Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 52, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF WEAK SOLUTIONS TO SUPERLINEAR ELLIPTIC SYSTEMS WITHOUT THE AMBROSETTI-RABINOWITZ CONDITION

XIAOHUI WANG, PEIHAO ZHAO

Abstract. In this article, we study the existence of the weak solution for superlinear elliptic equations and systems without the Ambrosetti-Rabinowitz condition. The Ambrosetti-Rabinowitz condition guarantees the boundedness of the PS sequence of the functional I for the corresponding problem. We establish the existence of the weak solution for the superlinear elliptic equation by using $(PS)_c$ form of the Mountain pass lemma, and the existence of the weak solution for the superlinear elliptic system by using $(PS)_c^*$ form of the Linking theorem.

1. Introduction and statement of main results

In this article, we investigate the existence of the nontrivial weak solution for the superlinear elliptic problems. We first consider the p-Laplacian equation

$$\begin{gathered} -\Delta_p u = \lambda f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $p > 1$, $\lambda > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded domain, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and for $1 < p < \infty$, the p-Laplacian operator is

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du) \quad \text{for } u \in W^{1,p}(\Omega).$$

We shall say the function f satisfies the well-known Ambrosetti-Rabinowitz (AR) condition, if there are constants $\theta > p$ and $r > 0$ such that

$$0 < \theta F(x,t) \le f(x,t)t \quad \text{for all } |t| \ge r \text{ and } x \in \Omega,$$

where

$$F(x,t) = \int_0^t f(x,s)ds.$$

Since 1973 when Ambrosetti and Rabinowitz [2] established the Mountain pass lemma under the AR condition, many researchers have studied the superlinear elliptic problems under the AR condition. The AR condition guarantees the boundedness of the PS sequence of the functional I given by the corresponding problem, which plays a key role in the application of the critical point theory. Although the AR condition is convenient, it is very restrictive and excludes a lot of nonlinear

2010 *Mathematics Subject Classification.* 35J20, 35J47, 35J15, 35A15.
Key words and phrases. Superlinear elliptic system; weak solution; mountain pass lemma; linking theorem.

Submitted May 8, 2018. Published May 27, 2020.

problems. Therefore, many researchers have been studied various problems without the AR condition.

In 2004, Schechter and Zou [20] established the existence of nontrivial weak solution for the problem (1.1) without the AR condition when $p = 2$. In this paper, for a general p $(1 < p < \infty)$, we will establish the existence of the nontrivial weak solution for the p-Laplacian superlinear elliptic boundary value problem (1.1) without the AR condition. The AR condition implies that there exist positive constants c_1 and c_2 such that

$$F(x,t) \geq c_1|t|^\theta - c_2 \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

Although this condition is weaker, it still eliminates many superlinear problems. A much weaker condition implies that superlinearity is
either

$$\lim_{t\to+\infty} \frac{F(x,t)}{|t|^p} = +\infty \quad \text{a.e. in } \Omega,$$

or

$$\lim_{t\to-\infty} \frac{F(x,t)}{|t|^p} = +\infty \quad \text{a.e. in } \Omega.$$

Our first objective is to establish the existence of the nontrivial weak solution for the p-Laplacian superlinear elliptic equation (1.1) under the weaker condition than the AR condition in this paper. Let us state the main result for the elliptic equation as follows.

In the next theorem we use the following assumptions:

(H1) $f \in C^0(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x,0) = 0$,

$$\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0$$

uniformly a.e. in Ω;

(H2) There exist positive constants a and b such that

$$|f(x,t)| \leq a + b|t|^{q-1} \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where

$$q \in [1, p^*), \quad p^* = \begin{cases} \frac{np}{n-p} & \text{if } 1 < p < n, \\ +\infty & \text{if } p \geq n; \end{cases}$$

(H3) Either

$$\lim_{t\to+\infty} \frac{F(x,t)}{|t|^p} = +\infty, \quad \text{or} \quad \lim_{t\to-\infty} \frac{F(x,t)}{|t|^p} = +\infty,$$

uniformly a.e. in Ω;

(H4) There exist $\mu > p$ and $r > 0$ such that

$$\mu F(x,t) - tf(x,t) \leq C(|t|^p + 1) \quad \text{for all } |t| \geq r \text{ and } x \in \Omega.$$

Theorem 1.1. *If f satisfies* (H1)–(H4)*, then for each $\lambda > 0$, problem* (1.1) *has at least one nontrivial solution.*

Secondly, we consider the non-cooperative elliptic system

$$\begin{gathered} -\triangle u = H_u(x,u,v) \quad x \in \Omega, \\ -\triangle v = -H_v(x,u,v) \quad x \in \Omega, \\ u(x) = v(x) = 0 \quad x \in \partial\Omega, \end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a smooth bounded domain, $H : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function, H_u denotes the partial derivative of H with respect to the variable u. We write $z := (u, v)$, we suppose $H(x, 0) \equiv 0$ and $H_z(x, 0) \equiv 0$, then $z = 0$ is a trivial solution for this system. We will also establish the existence of the nontrivial solution for the elliptic system (1.2) in this paper. Roughly speaking, we are mainly interested in the class of Hamiltonians H such that

$$H(x,u,v) \sim |u|^p + |v|^q + R(x,u,v) \quad \text{with} \lim_{|z|\to\infty} \frac{R(x,u,v)}{|u|^p + |v|^q} = 0,$$

where $1 < p < 2^* := \frac{2n}{n-2}$ and $q > 1$.

For elliptic system, we shall say H satisfies the AR condition, if there exist $\mu > 2,\ \nu > 1$ and $R \geq 0$ such that

$$\frac{1}{\mu} H_u(x,z)u + \frac{1}{\nu} H_v(x,z)v \geq H(x,z) \quad \text{whenever } |z| \geq R,$$

with the provision that $\nu = \mu$ if $q > 2$.

In 1995, by using variational method, Costa and Magalhaes [6] established the existence of the nontrivial weak solution for the subcritical non-cooperative elliptic system without the AR condition. In 2004, Lam and Lu [14] obtained the existence of the nontrivial weak solution for the critical and subcritical superlinear cooperative elliptic system without the AR condition. In 2003, De Figueiredo and Ding [7] obtained the existence of the nontrivial weak solution for the supercritical superlinear non-cooperative elliptic system when $2 < p < 2^*$ under the AR condition.

As we mentioned above, many researchers have studied the existence of the nontrivial weak solution for the superlinear elliptic systems, such as, the subcritical non-cooperate elliptic system without the AR condition, the critical and subcritical superlinear cooperative elliptic system without the AR condition and the supercritical superlinear non-cooperative elliptic system under the AR condition. Our another aim in this paper is to prove the existence of the nontrivial weak solution for the supercritical superlinear non-cooperative elliptic system without the AR condition, that is, we are going to study the system (1.2) without the AR condition when $p \in (2, 2^*)$ and $q \in (2^*, +\infty)$.

We would like to mentioned that the main difficulty is to establish the boundedness of the $(PS)_c^*$ sequence for the non-cooperative elliptic system without the AR condition.

Let us state the main result for the elliptic system, using the following assumptions:

(H5) There exist $p \in (2, 2^*)$ and $q \in (2^*, +\infty)$ such that

$$|H_u(x,u,v)| \leq \gamma_0 \Big(1 + |u|^{p-1} + |v|^{\frac{q}{2}-1}\Big),$$
$$|H_v(x,u,v)| \leq \gamma_0 \Big(1 + |u|^{p-1} + |v|^{q-1}\Big),$$

for all (x, z). In all hypotheses on H the γ_i denote positive constants independent of (x, z);

(H6) $\lim_{z\to\infty} H(x,z)/|z| = +\infty$ uniformly in Ω;

(H7) There exist $\mu > 2$ and $R_1 > 0$ such that

$$\mu H(x,z) - zH_z(x,z) \leq C(|z|^p + 1) \quad \text{whenever } |z| \geq R_1;$$

(H8) For p and q as above,
$$H(x,z) \geq \gamma_1(|u|^p + |v|^q) - \gamma_2 \quad \text{for all } (x,z);$$
(H9) $H(x,0,v) \geq 0$ and $H_u(x,u,0) = o(|u|)$ uniformly with respect to x, as $u \to 0$.

Theorem 1.2. *Suppose H satisfies* (H5)–(H8)*. Then the superlinear elliptic system* (1.2) *has at least one nontrivial weak solution.*

The rest of this paper is organized as follows. In section 2, we will discuss the superlinear elliptic equation (1.1) by a variational method, and establish the existence of the nontrivial weak solution for this superlinear elliptic equation. Furthermore, we will investigate the superlinear non-cooperative elliptic system (1.2) by variational method in section 3, and establish the existence of the nontrivial weak solution for this superlinear non-cooperative elliptic system.

