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MULTIPLE POSITIVE SOLUTIONS FOR BIHARMONIC EQUATION OF KIRCHHOFF TYPE INVOLVING CONCAVE-CONVEX NONLINEARITIES

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Abstract. In this article, we study the multiplicity of positive solutions for the biharmonic equation of Kirchhoff type involving concave-convex nonlinearities,

$$\Delta^2 u - \Big(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)\Delta u + V(x)u = \lambda f_1(x)|u|^{q-2}u + f_2(x)|u|^{p-2}u.$$

Using the Nehari manifold, Ekeland variational principle, and the theory of Lagrange multipliers, we prove that there are at least two positive solutions, one of which is a positive ground state solution.

1. Introduction

In this article, we are concerned with the multiplicity of positive solutions for the biharmonic equations of Kirchhoff type

$$\Delta^2 u - \Big(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

with $u \in H^2(\mathbb{R}^N)$.

This problem is often referred to be nonlocal because of the presence of the term $\int_{\mathbb{R}^N} |\nabla u|^2 dx\Delta u$, which implies the problem is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties, which make the study of such a class of problem particularly interesting. So there are many papers presented to study the nonlocal problems. We refer the reader to [2, 10, 13, 15, 16, 18, 20, 22, 30].

The motivation of this paper is from the studies on the dynamical system. Problem (1.1) is related to the stationary analog of the dissipative evolutionary equation

$$u_{tt} + h(u_t) + \Delta^2 u - \Big(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)\Delta u + f(x,u) = 0. \tag{1.2}$$

This equation arises as an evolutionary mathematical model in various systems for relevant physical applications, see [20] and the references therein.

The understanding of the asymptotic behavior of dynamical systems generated by dissipative evolutionary equation is an important problem of modern mathematical physics. The main method to treat this problem for a dissipative system is to consider the existence of a global attractor and analyze the structure of the global

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attractor, which is an invariant compact set and attracts all bounded subsets in some space.

Recently, the authors in [19, 21, 33, 35] proved the existence of the multiple equilibrium points in the global attractors for the symmetric dynamical systems by estimating the lower bound of Z_2 index of two disjoint subsets of the global attractor for which one subset is located in the area where the Lyapunov function F is positive and the other subset is located in the area where the Lyapunov function F is negative. By the way, a fixed point, or a stationary point, or an equilibrium point for a semigroup of an evolutionary equation corresponds to the solution of the related stationary equation [24].

To have a better understanding of the asymptotic behavior of the dissipative system (1.2) for future studies, our aim in the present paper is to find the multiplicity of positive solutions for the corresponding stationary equation (1.1) based on the variational methods.

For $a = 1, b = 0$ in (1.1), we obtain the fourth-order elliptic equation

$$\Delta^2 u - \Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

with $u \in H^2(\mathbb{R}^N)$. The existence and multiplicity of positive, negative, sign-changing and high energy solutions of (1.3) have been the subject of extensive mathematical studies in recent years, see [1, 4, 31, 32] and the references therein.

Without the term $\Delta^2 u$, system (1.1) becomes the equation

$$-\Big(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

The solvability of the Kirchhoff-type Equation (1.4) has been widely studied by various authors. For example, Ma and Muñoz Rivera [18] investigated the existence of positive solutions of such problems by using variational methods. Perera and Zhang [22] obtained a nontrivial solution of (1.4) via Yang index and critical group. He and Zou [13] studied (1.4) the existence of infinitely many solutions by using the local minimum methods and the fountain theorems. Chen, Kuo and Wu [10] considered problem (1.4) with concave and convex nonlinearities by using Nehari manifold and fibering map methods, and the existence of multiple positive solutions were obtained. Li and Ye [16] considered (1.4) with super linear nonlinearities by using a monotonicity trick and a new version of global compactness lemma, and obtained the existence of positive ground state solutions. For other important results, see [2, 15, 30] and the references therein.

By using the mountain pass techniques and the truncation method, Wang, Avci and An [26] considered the existence of nontrivial solutions for problem (1.1) in bounded domain. By using variational methods and the truncation method, Wang, An and An [25] studied the positive solutions of (1.1) with $V(x) = 1$. Very recently, Khoutir and Chen [14] obtained the existence of ground state solutions and a least energy sign-changing solution of (1.1) by using the variational methods and the Nehari method with $f(x,u) = |u|^{p-2}u$, Wang, Ru and An [27] investigated the existence of nontrivial solutions of (1.1) via Galerkin method with $V(x) = 1$.

Concerning the concave-convex nonlinearity, there is a considerable literature that takes into account different type of problem, see for instance, the pioneering paper by Ambrosetti, Brezis and Cerami [3] for the elliptic problems in bounded domain, Wu [29] for the elliptic problem in unbounded domain, Liu and Wang [17] and Cao and Xu [9] for the Schrödinger equations, Zhang, Xu and Zhang [34] for the Choquard equation. However, very little work has been done for problem (1.1) in $\mathbb{R}^N$

Inspired by the above-mentioned papers, we are focus on system (1.1) in the case of $f(x,u)$ involving a combination of convex and concave terms,

$$\Delta^2 u - \Big(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)\Delta u + V(x)u = \lambda f_1(x)|u|^{q-2}u + f_2(x)|u|^{p-2}u, \quad (1.5)$$

with $u \in H^2(\mathbb{R}^N)$, where $N \le 7$, a, b are positive constants, $\lambda > 0$ is a parameter $1 < q < 2$, $4 < p < 2*$ ($2* = \infty$ if $N \le 4$ and $2* = \frac{2N}{N-4}$ if $N = 5, 6, 7$).

We assume that $V(x)$ and $f(x)$ satisfy the following hypotheses:

- (H1) $V \in C(\mathbb{R}^N, \mathbb{R}), \inf_{x\in\mathbb{R}^N} V(x) \ge a_0 > 0$, where a_0 is a constant. Moreover, for every $M > 0$, $\text{meas}\{x \in \mathbb{R}^N : V(x) \le M\} < \infty$, where meas denotes the Lebesgue measure in $\mathbb{R}^N$;
- (H2) $f_1 \in L^{q^*}(\mathbb{R}^N) \cap C(\mathbb{R}^N) \setminus \{0\}$, where $q^* = p/(p-q)$;
- (H3) $f_2 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $f_2(x) > 0$ for almost every $x \in \mathbb{R}^N$.

Our main result reads as follows.

Theorem 1.1. *Suppose the conditions* (H1)–(H3) *hold. Let*

$$\lambda_0 = \frac{(p-2)S_p^{q/2}}{(p-q)|f_1|_{q^*}}\Big[\frac{(2-q)S_p^{p/2}}{(p-q)|f_2|_\infty}\Big]^{\frac{2-q}{p-2}},$$

with S_p defined in (2.1) *below. Then we have:*

1. *for each $0 < \lambda < \lambda_0$, problem* (1.5) *has at least two positive solutions, one of which has negative energy;*
2. *if $0 < \lambda < \frac{q}{p-2}\lambda_0$, the solution corresponding to the negative energy is a positive ground state solution and the other one corresponds to positive energy.*

When we restrict the space dimension to $N \le 7$, because $4 < p < 2_* = \frac{2N}{N-4}$, then $\frac{2N}{N-4} > 4$, hence $N < 8$. In the present study, we mainly focus on the case $p > 4$, whereas the different solution structure will appear if $p < 4$ as in [10], which we will investigate in the future study.

