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POSITIVE SOLUTIONS FOR ASYMPTOTICALLY 3-LINEAR QUASILINEAR SCHRÖDINGER EQUATIONS

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Abstract. In this article, we study the quasilinear Schrödinger equation

$$-\Delta u+V(x)u-\frac{\kappa}{2}[\Delta(1+u^2)^{1/2}]\frac{u}{(1+u^2)^{1/2}}=h(u),\quad x\in\mathbb{R}^N,$$

where $N\geq 3$, $\kappa>0$ is a parameter, $V:\mathbb{R}^N\to\mathbb{R}$ is a given potential. The nonlinearity $h\in C(\mathbb{R},\mathbb{R})$ is asymptotically 3-linear at infinity. We obtain the nonexistence of a least energy solution and the existence of a positive solution, via the Pohožaev manifold and a linking theorem. Our results improve recent results in [4, 22].

1. Introduction and statement of main results

In this article, we study the quasilinear Schrödinger equation

$$-\Delta u+V(x)u-\frac{\kappa}{2}[\Delta(1+u^2)^{1/2}]\frac{u}{(1+u^2)^{1/2}}=h(u),\quad x\in\mathbb{R}^N, \tag{1.1}$$

where $N\geq 3$, $\kappa>0$ is a parameter, $V:\mathbb{R}^N\to\mathbb{R}$ is a given potential and h is a real function. Solutions of (1.1) are related to standing waves for the following quasilinear Schrödinger equation

$$iz_t=-\Delta z+W(x)z-a(x)\eta(|z|^2)z-\kappa[\Delta(\varphi(|z|^2)\varphi'(|z|^2)]z,\quad x\in\mathbb{R}^N, \tag{1.2}$$

where $z:\mathbb{R}\times\mathbb{R}^N\to\mathbb{C}$, $W:\mathbb{R}^N\to\mathbb{R}$ is a given potential and $a,\varphi,\eta:\mathbb{R}\to\mathbb{R}$ are real functions. Note that (1.2) is a generalized nonlinear Schrödinger equation, which has been derived as mathematical models of several physical phenomena corresponding to various types of the nonlinear terms φ and η, see [3, 8, 9, 12, 24]. Substituting $z(t,x)=\exp(-iEt)u(x)$ into (1.2), we obtain the equation

$$-\Delta u+V(x)u-\kappa[\Delta(\varphi(|u|^2))\varphi'(|u|^2)u=a(x)\eta(|u|^2)u,\quad x\in\mathbb{R}^N, \tag{1.3}$$

where $V(x):=W(x)-E$ is the new potential function. Setting $h(t):=\eta(t^2)t$, then if $\varphi(t)=\sqrt{1+t}$, $a(x)=1$, Equation (1.3) turns into (1.1) and if $\varphi(t)=t$ and $\kappa=1$, Equation (1.3) becomes the quasilinear problem

$$-\Delta u+V(x)u-\Delta(u^2)u=a(x)h(u),\quad x\in\mathbb{R}^N. \tag{1.4}$$

Since the behavior of h at infinity plays an important role in searching the weak solutions of (1.3), many authors have studied (1.3) with particular forms of φ via

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variational methods under various conditions on the nonlinearity h; e.g. h hass superlinear growth [5, 6, 15, 17, 18, 23, 25] or has asymptotically linear growth [16] at infinity. Moreover, the existence of positive solutions for (1.4) was obtained in [4, 22] for asymptotically 2-linear growth of the nonlinear term $h(t)$ at infinity, where $h(t)$ was assumed to satisfy

- (H1) $h \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and $\lim_{t\to 0^+} \frac{h(t)}{t} = 0$;
- (H2) $\lim_{t\to\infty} \frac{h(t)}{t^2} = 1$;
- (H3) If $Q(t) = \frac{1}{4}h(t)t - H(t), H(t) = \int_0^t h(s)ds$, and thus a constant $D \geq 1$ exists such that $0 < Q(s) \leq DQ(t)$ for $0 < s \leq t$, and $\lim_{t\to\infty} Q(t) = +\infty$.

We note that if $h(t)$ is positive for $t > 0$, then from (H1) and (H3) it follows that

$$H(t) > ct^4$$

for some positive constant c and large $t > 0$, which means that (H2) does not occur when $h(t)$ satisfies assumptions (H1) and (H3).

The purpose of this article is to investigate the existence of positive solutions to (1.1) for the nonlinear term $h(t)$ satisfying the modified assumptions (H1)-(H3). More precisely, we suppose that h satisfies the following assumptions

- (H1') $h \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and $\lim_{t\to 0^+} \frac{h(t)}{t} = 0$;
- (H2') $\lim_{t\to+\infty} \frac{h(t)}{t^3} = 1$;
- (H3') $Q(t) = \frac{1}{4}h(t)t - H(t) > 0$ for all $t > 0$, where $H(t) = \int_0^t h(s)ds$.

Also, we assume that the following conditions on the potential function $V(x)$

- (H4) $V \in C^2(\mathbb{R}^N, \mathbb{R})$;
- (H5) $\lim_{|x|\to+\infty} V(x) = V_\infty < 1$, $1/2 < V_\infty < V(x)$ for all $x \in \mathbb{R}^N$;
- (H6) $\langle \nabla V(x), x\rangle \leq 0$ for all $x \in \mathbb{R}^N$ with the strict inequality holding on a subset of positive Lebesgue measure of $\mathbb{R}^N$;
- (H7) $NV(x) + \langle \nabla V(x), x\rangle \geq NV_\infty$ for all $x \in \mathbb{R}^N$;
- (H8) $\frac{xH_V(x)x}{N} + \langle \nabla V(x), x\rangle \leq 0$ for all $x \in \mathbb{R}^N$, where H_V is the Hessian matrix of the function $V(x)$.

