_+$, the function $s \mapsto ts - G(s)$ reaches its maximum on R_+ at the unique point $(G')^{-1}(t)$. Therefore

$$G^*(t) = t(G')^{-1}(t) - G((G')^{-1}(t)), \quad t \in R_+.$$

Using the general Young's inequality: $s_1 s_2 \leq G(s_1) + G^*(s_2)$, for

$$s_1 = G^{-1}\Big(- \delta_2 g_2'(s) \int_\Omega a_2(x)(\nabla u(t) - \nabla u(t-s))^2 dx \Big),$$
$$s_2 = \frac{A\delta_1 G'(\varepsilon_0 E(t)) g_2(s)}{G^{-1}(-A\delta_2 g_2'(s))},$$

we obtain

$$\begin{aligned}
&(g_2 \circledcirc \nabla u)(t) \\
&\leq \frac{1}{\delta_1 G'(\varepsilon_0 E(t))} \int_0^\infty G^*\Big(\frac{A\delta_1 G'(\varepsilon_0 E(t)) g_2(s)}{G^{-1}(-A\delta_2 g_2'(s))}\Big) ds - \frac{\delta_2}{\delta_1 G'(\varepsilon_0 E(t))} (g_2' \circledcirc \nabla u)(t).
\end{aligned}$$

Using that $G^*(s) \le s(G')^{-1}(s)$ and the definition of $E'(t)$, we obtain

$$
\begin{aligned}
&(g_2 \circledcirc \nabla u)(t)\\
&\le \int_0^\infty \frac{Ag_2(s)}{G^{-1}(-A\delta_2 g_2'(s))}(G')^{-1}\Big(\frac{A\delta_1 G'(\varepsilon_0 E(t) g_2(s))}{G^{-1}(-A\delta_2 g_2'(s))}\Big)ds - \frac{2\delta_2}{\delta_1 G'(\varepsilon_0 E(t))}E'(t).
\end{aligned}
$$

Condition (2.2) implies

$$
\sup_{s\in R_+} \frac{g_2(s)}{G^{-1}(-g_2'(s))} = m_2 < +\infty.
$$

Then, using that $(G')^{-1}$ is non-decreasing, for $\delta_2 = \frac{1}{A}$ we obtain

$$
\begin{aligned}
&(g_2 \circledcirc \nabla u)(t)\\
&\le \int_0^\infty \frac{Ag_2(s)}{G^{-1}(-g_2'(s))}(G')^{-1}(m_2 A\delta_1 G'(\varepsilon_0 E(t)))ds - \frac{2}{A\delta_1 G'(\varepsilon_0 E(t))}E'(t).
\end{aligned}
$$

Choosing $\delta_1 = \frac{1}{m_2 A}$ and using that

$$
\int_0^\infty \frac{Ag_2(s)}{G^{-1}(-g_2'(s))}ds = m_3 < +\infty,
$$

we obtain

$$
(g_2 \circledcirc \nabla u)(t) \le \frac{-2m_1}{G'(\varepsilon_0 E(t))}E'(t) + m_3\varepsilon_0 E(t).
$$

Thus, (3.20) holds. □

We define the following partition of Ω

$$
\Omega_+ := \{x \in \Omega : |u_t| \ge \epsilon_1\}, \quad \Omega_- := \{x \in \Omega : |u_t| < \epsilon_1\}.
$$

Now we state our main result of this article.

Theorem 3.9. *Assume that* (1.1) *and* (A1)–(A7) *are satisfied. Then there exist positive constants* ε_0, τ_0, c' *and* c'' *such that the solution of* (1.1) *satisfies*

$$
E(t) \le c''(G_2)^{-1}\Big(\int_0^t c'\zeta(s)ds\Big), \quad t \ge 0, \tag{3.21}
$$

where

$$
G_2(t) = \int_t^1 \frac{1}{H_1(s)}ds,
$$

$$
G_1(s) = \begin{cases} s, & \text{if (2.1) holds},\\ sG'(\varepsilon_0 s), & \text{if (2.2) holds}, \end{cases}
$$

$$
H_1(s) = \begin{cases} G_1(s), & \text{if } H \text{ is linear on } [0,\epsilon_1],\\ G_1(s)H'(\tau_0 G_1(s)), & \text{otherwise}, \end{cases}
$$

$$
\zeta(t) = \min\{\xi(t), \gamma(t)\}.
$$

Proof. Case (2.1) holds. At this point we use (2.10) to obtain

$$
(g_2 \circledcirc \nabla u)(t) \le -\frac{1}{\kappa}(g_2' \circledcirc \nabla u)(t) \le -\frac{2}{\kappa}E'(t). \tag{3.22}
$$

Case (2.2) holds. We have (3.20).