Since problem (1.5) is defined in $\mathbb{R}^N$ which is unbounded, the lack of compactness of the Sobolev embedding becomes more delicate by using variational techniques. To overcome the lack of compactness, the condition (V) , which was first introduced by Bartsch and Wang in [5], is always assumed to preserve the compactness of embedding of the working space. From [11], we know that under the assumption (H1), the continuous embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $2 \le s < 2_*$, where E is denoted in Section 2.1.

Since the functional of (1.5) is concave-convex, it may have several critical points in the direction of nontrivial u. Hence the standard method of Nehari manifold is invalid. Motivated by [8, 10, 28, 29], we will make a partition of the Nehari manifold and figure out two non-degenerate submanifolds, and then consider minimization problems in the two submanifolds respectively to obtain one positive energy solution and one negative energy solution.

The remainder of this paper is organized as follows. After presenting some preliminary results in Section 2, we give the proof of our main result in Section 3.

2. Preliminaries

In the sequel, we shall use the following notation:

• $H = H^2(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : |\nabla u|, \Delta u \in L^2(\mathbb{R}^N)\}$ is the usual Sobolev space endowed with the scalar product and norm

$$\langle u, v\rangle_H = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \nabla v + uv)dx, \quad \|u\|_H = \langle u, u\rangle_H^{1/2}.$$

In $L^s(\mathbb{R}^N)$, we define the norm

$$|u|_s = \Big(\int_{\mathbb{R}^N} |u|^s dx)^{1/s} \quad \text{for } 0 < s \le \infty.$$

• $E = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\}$, with inner in E and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + a\nabla u \nabla v + V(x)uv)dx, \quad \|u\| = \langle u, u\rangle^{1/2}.$$

• Denote by S_p the best Sobolev constant for the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ which is given by

$$S_p = \inf_{E\setminus\{0\}} \frac{\|u\|^2}{\big(\int_{\mathbb{R}^N} |u|^p dx)^{2/p}} > 0\,. \tag{2.1}$$

Then we have

$$|u|_p \le S_p^{-1/2}\|u\|, \quad \forall u \in E. \tag{2.2}$$

• C, C_i denote different positive constants whose exact valued is inessential.

The energy functional corresponding to (1.5) is

$$\begin{aligned} I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\Delta u|^2 + a|\nabla u|^2 + V(x)u^2)dx + \frac{b}{4}|\nabla u|_2^4 \\ - \frac{\lambda}{q}\int_{\mathbb{R}^N} f_1(x)|u|^q dx - \frac{1}{p}\int_{\mathbb{R}^N} f_2(x)|u|^p dx, \quad u \in E. \end{aligned}$$

By the assumptions (H1)–(H3), one has that $I_\lambda(u) \in C^1(E, \mathbb{R})$ and for any $u, v \in E$,

$$\begin{aligned} \langle I_\lambda'(u), v\rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + a\nabla u \nabla v + V(x)uv)dx + b\int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla v\, dx \\ - \lambda \int_{\mathbb{R}^N} f_1(x)|u|^{q-2}uv\, dx - \int_{\mathbb{R}^N} f_2(x)|u|^{p-2}uv\, dx. \end{aligned}$$

It is well-known that u is a solution of system (1.5) if and only if $u \in E$ is a critical point of I_λ.

It is easy to verify that the energy functional I_λ is not bounded from below on E. However, it is convenient to consider the functional restricted to a natural constraint, the Nehari manifold

$$\mathcal{N}_\lambda = \{u \in E \setminus \{0\} : \langle I_\lambda'(u), u\rangle = 0\}.$$

Thus $u \in \mathcal{N}_\lambda$, if and only if

$$\|u\|^2 + b|\nabla u|_2^4 = \lambda \int_{\mathbb{R}^N} f_1(x)|u|^q dx + \int_{\mathbb{R}^N} f_2(x)|u|^p dx.$$

Obviously, $\mathcal{N}_\lambda$ contains every nontrivial solution of (1.5). The following result is readily established.

Lemma 2.1. *The energy functional I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Proof. By Hölder's inequality and (2.1), we have

$$\begin{aligned}\int_{\mathbb{R}^N} f_1(x)|u|^q dx &\le \Big(\int_{\mathbb{R}^N} |f_1(x)|^{q^*} dx\Big)^{1/q^*} \Big(\int_{\mathbb{R}^N} |u|^p dx\Big)^{q/p} \\ &= |f_1|_{q^*}|u|_p^q \le |f_1|_{q^*} S_p^{-q/2}\|u\|^q.\end{aligned} \tag{2.3}$$

For $u \in \mathcal{N}_\lambda$, noting (2.3), we can conclude that

$$\begin{aligned}I_\lambda(u) &= I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u), u\rangle \\ &= \frac{1}{4}\|u\|^2 - \lambda(\frac{1}{q} - \frac{1}{4}) \int_{\mathbb{R}^N} f_1(x)|u|^q dx + (\frac{1}{4} - \frac{1}{p}) \int_{\mathbb{R}^N} f_2(x)|u|^p dx \\ &\ge \frac{1}{4}\|u\|^2 - \lambda(\frac{1}{q} - \frac{1}{4})|f_1|_{q^*} S_p^{-q/2}\|u\|^q.\end{aligned}$$

Combining $1 < q < 2 < 4 < p < 2*$, we know that I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$. □

The Nehari manifold $\mathcal{N}_\lambda$ is closely linked to the behavior of fibering maps which is given by $K_u(t) = I_\lambda(tu)$ for $t > 0$. The fibering map has been introduced by Drábek and Pohozaev in [12] and are also discussed in Brown and Zhang [8] and Brown and Wu [7]. If $u \in E$, we have

$$\begin{aligned}K_u(t) &= \frac{1}{2}t^2\|u\|^2 + \frac{bt^4}{4}|\nabla u|_2^4 - \frac{\lambda t^q}{q} \int_{\mathbb{R}^N} f_1(x)|u|^q dx - \frac{t^p}{p} \int_{\mathbb{R}^N} f_2(x)|u|^p dx; \\ K'_u(t) &= t\|u\|^2 + bt^3|\nabla u|_2^4 - \lambda t^{q-1} \int_{\mathbb{R}^N} f_1(x)|u|^q dx - t^{p-1} \int_{\mathbb{R}^N} f_2(x)|u|^p dx; \\ K''_u(t) &= \|u\|^2 + 3bt^2|\nabla u|_2^4 - \lambda(q-1)t^{q-2} \int_{\mathbb{R}^N} f_1(x)|u|^q dx \\ &\quad - (p-1)t^{p-2} \int_{\mathbb{R}^N} f_2(x)|u|^p dx.\end{aligned}$$