We want to point out that similar method can be applied to (1.4). Now, we study the quasilinear Equation (1.1). The associated energy functional of the Euler-Lagrange equation (1.1) is

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big[1 + \frac{\kappa u^2}{2(1+u^2)}\big]|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx - \int_{\mathbb{R}^N} H(u)dx.$$

When $V(x) \equiv V_\infty$, we are led to the limiting problem of (1.1),

$$-\Delta u + V_\infty u - \frac{\kappa}{2}[\Delta(1+u^2)^{1/2}]\frac{u}{(1+u^2)^{1/2}} = h(u), \quad x \in \mathbb{R}^N. \tag{1.5}$$

The associated energy functional of (1.5) is

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big[1 + \frac{\kappa u^2}{2(1+u^2)}\big]|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty|u|^2 dx - \int_{\mathbb{R}^N} H(u)dx.$$

Making a change of variable, i.e. using the dual approach (cf. [7]), we can reduce the quasilinear Schrödinger equation into a semilinear equation like the case of $\kappa = 0$. Let

$$v = G(u) = \int_0^u g(t)dt \tag{1.6}$$

with G satisfying

$$(G^{-1}(t))' = \frac{1}{g(G^{-1}(t))} = \frac{1}{\sqrt{1+\frac{\kappa(G^{-1}(t))^2}{2(1+(G^{-1}(t))^2)}}}$$

for $t \in [0,+\infty)$ and $G^{-1}(t) = -G^{-1}(-t)$ for $t \in (-\infty, 0]$. Then, (1.1) and (1.5) will be reduced to the semilinear equation

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{h(G^{-1}(v))}{g(G^{-1}(v))}, x \in \mathbb{R}^N, \tag{1.7}$$

and

$$-\Delta v + V_\infty \frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{h(G^{-1}(v))}{g(G^{-1}(v))}, \quad x \in \mathbb{R}^N. \tag{1.8}$$

Clearly, weak solutions of (1.7) and (1.8) correspond to critical points of the energy functional

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx - \int_{\mathbb{R}^N} H(G^{-1}(v))dx, \tag{1.9}$$

and

$$J_\infty(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty|G^{-1}(v)|^2 dx - \int_{\mathbb{R}^N} H(G^{-1}(v))dx. \tag{1.10}$$

Moreover, for $\psi \in H^1(\mathbb{R}^N)$, the derivative of J in the direction ψ at v is

$$\langle J'(v), \psi\rangle = \int_{\mathbb{R}^N} \nabla v \nabla \psi dx + \int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\psi dx - \int_{\mathbb{R}^N} \frac{h(G^{-1}(v))}{g(G^{-1}(v))}\psi dx.$$

If $v \in H^1(\mathbb{R}^N)$ is a weak solution of (1.7), then v satisfies the Pohožaev identity $\gamma(v) = 0$, where

$$\begin{aligned}\gamma(v) = &\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{N}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx \\ &+ \frac{1}{2}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |G^{-1}(v)|^2 dx - N\int_{\mathbb{R}^N} H(G^{-1}(v))dx.\end{aligned}$$

Furthermore, we defined the Pohožaev manifold associated with (1.7) by

$$\mathcal{P} := \{v \in H^1(\mathbb{R}^N)\backslash\{0\} : \gamma(v) = 0\}.$$

Motivated by [11], we will employ the minimization methods restricted to the Pohožaev manifold to obtain the existence of positive solutions for (1.1). Now, we state our first result.

Theorem 1.1. *Assume that* (H1')–(H3'), (H4)–(H8) *hold. Then $\mathcal{P}$ is a natural constraint of* (1.1)*, i.e. any critical point of $J|_{\mathcal{P}}$ is a critical point of J in $H^1(\mathbb{R}^N)$. Moreover, $p = \inf_{v\in\mathcal{P}} J(v)$ is not a critical level for the function J.*

We define

$$\Gamma_\infty = \{\xi \in C([0,1], H^1(\mathbb{R}^N)) : \xi(0) = 0 \neq \xi(1), J_\infty(\xi(1)) < 0\},$$

as well as the mountain pass min-max level

$$c_\infty = \inf_{\xi\in\Gamma_\infty} \max_{t\in[0,1]} J_\infty(\xi(t)),$$

where J_∞ is defined by (1.10). We will use the linking theorem and the barycenter function is restricted to the Pohožaev manifold to obtain the existence of weak solution for (1.1). Here is our second result.

Theorem 1.2. *Assume that* (H1')–(H3'), (H4)–(H8) *and the following conditions hold:*

(1) $h \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R}^+, \mathbb{R}^+)$*;*
(2) $\|V(x) - V_\infty\|_\infty$ *is sufficiently small;*
(3) *the least energy level* c_∞ *of* J_∞ *is an isolated radial critical level or equation* (1.5) *admits a unique positive solution which is radially symmetric about some point.*

Then (1.1) *admits a positive solution whose energy is above* c_∞*.*

There are functions satisfying (H1')–(H3'), for example $h(t) = \frac{t^5}{1+t^2}$. There are also functions satisfying (H4)–(H8), for example $V(x) = c_1 + \frac{c_2}{1+|x|^2}$, where $1/2 < c_1 < 1, c_2 > 0$. In fact, since $N \geq 3$, we know that

$$\langle \nabla V(x), x\rangle = -\frac{2c_2|x|^2}{(1+|x|^2)^2} < 0,$$

$$NV(x) + \langle \nabla V(x), x\rangle = Nc_1 + \frac{Nc_2 + c_2|x|^2(N-2)}{(1+|x|^2)^2} \geq Nc_1 = NV_\infty,$$

$$\begin{aligned}\frac{xH_V(x)x}{N} + \langle \nabla V(x), x\rangle &= \frac{4c_2(|x|^4 - |x|^2)}{N(1+|x|^2)^3} - \frac{2c_2|x|^2}{(1+|x|^2)^2} \\ &= \frac{2c_2|x|^2}{(1+|x|^2)^3}\big[\big(\frac{2}{N} - 1\big)|x|^2 - \big(\frac{2}{N} + 1\big)\big] \leq 0.\end{aligned}$$

In this article: $\|u\|_q$ $(1 \leq q \leq \infty)$ denotes the standard norm in $L^q(\mathbb{R}^N)$. $\langle \cdot, \cdot \rangle$ denotes the duality pairing between a Banach space and its dual space. $\to$ and $\rightharpoonup$ denote strong convergence and weak convergence in the related function space, respectively. $o_n(1)$ denotes the quantities tending to 0 as $n \to \infty$. $C, C_0, C_1, \dots$ denote positive constants. $B_R(0)$ denotes a ball centered at the origin with radius $R > 0$.