For the two cases, from (3.20) and (3.22) we deduce that

$$\frac{G_1(E(t))}{E(t)}(g_2 \circledcirc \nabla u)(t) \leq -cE'(t) + c\varepsilon_0 G_1(E(t)). \tag{3.23}$$

Therefore, multiplying (3.19) by $\frac{G_1(E(t))}{E(t)}$, and using (3.23) we obtain

$$\begin{aligned}\frac{G_1(E(t))}{E(t)}F_1'(t) \leq & -\kappa_1 G_1(E(t)) + \kappa_2 \frac{G_1(E(t))}{E(t)}(g_1 \circ \nabla u)(t) - \kappa_2 cE'(t) \\ & + \kappa_3 \frac{G_1(E(t))}{E(t)} \int_\Omega h^2(u_t)dx + \kappa_2 c\varepsilon_0 G_1(E(t)).\end{aligned}$$

Choosing ε_0 small enough, we arrive at

$$\begin{aligned}\frac{G_1(E(t))}{E(t)}F_1'(t) + \kappa_2 cE'(t) \leq & -cG_1(E(t)) + \kappa_2 \frac{G_1(E(t))}{E(t)}(g_1 \circ \nabla u)(t) \\ & + \kappa_3 \frac{G_1(E(t))}{E(t)} \int_\Omega h^2(u_t)dx.\end{aligned} \tag{3.24}$$

Let

$$F_2(t) = \frac{G_1(E(t))}{E(t)}F_1(t) + \kappa_2 cE(t).$$

By recalling that $t \to \frac{G_1(E(t))}{E(t)}$ is non-increasing, we deduce that $F_2(t) \sim E(t)$ and by exploiting (3.24), we conclude that for $t \geq t_0$,

$$F_2'(t) \leq -cG_1(E(t)) + \kappa_2 \frac{G_1(E(t))}{E(t)}(g_1 \circ \nabla u)(t) + \kappa_3 \frac{G_1(E(t))}{E(t)} \int_\Omega h^2(u_t)dx. \tag{3.25}$$

We use (A3) to obtain

$$\begin{aligned}\zeta(t)(g_1 \circ \nabla u)(t) &\leq \xi(t)(g_1 \circ \nabla u)(t) \\ &= \int_\Omega a_1(x) \int_0^t \xi(t) g_1(t-s)(\nabla u(t) - \nabla u(s))^2 \, ds \, dx \\ &\leq \int_\Omega a_1(x) \int_0^t \xi(t-s) g_1(t-s)(\nabla u(t) - \nabla u(s))^2 \, ds \, dx \\ &\leq -c(g_1' \circ \nabla u)(t) \\ &\leq -cE'(t).\end{aligned}$$

Multiplying (3.25) by $\zeta(t)$ and using that $t \to \frac{G_1(E(t))}{E(t)}$ is non-increasing, we obtain

$$\begin{aligned}\zeta(t)F_2'(t) \leq & -c\zeta(t)G_1(E(t)) + \kappa_2 \frac{G_1(E(t))}{E(t)}\zeta(t)(g_1 \circ \nabla u)(t) \\ & + \kappa_3 \frac{G_1(E(t))}{E(t)}\zeta(t) \int_\Omega h^2(u_t)dx \\ \leq & -c\zeta(t)G_1(E(t)) - cE'(t) + c\zeta(t) \int_\Omega h^2(u_t)dx.\end{aligned} \tag{3.26}$$

Using that $\zeta(t)$ is a non-increasing continuous function, $\xi(t)$ and $\eta(t)$ are non-increasing, and $\zeta(t)$ is differentiable, with $\zeta'(t) \leq 0$, then the functional

$$F_3(t) := \zeta(t)F_2(t) + cE(t)$$

satisfies $F_3(t) \sim E(t)$, and

$$F_3'(t) \le -c\zeta(t)G_1(E(t)) + c\zeta(t)\int_\Omega h^2(u_t)dx. \tag{3.27}$$

Case 1. H is linear on $[0,\epsilon_1]$. In this case, there exists $\kappa_5 > 0$ such that

$$\zeta(t)\int_\Omega h^2(u_t)dx \le \kappa_5\gamma(t)\int_\Omega u_t h(u_t)dx \le -\kappa_5 E'(t),$$

which together with (3.27) implies

$$(F_3(t) + cE(t))' \le -c\zeta(t)G_1(E(t)), \tag{3.28}$$

which gives $J(t) := (F_3(t) + cE(t))\delta_0 \sim E(t)$ and

$$J'(t) \le -c\delta_0\zeta(t)G_1(J(t)) =: -c'\zeta(t)H_1(J(t)). \tag{3.29}$$

We choose δ_0 small enough so that

$$J(t) \le E(t) \quad \text{and} \quad G_2(J(t_0)) - c'\int_0^{t_0}\zeta(s)ds > 0.$$

By integrating (3.29) over (t_0,t) and noting that G_2 is non-increasing, we deduce that

$$J(t) \le G_2^{-1}(G_2(J(t_0)) + c'\int_0^t \zeta(s)ds - c'\int_0^{t_0}\zeta(s)ds) \le G_2^{-1}(c'\int_0^t \zeta(s)ds).$$