It is easy to see that for $u \in E \setminus \{0\}$ and $t > 0, K'_u(t) = 0$ if and only if $tu \in \mathcal{N}_\lambda$, i.e., positive critical points of K_u correspond to points on the Nehari manifold. In particular, $K'_u(1) = 0$ if and only if $u \in \mathcal{N}_\lambda$. Since $K_u(t) \in C^2(\mathbb{R}^+, \mathbb{R})$, we split $\mathcal{N}_\lambda$ into three parts corresponding to local minima, points of inflection and local maxima.

$$\begin{aligned}\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : K''_u(1) > 0\}, \\ \mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : K''_u(1) = 0\}, \\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : K''_u(1) < 0\}.\end{aligned}$$

For each $u \in \mathcal{N}_\lambda$, we have

$$K''_u(1) = \|u\|^2 + 3b|\nabla u|_2^4 - \lambda(q-1) \int_{\mathbb{R}^N} f_1(x)|u|^q dx - (p-1) \int_{\mathbb{R}^N} f_2(x)|u|^p dx \tag{2.4}$$

$$= K''_u(1) - (q-1)\langle I'_\lambda(u), u\rangle$$

$$= (2-q)\|u\|^2 + b(4-q)|\nabla u|_2^4 - (p-q) \int_{\mathbb{R}^N} f_2(x)|u|^p dx \tag{2.5}$$

$$= K''_u(1) - (p-1)\langle I'_\lambda(u), u\rangle$$

$$= (2-p)\|u\|^2 + b(4-p)|\nabla u|_2^4 + \lambda(p-q)\int_{\mathbb{R}^N} f_1(x)|u|^q dx. \tag{2.6}$$

We now derive some basic properties of $\mathcal{N}_\lambda^0$, $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^-$.

Lemma 2.2. *If $\lambda \in (0, \lambda_0)$, then $\mathcal{N}_\lambda^0 = \emptyset$*

Proof. Suppose the contrary, then there exists a $\lambda^* \in (0, \lambda_0)$ such that $\mathcal{N}_{\lambda^*}^0 \neq \emptyset$. Hence there at least exists a $u_0 \in \mathcal{N}_{\lambda^*}^0$ satisfying $K''_{u_0}(1) = 0$. From (2.5) and the Hölder and Sobolev inequalities, we have

$$\begin{aligned}(2-q)\|u_0\|^2 + b(4-q)|\nabla u_0|_2^4 &= (p-q)\int_{\mathbb{R}^N} f_2(x)|u_0|^p dx \\ &\le (p-q)|f_2|_\infty S_p^{-p/2}\|u_0\|^p,\end{aligned} \tag{2.7}$$

which implies

$$(2-q)\|u_0\|^2 \le (p-q)|f_2|_\infty S_p^{-p/2}\|u_0\|^p,$$

hence, we derive that

$$\|u_0\| \ge \Big(\frac{(2-q)S_p^{p/2}}{(p-q)|f_2|_\infty}\Big)^{\frac{1}{p-2}}. \tag{2.8}$$

Similarly, using (2.6) and Hölder and Sobolev inequalities, we have

$$\begin{aligned}(p-2)\|u_0\|^2 + b(p-4)|\nabla u_0|_2^4 &= \lambda^*(p-q)\int_{\mathbb{R}^N} f_1(x)|u|^q dx \\ &\le \lambda^*(p-q)|f_1|_{q^*} S_p^{-q/2}\|u_0\|^q,\end{aligned} \tag{2.9}$$

thus

$$(p-2)\|u_0\|^2 \le \lambda^*(p-q)|f_1|_{q^*} S_p^{-q/2}\|u_0\|^q,$$

which implies

$$\|u_0\| \le \Big(\frac{\lambda^*(p-q)|f_1|_{q^*}}{(p-2)S_p^{q/2}}\Big)^{\frac{1}{2-q}}. \tag{2.10}$$

Combining with (2.8) and (2.10), we deduce that

$$\lambda^* \ge \frac{(p-2)S_p^{q/2}}{(p-q)|f_1|_{q^*}}\Big[\frac{(2-q)S_p^{p/2}}{(p-q)|f_2|_\infty}\Big]^{\frac{2-q}{p-2}} = \lambda_0$$

which contradicts the assumptions. The proof is complete. □

to have a better understanding of the Nehari manifold $\mathcal{N}_\lambda$ and the fibering maps $K_u(t)$, we consider the function $h_b(t) : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$h_b(t) = t^{2-q}\|u\|^2 + bt^{4-q}|\nabla u|_2^4 - t^{p-q}\int_{\mathbb{R}^N} f_2(x)|u|^p dx\,.$$

Then

$$K'_u(t) = t^{q-1}\Big(h_b(t) - \lambda\int_{\mathbb{R}^N} f_1(x)|u|^q dx\Big).$$

Clearly, $tu \in \mathcal{N}_\lambda$ if and only if $h_b(t) = \lambda\int_{\mathbb{R}^N} f_1(x)|u|^q dx$, $tu \in \mathcal{N}_\lambda^+$ (or $\mathcal{N}_\lambda^-$) if and only if $h'_b(t) > 0$ (or < 0).

For $u\in E\backslash\{0\}$ with $\int_{\mathbb{R}^N} f_2(x)|u|^p dx>0$, it is obviously that $h_b(0)=0, h_b(t)>0$ for t is small enough and $h_b(t)\to-\infty$ as $t\to\infty$. Note that $1<q<2, 4<p<2_*$, and from

$$h_b'(t)=t^{p-q-1}\big((2-q)t^{2-q}\|u\|^2+b(4-q)t^{4-p}|\nabla u|_2^4-(p-q)\int_{\mathbb{R}^N} f_2(x)|u|^p dx\big)=0,$$

we can infer that there is a unique $t_{b,\max}>0$ such that $h_b(t)$ achieves its maximum at $t_{b,\max}$, increasing for $t\in[0,t^-_{b,\max})$ and decreasing for $t\in(t^+_{b,\max},\infty)$ with $\lim_{t\to\infty}h_b(t)=-\infty$ where $t^-_{b,\max}\le t_{b,\max}\le t^+_{b,\max}$.