2. Superlinear elliptic equation

In this section, we establish the existence of the nontrivial weak solution for the superlinear elliptic boundary value problem (1.1) of p-Laplacian type.

2.1. **Preliminaries.** Throughout this section, let Ω be a bounded domain in $\mathbb{R}^n$. For $1 < p < +\infty$, we denote by $\|u\| = \big(\int_\Omega |\nabla u|^p dx\big)^{1/p}$ the norm in the Sobolev space $W_0^{1,p}(\Omega)$, by $\|\cdot\|_*$ the norm in $W^{-1,p'}(\Omega)$ which is the dual space of $W_0^{1,p}(\Omega)$, by $\|u\|_p = \big(\int_\Omega |u|^p dx\big)^{1/p}$ the usual L^p norm, by $|E|$ the n-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^n$. Moreover, we use " $\to$ " and " $\rightharpoonup$ " denote the strong and weak convergence respectively, " $\hookrightarrow$ " and " $\hookrightarrow\hookrightarrow$ " denote imbedding and compact imbedding respectively. We denote the subsequence of a sequence $\{u_n\}$ as $\{u_n\}$ to simplify the notion unless specified. And X denotes a Banach space.

Definition 2.1. We shall say that the convex function $A : X \to \mathbb{R}$ is uniformly convex on the set (convex) $S \subset X$, if for any $\epsilon_1 > 0$, there exists $\delta(\epsilon_1) > 0$ such that
$$A\Big(\frac{x+y}{2}\Big) \leq \frac{1}{2}A(x) + \frac{1}{2}A(y) - \delta(\epsilon_1),$$
for $x, y \in S$ with $\|x-y\| > \epsilon_1$. If A is uniformly convex on every ball of X, we shall say that A is locally uniformly convex, i.e., if for any $\epsilon_2 > 0$, there exists $\delta(\epsilon_2) > 0$ such that $x,\ y \in X,\ |A(x)| \leq 1,\ |A(y)| \leq 1$ and $|A(x-y)| > \epsilon_2$, then
$$\big|A\big(\frac{x+y}{2}\big)\big| < 1 - \delta(\epsilon_2).$$

Remark 2.2 ([18])**.** X is uniformly convex if and only if its norm is locally uniformly convex.

Remark 2.3 ([18])**.** The Banach space $W_0^{1,p}(\Omega)$ with norm $\|u\| = \big(\int_\Omega |\nabla u|^p dx\big)^{1/p}$ is uniformly convex.

Remark 2.4 ([3])**.** Every uniformly convex Banach space is reflexive. That is, the Banach space $W_0^{1,p}(\Omega)$ is reflexive.

Remark 2.5. [24] Let X be a reflexive Banach space, $\{x_n\}$ is a bounded sequence in X. Then $\{x_n\}$ has weak convergent subsequence. That is, the bounded sequence in reflexive Banach space $W_0^{1,p}(\Omega)$ has weak convergent subsequence.

Definition 2.6. Let I be a functional defined in Banach space X. We say that I is weakly lower semicontinuous, if for any sequence $\{x_n\}$ such that $x_n \rightharpoonup x$ weakly, then we have

$$\liminf_{n\to\infty} I(x_n) \geq I(x).$$

Definition 2.7. An operator $I' : X \to X^*$ satisfies the (S_+) condition, if for every sequence $\{x_n\} \subset X$ such that $x_n \rightharpoonup x$ and

$$\limsup_{n\to+\infty} \langle I'(x_n), x_n - x\rangle \leq 0,$$

we have $x_n \to x$ strongly.

We would like to mentioned that the (S_+) condition is used to prove that the weak convergent sequence obtained is actually strongly convergent. Next, we verify that the relevant functional satisfies the (S_+) condition.

Proposition 2.8. *Let X be a Banach space. We denote $I(x) = \|x\|^p$, where $p \geq 1$, $x \in X$, then $I : X \to \mathbb{R}$ is C^1 and $I' : X \to X^*$ satisfies the (S_+) condition.*

Proof. It is easy to verify that $I : X \to \mathbb{R}$ is a C^1 functional. Let $\{x_n\}$ be a sequence in X such that $x_n \rightharpoonup x$ and

$$\limsup_{n\to+\infty} \langle I'(x_n), x_n - x\rangle \leq 0.$$

Claim: $x_n \to x$ in X. Indeed, since $\{x_n\}$ is weakly convergent, it is bounded. That is, there is a large enough $R > 0$ such that $\|x_n\| < R$. In view of Remark 2.3, we obtain that I is locally uniformly convex. Then I is locally bounded, and therefore, $I(x_n)$ is bounded. For a subsequence $\{x_n\}$, we assume that $I(x_n) \to c$. Since $\|\cdot\|$ is continuous and convex, we know that I is weakly lower semicontinuous. Also by the definition of weakly lower semicontinuous, we have

$$I(x) \leq \liminf I(x_n) = c.$$

On the other hand, since I is convex, its graphic lies above the tangent hyperplane at x_n, that is,

$$I(x) \geq I(x_n) + \langle I'(x_n), x - x_n\rangle.$$

Using that

$$\limsup_{n\to+\infty} \langle I'(x_n), x_n - x\rangle \leq 0,$$

we deduce that $I(x) \geq c$. Then $I(x) = c$. Also we have that $\frac{x_n+x}{2} \rightharpoonup x$, and again by weakly lower semicontinuity, we obtain

$$c = I(x) \leq \liminf I\big(\frac{x + x_n}{2}\big). \tag{2.1}$$

If we suppose that $\{x_n\}$ does not convergence strongly to x, then there exists an $\epsilon > 0$ and a subsequence $\{x_n\}$ that verifies $\|x - x_n\| \geq \epsilon$. Using the uniform convexity of I over ball $B(0, R)$, we obtain that there exists a $\delta(\epsilon) > 0$ such that

$$\frac{1}{2} I(x) + \frac{1}{2} I(x_n) - I\big(\frac{x + x_n}{2}\big) \geq \delta(\epsilon).$$

Taking $n \to +\infty$, we have

$$\limsup I\big(\frac{x + x_n}{2}\big) \leq c - \delta(\epsilon),$$

which contradicts (2.1). Then the desired conclusion follows from the claim. $\square$

Definition 2.9. Let $(X, \|\cdot\|_X)$ be a real Banach space with dual space $(X^*, \|\cdot\|_{X^*})$, and $I \in C^1(X, \mathbb{R})$. For $c \in \mathbb{R}$, we shall say I satisfies the $(PS)_c$ condition, if for any sequence $\{x_n\} \subset X$ such that $I(x_n) \to c$ and $I'(x_n) \to 0$, we have that $\{x_n\}$ is strongly convergent in X.

Theorem 2.10 (Mountain pass lemma [24])**.** *Let X be a real Banach space, $I \in C^1(X, \mathbb{R})$ satisfies*

(1) $I(0) \leq 0$;
(2) *There exist constants ρ, $\alpha > 0$ such that $I(u) \geq \alpha$, when $\|u\| = \rho$;*
(3) *There exists an $e \in E \setminus B_\rho$ such that $I(e) < 0$.*

Denote $c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t))$, where

$$\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Then $c > 0$ and there is a sequence $\{x_n\} \subset X$ such that

$$I(x_n) \to c, \quad I'(x_n) \to 0.$$

Furthermore, if f satisfies the $(PS)_c$ condition, then c is the critical value of I.

2.2. **Existence of a nontrivial weak solution to the elliptic equation.** In this subsection, we establish the existence of the nontrivial weak solution for the elliptic equation. We firstly introduce the energy functional corresponding to the elliptic equation (1.1).

If $\Omega \subset \mathbb{R}^n$ is a bounded domain and f satisfies $(H1)$ and $(H2)$, then we define functional in $W_0^{1,p}(\Omega)$,

$$I_\lambda(u) = \frac{1}{p} \int_\Omega |\nabla u|^p dx - \lambda \int_\Omega F(x, u) dx. \tag{2.2}$$

For any $\lambda \in \mathbb{R}^1$, a straightforward computation yields that $I_\lambda \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$, and

$$\langle I'_\lambda(u), v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx - \lambda \int_\Omega f(x, u) v \, dx, \tag{2.3}$$

for any $u \in W_0^{1,p}(\Omega)$. Next, we prove that the functional I_λ satisfies the mountain pass geometry as follows.

Lemma 2.11. *If $\lambda > 0$ and f satisfies* (H1)–(H3)*, then*

(1) *$I_\lambda(u)$ is unbounded from below in $W_0^{1,p}(\Omega)$;*
(2) *$u = 0$ is a strictly local minimum for $I_\lambda(u)$.*

Proof. For any $M > 0$, it follows from (H3) that there is a $C_M > 0$ such that

$$F(x, t) \geq M t^p - C_M \quad \text{for all } t \geq 0 \text{ and all } x \in \Omega. \tag{2.4}$$

Indeed, for any $M > 0$, there is a $s_0 > 0$ such that

$$\frac{F(x, t)}{t^p} \geq M \quad \text{whenever } t > s_0.$$

That is, $F(x, t) \geq M t^p$ whenever $t > s_0$.