2. Preliminaries

We shall work in the space $H^1(\mathbb{R}^N)$ with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx,$$

because of (H4) and (H5), this norm is equivalent to the standard $H^1(\mathbb{R}^N)$ norm.

If u is a solution of (1.1), then for all $\varphi \in H^1(\mathbb{R}^N)$ we have

$$\begin{aligned}\langle I'(u), \varphi\rangle = &\int_{\mathbb{R}^N} [g^2(u)\nabla u \nabla \varphi + g(u)g'(u)|\nabla u|^2 \varphi] dx + \int_{\mathbb{R}^N} V(x) u\varphi dx \\ &- \int_{\mathbb{R}^N} h(u)\varphi dx = 0, \quad u \in H^1(\mathbb{R}^N),\end{aligned} \tag{2.1}$$

where $g(t) = \sqrt{1 + \frac{\kappa t^2}{2(1+t^2)}}$.

On the one hand, if we choose $\varphi = \frac{\psi}{g(u)}$ in (2.1), combining (1.6) and (1.9), we obtain

$$\begin{aligned}\langle J'(v), \psi\rangle = &\int_{\mathbb{R}^N} \nabla v \nabla \psi dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi dx \\ &- \int_{\mathbb{R}^N} \frac{h(G^{-1}(v))}{g(G^{-1}(v))} \psi dx = 0.\end{aligned} \tag{2.2}$$

On the other hand, let $\psi = g(u)\varphi$ in (2.2), we obtain (2.1). Thus (2.1) is equivalent to (2.2). Hence, u is a weak solution of (1.1) if and only if v is a critical point of the functional J.

Note that a function $v \in H^1(\mathbb{R}^N)$ is a least energy solution if and only if v is a solution of (1.5) and $J_\infty(v) = m_\infty$, where

$$m_\infty = \inf \big\{ J_\infty(v) : v \in H^1(\mathbb{R}^N)\backslash\{0\} \text{ is a solution of (1.5)} \big\}.$$

To see the smoothness of J, we need the following lemma.

Lemma 2.1. *The functions $g(t)$ and $G(t) = \int_0^t g(s)ds$ satisfy the following properties:*

(1) *$G(t)$ and $G^{-1}(t)$ are odd functions.*
(2) $1 \le g(t) \le \sqrt{1 + \frac{\kappa}{2}}$.
(3) $0 \le \frac{t}{g(t)} g'(t) \le \frac{\sqrt{2(2+\kappa)}-2}{\sqrt{2(2+\kappa)}+2}$ *for all* $t \ge 0$.
(4) $\sqrt{\frac{2}{2+\kappa}}|t| \le |G^{-1}(t)| \le |t|$ *for all* $t \in \mathbb{R}$.

Proof. From the definition of $G(t)$ we can prove (1) and (2).

(3) Setting $Z(t) = \frac{t}{g(t)} g'(t)$, direct computations show that

$$Z(t) = \frac{\kappa t^2}{(1+t^2)[2 + (2+\kappa)t^2]} = \Phi(t^2).$$

Then $0 \le Z(t)$ for $\kappa > 0$. Moreover, $\Phi(r)$ attains its maximum at $r_0 = \sqrt{\frac{2}{2+\kappa}}$ and

$$Z_{\max}(t) = Z(t)\big|_{t^2 = \sqrt{\frac{2}{2+\kappa}}} = \frac{\sqrt{2(2+\kappa)} - 2}{\sqrt{2(2+\kappa)} + 2}.$$

Then, (3) holds.

(4) Since $g(t)$ is nondecreasing for $t \ge 0$, by the differential mean value theorem, we know that

$$t = g(0)t \le G(t) = \int_0^t g(s)ds = g(\xi)t \le g(t)t \le g(\infty)t = \sqrt{1 + \frac{\kappa}{2}} t, \xi \in [0, t].$$

Then, $\sqrt{2/(2+\kappa)}t \le G^{-1}(t) \le t$. When $t < 0$, it deduce from the oddness of $G^{-1}(t)$ that $t \le G^{-1}(t) \le \sqrt{\frac{2}{2+\kappa}}t$. Thus the proof is complete. □

Now, by Lemma 2.1, J is well defined and is of C^1 if $h(t)$ satisfies the conditions (H1')–(H3'). Next, we show another property of the change of variable G which will play important roles in proving our results.

Lemma 2.2. *For $t > 0$, it holds*

$$\frac{1}{2} G^{-1}(t) g(G^{-1}(t)) \le t \le G^{-1}(t) g(G^{-1}(t)).$$

Proof. Let $\eta(s) = G(s) - \frac{1}{2} s g(s)$, then by Lemma 2.1(3), we have

$$\eta'(s) = g(s) - \frac{1}{2} g(s) - \frac{1}{2} s g'(s) = \frac{1}{2} g(s)\big[1 - \frac{s g'(s)}{g(s)}\big] \ge 0.$$

Thus $\eta(s) \ge \eta(0)$, let $s = G^{-1}(t)$, we have $\frac{1}{2} G^{-1}(t) g(G^{-1}(t)) \le t$ for $t > 0$. Moreover, we set

$$\theta(t) = G^{-1}(t) g(G^{-1}(t)) - t.$$

Direct computation shows that

$$
\begin{aligned}
\theta'(t) &= \frac{1}{g(G^{-1}(t))} g(G^{-1}(t)) + G^{-1}(t) g'(G^{-1}(t)) - 1 \\
&= G^{-1}(t) \Big(\sqrt{1 + \frac{\kappa (G^{-1}(t))^2}{2(1 + (G^{-1}(t))^2)}} \Big)' \\
&= G^{-1}(t) \frac{\kappa G^{-1}(t)}{2[1 + \frac{\kappa (G^{-1}(t))^2}{2(1+(G^{-1}(t))^2)}][1 + (G^{-1}(t))^2]^2} \\
&\geq 0, \quad \forall \kappa > 0,\ t > 0.
\end{aligned}
$$

Then $\theta(t) \geq \theta(0)$, which implies that $t \leq G^{-1}(t) g(G^{-1}(t))$ for $t > 0$. □

3. Pohožaev manifold

In this section, we will show the nonexistence of solution for (1.1). First, for the Pohožaev manifold $\mathcal{P}$, we have the following properties.