Lemma 2.3. *Suppose that $0<\lambda<\lambda_0$ and $u\in E\setminus\{0\}$. Then*

(i) *if $\int_{\mathbb{R}^N} f_1(x)|u|^q dx\le 0$, then there at least exists a $t^->t_{b,\max}$ such that $t^-u\in\mathcal{N}_\lambda^-$, and*
$$I_\lambda(t^-u)=\sup_{t\ge0} I_\lambda(tu);$$

(ii) *if $\int_{\mathbb{R}^N} f_1(x)|u|^q dx>0$, then there at least exists a $t^->t_{b,\max}$ and a $t^+<t_{b,\max}$, i.e. $0<t^+<t_{b,\max}<t^-$ such that $t^+u\in\mathcal{N}_\lambda^+$ and $t^-u\in\mathcal{N}_\lambda^-$, satisfying*
$$I_\lambda(t^+u)=\inf_{t_{b,\max}\ge t\ge0} I_\lambda(tu),\quad I_\lambda(t^-u)=\sup_{t\ge t_{b,\max}} I_\lambda(tu).$$

Proof. Denote $h_0(t)\doteq h_b(t)|_{b=0}$. Note that $b>0$, we have

$$h_b(t)>h_0(t)=t^{2-q}\|u\|^2-t^{p-q}\int_{\mathbb{R}^N} f_2(x)|u|^p dx.$$

It is easy to see that $h_0(t)$ has a unique critical point

$$t_{0,\max}=\big(\frac{(2-q)\|u\|^2}{(p-q)\int_{\mathbb{R}^N} f_2(x)|u|^p dx}\big)^{\frac{1}{p-2}},$$

and

$$\begin{aligned}
h_0(t_{0,\max})&=\Big(\frac{(2-q)\|u\|^2}{(p-q)\int_{\mathbb{R}^N} f_2(x)|u|^p dx}\Big)^{\frac{2-q}{p-2}}\|u\|^2\\
&\quad-\Big(\frac{(2-q)\|u\|^2}{(p-q)\int_{\mathbb{R}^N} f_2(x)|u|^p dx}\Big)^{\frac{p-q}{p-2}}\int_{\mathbb{R}^N} f_2(x)|u|^p dx\\
&=\|u\|^q\Big(\frac{\|u\|^p}{\int_{\mathbb{R}^N} f_2(x)|u|^p dx}\Big)^{\frac{2-q}{p-2}}\big(\frac{2-q}{p-q}\big)^{\frac{2-q}{p-2}}\frac{p-2}{p-q}\\
&\ge\|u\|^q\Big(\frac{\|u\|^p}{|f_2|_\infty S_p^{-p/2}\|u\|^p}\Big)^{\frac{2-q}{p-2}}\big(\frac{2-q}{p-q}\big)^{\frac{2-q}{p-2}}\frac{p-2}{p-q}\\
&=\|u\|^q\Big[\frac{(2-q)S_p^{p/2}}{(p-q)|f_2|_\infty}\Big]^{\frac{2-q}{p-2}}\frac{p-2}{p-q}>0.
\end{aligned}$$

Hence

$$h_b(t_{b,\max})\ge h_b(t_{0,\max})>h_0(t_{0,\max})>0.$$

From $0<\lambda<\lambda_0$,

$$\begin{aligned}
\lambda\int_{\mathbb{R}^N} f_1(x)|u|^q dx&\le\lambda\Big(\int_{\mathbb{R}^N}|f_1(x)|^{q^*}dx\Big)^{1/q^*}\Big(\int_{\mathbb{R}^N}|u|^p dx\Big)^{q/p}\\
&\le\lambda|f_1|_{q^*}|u|_p^q\le\lambda|f_1|_{q^*}S_p^{-q/2}\|u\|^q
\end{aligned}$$

$$
\begin{aligned}
&\le \|u\|^q\Big[\frac{(2-q)S_p^{p/2}}{(p-q)|f_2|_\infty}\Big]^{\frac{2-q}{p-2}}\frac{p-2}{p-q}\\
&\le h_0(t_{0,\max}) < h_b(t_{b,\max}).
\end{aligned}
$$

The rest of the proof is similar to the one in [29, Lemma 2.6], we omit it here. □

From Lemma 2.3, we can see that for $0<\lambda<\lambda_0$, $\mathcal{N}_\lambda^+\neq\emptyset$ and $\mathcal{N}_\lambda^-\neq\emptyset$. Combining with Lemma 2.1, we define

$$
\alpha_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u), \quad \alpha_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} I_\lambda(u).
$$

In the following part, we will drive some basic properties of $\alpha_\lambda^+,\alpha_\lambda^-$.

Lemma 2.4. *For given λ_0 in Theorem 1.1, we have*

(i) $\alpha_\lambda^+<0$ *for* $0<\lambda<\lambda_0$*;*

(ii) $\alpha_\lambda^->0$*, for* $0<\lambda<\frac{q}{p-2}\lambda_0$.

Proof. (i) For each $u\in\mathcal{N}_\lambda^+$, $K_u''(1)>0$. From (2.6), we have

$$
\lambda(p-q)\int_{\mathbb{R}^N} f_1(x)|u|^q dx > (p-2)\|u\|^2 + b(p-4)|\nabla u|_2^4.
$$

Then for each $u\in\mathcal{N}_\lambda^+$,

$$
\begin{aligned}
I_\lambda(u) &= I_\lambda(u) - \frac{1}{p}\langle I_\lambda'(u),u\rangle\\
&= \frac{p-2}{2p}\|u\|^2 + \frac{p-4}{4p}b|\nabla u|_2^4 - \lambda\frac{p-q}{pq}\int_{\mathbb{R}^N} f_1(x)|u|^q dx\\
&< \frac{p-2}{2p}\|u\|^2 + \frac{p-4}{4p}b|\nabla u|_2^4 - \frac{1}{pq}((p-2)\|u\|^2+(p-4)b|\nabla u|_2^4)\\
&= \frac{(p-2)(q-2)}{2pq}\|u\|^2 + \frac{(p-4)(q-4)}{4pq}b|\nabla u|_2^4 < 0,
\end{aligned}
$$

it follows that $\alpha_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u) < 0$.

(ii) Let $u\in\mathcal{N}_\lambda^-$, by (2.5) and similar to the proof of (2.8), we have

$$
\|u\| \ge \Big(\frac{(2-q)S_p^{p/2}}{(p-q)|f_2|_\infty}\Big)^{\frac{1}{p-2}}. \tag{2.11}
$$

Then

$$
\begin{aligned}
I_\lambda(u) &= I_\lambda(u) - \frac{1}{4} < I_\lambda'(u),u >\\
&= \frac{1}{4}\|u\|^2 - \lambda(\frac{1}{q}-\frac{1}{4})\int_{\mathbb{R}^N} f_1(x)|u|^q dx + (\frac{1}{4}-\frac{1}{p})\int_{\mathbb{R}^N} f_2(x)|u|^p dx\\
&\ge \frac{1}{4}\|u\|^2 - \lambda(\frac{1}{q}-\frac{1}{4})|f_1|_{q^*}S_p^{-1/2}\|u\|^q\\
&= \|u\|^q\big(\frac{1}{4}\|u\|^{2-q} - \lambda(\frac{1}{q}-\frac{1}{4})|f_1|_{q^*}S_p^{-1/2}\big)\\
&\ge \Big(\frac{(2-q)S_p^{p/2}}{(p-q)|f_2|_\infty}\Big)^{\frac{q}{p-2}}\Big(\frac{1}{4}\Big(\frac{(2-q)S_p^{p/2}}{(p-q)|f_2|_\infty}\Big)^{\frac{2-q}{p-2}} - \lambda\frac{p-q}{4q}|f_1|_{q^*}S_p^{-1/2}\Big).
\end{aligned}
$$

Thus, if $0<\lambda<\frac{q}{p-2}\lambda_0$, we have $I_\lambda(u)>c_0$ for some $c_0>0$, which implies that $\alpha_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} I_\lambda(u) > 0$. This completes the proof. □

Lemma 2.5. *If $0<\lambda<\lambda_0$, then $\mathcal{N}_\lambda^-$ is closed in E.*

Proof. Let $u_n\in\mathcal{N}_\lambda^-$ such that $u_n\to u$ in E. We need to prove $u\in\mathcal{N}_\lambda^-$. Note that $\langle I_\lambda'(u_n),u_n\rangle$=0 and

$$\begin{aligned}&\langle I_\lambda'(u_n),u_n\rangle-\langle I_\lambda'(u),u\rangle\\&=\langle I_\lambda'(u_n)-I_\lambda'(u),u\rangle-\langle I_\lambda'(u_n),u_n-u\rangle\to 0,\quad \text{as } n\to\infty,\end{aligned}\tag{2.12}$$

we have $\langle I_\lambda'(u),u\rangle=0$, which implies $u\in\mathcal{N}_\lambda$.