Furthermore, thanks to F being continuous on $\overline{\Omega} \times [0, s_0]$, we have

$$\max_{x \in \overline{\Omega}, 0 \leq t \leq s_0} \{F(x, t) - M t^p\} \leq C_M.$$

Also since

$$F(x, t) - M t^p + \max_{x \in \overline{\Omega}, 0 \leq t \leq s_0} \{F(x, t) - M t^p\} \geq 0,$$

we obtain $F(x,t) \ge Mt^p - C_M$, for any $x \in \overline{\Omega}$ and $0 \le t \le s_0$. To sum up, we obtain (2.4). Taking $\phi \in W_0^{1,p}(\Omega)$ with $\phi > 0$, and $t \ge 0$. Then for any $\lambda > 0$, we have

$$\begin{aligned} I_\lambda(t\phi) &= \frac{1}{p}t^p \int_\Omega |\nabla\phi|^p dx - \lambda \int_\Omega F(x,t\phi)dx \\ &\le \frac{1}{p}t^p\|\phi\|^p - \lambda t^p M \int_\Omega \phi^p dx + \lambda C_M|\Omega| \\ &= t^p\Big(\frac{1}{p}\|\phi\|^p - \lambda M \int_\Omega \phi^p dx\Big) + \lambda C_M|\Omega|. \end{aligned}$$

If M is large enough such that

$$\frac{1}{p}\|\phi\|^p - \lambda M \int_\Omega \phi^p dx < 0,$$

then $\lim_{t\to+\infty} I_\lambda(t\phi) = -\infty$, which is equivalent to (1).

On the other hand, for any $\epsilon > 0$, by using (H1) and (H2), it is easy to see that there exists a $C_\epsilon > 0$ such that

$$|f(x,t)| \le \epsilon|t|^{p-1} + C_\epsilon|t|^{q-1} \quad \text{for all } (x,t) \in \overline{\Omega}\times\mathbb{R}.$$

That is,

$$|F(x,t)| \le \epsilon|t|^p + C_\epsilon|t|^q \quad \text{for all } (x,t) \in \overline{\Omega}\times\mathbb{R}, \tag{2.5}$$

where $q \in (p,p^*)$. Indeed, in view of $(H1)$, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\frac{|f(x,t)|}{|t|^{p-1}} < \epsilon \quad \text{for any } |t| < \delta.$$

That is, $|f(x,t)| < \epsilon|t|^{p-1}$ for any $|t| < \delta$.

Furthermore, from (H2), we have

$$\begin{gathered} |f(x,t)| \le a + b|t|^{q-1} \le a|t|^{q-1} + b|t|^{q-1} = (a+b)|t|^{q-1} \quad \text{for } |t| > 1, \\ |f(x,t)| \le a + b|t|^{q-1} = (a|t|^{1-q} + b)|t|^{q-1} \le (a|\delta|^{1-q} + b)|t|^{q-1} \quad \text{for } \delta \le |t| \le 1. \end{gathered}$$

Therefore, for any $\epsilon > 0$, there is a $C_\epsilon > 0$ such that

$$|f(x,t)| \le \epsilon|t|^{p-1} + C_\epsilon|t|^{q-1} \quad \text{for all } (x,t) \in \overline{\Omega}\times\mathbb{R}, \tag{2.6}$$

where $C_\epsilon = \max\{(a+b), a|\delta|^{1-q} + b\}$. It follows from (2.5) and the Poincaré inequality that

$$\|u\|_p^p \le \frac{1}{\lambda_1}\|u\|^p,$$

where

$$0 < \lambda_1 = \inf_{u\in W_0^{1,p}(\Omega), u\ne 0} \frac{\|u\|^p}{\|u\|_p^p}.$$

And therefore, for any $\lambda > 0$ and $\epsilon > 0$ small enough such that $\frac{1}{p} - \frac{\lambda\epsilon}{\lambda_1} > 0$, the Hölder inequality implies

$$
\begin{aligned}
I_\lambda(u) &= \frac{1}{p}\|u\|^p - \lambda \int_\Omega F(x,u)dx \\
&\geq \frac{1}{p}\|u\|^p - \lambda\epsilon \int_\Omega |u|^p dx - \lambda C_\epsilon \int_\Omega |u|^q dx \\
&\geq \frac{1}{p}\|u\|^p - \lambda\epsilon \int_\Omega |u|^p dx - \lambda C_\epsilon |\Omega|^{\frac{p-q}{p}} \Big(\int_\Omega |u|^p dx\Big)^{q/p} \\
&\geq \Big(\frac{1}{p} - \frac{\lambda\epsilon}{\lambda_1}\Big)\|u\|^p - \lambda C_\epsilon |\Omega|^{\frac{p-q}{p}} \Big(\frac{1}{\lambda_1}\|u\|^p\Big)^{q/p} \\
&= \Big(\frac{1}{p} - \frac{\lambda\epsilon}{\lambda_1}\Big)\|u\|^p - \frac{\lambda C_\epsilon}{\lambda_1^{q/p}} |\Omega|^{\frac{p-q}{p}} \|u\|^q \\
&= \Big(\frac{1}{p} - \frac{\lambda\epsilon}{\lambda_1} - \frac{\lambda C_\epsilon}{\lambda_1^{q/p}} |\Omega|^{\frac{p-q}{p}} \|u\|^{q-p}\Big)\|u\|^p \\
&\geq \frac{1}{2}\Big(\frac{1}{p} - \frac{\lambda\epsilon}{\lambda_1}\Big)\|u\|^p,
\end{aligned}
\tag{2.7}
$$

provided $\|u\| = \rho$ is sufficiently small such that

$$
\frac{\lambda C_\epsilon}{\lambda_1^{q/p}} |\Omega|^{\frac{p-q}{p}} \|u\|^{q-p} < \frac{1}{2}\Big(\frac{1}{p} - \frac{\lambda\epsilon}{\lambda_1}\Big),
$$

when $q \in (p, p^*)$. Therefore $u = 0$ is a strictly local minimum for $I_\lambda(u)$. □

Lemma 2.12. *Assume f satisfies* (H1)–(H3) *and $0 < \lambda_0 < \mu_0$. Then $I_\lambda(u)$ possesses uniform mountain pass geometric structure around $u = 0$ for $\lambda \in [\lambda_0, \mu_0]$, i.e., there is an $e \in W_0^{1,p}(\Omega)$ such that $I_\lambda(e) < 0$ for any $\lambda \in [\lambda_0, \mu_0]$, and there exist constants ρ, $\alpha > 0$ such that $I_\lambda(u) \geq \alpha$ for any $\lambda \in [\lambda_0, \mu_0]$, and $u \in W_0^{1,p}(\Omega)$ with $\|u\| = \rho$.*

Proof. Fix $\epsilon > 0$ small enough, in view of (2.7), we have

$$
I_\lambda(u) \geq \frac{1}{2}\Big(\frac{1}{p} - \frac{\mu_0\epsilon}{\lambda_1}\Big)\|u\|^p,
$$

for any $\lambda \in [\lambda_0, \mu_0]$ and $u \in W_0^{1,p}(\Omega)$. Thus there is a $\rho = \rho(\mu_0, \epsilon) > 0$, taking

$$
\alpha = \frac{1}{2}\Big(\frac{1}{p} - \frac{\mu_0\epsilon}{\lambda_1}\Big)\|\rho\|^p,
$$

we have $I_\lambda(u) \geq \alpha$, for any $\lambda \in [\lambda_0, \mu_0]$ and $u \in W_0^{1,p}(\Omega)$ with $\|u\| = \rho$. Let us take $\phi \in W_0^{1,p}(\Omega)$ with $\phi > 0$, and $M > 0$ large enough such that

$$
\frac{1}{p}||\phi||^p - \lambda_0 M \int_\Omega \phi^p dx < 0.
$$

As a consequence of (2.4), for any $t > 0$, we have

$$
\begin{aligned}
I_{\lambda_0}(t\phi) &\leq \frac{1}{p}t^p\|\phi\|^p - \lambda_0 t^p M \int_\Omega \phi^p dx + \lambda_0 C_M |\Omega| \\
&\leq t^p\Big(\frac{1}{p}\|\phi\|^p - \lambda_0 M \int_\Omega \phi^p dx\Big) + \lambda_0 C_M |\Omega|.
\end{aligned}
$$

Furthermore, taking $e = t_0\phi$ with t_0 large enough such that $I_{\lambda_0}(e) < 0$, for any $0 < \lambda_0 < \lambda$, we have

$$I_\lambda(e) < I_{\lambda_0}(e) < 0.$$

This means that $I_\lambda(e) < 0$. □

By Lemmas 2.11, 2.12 and the Mountain pass lemma (Theorem 2.10), there is a $(PS)_c$ sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ satisfies

$$I_\lambda(u_n) \to c, \quad I'_\lambda(u_n) \to 0. \tag{2.8}$$

Next, we prove that the $(PS)_c$ sequence is actually bounded.