Lemma 3.1. *The functional $\gamma : H^1(\mathbb{R}^N) \to \mathbb{R}$ and the Pohožaev manifold $\mathcal{P}$ satisfy:*

1. *$\{v \equiv 0\}$ is an isolated point of $\gamma^{-1}(\{0\})$;*
2. *$\mathcal{P}$ is a closed set;*
3. *$\mathcal{P}$ is a C^1 manifold;*
4. *there exists $\sigma > 0$ such that $\|v\| > \sigma$ for all $v \in \mathcal{P}$.*

Proof. (1) By (H1') and (H2'), we can deduce that for any $\varepsilon > 0$ and $4 \leq q \leq 2^*$, there is C_ε such that

$$
|H(s)| \leq \frac{\varepsilon}{2} |s|^2 + \frac{C_\varepsilon}{q} |s|^q, \tag{3.1}
$$

and $|h(s)| \leq \varepsilon |s| + C_\varepsilon |s|^{q-1}$ for all $s \in \mathbb{R}$. Thanks to (H5), (H7) and Lemma 2.1(2), (4), if we choose $\varepsilon = V_\infty/2$, then

$$
\begin{aligned}
\gamma(v) &= \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} V(x) |G^{-1}(v)|^2 dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} \langle \nabla V(x), x \rangle |G^{-1}(v)|^2 dx - N \int_{\mathbb{R}^N} H(G^{-1}(v)) dx \\
&\geq \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} V_\infty |G^{-1}(v)|^2 dx \\
&\quad - N \int_{\mathbb{R}^N} \big[\frac{\varepsilon}{2} |G^{-1}(v)|^2 + \frac{C_\varepsilon}{q} |G^{-1}(v)|^q \big] dx \\
&\geq \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} V_\infty |G^{-1}(v)|^2 dx - \frac{N}{2} \int_{\mathbb{R}^N} \frac{V_\infty}{2} |G^{-1}(v)|^2 dx \\
&\quad - \frac{N C_\varepsilon}{q} \int_{\mathbb{R}^N} |G^{-1}(v)|^q dx \\
&\geq \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{N}{2(2+\kappa)} \int_{\mathbb{R}^N} V_\infty |v|^2 dx - \frac{N C_\varepsilon}{q} \int_{\mathbb{R}^N} |v|^q dx \\
&\geq \min \big\{ \frac{N-2}{2}, \frac{N}{2(2+\kappa)} \big\} C \|v\|^2 - \frac{N C_\varepsilon}{q} \|v\|^q.
\end{aligned}
$$

Let $\|v\| = \rho > 0$ be small enough such that $\min\big\{\frac{N-2}{2}, \frac{N}{2(2+\kappa)}\big\}C\rho^2 > 2NC_\varepsilon \frac{\rho^q}{q}$, we obtain

$$\gamma(v) \geq \min\big\{\frac{N-2}{2}, \frac{N}{2(2+\kappa)}\big\}C\rho^2 - \frac{NC_\varepsilon}{q}\rho^q > \frac{1}{2}\big\{\frac{N-2}{2}, \frac{N}{2(2+\kappa)}\big\}C\rho^2 > 0.$$

(2) The functional $\gamma(v)$ is a C^1 functional, thus $\mathcal{P} \cup \{0\} = \gamma^{-1}(0)$ is a closed subset. Moreover, $\{v \equiv 0\}$ is an isolated point in $\gamma^{-1}(\{0\})$ and the assertion follows.

(3) Since $v \in \mathcal{P}$, we have

$$\begin{aligned}
&(N-2)\int_{\mathbb{R}^N} |\nabla v|^2 dx + N\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx \\
&+ \int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |G^{-1}(v)|^2 dx \\
&= 2N\int_{\mathbb{R}^N} H(G^{-1}(v))dx.
\end{aligned} \tag{3.2}$$

Combining (H3',) (H7), and Lemma 2.2, we obtain

$$\begin{aligned}
&\langle \gamma'(v), v\rangle \\
&= (N-2)\int_{\mathbb{R}^N} |\nabla v|^2 dx + N\int_{\mathbb{R}^N} V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}v\,dx \\
&\quad + \int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle \frac{G^{-1}(v)}{g(G^{-1}(v))}v\,dx - N\int_{\mathbb{R}^N} \frac{h(G^{-1}(v))}{g(G^{-1}(v))}v\,dx \\
&= \int_{\mathbb{R}^N} (NV(x) + \langle \nabla V(x), x\rangle)\big[\frac{G^{-1}(v)}{g(G^{-1}(v))}v - |G^{-1}(v)|^2\big]dx \\
&\quad + N\int_{\mathbb{R}^N} \big[2H(G^{-1}(v)) - \frac{h(G^{-1}(v))}{g(G^{-1}(v))}v\big]dx \\
&\leq \int_{\mathbb{R}^N} (NV(x) + \langle \nabla V(x), x\rangle)\big[\frac{G^{-1}(v)}{g(G^{-1}(v))}g(G^{-1}(v))G^{-1}(v) - |G^{-1}(v)|^2\big]dx \\
&\quad + N\int_{\mathbb{R}^N} \big[2H(G^{-1}(v)) - \frac{h(G^{-1}(v))}{g(G^{-1}(v))}\frac{1}{2}g(G^{-1}(v))G^{-1}(v)\big]dx < 0.
\end{aligned}$$

This shows that $\mathcal{P}$ is a C^1 manifold.