For any $u\in\mathcal{N}_\lambda^-$, by (2.6), we obtain that $\mathcal{N}_\lambda^-$ is bound away from 0.

By (2.5), it follows that $K_{u_n}''(1)\to K_u''(1)$, noting that $K_{u_n}''(1)<0$, we have $K_u''(1)\le 0$. By Lemma 2.3, for $\lambda<\lambda_0$, $K_u''(1)<0$. Therefore $u\in\mathcal{N}_\lambda^-$. □

The following lemma aims to find the critical point of I_λ on the whole space from the minimizer for I_λ on Nehari manifold.

Lemma 2.6. *For $\lambda\in(0,\lambda_0)$, if u_0 is a local minimizer for I_λ on $\mathcal{N}_\lambda$, then $I_\lambda'(u_0)=0$ in $H^{-2}(\mathbb{R}^N)$, where $H^{-2}(\mathbb{R}^N)$ is the dual space of $H^2(\mathbb{R}^N)$.*

Proof. From Lemma 2.3, we know that $u_0\notin\mathcal{N}_\lambda^0$. The rest of the proof is essentially the same as that in [8], see also in [6, 28] we omit it here. □

By the above lemma, we know that the problem of finding solutions of (1.5) can be translated into that finding minimizers of I_λ on $\mathcal{N}_\lambda$.

3. Proof of the main result

We first establish a lemma for locally compactness.

Lemma 3.1. *Under assumptions* (H1)–(H3), *I_λ satisfies the $(PS)_c$ condition with $c\in\mathbb{R}$ on $\mathcal{N}_\lambda^+$ (or $\mathcal{N}_\lambda^-$), i.e. if $u_n\in\mathcal{N}_\lambda^+$ (or $\mathcal{N}_\lambda^-$) such that $I_\lambda(u_n)\to c$ and $I_\lambda'(u_n)\to 0$, then there exists a convergent subsequence of u_n.*

Proof. Assume that $u_n\in\mathcal{N}_\lambda^+$ (or $\mathcal{N}_\lambda^-$) such that $I_\lambda(u_n)\to c$ and $I_\lambda'(u_n)\to 0$. By Lemma 2.1, we infer that u_n is bounded in E. Up to a subsequence, we may assume that

$$u_n\rightharpoonup u\quad\text{in } E,\quad u_n\to u\quad\text{in } L^s(\mathbb{R}^n),\quad s\in[2,2^*).$$

It follows that

$$b\Big(\int_{\mathbb{R}^N}|\nabla u|^2dx-\int_{\mathbb{R}^N}|\nabla u_n|^2dx\Big)\int_{\mathbb{R}^N}\nabla u\nabla(u_n-u)dx\to 0\quad\text{as } n\to\infty.\tag{3.1}$$

By using twice the Hölder inequality, the corresponding exponents are $(\frac{p}{p-q},\frac{p}{q})$ and $(q,\frac{q}{q-1})$ respectively, we obtain

$$\begin{aligned}&\big|\lambda\int_{\mathbb{R}^N}f_1(x)(|u_n|^{q-2}u_n-|u|^{q-2}u)(u_n-u)dx\big|\\&\le\lambda\Big(\int_{\mathbb{R}^N}|f_1(x)|^{q^*}dx\Big)^{1/q^*}\Big(\int_{\mathbb{R}^N}||u_n|^{q-2}u_n-|u|^{q-2}u|^{p/q}|u_n-u|^{p/q}dx\Big)^{q/p}\\&\le\lambda C|f_1|_{q^*}\big(|u_n|_p^{q-1}+|u|_p^{q-1}\big)|u_n-u|_p\to 0,\quad\text{as } n\to\infty,\end{aligned}$$

where C is a positive constant. Similarly,

$$\big|\int_{\mathbb{R}^N}f_2(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)dx\big|\to 0,\quad\text{as } n\to\infty.$$

Then

$$\begin{aligned} o(1) &= \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \\ &= \|u_n - u\|^2 + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 dx \\ &\quad - b\Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\Big) \int_{\mathbb{R}^N} \nabla u \nabla(u_n - u) dx \\ &\quad - \lambda \int_{\mathbb{R}^N} f_1(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u) dx \\ &\quad - \int_{\mathbb{R}^N} f_2(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) dx \\ &= \|u_n - u\|^2 + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 dx + o(1) \end{aligned}$$

Thus $u_n \to u$ in E. □

Now we use an idea in [23] to extract a $(PS)_{\alpha_\lambda^+}$ sequence from the minimizing sequence of the energy functional I_λ on Nehari manifold $\mathcal{N}_\lambda$.

Lemma 3.2. *Suppose that $u \in \mathcal{N}_\lambda^+$, there exist $\epsilon = \epsilon(u) > 0$ and a differentiable function $\psi^+ : B_\epsilon(0) \to \mathbb{R}^+ := (0, +\infty)$ such that*

1. *$\psi^+(0) = 1$;*
2. *$\psi^+(w)(u - w) \in \mathcal{N}_\lambda^+, \forall w \in B_\epsilon(0)$;*
3. *$\langle(\psi^+)(0), w\rangle = \frac{L(u,w)}{K''_u(1)}$, where*

$$\begin{aligned} L(u, w) &= 2(u, w) + 4b\int_{\mathbb{R}^N} |\nabla u|^2 \nabla u \nabla w\, dx - q\int_{\mathbb{R}^N} f_1(x)|u|^{q-2}uw\, dx \\ &\quad - p\int_{\mathbb{R}^N} f_2(x)|u|^{p-2}uw\, dx. \end{aligned}$$

Moreover, for any $C_1, C_2 > 0$, there exists $C > 0$ such that if $C_1 \le \|u\| \le C_2$, then $|\langle(\psi^+)'(0), w\rangle| \le C\|w\|$.