Lemma 2.13. *Assume f satisfies* (H1)–(H4)*, then the $(PS)_c$ sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ for the functional I_λ defined in* (2.2) *is bounded.*

Proof. Suppose towards a contradiction that

$$\|u_n\| \to +\infty. \tag{2.9}$$

Denote

$$w_n = \frac{u_n}{\|u_n\|}.$$

It is obvious that $w_n \in W_0^{1,p}(\Omega)$ with $\|w_n\| = 1$, and therefore, it follows from the Remark 2.5 that there exists a $w \in W_0^{1,p}(\Omega)$ such that $w_n \rightharpoonup w$ in reflexive Banach space $W_0^{1,p}(\Omega)$. Since Ω is bounded, the Sobolev's compact imbedding theorem implies that

$$w_n \to w \quad \text{in } L^q(\Omega) \text{ and } L^1(\Omega),$$

and therefore $w_n(x) \to w(x)$ a.e. in Ω. Set $\Omega_0 = \{x \in \Omega, w(x) \neq 0\}$. Then

$$\lim_{n\to+\infty} \frac{u_n}{\|u_n\|} = \lim_{n\to+\infty} w_n = w \neq 0 \quad \text{in } \Omega_0.$$

And in view of (2.9), we have $|u_n| \to +\infty$ a.e. in Ω_0. By (H3), it is easy to see that

$$\lim_{n\to+\infty} \frac{F(x,u_n)}{|u_n|^p} = +\infty \quad \text{a.e. in } \Omega_0,$$

which implies

$$\lim_{n\to+\infty} \frac{F(x,u_n)}{|u_n|^p}|w_n|^p = +\infty \quad \text{a.e. in } \Omega_0. \tag{2.10}$$

It follows from (H3) that there is a $N_0 > 0$ such that

$$\frac{F(x,u_n)}{|u_n|^p} > 1, \tag{2.11}$$

for any $x \in \Omega$ and $|u_n| \geq N_0$. Since F is continuous on $\overline{\Omega} \times [-N_0, N_0]$, there is a $M > 0$ such that

$$|F(x,u_n)| \leq M \quad \text{for all } (x,u_n) \in \overline{\Omega} \times [-N_0, N_0]. \tag{2.12}$$

Combining (2.11) with (2.12), we deduce that there is a constant C such that $F(x,u_n) \geq C$ for all $(x,u_n) \in \overline{\Omega} \times \mathbb{R}$, which shows that

$$\frac{F(x,u_n) - C}{\|u_n\|^p} \geq 0. \tag{2.13}$$

Thanks to (2.8), we have

$$c = I_\lambda(u_n) + o(1) = \frac{1}{p}\|u_n\|^p - \lambda \int_\Omega F(x,u_n)dx + o(1).$$

So we obtain

$$\|u_n\|^p = pc + p\lambda \int_\Omega F(x, u_n)dx + o(1). \tag{2.14}$$

In accordance with (2.8) and (2.14), we obtain

$$\int_\Omega F(x, u_n)dx \to +\infty. \tag{2.15}$$

Next, we claim that $|\Omega_0| = 0$. In fact, if $|\Omega_0| \neq 0$, then by using (2.10), (2.14) and the Fatou's lemma, we have

$$\begin{aligned}
+\infty &= \int_{\Omega_0} \liminf_{n\to+\infty} \frac{F(x, u_n)}{|u_n|^p}|w_n|^p dx - \int_{\Omega_0} \limsup_{n\to+\infty} \frac{C}{\|u_n\|^p} \\
&= \int_{\Omega_0} \liminf_{n\to+\infty} \Big(\frac{F(x, u_n)}{|u_n|^p}|w_n|^p - \frac{C}{\|u_n\|^p}\Big)dx \\
&\leq \liminf_{n\to+\infty} \int_{\Omega_0} \Big(\frac{F(x, u_n)}{|u_n|^p}|w_n|^p - \frac{C}{\|u_n\|^p}\Big)dx \\
&\leq \liminf_{n\to+\infty} \int_{\Omega} \Big(\frac{F(x, u_n)}{|u_n|^p}|w_n|^p - \frac{C}{\|u_n\|^p}\Big)dx \\
&= \liminf_{n\to+\infty} \int_{\Omega} \frac{F(x, u_n)}{|u_n|^p}|w_n|^p dx - \limsup_{n\to+\infty} \int_\Omega \frac{C}{\|u_n\|^p}dx \\
&= \liminf_{n\to+\infty} \int_{\Omega} \frac{F(x, u_n)}{|u_n|^p}|w_n|^p dx - \limsup_{n\to+\infty} \frac{C|\Omega|}{\|u_n\|^p} \\
&= \liminf_{n\to+\infty} \int_{\Omega} \frac{F(x, u_n)}{|u_n|^p}|w_n|^p dx \\
&= \liminf_{n\to+\infty} \frac{\int_\Omega F(x, u_n)dx}{pc + p\lambda \int F(x, u_n)dx + o(1)}.
\end{aligned} \tag{2.16}$$

Therefore, it follows from (2.15) and (2.16) that $+\infty \leq \frac{1}{p\lambda}$. This is a contradiction, which implies that $|\Omega_0| = 0$. Hence we obtain that $w(x) = 0$ a.e. in Ω. From (2.8), we have

$$I_\lambda(u_n) = \frac{1}{p}\|u\|^p - \lambda \int_\Omega F(x, u_n)dx \to c.$$

Then

$$\frac{I_\lambda(u_n)}{\|u_n\|^p} = \frac{1}{p} - \lambda \int_\Omega \frac{F(x, u_n)}{|u_n|^p}|w_n|^p dx,$$

that is,

$$\int_\Omega \frac{F(x, u_n)}{|u_n|^p}|w_n|^p dx \to \frac{1}{p\lambda},$$

Again by (2.8), we have

$$\langle I_\lambda'(u_n), u_n\rangle = \|u_n\|^p - \lambda \int_\Omega f(x, u_n)u_n dx = o(1),$$

where $o(1) \to 0$, as $n \to \infty$. Then

$$1 - \lambda \int_\Omega \frac{u_n f(x, u_n)}{|u_n|^p}|w_n|^p dx = \frac{\langle I_\lambda'(u_n), u_n\rangle}{\|u_n\|^p} \leq \frac{\|I_\lambda'(u_n)\| \cdot \|u_n\|}{\|u_n\|^p} = \frac{\|I_\lambda'(u_n)\|}{\|u_n\|^{p-1}} \to 0,$$

that is,

$$\int_\Omega \frac{u_n f(x, u_n)}{|u_n|^p}|w_n|^p dx \to \frac{1}{\lambda}.$$

Therefore,

$$\int_\Omega \frac{\mu F(x,u_n) - u_n f(x,u_n)}{|u_n|^p}|w_n|^p dx \to \frac{\mu}{p\lambda} - \frac{1}{\lambda}.$$

However, the hypothesis (H4) implies

$$\limsup \frac{\mu F(x,u_n) - u_n f(x,u_n)}{|u_n|^p}|w_n|^p \le \limsup C\frac{|u_n|^p+1}{|u_n|^p}|w_n|^p = 0.$$

Therefore,

$$\frac{\mu}{p\lambda} - \frac{1}{\lambda} \le 0,$$

which leads to a contradiction. Hence $\{u_n\}$ is bounded, i.e., there is a $C>0$ such that $\|u_n\| \le C < +\infty$. □

Lemma 2.14. *Assume f satisfies* (H2). *Then the $(PS)_c$ sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ for the functional I_λ defined in* (2.2) *has a convergent subsequence.*

Proof. Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a $(PS)_c$ sequence for the functional I_λ. Using Lemma 2.13, we deduce that $\{u_n\}$ is bounded. Therefore, there exists a $u \in W_0^{1,p}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega). \tag{2.17}$$

Furthermore, the Sobolev's compact imbedding implies $u_n \to u \quad \text{in } L^q(\Omega)$.