(4) Since 0 is an isolated point in $\gamma^{-1}(\{0\})$, there must be a ball $\|v\| \leq \sigma$ which doesn't intersect $\mathcal{P}$ and the assertion is proved. □

Next, we obtain relations between the Pohožaev manifold $\mathcal{P}$ associated with (1.7) and the Pohožaev manifold $\mathcal{P}_\infty$ associated with limiting problem (1.8). Recall that

$$\mathcal{P}_\infty := \{v \in H^1(\mathbb{R}^N) \setminus \{0\} : \gamma_\infty(v) = 0\},$$

where

$$\begin{aligned}
\gamma_\infty(v) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{N}{2}\int_{\mathbb{R}^N} V_\infty |G^{-1}(v)|^2 dx \\
- N\int_{\mathbb{R}^N} H(G^{-1}(v))dx.
\end{aligned} \tag{3.3}$$

Next, we obtain the Lemmas 3.2–3.6 and 3.8 which will be used for proving Theorem 1.1. Their proofs can be found in [11] and [14].

Lemma 3.2. *Assume that* $\int_{\mathbb{R}^N}[H(G^{-1}(v)) - \frac{1}{2}V_\infty|G^{-1}(v)|^2]dx > 0$, *then there exist unique* $t_1 > 0$ *and* $t_2 > 0$ *such that* $v\big(\frac{\cdot}{t_1}\big) \in \mathcal{P}$ *and* $v\big(\frac{\cdot}{t_2}\big) \in \mathcal{P}_\infty$.

If $v \in \mathcal{P}$, *then there exists* $0 < t_v < 1$ *such that* $v\big(\frac{\cdot}{t_v}\big) \in \mathcal{P}_\infty$.

If $w \in \mathcal{P}_\infty$, *then there exists* $t_w > 1$ *such that* $w\big(\frac{\cdot}{t_w}\big) \in \mathcal{P}$.

Lemma 3.3. *Assume that*

$$\Omega = \{v \in H^1(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N}[H(G^{-1}(v)) - \frac{1}{2}V_\infty|G^{-1}(v)|^2]dx > 0\}.$$

Then the function $t_1 : \Omega \to \mathbb{R}^+$ *is given by* $v \mapsto t_1(v)$ *such that* $v\big(\frac{\cdot}{t_1(v)}\big) \in \mathcal{P}$ *is continuous.*

Lemma 3.4. *Assume that* $v \in \mathcal{P}_\infty$, *then for all* $y \in \mathbb{R}^N$. *Then* $v(\cdot - y) \in \mathcal{P}_\infty$. *Moreover, there exists* $t_y > 1$ *such that*

$$v\big(\frac{\cdot - y}{t_y}\big) \in \mathcal{P} \quad \text{and} \quad \lim_{|y|\to\infty} t_y = 1.$$

Lemma 3.5. *It holds* $\sup_{y\in\mathbb{R}^N} t_y := \bar{t} < +\infty$ *and* $\bar{t} > 1$.

Lemma 3.6. *There exists a real number* $\hat{\sigma} > 0$ *such that* $\inf_{v\in\mathcal{P}} \|\nabla v\|_2 \geq \hat{\sigma}$.

Lemma 3.7. *If* $v \in H^1(\mathbb{R}^N)$ *satisfies* $\int_{\mathbb{R}^N}[H(G^{-1}(v)) - \frac{1}{2}V_\infty|G^{-1}(v)|^2]dx > 0$ *and* $t_v > 0$ *are such that* $v\big(\frac{\cdot}{t_v}\big) \in \mathcal{P}_\infty$, *then*

$$J_\infty\Big(v\big(\frac{x}{t_v}\big)\Big) = \frac{t_v^{N-2}}{N}\int_{\mathbb{R}^N}|\nabla v|^2 dx.$$

Proof. If $v\big(\frac{\cdot}{t_v}\big) \in \mathcal{P}_\infty$, by (3.3), we know that

$$\frac{N-2}{2}t_v^{N-2}\int_{\mathbb{R}^N}|\nabla v|^2dx + \frac{N}{2}t_v^N\int_{\mathbb{R}^N}V_\infty|G^{-1}(v)|^2dx = Nt_v^N\int_{\mathbb{R}^N}H(G^{-1}(v))dx.$$

Then

$$\begin{aligned}
&J_\infty\Big(v\big(\frac{x}{t_v}\big)\Big)\\
&= \frac{t_v^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx + \frac{t_v^N}{2}\int_{\mathbb{R}^N}V_\infty|G^{-1}(v)|^2dx - t_v^N\int_{\mathbb{R}^N}H(G^{-1}(v))dx\\
&= \Big(\frac{1}{2} - \frac{N-2}{2N}\Big)t_v^{N-2}\int_{\mathbb{R}^N}|\nabla v|^2dx\\
&= \frac{t_v^{N-2}}{N}\int_{\mathbb{R}^N}|\nabla v|^2dx.
\end{aligned} \tag{3.4}$$

□

Lemma 3.8. *It holds* $p = \inf_{v\in\mathcal{P}} J(v) > 0$ *and* $p = c_\infty$.