Proof. Define a C^1 mapping $J : \mathbb{R}^+ \times E \to \mathbb{R}$ by $J(t, w) = K'_{u-w}(t)$, that is

$$\begin{aligned} J(t, w) &= t\|u - w\|^2 + bt^3(\int_{\mathbb{R}^N} |\nabla u - w|^2 dx)^2 - \lambda t^{q-1}\int_{\mathbb{R}^N} f_1(x)|u - w|^q dx \\ &\quad - t^{p-1}\int_{\mathbb{R}^N} f_2(x)|u - w|^p dx. \end{aligned}$$

Note that $u \in \mathcal{N}_\lambda^+$, then $J(1, 0) = 0$ and $J_t(1, 0) = K''_u(1) > 0$. Applying the implicit function theorem at point $(1, 0)$, we obtain that there exist $\epsilon = \epsilon(u) > 0$ and a differentiable function $\psi^+ : B_\epsilon(0) \to \mathbb{R}^+ := (0, +\infty)$ such that

$$\psi^+(0) = 1, \quad J(\psi^+(w), w) = 0, \quad \forall w \in B_\epsilon(0).$$

Next, we prove that $\psi^+(u - w) \in \mathcal{N}_\lambda^+$ for all $w \in B_\epsilon(0)$. Indeed, by $u \in \mathcal{N}_\lambda^+$ and the set $\mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0$ is closed, we can get $\operatorname{dist}(u, \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0) > 0$. Note that $\psi^+(w)(u - w)$ is continuous with respect to w, choose $\epsilon = \epsilon(u) > 0$ small enough, such that

$$\|\psi^+(w)(u - w) - u\| < \frac{1}{2}\operatorname{dist}(u, \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0), \ \forall w \in B_\epsilon(0).$$

Hence

$$\begin{aligned}\|\psi^+(w)(u-w)-\mathcal{N}_\lambda^-\cup\mathcal{N}_\lambda^0\| &\geq \operatorname{dist}(u,\mathcal{N}_\lambda^-\cup\mathcal{N}_\lambda^0)-\operatorname{dist}(\psi^+(w)(u-w),u)\\ &\geq \frac{1}{2}\operatorname{dist}(u,\mathcal{N}_\lambda^-\cup\mathcal{N}_\lambda^0)>0,\end{aligned}$$

thus $\psi^+(w)(u-w)\in\mathcal{N}_\lambda^+$, for all $w\in B_\epsilon(0)$.

By the differentiability of the implicit function theorem, we have

$$\langle(\psi^+)'(0),w\rangle=-\frac{\langle J_w(1,0),w\rangle}{J_t(1,0)}.$$

Note that $L(u,w)=-\langle J_w(1,0),w\rangle$, and $K_u''(1)=J_t(1,0)$, therefore, we have

$$\langle(\psi^+)(0),w\rangle=\frac{L(u,w)}{K_u''(1)}.$$

Finally, we verify that there exists $\delta>0$ such that $K_u''(1)\geq\delta>0$ with $C_1\leq\|u\|\leq C_2$, $u\in\mathcal{N}_\lambda^+$, where $C_1,C_2>0$. We will prove that by contradiction. Otherwise, if there exists a sequence $\{u_n\}\in\mathcal{N}_\lambda^+$ with $C_1\leq\|u_n\|\leq C_2$, satisfying $K_{u_n}''(1)\leq\delta_n$ for any δ_n sufficiently small and $\delta_n\to0$ as $n\to\infty$. From (2.5), we have

$$(2-q)\|u_n\|^2+b(4-q)\Big(\int_{\mathbb{R}^N}|\nabla u_n|^2dx\Big)^2=(p-q)\int_{\mathbb{R}^N}f_2(x)|u_n|^pdx+o(\delta_n),$$

where $o(\delta_n)\to0$ as $n\to\infty$. Noting that $1<q<2, 4<p<2_*$, $C_1\leq\|u_n\|\leq C_2$ and (2.7), we have

$$(2-q)\|u_n\|^2\leq(p-q)|f_2|_\infty S_p^{-p/2}\|u_n\|^p+o(\delta_n),$$

and hence

$$\|u_n\|\geq\Big(\frac{(2-q)S_p^{p/2}}{(p-q)|f_2|_\infty}\Big)^{\frac{1}{p-2}}+o(\delta_n). \tag{3.2}$$

From (2.6), we have

$$(p-2)\|u_n\|^2+b(p-4)\Big(\int_{\mathbb{R}^N}|\nabla u_n|^2dx\Big)^2=\lambda(p-q)\int_{\mathbb{R}^N}f_1(x)|u_n|^qdx+o(\delta_n).$$

In view of (2.9), we have

$$(p-2)\|u_n\|^2\leq\lambda(p-q)|f_1|_{q^*}S_p^{-q/2}\|u_n\|^q+o(\delta_n),$$

which implies

$$\|u_n\|\leq\Big(\frac{\lambda_0(p-q)|f_1|_{q^*}}{(p-2)S_p^{q/2}}\Big)^{\frac{1}{2-q}}+o(\delta_n), \tag{3.3}$$

Combing this with (3.2) and (3.3), as $n\to\infty$, we deduce a contradiction.

Therefore, if there exists $C>0$ such that if $C_1\leq\|u\|\leq C_2$, then $|\langle(\psi^+)'(0),w\rangle|\leq C\|w\|$. This ends the proof of Lemma 3.2. □

In the same way, to extract a $(PS)_{\alpha_\lambda^-}$ sequence from the minimizing sequence of problem, we establish the following lemma.

Lemma 3.3. *Suppose that $u\in\mathcal{N}_\lambda^-$, there exist $\epsilon=\epsilon(u)>0$ and a differentiable function $\psi^-:B_\epsilon(0)\to\mathbb{R}^+:=(0,+\infty)$ such that*

(1) *$\psi^-(0)=1$;*
(2) *$\psi^-(w)(u-w)\in\mathcal{N}_\lambda^-$ for all $w\in B_\epsilon(0)$;*

(3) $\langle(\psi^-)(0), w\rangle = \frac{L(u,w)}{K_u''(1)}$, *where*

$$L(u,w) = 2(u,w) + 4b\int_{\mathbb{R}^N} |\nabla u|^2 \nabla u \nabla w\,dx - q\int_{\mathbb{R}^N} f_1(x)|u|^{q-2}uw\,dx \\ - p\int_{\mathbb{R}^N} f_2(x)|u|^{p-2}uw\,dx.$$

Moreover, for any $C_1, C_2 > 0$, there exists $C > 0$ such that if $C_1 \le \|u\| \le C_2$, then $|\langle(\psi^-)'(0), w\rangle| \le C\|w\|$.

We are now ready to construct the $(PS)_{\alpha_\lambda^+}$ (or $(PS)_{\alpha_\lambda^-}$) sequence from the minimizing sequence of the energy functional I_λ on the Nehari manifold $\mathcal{N}_\lambda^+$ (or $\mathcal{N}_\lambda^-$).