Denote $\epsilon_n = \|I'_\lambda(u_n)\|_*$. It is easy to check that $\epsilon_n \to 0$ and

$$|\langle I'_\lambda(u_n), v\rangle| = \Big|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v\,dx - \lambda\int_\Omega f(x,u_n)v\,dx\Big| \le \epsilon_n\|v\|, \tag{2.18}$$

for any $v \in W_0^{1,p}(\Omega)$. Thanks to (H2), we have

$$\int_\Omega (f(x,u_n) - f(x,u))(u_n - u)dx \to 0. \tag{2.19}$$

In fact, it follows from the Hölder inequality that

$$\begin{aligned} &|\int_\Omega (f(x,u_n) - f(x,u))(u_n-u)dx| \\ &\le \Big(\int_\Omega |f(x,u_n) - f(x,u)|^p dx\Big)^{1/p}\Big(\int_\Omega |u_n-u|^q dx\Big)^{1/q}. \end{aligned}$$

Since $\|u_n\| \le C$ (see Lemma 2.13) and f is continuous on $\overline{\Omega}\times[-C,C]$, there is a $M>0$ such that

$$|f(x,u_n)| \le M \quad \text{for all } (x,u_n) \in \overline{\Omega}\times[-C,C].$$

Therefore,

$$\Big(\int_\Omega |f(x,u_n) - f(x,u)|^p dx\Big)^{1/p} \le \Big(\int_\Omega (2M)^p dx\Big)^{1/p} = 2M|\Omega|^{1/p},$$
$$\Big(\int_\Omega |u_n - u|^q dx\Big)^{1/q} \to 0,$$

since $u_n \to u$ in $L^q(\Omega)$. Hence

$$\int_\Omega (f(x,u_n) - f(x,u))(u_n-u)dx \le 2M|\Omega|^{1/p}\Big(\int_\Omega |u_n-u|^q dx\Big)^{1/q} \to 0,$$

as $n \to +\infty$. Taking $v = u_n - u$ in (2.18), and it follows from (2.19) that

$$\begin{aligned}\langle I'(u_n), u_n - u\rangle &= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)dx \\ &= \langle I'(u_n), u_n - u\rangle + \int_\Omega f(x, u_n)(u_n - u)dx \\ &\le \epsilon_n \|u_n - u\| + \int_\Omega f(x, u_n)(u_n - u)dx \to 0.\end{aligned}$$

By using the (S_+) property of I'_λ, we conclude that $u_n \to u$ in $W_0^{1,p}(\Omega)$. □

roof of Theorem 1.1. Firstly, in view of Lemmas 2.11, 2.12 and the Mountain pass lemma (Theorem 2.10), there is a $(PS)_c$ sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ that satisfies $I(u_n) \to c$ and $I'(u_n) \to 0$.

Secondly, in accordance to Lemma 2.14, we deduce that $\{u_n\}$ converges strongly to some function $u \in W_0^{1,P}(\Omega)$. Clearly, u is a weak solution for the problem (1.1). This completes the proof. □

3. Existence of the nontrivial weak solution for the superlinear elliptic system

In this section, we establish the existence of the nontrivial solution for the supercritical superlinear (i.e., $p \in (2, 2^*)$, $q \in (2^*, +\infty)$) elliptic system (1.2) without the AR condition.

3.1. **Preliminaries.** The key point is to show the boundedness of the $(PS)_c^*$ sequence of the energy functional. We denote by $|\cdot|_t$ the usual $L^t(\Omega)$ norm for all $t \in [1, \infty]$. For $q > 2^*$, let $V_q = H_0^1(\Omega) \cap L^q(\Omega)$ and the Banach space V_q equipped with the norm $\|v\|_{V_q} = (|\nabla v|_2^2 + |v|_q^2)^{\frac{1}{2}}$. Let E_q be the product space $H_0^1(\Omega) \times V_q$ with elements denoted by $z = (u, v)$ and the norm in E_q by $\|z\|_q = (|\nabla u|_2^2 + \|v\|_{V_q}^2)^{\frac{1}{2}}$. We also denote $|z| = |u| + |v|$. E_q has the direct sum decomposition

$$E_q = E_q^- \oplus E^+, \quad z = z^- + z^+,$$

where $E_q^- = \{0\} \times V_q$ and $E^+ = H_0^1(\Omega) \times \{0\}$. For simplicity, write $z^+ = u$, $z^- = v$.

If $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and H satisfies $(H5)$, then we define the functional on E_q as

$$I(z) := \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - |\nabla v|^2\right) dx - \int_\Omega H(x, z)dx. \tag{3.1}$$

By a straightforward computation, we obtain that I is a C^1 functional, and

$$\langle I'(z), w\rangle = \int_\Omega \nabla u \nabla \varphi dx - \int_\Omega H_u(x, z)\varphi dx + \int_\Omega H_v(x, z)\psi dx. \tag{3.2}$$

It is not difficult to verify that the critical point of I is the solution of the elliptic system (1.2).

Next, we show that the Frechet derivative of the functional I is weakly sequence continuous .

Lemma 3.1. *Assume* (H5) *holds. Then I' is weakly sequence continuous, that is, $I'(z_n) \rightharpoonup I'(z)$, as $z_n \rightharpoonup z$.*

Proof. Suppose $z_n \rightharpoonup z$ in E_q. We claim that $I'(z_n) \rightharpoonup I'(z)$, that is,

$$\langle I'(z_n), w\rangle \to \langle I'(z), w\rangle,$$

for any $w = (\varphi, \psi) \in E_q$. Since $z_n \rightharpoonup z$, we have $u_n \rightharpoonup u$ in H_0^1, and $v_n \rightharpoonup v$ in V_q. Thus $(u_n, \varphi) \to (u, \varphi)$, that is,

$$\int_\Omega \nabla u_n \nabla \varphi dx \to \int_\Omega \nabla u \nabla \varphi dx.$$

Similarly, we have

$$\int_\Omega \nabla v_n \nabla \psi dx \to \int_\Omega \nabla v \nabla \psi dx.$$

Therefore,

$$\int_\Omega (\nabla u_n \nabla \varphi - \nabla v_n \nabla \psi)\, dx \to \int_\Omega (\nabla u \nabla \varphi - \nabla v \nabla \psi)\, dx.$$

Next, we verify the following two equalities

$$\lim_{n\to\infty} \int_\Omega H_u(x, z_n)\varphi dx = \int_\Omega H_u(x, z)\varphi dx, \quad \text{for any } \varphi \in H_0^1(\Omega), \tag{3.3}$$

$$\lim_{n\to\infty} \int_\Omega H_v(x, z_n)\psi dx = \int_\Omega H_v(x, z)\psi dx, \quad \text{for any } \psi \in V_q. \tag{3.4}$$

It follows from the Sobolev's compact imbedding theorem and the Interpolation theorem that

$$\begin{aligned} u_n \to u \quad &\text{in } L^t \text{ for any } t \in [1, 2^*), \\ v_n \to v \quad &\text{in } L^t \text{ for any } t \in [1, q). \end{aligned}$$

By $(H5)$, we have

$$|H_u(x, z_n)\varphi| \le \gamma_0 \Big(|\varphi| + |u_n|^{p-1}|\varphi| + |v_n|^{\frac{q}{2}-1}|\varphi| \Big)$$

and

$$\begin{aligned} &\int_\Omega \Big(|\varphi| + |u_n|^{p-1}|\varphi| + |v_n|^{\frac{q}{2}-1}|\varphi| \Big) dx \\ &\le \int_\Omega |\varphi| dx + \Big(\int_\Omega |u_n|^{(p-1)\frac{p}{p-1}} dx \Big)^{\frac{p-1}{p}} \Big(\int_\Omega |\varphi|^p dx \Big)^{1/p} \\ &\quad + \Big(\int_\Omega |v_n|^{(\frac{q}{2}-1)\cdot 2_*} dx \Big)^{1/2_*} \Big(\int_\Omega |\varphi|^{2^*} dx \Big)^{1/2^*} \\ &= |\varphi|_1 + |u_n|_p^{p-1}|\varphi|_p + |v_n|_{2_*(\frac{q}{2}-1)}^{\frac{q}{2}-1} |\varphi|_{2^*}. \end{aligned}$$

Thanks to $\varphi \in H_0^1(\Omega) \hookrightarrow L^{2^*}$, and

$$\big(\frac{q}{2} - 1\big) 2_* = \big(\frac{q}{2} - 1\big) \frac{2^*}{2^* - 1} < \big(\frac{q}{2} - 1\big) 2 < q.$$

Then we obtain (3.3). Furthermore, (3.4) is obvious for $\psi \in L^\infty$. In fact,

$$\begin{aligned} \Big| \int_\Omega H_v(x, z_n)\psi dx - \int_\Omega H_v(x, z)\psi dx \Big| &= \Big| \int_\Omega (H_v(x, z_n) - H_v(x, z))\psi dx \Big| \\ &\le |\psi|_\infty \int_\Omega |H_v(x, z_n) - H_v(x, z)| dx \to 0. \end{aligned}$$

Generally, for $\psi \in V_q$, there is $\psi_m \in L^\infty$ such that $\psi_m \to \psi (m \to \infty)$ in L^q, since L^∞ is dense in L^q. In the light of (H5), we have

$$|H_v(x,u,v)| \le \gamma_0 \left(1 + |u|^{p-1} + |v|^{q-1}\right).$$

And z_n is bounded in E_q, then

$$\begin{aligned}
&\Big|\int_\Omega H_v(x,z_n)\psi dx\Big| \\
&= \Big|\int_\Omega H_v(x,z_n)(\psi_m + (\psi - \psi_m))dx\Big| \\
&\le \Big|\int_\Omega H_v(x,z_n)\psi_m dx\Big| + \Big|\int_\Omega H_v(x,z_n)(\psi - \psi_m)dx\Big| \\
&\le \Big|\int_\Omega H_v(x,z_n)\psi_m dx\Big| + c_1\left(|\psi - \psi_m|_1 + |u_n|_p^{p-1}|\psi - \psi_m|_p + |v_n|_q^{q-1}|\psi - \psi_m|_q\right) \\
&\le \Big|\int_\Omega H_v(x,z_n)\psi_m dx\Big| + c_2\left(|\psi - \psi_m|_1 + |\psi - \psi_m|_p + |\psi - \psi_m|_q\right).
\end{aligned}$$

Therefore, we obtain (3.4). Then $\langle I'(z_n), w\rangle \to \langle I'(z), w\rangle$ for all $w \in E_q$. □

Next, we introduce the Linking theorem, which is the basic tool for the existence of the nontrivial weak solution for the elliptic system.