Proof of Theorem 1.1. Arguing by contradiction, we suppose that there is $v \in H^1(\mathbb{R}^N)$ such that $J(v) = p$ and $J'(v) = 0$. Then $v \in \mathcal{P}$. By Lemma 3.2, there is $0 < t_v < 1$ such that $v\big(\frac{\cdot}{t_v}\big) \in \mathcal{P}_\infty$. From (3.4) and (H6) we obtain

$$\begin{aligned}
p = J(v) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}V(x)|G^{-1}(v)|^2dx - \int_{\mathbb{R}^N}H(G^{-1}(v))dx\\
&= (\frac{1}{2} - \frac{N-2}{2N})\int_{\mathbb{R}^N}|\nabla v|^2dx - \frac{1}{2N}\int_{\mathbb{R}^N}\langle\nabla V(x), x\rangle|G^{-1}(v)|^2dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{1}{2N}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |G^{-1}(v)|^2 dx \\
&> \frac{t_v^{N-2}}{N}\int_{\mathbb{R}^N} |\nabla v|^2 dx \\
&= J_\infty\Big(v(\frac{\cdot}{t_v})\Big) > c_\infty,
\end{aligned}
$$

which contradicts Lemma 3.8. Moreover, by [22, Lemma 2.3], any critical point of $J|_{\mathcal{P}}$ is a critical point of J in $H^1(\mathbb{R}^N)$, then p is not a critical level for the function J. □

4. Existence of a positive solution

In this section, we will show the existence of a positive solution for (1.1). Similarly to what done in [11], first, we prove the existence of a positive solution for limiting problem (1.8) by a global compactness lemma. Second, we prove the existence of positive for (1.1) using barycenter constrains and a version of the Linking Theorem.

Lemma 4.1. (1) *There exist $\rho, a > 0$ such that $J(v) \geq a, \|v\| = \rho$.*
(2) *There exists $e \in H^1(\mathbb{R}^N)$ with $\|e\| > \rho$ such that $J(e) < 0$.*

Proof. (1) By (3.1), (H5), Lemma 2.1 (4) and Sobolev embedding, select $\varepsilon = \frac{V_\infty}{2+\kappa}$, we know that

$$
\begin{aligned}
J(v) &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2+\kappa}\int_{\mathbb{R}^N} V_\infty v^2 dx - \frac{\varepsilon}{2}\int_{\mathbb{R}^N} v^2 dx - \frac{C}{q}\int_{\mathbb{R}^N} |v|^q dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2(2+\kappa)}\int_{\mathbb{R}^N} V_\infty v^2 dx - \frac{C}{q}\int_{\mathbb{R}^N} |v|^q dx \\
&\geq \frac{C}{2(2+\kappa)}\|v\|^2 - \frac{C_1}{q}\|v\|^q.
\end{aligned}
$$

Thereby, choosing $\|v\| = \rho$ is small enough, we have $J(v) \geq \frac{C}{2(2+\kappa)}\rho^2 - \frac{C_1}{q}\rho^q > 0$.

(2) Let $w \in H^1(\mathbb{R}^N)$ be a least energy solution of (1.8), motivated by [10, Lemma 2.2], we define a continuous path $\alpha : [0, +\infty) \to H^1(\mathbb{R}^N)$ by setting $\alpha(t)(x) = w(\frac{x}{t})$, if $t > 0$ and $\alpha(0) = 0$. Then $J_\infty(0) = 0$ and

$$
\begin{aligned}
J_\infty(\alpha(t)) &= \frac{1}{2}\int_{\mathbb{R}^N} \big|\nabla w(\frac{x}{t})\big|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty \big|G^{-1}\big(w(\frac{x}{t})\big)\big|^2 dx \\
&\quad - \int_{\mathbb{R}^N} H\Big(G^{-1}\big(w(\frac{x}{t})\big)\Big) dx \\
&= \frac{1}{2}t^{N-2}\int_{\mathbb{R}^N} |\nabla w(x)|^2 dx + \frac{1}{2}t^N \int_{\mathbb{R}^N} V_\infty |G^{-1}(w(x))|^2 dx \\
&\quad - t^N \int_{\mathbb{R}^N} H(G^{-1}(w(x))) dx.
\end{aligned}
$$

Taking the derivative, we have

$$
\begin{aligned}
\frac{d}{dt}J_\infty(\alpha(t)) =& \frac{N-2}{2}t^{N-3}\int_{\mathbb{R}^N} |\nabla w|^2 dx + \frac{N}{2}t^{N-1}\int_{\mathbb{R}^N} V_\infty |G^{-1}(w)|^2 dx \\
&- Nt^{N-1}\int_{\mathbb{R}^N} H(G^{-1}(w)) dx.
\end{aligned}
$$

Since w is a solution of (1.8), it satisfies the Pohožaev identity

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla w|^2dx+\frac{N}{2}\int_{\mathbb{R}^N}V_\infty|G^{-1}(w)|^2dx=N\int_{\mathbb{R}^N}H(G^{-1}(w))dx.$$

Therefore,

$$\frac{d}{dt}J_\infty(\alpha(t))=\frac{N-2}{2}t^{N-3}(1-t^2)\int_{\mathbb{R}^N}|\nabla w|^2dx.$$

Since $N\geq 3$, the map $t\mapsto J_\infty(\alpha(t))$ achieves the maximum value at $t=1$. Choosing $L>0$ is sufficiently large, we have $J_\infty(\alpha(L))<0$.

Taking $\zeta(t)=\alpha(tL)$, we have $\zeta\in\Gamma_\infty$. If $\zeta_y(t)=w\left(\frac{\cdot-y}{tL}\right)$, by ($V_2$) and Lebesgue Dominated Convergence Theorem, we know

$$\begin{aligned}
J(\zeta_y(1))&=\frac{1}{2}\int_{\mathbb{R}^N}\big|\nabla w(\frac{x-y}{L})\big|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)\big|G^{-1}\big(w(\frac{x-y}{L})\big)\big|^2dx\\
&\quad-\int_{\mathbb{R}^N}H\Big(G^{-1}\big(w(\frac{x-y}{L})\big)\Big)dx\\
&=\frac{1}{2}\int_{\mathbb{R}^N}\big|\nabla w(\frac{x}{L})\big|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x+y)|G^{-1}\big(w(\frac{x}{L})\big)|^2dx\\
&\quad-\int_{\mathbb{R}^N}H\Big(G^{-1}\big(w(\frac{x}{L})\big)\Big)dx\\
&=J_\infty(\zeta_y(1))+\frac{1}{2}\int_{\mathbb{R}^N}(V(x+y)-V_\infty)|G^{-1}(\zeta_y(1))|^2dx\\
&<0,\quad\text{for } |y| \text{ large.}
\end{aligned}$$