Consequently, the relation between of $J(t)$ and $E(t)$ yields

$$E(t) \le c''G_2^{-1}(c'\int_0^t \zeta(s)ds).$$

Case 2. $H'(0) = 0$ and $H'' > 0$ on $(0,\epsilon_1]$. In this case, we first estimate $\int_\Omega h^2(u_t)dx$ on the right-hand of (3.27). Noting that H^{-1} is concave and increasing, and using Jensen's inequality, (A5) and (2.10), we deduce that

$$\begin{aligned}
\int_\Omega h^2(u_t)dx &= \int_{\Omega_+} h^2(u_t)dx + \int_{\Omega_-} h^2(u_t)dx \\
&\le M_1\int_{\Omega_+} u_t h(u_t)dx + \int_{\Omega_-} h^2(u_t)dx \\
&\le -M_1E'(t) + \int_{\Omega_-} H^{-1}(u_t h(u_t))dx \\
&\le -M_1E'(t) + cH^{-1}(S(t)),
\end{aligned} \tag{3.30}$$

where $S(t) = \frac{1}{vol(\Omega_-)}\int_{\Omega_-} u_t h(u_t)dx$. Hence (3.27) becomes

$$F_3'(t) \le -c\zeta(t)G_1(E(t)) - cM_1\zeta(t)E'(t) + c\zeta(t)H^{-1}(S(t)). \tag{3.31}$$

Now, we define $F_4(t) := F_3(t) + cM_1\zeta(t)E(t)$. Then, we have

$$F_4'(t) \le -c\zeta(t)G_1(E(t)) + c\zeta(t)H^{-1}(S(t)). \tag{3.32}$$

We define

$$F_5(t) := H'(\tau_0 G_1(E(t)))F_4(t) + \kappa_6 E(t), \tag{3.33}$$

where $\tau_0 > 0$ and $\kappa_6 > 0$ to be determined later. Then, using $E'(t) \le 0$, $G_1'(t) \ge 0$, $H''(t) \ge 0$, and (3.31), we obtain

$$\begin{aligned} &F_5'(t) \\ &= \tau_0 G_1'(E(t))E'(t)H''(\tau_0 G_1(E(t)))F_4(t) + H'(\tau_0 G_1(E(t)))F_4'(t) + \kappa_6 E'(t) \\ &\le H'(\tau_0 G_1(E(t)))F_4'(t) + \kappa_6 E'(t) \\ &\le -c\zeta(t)G_1(E(t))H'(\tau_0 G_1(E(t))) + c\zeta(t)H^{-1}(S(t))H'(\tau_0 G_1(E(t))) \\ &\quad + \kappa_6 E'(t). \end{aligned} \tag{3.34}$$

Let H_* be the convex conjugate of H in the sense of Young, then

$$H^*(s) = s(H')^{-1}(s) - H\left((H')^{-1}(s)\right), \quad s \in R^+, \tag{3.35}$$

and H^* satisfies

$$AB \le H^*(A) + H(B), \quad A, B \ge 0. \tag{3.36}$$

Furthermore, using (3.35) and $H'(0) = 0$, $(H')^{-1}$ is increasing, and H is also increasing yield

$$H^*(s) \le s(H')^{-1}(s), \quad s \ge 0. \tag{3.37}$$

Taking $H'(\tau_0 G_1(E(t))) = A$ and $H^{-1}(S'(t)) = B$ in (3.34), applying (3.36) and (3.37), and noting that $0 \le H'(\tau_0 G_1(E(t))) \le H'(\tau_0 G_1(E(0)))$ due to H' is increasing, we obtain

$$\begin{aligned} F_5'(t) \le &- c\zeta(t)G_1(E(t))H'(\tau_0 G_1(E(t))) + c\zeta(t)H^*(H'(\tau_0 G_1(E(t)))) \\ &+ c\zeta(t)S(t) + \kappa_6 E'(t) \\ \le &- c\zeta(t)G_1(E(t))H'(\tau_0 G_1(E(t))) + c\zeta(t)\tau_0 G_1(E(t))H'(\tau_0 G_1(E(t))) \\ &- cE'(t) + \kappa_6 E'(t). \end{aligned}$$

Consequently, with a suitable choice of τ_0 and κ_6, we obtain

$$F_5'(t) \le -c\zeta(t)G_1(E(t))H'(\tau_0 G_1(E(t)) =: -c\zeta(t)H_1(E(t)). \tag{3.38}$$

Thus, by defining $R(t) = \delta_3 F_5(t) \sim E(t)$, and by computation, we have

$$R'(t) \le -c\delta_3\zeta(t)G_1(R(t))H'(\tau_0 G_1(R(t))) =: -c'\zeta(t)H_1(R(t)). \tag{3.39}$$

We choose δ_3 small enough so that

$$R(t) \le E(t) \quad \text{and} \quad G_2(R(t_0)) - c'\int_0^{t_0} \zeta(s)ds > 0.$$

By integrating (3.39) over (t_0, t) and noting that G_2 is non-increasing, we deduce

$$R(t) \le G_2^{-1}(G_2(R(t_0) + c'\int_0^t \zeta(s)ds - c'\int_0^{t_0} \zeta(s)ds) \le G_2^{-1}(c'\int_0^t \zeta(s)ds).$$

Consequently, the equality relation between of $R(t)$ and $E(t)$ yields

$$E(t) \le c'' G_2^{-1}(c'\int_0^t \zeta(s)ds).$$

This completes the proof. □

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