Let E be a Banach space with the norm $\|\cdot\|$. Suppose E has the direct sum decomposition $E = E^1 \oplus E^2$, where E^1 and E^2 are both infinite dimension. Assume (e_n^1) and (e_n^2) are the basis of E^1 and E^2 respectively. Let

$$X_n := \operatorname{span}\{e_1^1, \dots, e_n^1\} \oplus E^2, \quad X^m := E^1 \oplus \operatorname{span}\{e_1^2, \dots, e_m^2\},$$

and $(X^m)^\perp$ denote the supplement of X^m in E. For a functinal $I \in C^1(E, \mathbb{R})$, let $I_n := I\big|_{X_n}$ denote the restriction of I to X_n.

Definition 3.2. Let E be a Banach space, and $I \in C^1(E, \mathbb{R})$. We shall say $\{z_j\} \subset E$ is a $(PS)_c^*$ sequence, if $z_j \in X_{n_j}$ satisfies

$$I(z_j) \to c, \quad I'_{n_j}(z_j) \to 0,$$

as $n_j \to \infty$. Furthermore, we shall say I satisfies $(PS)_c^*$ condition, if any $(PS)_c^*$ sequence has a convergent subsequence.

Definition 3.3. Let E be a Banach space, Q, Q_0 and S are the closed subset of E with $Q_0 \subset Q$. We say (Q, Q_0) links with S, if

(1) $Q_0 \cap S = \emptyset$;
(2) For any continuous map $\gamma : Q \to E$ satisfies $\gamma\,|_{Q_0} = \mathrm{id}\,\big|_{Q_0}$, we have

$$\gamma(Q) \cap S \ne \emptyset.$$

Remark 3.4 ([24])**.** Let (Q, Q_0) link with S. Define the subset family of E as

$$\Gamma = \{\gamma \in C(Q, X) : \gamma\,|_{Q_0} = \mathrm{id}\,|_{Q_0}\}.$$

If I is a C^1 functional on E, set

$$c = \inf_{\gamma \in \Gamma} \sup_{x \in \gamma(Q)} I(x), \tag{3.5}$$

then under suitable conditions, we can demonstrate c is the critical value of I.

Theorem 3.5 (Linking theorem [24]). *Assume E is a Banach space, Q, Q_0 and S are the closed subset of E with $Q_0 \subset Q$, and (Q, Q_0) links with S. Moreover, assume $I \in C^1(E, \mathbb{R})$ satisfies*

(1) $\sup_{x\in Q} I(x) < r < +\infty$;
(2) *There exists a constant $\beta > \alpha$ such that*

$$\sup_{x\in Q_0} I(x) \le \alpha, \quad \inf_{x\in S} I(x) \ge \beta.$$

Then there is a sequence $\{x_n\} \subset E$ such that

$$I(x_n) \to c, \quad I_n'(x_n) \to 0,$$

where c is defined in (3.5).

3.2. **Existence of a nontrivial weak solution for the elliptic system.** Now we set $E^{'} = E_q^-$, $E^2 = E^+$ and $e_n^1 = e_n^-$, $e_n^2 = e_n^+$ for all $n \in N$, and therefore, $E_q = E^1 \oplus E^2$. We will show that the functional I defined in (3.1) satisfies the linking geometry.

Lemma 3.6. *Suppose H satisfies* (H5) *and* (H9). *Then there exist constants r and $\rho > 0$ such that*

$$\inf I(\partial B_r^+) \ge \rho,$$

where $B_r^+ = B_r(0) \cap E^+$.

Proof. Recalling (H5) and (H9), for any $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$H(x, u, 0) \le \epsilon |u|^2 + C_\epsilon |u|^{2^*}.$$

In fact, it follows from $(H5)$ that

$$|H_u(x, u, 0)| \le \gamma_0 \left(1 + |u|^{p-1}\right). \tag{3.6}$$

Furthermore, in the light of (H9), we have $H_u(x, u, 0) = o(|u|)$ as $u \to 0$. Then for any $\epsilon > 0$, there is a constant $\bar{c} > 0$ such that

$$|H_u(x, u, 0)| \le \epsilon |u| \quad \text{whenever } |u| < \bar{c}.$$

And there exists $C > 0$ such that

$$|H_u(x, u, 0)| \le \gamma_0 \left(1 + |u|^{p-1}\right) \le C|u|^{p-1} \quad \text{whenever } |u| > \bar{c}.$$

To sum up, we have

$$|H_u(x, u, 0)| \le \epsilon |u| + C_\epsilon |u|^{p-1}.$$

That is,

$$|H(x, u, 0)| \le \epsilon |u|^2 + C_\epsilon |u|^p < \epsilon |u|^2 + C_\epsilon |u|^{2^*}.$$

Therefore,

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega H(x, u, 0) dx \ge \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \epsilon |u|_2^2 - C_\epsilon |u|_{2^*}^{2^*}.$$

Then we obtain the conclusion. ∎

Assume $e \in E^+$ with $|\nabla e|_2^2 = 1$, and let

$$Q = \{(se, v) : 0 \le s \le r_1, \ \|v\|_q \le r_2\}.$$

Lemma 3.7. *Suppose H satisfies* (H8) *and* (H9). *Then there are constants $r_1, r_2 > 0$ with $r_1 > r$ such that*

$$I(z) \le 0 \quad \text{for all } z \in \partial Q.$$

Proof. In view of (H9) and $H(x,0,v) \geq 0$, we have

$$I(z) := -\frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega H(x,0,v)dx \leq 0,$$

when $z \in E_q^-$. By (H8), we obtain

$$\begin{aligned} I((se,v)) &= \frac{s^2}{2}\int_\Omega |\nabla e|^2 dx - \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega H(x,se,v)dx \\ &\leq \frac{s^2}{2} - \frac{1}{2}|\nabla v|_2^2 - \int_\Omega \left(\gamma_1(|se|^p + |v|^q) - \gamma_2\right) dx \\ &\leq \frac{s^2}{2} - \frac{1}{2}|\nabla v|_2^2 - c_1\int_\Omega \left(|se|^p + |v|^q\right) dx + c_2. \end{aligned}$$

Since $p > 2$, we obtain the conclusion. □

Next, we establish the boundedness of the $(PS)_c^*$ sequence, which plays an important role in the existence theory of the nontrivial weak solution.