Lemma 3.4. *Suppose* (H1)–(H3) *hold, and $0 < \lambda < \lambda_0$, then there exists a sequence $u_n \in \mathcal{N}_\lambda^+$ such that $I_\lambda(u_n) \to \alpha_\lambda^+$ and $I_\lambda'(u_n) \to 0$ as $n \to \infty$.*

Proof. By Lemma 2.1 and the Ekeland Variational Principle on $\mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$ such that

$$\inf_{u\in\mathcal{N}_\lambda^+\cup\mathcal{N}_\lambda^0} I_\lambda(u) \le I_\lambda(u_n) < \inf_{u\in\mathcal{N}_\lambda^+\cup\mathcal{N}_\lambda^0} I_\lambda(u) + \frac{1}{n},$$

$$I_\lambda(v) \ge I_\lambda(u_n) - \frac{1}{n}\|v - u_n\|, \quad \forall v \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0.$$

Observe that $\mathcal{N}_\lambda^0 = \emptyset$, then we have $\inf_{u\in\mathcal{N}_\lambda^+\cup\mathcal{N}_\lambda^0} I_\lambda(u) = \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u) = \alpha_\lambda^+$. Thus $I_\lambda(u_n) \to \alpha_\lambda^+$, and we may assume that $u_n \in \mathcal{N}_\lambda^+$. By Lemma 2.4, we know that $\alpha_\lambda^+ < 0$.

To finish the proof, we only need to verify that $I_\lambda'(u_n) \to 0$. Applying Lemma 3.2 with u_n to obtain the function $\psi_n^+ : B_{\epsilon_n}(0) \to \mathbb{R}^+$ such that

$$\psi_n^+(w)(u_n - w) \in \mathcal{N}_\lambda^+, \quad \forall w \in B_{\epsilon_n}(0).$$

By the continuity of $\psi_n^+(w)$ and $\psi_n^+(0) = 1$, without loss of generality, we can assume ϵ_n is sufficiently small such that $1/2 \le \psi_n^+(w) \le 3/2$ for $\|w\| \le \epsilon_n$. From $\psi_n^+(w)(u_n - w) \in \mathcal{N}_\lambda^+$ and (b), we have

$$I_\lambda(\psi_n^+(w)(u_n - w)) - I_\lambda(u_n) \ge -\frac{1}{n}\|\psi_n^+(w)(u_n - w) - u_n\|,$$

and by the mean value theorem, we have

$$\langle I_\lambda'(u_n), \psi_n^+(w)(u_n - w) - u_n\rangle + o(\|\psi_n^+(w)(u_n - w) - u_n\|) \\ \ge -\frac{1}{n}\|\psi_n^+(w)(u_n - w) - u_n\|.$$

Consequently,

$$\psi_n^+(w)\langle I_\lambda'(u_n), w\rangle + (1 - \psi_n^+(w))\langle I_\lambda'(u_n), u_n\rangle \\ \le \frac{1}{n}\|(\psi_n^+(w) - 1)u_n - \psi_n^+(w)w\| + o\big(\|\psi_n^+(w)(u_n - w) - u_n\|\big).$$

By the choice of ϵ_n and $\frac{1}{2} \le \psi_n^+(w) \le \frac{3}{2}$, we infer that there exists $C_3 > 0$ such that

$$|\langle I_\lambda'(u_n), w\rangle| \le \frac{1}{n}\|\langle(\psi_n^+)'(0), w\rangle u_n\| + \frac{C_3}{n}\|w\| + o\big(|\langle(\psi_n^+)'(0), w\rangle|(\|u_n\| + \|w\|)\big).$$

For $u_n \in \mathcal{N}_\lambda^+$, we claim that $\inf_{n\in\mathbb{N}} \|u_n\| \geq C_1 > 0$, where C_1 is a constant. Otherwise, $I_\lambda(u_n)$ would converge to zero, which contradict with $I_\lambda(u_n) \to \alpha_\lambda^+ < 0$. Moreover, by Lemma 2.1 we know that I_λ is coercive on $\mathcal{N}_\lambda^+$, $\|u_n\|$ is bounded in E. Thus, there exists $C_2 > 0$ such that $0 < C_1 \leq \|u_n\| \leq C_2$. From Lemma 3.2, $|\langle(\psi_n^+)'(0), w\rangle| \leq C\|w\|$. Hence

$$
\begin{gathered}
|\langle I_\lambda'(u_n), w\rangle| \leq \frac{C}{n}\|w\| + \frac{C}{n}\|w\| + o(\|w\|), \\
\|I_\lambda'(u_n)\| = \sup_{w\in E\setminus\{0\}} \frac{|\langle I_\lambda'(u_n), w\rangle|}{\|w\|} \leq \frac{C}{n} + o(1).
\end{gathered}
$$

Then $\|I_\lambda'(u_n)\| \to 0$ as $n \to \infty$. Thus, $\{u_n\} \subset \mathcal{N}_\lambda^+$ is a $(PS)_{\alpha_\lambda^+}$ sequence for I_λ on E. Similarly, we can construct the $(PS)_{\alpha_\lambda^-}$ sequence. □

Lemma 3.5. *Suppose* (H1)–(H3) *hold, and* $0 < \lambda < \lambda_0$*, then there exists a sequence* $u_n \in \mathcal{N}_\lambda^-$ *such that* $I_\lambda(u_n) \to \alpha_\lambda^-$ *and* $I_\lambda'(u_n) \to 0$ *as* $n \to \infty$*.*

Now, we are in a position to give the proof of our main result.

Proof of Theorem 1.1. Firstly, we consider the minimization problem

$$
\alpha_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u).
$$

By Lemma 3.4, there exists $u_n \in \mathcal{N}_\lambda^+$ such that $I_\lambda(u_n) \to \alpha_\lambda^+$ and $I_\lambda'(u_n) \to 0$. From Lemma 3.1, there exists a strongly convergent subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, satisfying $u_n \to w_1$ in E. From the proof of Lemma 3.4 we know that there exist $C_1, C_2 > 0$ such that $0 < C_1 \leq \|u_n\| \leq C_2$, then $0 < C_1 \leq \|w_1\| \leq C_2$. Thus $w_1 \neq 0$. Next we prove $w_1 \in \mathcal{N}_\lambda^+$. Indeed, by (2.5), it follows that $K_{u_n}''(1) \to K_{w_1}''(1)$. From $K_{u_n}''(1) > 0$, we have $K_{w_1}''(1) \geq 0$. By Lemma 2.3, we know that $K_{w_1}''(1) > 0$. Hence

$$
w_1 \in \mathcal{N}_\lambda^+, \quad I(w_1) = \lim_{n\to\infty} I_\lambda(u_n) = \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u).
$$

Thus w_1 is a nontrivial solution of (1.5) by Lemma 2.6. Since $I_\lambda(w_1) = I_\lambda(|w_1|)$ and $|w_1| \in \mathcal{N}_\lambda^+$, we may assume that w_1 is a positive solution of (1.5). Therefore, we find a positive solution of (1.5).

Secondly, we consider the minimization problem $\alpha_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} I_\lambda(u)$. Similar to the above proof, we can also find a positive solution $w_2 \in \mathcal{N}_\lambda^-$.