Choosing $e=\zeta_y(1)$, we complete the proof. $\square$

From Lemma 4.1, the min-max mountain pass level for the function J is

$$c=\inf_{\xi\in\Gamma}\max_{t\in[0,1]}J(\xi(t)),$$

where

$$\Gamma=\{\xi\in C([0,1],H^1(\mathbb{R}^N)):\xi(0)=0\neq\xi(1),J(\xi(1))<0\}.$$

Then there is a Cerami sequence $\{v_n\}$ for the functional J at level c such that

$$J(v_n)\to c\quad\text{and}\quad\|J'(v_n)\|(1+\|v_n\|)\to 0.$$

Now, we state the following Lemma, the proof is similar to the proof of [11, Lemmas 4.1 and 4.2], we omit it.

Lemma 4.2. *It holds $c=c_\infty=p$.*

Lemma 4.3. *For all $\xi\in\Gamma$, there is $s\in(0,1)$ such that $\xi(s)\in\mathcal{P}$.*

Proof. Since

$$\begin{aligned}
\gamma(v)&=\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{N}{2}\int_{\mathbb{R}^N}V(x)|G^{-1}(v)|^2dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^N}\langle\nabla V(x),x\rangle|G^{-1}(v)|^2dx-N\int_{\mathbb{R}^N}H(G^{-1}(v))dx\\
&=NJ(v)-\int_{\mathbb{R}^N}|\nabla v|^2dx+\frac{1}{2}\int_{\mathbb{R}^N}\langle\nabla V(x),x\rangle|G^{-1}(v)|^2dx,
\end{aligned}$$

by (H6), we have $\gamma(v) < NJ(v)$, for every $v \in H^1(\mathbb{R}^N)$. If $\xi \in \Gamma$, we have $\gamma(\xi(0)) = 0$ and $\gamma(\xi(1)) < NJ(\xi(1)) < 0$. Now, there is $s \in (0,1)$ such that $\gamma(\xi(s)) = 0$ for $\|\xi(s)\| > \rho$. Then $\xi(s) \in \mathcal{P}$. □

Lemma 4.4. *If $\{v_n\} \subset H^1(\mathbb{R}^N)$ is a $(Ce)_c$ sequence with $c > 0$, then $\{v_n\}$ is bounded.*

Proof. Since $G^{-1}(v_n)g(G^{-1}(v_n)) \in H^1(\mathbb{R}^N)$, we have

$$\begin{aligned} J(v_n) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx - \int_{\mathbb{R}^N} H(G^{-1}(v_n))dx \\ &= c + o_n(1), \end{aligned}$$

and

$$\begin{aligned} &\langle J'(v_n), G^{-1}(v_n)g(G^{-1}(v_n))\rangle \\ &= \int_{\mathbb{R}^N} \big[1 + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n))\big] |\nabla v_n|^2 dx \\ &\quad + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx - \int_{\mathbb{R}^N} h(G^{-1}(v_n))G^{-1}(v_n) dx \\ &= o_n(1). \end{aligned}$$

Then, by (H3') and Lemma 2.1 (3), we obtain

$$\begin{aligned} &c + o_n(1) \\ &= J(v_n) - \frac{1}{4}\langle J'(v_n), G^{-1}(v_n)g(G^{-1}(v_n))\rangle \\ &= \frac{1}{4}\int_{\mathbb{R}^N} \big[1 - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n))\big] |\nabla v_n|^2 dx + \frac{1}{4}\int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx \\ &\quad - \int_{\mathbb{R}^N} \big[H(G^{-1}(v_n)) - \frac{1}{4} h(G^{-1}(v_n))G^{-1}(v_n)\big] dx \\ &\geq \frac{1}{4}\Big[1 - \frac{\sqrt{2(2+\kappa)} - 2}{\sqrt{2(2+\kappa)} + 2}\Big] \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{2(2+\kappa)} \int_{\mathbb{R}^N} V(x)|v_n|^2 dx \\ &\geq \min\Big\{\frac{4}{\sqrt{2(2+\kappa)} + 2}, \frac{1}{2(2+\kappa)}\Big\} \|v_n\|^2, \end{aligned}$$

hence, $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. □

Lemma 4.5 (Splitting)**.** *Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a bounded sequence such that*

$$J(v_n) \to c > 0 \quad \text{and} \quad (1 + \|v_n\|)\|J'(v_n)\| \to 0.$$

Replacing $\{v_n\}$ by a subsequence, if necessary, there exists a solution $\bar{v}$ of (1.1), a number $k \in \mathbb{N}\bigcup\{0\}, k$ functions $v^1, v^2, \dots, v^k$ and k sequence of points $\{y_n^j\} \in \mathbb{R}^N, 1 \leq j \leq k$, satisfying

(1) *$v_n \to \bar{v}$ in $H^1(\mathbb{R}^N)$ or*
(2) *v^j are nontrivial solutions of (1.8);*
(3) *$|y_n^j| \to \infty$ and $|y_n^j - y_n^i| \to \infty, i \neq j$;*
(4) *$v_n - \sum_{i=1}^k v^i(x - y_n^i) \to \bar{v}$;*
(5) *$J(v_n) \to J(\bar{v}) + \sum_{i=1}^k J_\infty(v^i)$.*

Proof. The proof is a version of concentration compactness of Lions in [13, 19], one can mimic the proof of [21, Theorem 8.4]. □

Corollary 4.6. *If $J(v_n) \to c_\infty$ and $\|J'(v_n)\|(1+\|v_n\|) \to 0$, then either $\{v_n\}$ is relatively compact or the splitting lemma 4.5 holds with $k=1$ and $\bar{v}=0$.*

Let

$$c_\sharp := \inf\{c > c_\infty : c \text{ is a radial critical value of } J_\infty\}.$$

Then we have the following lemma.