Lemma 3.8. *Assume H satisfies* (H6) *and* (H7). *Then the $(PS)_c^*$ sequence $\{z_n\} \subset E_q$ is bounded, where $z_n = (u_n, v_n)$.*

Proof. Without loss of generality, suppose $\|z_n\|_q \to +\infty$. By

$$\|z_n\|_q^2 = |\nabla u_n|_2^2 + |\nabla v_n|_2^2 + |v_n|_q^2,$$

we assume that

$$|\nabla u_n|_2 \to +\infty, \quad \frac{|\nabla v_n|}{|\nabla u_n|_2} \to a < 1.$$

Setting $Y_n = \frac{z_n}{\|z_n\|_q}$, then $Y_n \in E_q$ with $\|Y_n\|_q = 1$. Therefore, there is $Y \in E_q$ such that $Y_n \rightharpoonup Y$ in E_q. Then we have $Y_n(x) \to Y(x)$ a.e. in Ω. Denote

$$\Omega_0 = \{x \in \Omega,\ Y(x) \neq 0\}.$$

Then we have

$$\lim_{n\to+\infty} \frac{z_n}{\|z_n\|_q} = \lim_{n\to+\infty} Y_n = Y \neq 0 \quad \text{a.e. in } \Omega_0, \tag{3.7}$$

which implies $|z_n| \to +\infty$ a.e. in Ω_0. It follows from (H6) that

$$\lim_{n\to+\infty} \frac{H(x,z_n)}{|z_n|^2}|Y_n|^2 = +\infty \quad \text{a.e. in } \Omega_0.$$

Again by using (H6), there is $N_0 > 0$ such that, for any $x \in \Omega$, we have

$$\frac{H(x,z_n)}{|z_n|^2} > 1 \quad \text{whenever } |z_n| \geq N_0. \tag{3.8}$$

Since H is continuous on $\overline{\Omega} \times [-N_0, N_0] \times [-N_0, N_0]$, there exists an $M > 0$ such that

$$|H(x,z_n)| \leq M, \tag{3.9}$$

for any $(x, z_n) \in \overline{\Omega} \times [-N_0, N_0] \times [-N_0, N_0]$. From (3.8) and (3.9), we deduce that there exists constant C such that

$$H(x,z_n) \geq C \quad \text{for all } (x,z_n) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}.$$

This implies that

$$\frac{H(x,z_n)-C}{\|z_n\|_q^2}\geq 0.$$

Since

$$I(z_n)=\frac{1}{2}\int_\Omega \left(|\nabla u_n|^2-|\nabla v_n|^2\right)dx-\int_\Omega H(x,z_n)dx=c+o(1),$$

we have

$$\int_\Omega \left(|\nabla u_n|^2-|\nabla v_n|^2\right)dx=2c+2\int_\Omega H(x,z_n)dx+o(1).$$

In view of $|\nabla u_n|_2\to+\infty$, we have $\int_\Omega H(x,z_n)dx\to+\infty$. Therefore,

$$\begin{aligned}
+\infty&=\int_{\Omega_0}\liminf_{n\to+\infty}\frac{H(x,z_n)}{|z_n|^2}|Y_n|^2dx-\int_{\Omega_0}\limsup_{n\to+\infty}\frac{C}{\|z_n\|_q^2}dx\\
&=\int_{\Omega_0}\liminf_{n\to+\infty}\Big(\frac{H(x,z_n)}{|z_n|^2}|Y_n|^2-\frac{C}{\|z_n\|_q^2}\Big)dx\\
&\leq\liminf_{n\to+\infty}\int_{\Omega_0}\Big(\frac{H(x,z_n)}{|z_n|^2}|Y_n|^2-\frac{C}{\|z_n\|_q^2}\Big)dx\\
&\leq\liminf_{n\to+\infty}\int_{\Omega}\Big(\frac{H(x,z_n)}{|z_n|^2}|Y_n|^2-\frac{C}{\|z_n\|_q^2}\Big)dx\\
&=\liminf_{n\to+\infty}\int_\Omega\frac{H(x,z_n)}{\|z_n\|_q^2}dx-\limsup_{n\to+\infty}\int_\Omega\frac{C}{\|z_n\|_q^2}dx\\
&=\liminf_{n\to+\infty}\int_\Omega\frac{H(x,z_n)}{\|z_n\|_q^2}dx-\limsup_{n\to+\infty}\frac{C|\Omega|}{\|z_n\|_q^2}\\
&=\liminf_{n\to+\infty}\int_\Omega\frac{H(x,z_n)}{\|z_n\|_q^2}dx\\
&=\liminf_{n\to+\infty}\frac{\int_\Omega H(x,z_n)dx}{|\nabla u|_2^2+|\nabla v|_2^2+|v|_q^2}\\
&=\liminf_{n\to+\infty}\frac{\int_\Omega H(x,z_n)dx}{2c+2\int_\Omega H(x,z_n)dx+2|\nabla v|_2^2+|v|_q^2+o(1)}=\frac{1}{2},
\end{aligned}$$

which leads to a contradiction. Then $|\Omega_0|=0$, and therefore, $Y(x)=0$ a.e. in Ω_0. Since

$$I(z_n)=\frac{1}{2}\int_\Omega \left(|\nabla u_n|^2-|\nabla v_n|^2\right)dx-\int_\Omega H(x,z_n)dx,$$

we have

$$\frac{I(z_n)}{\|z_n\|_q^2}=\frac{1}{2}-\int_\Omega\frac{H(x,z_n)}{|z_n|^2}|Y_n|^2dx,$$

that is,

$$\int_\Omega\frac{H(x,z_n)}{|z_n|^2}|Y_n|^2dx\to\frac{1}{2}.$$

Moreover, thanks to

$$\langle I'(z_n),z_n\rangle=\int_\Omega|\nabla u_n|dx-\int_\Omega|\nabla v_n|^2dx-\int_\Omega H_z(x,z_n)z_n\,dx.$$

we have

$$1-\int_\Omega \frac{H_z(x,z_n)z_n}{\|z_n\|_q^2}dx=\frac{\langle I'(z_n),z_n\rangle}{\|z_n\|_q^2}\ leq\frac{\|I'(z_n)\|\cdot\|z_n\|_q}{\|z_n\|_q^2}=\frac{\|I'(z_n)\|}{\|z_n\|_q^2}\to 0,$$

that is,

$$\int_\Omega \frac{z_nH_z(x,z_n)}{|z_n|}|Y_n|^2dx\to 1.$$

Therefore,

$$\int_\Omega \frac{\mu H(x,z_n)-z_nH_z(x,z_n)}{|z_n|^2}|Y_n|dx\to \frac{\mu}{2}-1.$$

Then it follows from (H7) that

$$\limsup \frac{\mu H(x,z_n)-z_nH_z(x,z_n)}{|z_n|^2}|Y_n|^2\le \limsup C\frac{|z_n|^2+1}{|z_n|^2}|Y_n|^2=0;$$

that is, $\frac{\mu}{2}-1\le 0$. Hence, $\mu\le 2$. This is a contradiction. Therefore, $\{z_n\}$ is bounded in E_q. □

Lemma 3.9. *Let $\{z_n\}\subset X_n$ be a $(PS)_c^*$ sequence. Then there exists $z\in E_q$ such that along a subsequence, $z_n\rightharpoonup z$ with $I'(z)=0$ and $I(z)\ge c$.*

Proof. Since $\{z_n\}\subset X_n$ is a $(PS)_c^*$ subsequence, it follows from Lemma 3.8 that the $(PS)_c^*$ sequence is bounded, that is, $\{z_n\}$ is bounded. Thus, $\{z_n\}$ has weakly convergent subsequence, might as well suppose $z_n\rightharpoonup z$ in E_q. Then for any $1\le s<2^*$, the imbedding theorem implies that $z_n\to z$ in $(L^s(\Omega))^2$. As a result,

$$z_n(x)\to z(x)\quad \text{a.e. in } \Omega.$$

Moreover, by using Lemma 3.1, we know I' is weakly sequence continuous. Hence, we obtain $I'(z)=0$.

Let $w=(\varphi,\psi)=(u_n-u,0)$ in (3.2) and by $I_n'(z_n)\to 0$, we have

$$\begin{aligned}(\nabla u_n,\nabla u_n-\nabla u)_{L^2}&=I_n'(z_n)(u_n-u,0)+\int_\Omega H_u(x,z_n)(u_n-u)dx\\&=o(1)+\int_\Omega H_u(x,z_n)(u_n-u)dx.\end{aligned}$$

By using (H5), Hölder's inequality and $\frac{2q}{q+2}<2<2^*$, we have

$$\begin{aligned}&\Big|\int_\Omega H_u(x,z_n)(u_n-u)dx\Big|\\&\le \int_\Omega \gamma_0\left(1+|u_n|^{p-1}+|v_n|^{\frac{q}{2}-1}\right)|u_n-u|dx\\&\le \gamma_0\left(|u_n-u|_1+|u_n|_p^{p-1}|u_n-u|_p+|v_n|_q^{\frac{q}{2}-1}|u_n-u|_{\frac{2q}{q+2}}\right)=o(1).\end{aligned}$$

Therefore,

$$(\nabla u_n,\nabla u_n-\nabla u)_{L^2}=o(1),$$

that is,

$$|\nabla u_n|_2^2\to|\nabla u|_2^2.$$

Then $u_n\to u\quad$ in $H_0^1(\Omega)$.