From the above proof, we know if $0 < \lambda < \lambda_0$, then problem(1.5) has at least two positive solutions $w_1 \in \mathcal{N}_\lambda^+$ and $w_2 \in \mathcal{N}_\lambda^-$. Combining with Lemma 2.4 (i), we have $I_\lambda(w_1) < 0$. Moreover, by Lemma 2.4 (ii), if $0 < \lambda < \frac{q}{p-2}\lambda_0$, for any $u \in \mathcal{N}_\lambda^-, I_\lambda(u) > 0$, then $I_\lambda(w_2) > 0$. Hence if $0 < \lambda < \frac{q}{p-2}\lambda_0$, then $I_\lambda(w_1) = \inf_{u\in\mathcal{N}_\lambda} I_\lambda(u)$, w_1 is a positive ground state solution of (1.5). □

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References

[1] V. Alexiades, A. R. Elcrat, P .W. Schaefer; *Existence theorems for some nonlinear fourth-order elliptic boundary value problems*, Nonlinear Anal., 4 (4) (1980), 805-813.

[2] C. O. Alves, F. J. S. A. Corrêa, G. M. Figueiredo; *On a class of nonlocal elliptic problems with critical growth*, Differ. Equ. Appl., 2 (2010) 409-417.

[3] A. Ambrosetti, H. Brezis, G. Cerami; *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal., 122 (1994), 519-543.

[4] Y. An, R. Liu; *Existence of nontrivial solutions of an asymptotically linear fourth-order elliptical equation*, Nonlinear Anal., 68 (2008), 3325-3331.

[5] T. Bartsch, Z. Q. Wang; *Existence and multiplicity results for some superlinear elliptic problem on R^N*, Comm. Partial Differential Equations, 20 (1995), 1725-1741.

[6] P. A. Binding, P. Drábek, Y. X. Huang; *On Neumann boundary value problems for some quasilinear elliptic equations*, Electron. J. Differential Equations, 1997 (1997) no. 5, 1-11.

[7] K. J. Brown, T. Wu; *A fibering map approach to a potential operator equation and its applications*, Differential Integral Equations, 22 (2009), 1097-1114.

[8] K. J. Brown, Y. Zhang; *The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function*, J. Differential Equations, 193 (2003), 481-499.

[9] X. F. Cao, J. X. Xu; *Multiple positive solutions for Schröinger problems with concave and convex nonlinearities*, Elecron. J. Qual. Theory Differ. Equ. 2018, Paper No 68, 21 pp.

[10] C.Y. Chen, Y. C. Kuo, T. F. Wu; *The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions*, J. Differ. Equ., 250 (2011), 1876-1908.

[11] S. X. Chen, J. Liu, X. Wu; *Existence and multiplicity of nontrivial solutions for a class of modified nonlinear fourth-order elliptic equations on $\mathbb{R}^N$*, Appl. Math. Comput., 248 (2014), 593-601.

[12] P. Drábek, S. I. Pohozaev; *Positive solutions for the p-Laplacian: Application of the fibering method*, Proc. Roy. Soc. Edinburgh Sect. A, 127 (1997), 703-726.

[13] X. M. He, W. M. Zou; *Infinitely many positive solutions for Kirchhoff-type problems*, Nonlinear Anal. 70 (2009), 1407-1414.

[14] S. Khoutir, H. Chen; *Ground state solutions and least energy sign-changing solutions for a class of fourth order Kirchhoff-type equations in $\mathbb{R}^N$*, Arab J. Math. Sci. 23 (2017), 94-108.

[15] Y. H. Li, F. Y. Li, J. P. Shi; *Existence of a positive solution to Kirchhoff type problemswithout compactness conditions*, J. Differ. Equ., 253 (2012), 2285-2294.

[16] G. B. Li, H. Y. Ye; *Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$*, J. Differ. Equ., 257 (2014), 566-600.

[17] Z. Liu, Z. Q. Wang; *Schrödinger equations with concave and convex nonlinearities*, Z. Angew. Math. Phs., 56 (2005), 609-629.

[18] T. F. Ma, J. E. Muñoz Rivera; *Positive solutions for a nonlinear nonlocal elliptic transmission problem*, Appl. Math. Lett., 16 (2003), 243-248.

[19] F. J. Meng, C. C. Liu, C. Zhang; *Existence of multiple equilibrium points in global attractor for damped wave equation*, Boundary Value Problems, (2019), 2019:6, 9pp.

[20] F. J. Meng, J. Wu, C. X. Zhao; *Time-dependent global attractor for extensible Berger equation*, J. Math. Anal. Appl., 469 (2019), 1045-1069.

[21] F.J. Meng, C. K. Zhong; *Multiple equilibrium points in global attractor for the weakly damped wave equation with critical exponent.* Discrete Contin Dyn Syst Ser B., 19 (1) (2014), 217-230.

[22] K. Perera, Z. Zhang; *Nontrivial solutions of Kirchhoff-type problems via theYang index*, J. Differ. Equ. 221 (2006), 246-255.

[23] G. Tarantello; *On nonhomogeneous elliptic equations involving critical Sobolev exponent.* Ann. Inst. Henri Poincaré, Anal. Non Linéaire 9 (1992), 281-304.

[24] R. Temam; *Infinite-dimensional Systems in Mechanics and Physics*, Springer-Verlag, New York, 1997.

[25] F. L. Wang, T. Q. An, Y. K. An; *Existence of solutions for fourth order elliptic equations of Kirchhoff type on $\mathbb{R}^N$*, Electron. J. Qual. Theory Differ. Equ., 39 (2014), 1-11.

[26] F. L. Wang, M. Avci, Y. K. An; *Existence of solutions for fourth order elliptic equations of Kirchhoff type*, J. Math. Anal. Appl. 409 (2014), 140-146.

[27] F. L. Wang, Y. F. Ru, T. Q. An; *Nontrivial solutions for a fourth-order elliptic equation of Kirchhoff type via Galerkin method.* J. Fixed Point Theory Appl. 20 (2) (2018), 71, 9 pp

[28] T. F. Wu; *On semilinear elliptic equations involving concave-convex nonlinearities and sign-changing weight function*, J. Math. Anal. Appl., 318 (1) (2006), 253-270.
[29] T. F. Wu; *Multiple positive solutions for a class of concave-convex elliptic problems in R^N involving sign-changing weight*, J. Funct. Anal. 258 (2010), 99-131.
[30] J. Wang, L. X. Tian, J. X. Xu, F. B. Zhang; *Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth*, J. Differ. Equ., 253 (2012), 2314-2351.
[31] Y. W. Ye, C. L. Tang; *Existence and multiplicity of solutions for four-order elliptic equations in $\mathbb{R}^N$*, J. Math. Anal. Appl., 406 (2013), 335-351.
[32] Y. L. Yin, X. Wu; *High energy solutions and nontrivial solutions for fourth-order elliptic equations*, J. Math. Anal. Appl., 375 (2011), 699-705.
[33] J. Zhang, C. K. Zhong, B. You; *The existence of multiple equilibrium points in global attractors for some symmetric dynamical systems II*. Nonlinear Anal. RWA, 36 (2017), 44-55.
[34] H. Zhang, J. X. Xu, F. B. Zhang; *Bound and ground states for a concave-convex generalized Choquard equation*, Acta Appl. Math., 147 (2017), 81-93.
[35] C. K. Zhong, B. You, R. Yang; *The existence of multiple equilibrium points in global attractors for some symmetric dynamical systems*. Nonlinear Anal. RWA, 19 (2014), 31-44.

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