Let $p_n : E_q \to X_n$ denote the projection. Observe that $P_n z \to z$ in E_q for all $z \in E_q$. Moreover, using again $(H5)$ and Hölder's inequality, we deduce

$$
\begin{aligned}
&\Big| \int_\Omega H_v(x,z_n)(v - P_n v)dx \Big| \\
&\leq C\left(|v - P_n v|_1 + |u_n|_p^{p-1} |v - P_n v|_p + |v_n|_q^{q-1} |v - P_n v|_q \right) \to 0.
\end{aligned}
$$

On the other hand, lettin $w = (\varphi, \psi) = (0, v_n - P_n v)$ in (3.2), and by $I_n'(z_n) \to 0$, we obtain

$$
\begin{aligned}
&I_n'(z_n)(0, v_n - P_n v) \\
&= -\int_\Omega \nabla v_n \nabla (v_n - P_n v)dx + \int_\Omega H_v(x,z_n)(v_n - P_n v)dx \\
&= -\int_\Omega \nabla v_n \nabla (v_n - v + v - P_n v)dx + \int_\Omega H_v(x,z_n)(v_n - v + v - P_n v)dx \\
&= -(\nabla v_n \nabla v_n - \nabla v)_{L^2} - \int_\Omega \nabla v_n \nabla (v - p_n)dx + \int_\Omega H_v(x,z_n)(v_n - v)dx \\
&\quad + \int_\Omega H_v(x,z_n)(v - P_n v)dx \\
&= -(\nabla v_n \nabla v_n - \nabla v)_{L^2} + \int_\Omega H_v(x,z_n)(v_n - v)dx + o(1).
\end{aligned}
$$

Then

$$
\begin{aligned}
(\nabla v_n \nabla v_n - \nabla v)_{L^2} &= \int_\Omega H_v(x,z_n)(v_n - v)dx + o(1) \\
&= \int_\Omega H_z(x,z_n)(z_n - z)dx + \int_\Omega H_u(x,z_n)(u_n - u)dx + o(1) \\
&= \int_\Omega H_z(x,z_n)z_n dx - \int_\Omega H_z(x,z_n)z dx + o(1).
\end{aligned}
$$

It follows from the Lebesgue's theorem and the weak sequential continuity of H_z that

$$
\begin{aligned}
|\nabla v|_2^2 - \limsup_{n\to\infty} |\nabla v_n|_2^2 &= \liminf_{n\to\infty} \Big(\int_\Omega H_z(x,z_n)z_n dx - \int_\Omega H_z(x,z_n)z dx \Big) \\
&\geq \int_\Omega \liminf_{n\to\infty} \left(H_z(x,z_n)z_n - H_z(x,z_n)z \right) dx = 0,
\end{aligned}
$$

that is,

$$
|\nabla v|_2^2 \geq \limsup_{n\to\infty} |\nabla v_n|_2^2,
$$

which together with the weak lower semicontinuity of the norm implies that

$$
|\nabla v|_2 \leq \limsup_{n\to\infty} |\nabla v_n|_2.
$$

So $|\nabla v_n|_2 \to |\nabla v|_2$, that is, $v_n \to v$ in $H_0^1(\Omega)$. Observe that

$$
\begin{aligned}
I(z) - I(z_n) &= \frac{1}{2}\left(|\nabla u|_2^2 - |\nabla u_n|_2^2 \right) - \frac{1}{2}\left(|\nabla v|_2^2 - |\nabla v_n|_2^2 \right) \\
&\quad + \int_\Omega H(x,z_n)dx - \int_\Omega H(x,z)dx.
\end{aligned}
$$

The Lebesgue's theorem then yields

$$\begin{aligned} I(z) - C &= \liminf_{n\to\infty} \int_\Omega H(x, z_n)dx - \int_\Omega H(x, z)dx \\ &\geq \int_\Omega \liminf_{n\to\infty} H(x, z_n)dx - \int_\Omega H(x, z)dx = 0. \end{aligned}$$

Then we have $I(z) \geq C$. □

Proof of Theorem 1.2. From the above discussion, it follows from Lemmas 3.6 and 3.7 that I has the linking geometry. Let $Q_n := Q \cap X_n$ and define

$$c_n := \inf_{\gamma\in\Gamma_n} \sup_{x\in\gamma(Q_n)} I(x),$$

where $\Gamma_n := \{\gamma \in C(Q_n, X_n) : \gamma |_{\partial Q_n} = id\}$. Then

$$\rho \leq c_n \leq k := \sup I(\gamma(Q)).$$

Therefore, by the Linking theorem, there is $z_n \in X_n$ such that

$$|I(z_n) - c_n| \leq \frac{1}{n} \quad \text{and} \quad \|I_n'(z_n)\| \leq \frac{1}{n}.$$

So we obtain a $(PS)_c^*$ sequence $\{z_n\} \subset E_q$ with $c \in [\rho, k]$. Lemma 3.9 implies $z_n \rightharpoonup z$ with $I'(z) = 0$ and $I(z) \geq c$. As a result, the Theorem 1.2 is obtained. □

Acknowledgements. This research was supported by the National Science Foundation of China (No.11471147). The authors would like to thank the referees for helpful suggestions.

References

[1] R. Adams, J. Fournier; *Sobolev Spaces*, Academic Press, New York (2003).

[2] A. Ambrosetti, P. H. Rabinowitz; *Dual variational methods in critical point theory and applications*, Journal of Functional Analysis, **14**, No. 4(1973), 349-381.

[3] H. Brezis; *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York (2011).

[4] P. Cl ment, M. Garc a-Huidobro, Man sevich; *Mountain pass type solutions for quasilinear elliptic equations*, Calculus of Variations and Partial Differential Equations, **11**, No. 1(2000), 33-62.

[5] M. L. M. Carvalho, J. V. A. Goncalves, E. D. D. Silva; *On quasilinear elliptic problems without the Ambrosetti-Rabinowitz condition*, Journal of Mathematical Analysis and Applications, **426**, No. 1(2015), 466-483.

[6] D. G. Costa, C. A. Magalhaes; *A variational approach to noncooperative elliptic systems*, Nonlinear Analysis Theory Methods and Applications, **25**, No. 7(1995), 699-715.

[7] D. G. De Figueiredo, Y. H. Ding; *Strongly indefinite functionals and multiple solutions of elliptic systems*, Transactions of the American Mathematical Society, **355**, No. 7(2003), 2973-2989.

[8] D. G. De Figueiredo, P. L. Felmer; *On superquadratic elliptic systems*, Transactions of the American Mathematical Society, **343**, No. 1(1994), 99-116.

[9] G. Dinca, P. Jebelean, J. Mawhin; *A result of Ambrosetti-Rabinowitz type for p-Laplancian*, Qualitative problems for differential equations and control theory, World Sci. Publ., River Edge, NJ, (1995), 231-242.

[10] L. C. Evans; *Partial Differential Equations(second edition)*, University of Califolia, Berkeley, (2010).

[11] D. Gilbarg, N. Trudinge; *Elliptic Partial Differential Equation of Second Order*, Springer, New York (2001).

[12] Z. Hu, L. Wang, P. H. Zhao; *A priori estimates and existence for quasilinear elliptic equations with nonlinear Neumann boundary conditions*, Electronic Journal of Differential Equations, **2016**, No. 187 (2016), 1-8.

[13] P. L. Lions; *On the existence of positive solutions of semilinear elliptic equations*, Siam Review, **24**, No. 4 (1982), 441-467.

[14] N. Lam, G. Lu; *Elliptic equations and systems with subcritical and critical exponential growth without the Ambrosetti-Rabinowitz condition*, Journal of Geometric Analysis, **24**, No. 1(2014), 118-143.

[15] Z. L. Liu, Z. Q. Wang; *On the Ambrosetti-Rabinowitz superlinear condition*, Advanced Nonlinear Studies, **4**, No. 4(2004), 563-574.

[16] G. B. Li, C. Y. Yang; *The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti-Rabinowitz condition*, Nonlinear Analysis, **72**, (2010), 4602-4613.

[17] O. H. Miyagaki, M. A. S. Souto; *Superlinear problems without Ambrosetti and Rabinowitz growth condition*, Journal of Differential Equations, **245**, No. 12(2008), 3628-3638.

[18] P. D. N pol , M. C. Mariani; *Mountain pass solutions to equations of p-Laplacian type*, Nonlinear Analysis, **54**, No. 7(2003), 1205-1219.

[19] M. Struwe; *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems(fourth edition)*, Springer, 2012.

[20] M. Schechter, W. M. Zou; *Superlinear problem*, Pacific Journal of Mathematics, **214**, No. 1(2004), 145-160.

[21] J. Vazquez; *A strong maximum principle for some quasilinear elliptic equations*, Applied Mathematics and Optimization, **12**, No. 1(1984), 191-202.

[22] L. Wang, P. H. Zhao; *Infinitely many sign-changing solutions for concave-convex elliptic problem with nonlinear boundary condition*, Electronic Journal of Differential Equations, No. 266(2015), 1-9.

[23] Z. Q. Wu, J. X. Yin, C. P. Wang; *Introduction to elliptic and parabolic equations*, Science Press, Beijing (2003).

[24] C. K. Zhong, X. L. Fan, W. Y. Cheng; *Introduction to Nonlinear Functional Analysis*, Lanzhou University Press, Lanzhou, 2004.

Xiaohui Wang
School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China
Email address: `xiaohuiwang1@126.com`

Peihao zhao
School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China
Email address: `zhaoph@lzu.edu.cn`