Lemma 4.7. *Assume that c_∞ is an isolated radial critical level for J_∞. Then $c_\sharp > c_\infty$ and J satisfies condition (Ce) at level $d \in (c_\infty, \min\{c_\sharp, 2c_\infty\})$. Assume now that the limiting problem* (1.8) *admits a unique positive radial solution. Then J satisfies condition (Ce) at level $d \in (c_\infty, 2c_\infty)$.*

The proof of the above lemma is analogous to the proof of [11, Lemma 5.9], we omit it.

Lemma 4.8. *Let $J(v_j) \to d > 0$ and $\{v_j\} \subset \mathcal{P}$, then $\{v_j\}$ is bounded in $H^1(\mathbb{R}^N)$.*

Proof. Since $\{v_j\} \subset \mathcal{P}$, by (H6) and (3.2), we obtain

$$\begin{aligned} d+1 &\geq J(v_j) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_j|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v_j)|^2 dx - \int_{\mathbb{R}^N} H(G^{-1}(v_j))dx \\ &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_j|^2 dx - \frac{1}{2N}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |G^{-1}(v_j)|^2 dx \\ &\geq \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_j|^2 dx. \end{aligned}$$

Then, $\|\nabla v_j\|_2$ is bounded. By Sobolev inequality, the sequence $\|v_j\|_{2^*}$ is also bounded. Setting $\varepsilon = V_\infty/2$, combining this with (H1') and (H2'), (H5), Lemma 2.1 (4), we have

$$\begin{aligned} d+1 = J(v_j) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_j|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(v_j)|^2 dx - \int_{\mathbb{R}^N} H(G^{-1}(v_j))dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_j|^2 dx + \frac{V_\infty}{2}\int_{\mathbb{R}^N} |G^{-1}(v_j)|^2 dx - \frac{\varepsilon}{2}\int_{\mathbb{R}^N} |G^{-1}(v_j)|^2 dx \\ &\quad - \frac{C_\varepsilon}{q}\int_{\mathbb{R}^N} |G^{-1}(v_j)|^{2^*} dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_j|^2 dx + \frac{V_\infty}{2+\kappa}\int_{\mathbb{R}^N} |v_j|^2 dx - \frac{\varepsilon}{2}\int_{\mathbb{R}^N} |v_j|^2 dx - \frac{C_\varepsilon}{q}\int_{\mathbb{R}^N} |v_j|^{2^*} dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_j|^2 dx + \frac{V_\infty}{2(2+\kappa)}\int_{\mathbb{R}^N} |v_j|^2 dx - \frac{C_\varepsilon}{q}\int_{\mathbb{R}^N} |v_j|^{2^*} dx. \end{aligned}$$

If $\|v_j\|_2 \to \infty$, we obtain a contradiction. □

Next, we introduce the barycenter function, see [1, 20], which is crucial for proving the existence of a solution for (1.1).

Definition 4.9. The barycenter function of a function $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ is defined by

$$\mu(u)(x) := \frac{1}{|B_1|}\int_{B_1(x)} |u(y)|dy.$$

It follows that $\mu(u) \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. Subsequently, take

$$\hat{u}(x) := \big[\mu(u)(x) - \frac{1}{2}\max \mu(u)\big]^+,$$

we know that $\hat{u} \in C_0(\mathbb{R}^N)$. Now we define the barycenter of u by

$$\beta(u) = \frac{1}{\|\hat{u}\|_1} \int x\hat{u}(x)dx.$$

Since $\mu(u)$ has compact support, by definition, $\beta(u)$ is well defined. β satisfies the following properties

(1) β is a continuous function in $H^1(\mathbb{R}^N)\backslash\{0\}$.
(2) If u is radially symmetric, then $\beta(u) = 0$.
(3) Given $y \in \mathbb{R}^N$ and setting $u_y(x) := u(x-y)$, we have $\beta(u_y) = \beta(u) + y$.

Lemma 4.10. *Let $\{u_n\}, \{v_n\} \subset H^1(\mathbb{R}^N)$ be such that $\|u_n - v_n\| \to 0$ and $J'(v_n) \to 0$ as $n \to \infty$. Then $J'(u_n) \to 0$ as $n \to \infty$.*

Proof. For each $\varphi \in H^1(\mathbb{R}^N)$, we have

$$\begin{aligned}
&\langle J'(u_n) - J'(v_n), \varphi\rangle \\
&= \int_{\mathbb{R}^N} \nabla(u_n - v_n)\nabla\varphi dx + \int_{\mathbb{R}^N} V(x)\big[\frac{G^{-1}(u_n)}{g(G^{-1}(u_n))} - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\big]\varphi dx \\
&\quad - \int_{\mathbb{R}^N} \big[\frac{h(G^{-1}(u_n))}{g(G^{-1}(u_n))} - \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}\big]\varphi dx.
\end{aligned}$$

From $\|u_n - v_n\| \to 0$, we have

$$\int_{\mathbb{R}^N} \nabla(u_n - v_n)\nabla\varphi dx \le \Big(\int_{\mathbb{R}^N} |\nabla(u_n - v_n)|^2 dx\Big)^{1/2} \Big(\int_{\mathbb{R}^N} |\nabla\varphi|^2 dx\Big)^{1/2} \to 0,$$

as $n \to \infty$. Since the function $\frac{G^{-1}(s)}{g(G^{-1}(s))}$ is continuous for s, by (H5), when u_n is sufficiently close to v_n, we can conclude that

$$\int_{\mathbb{R}^N} V(x)\Big[\frac{G^{-1}(u_n)}{g(G^{-1}(u_n))} - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\Big]\varphi dx \to 0.$$

Moreover, by the assumption $h \in \mathrm{Lip}(\mathbb{R}^+, \mathbb{R}^+)$, we deduce from Lemma 2.1 (2), (4) that

$$\begin{aligned}
&\Big|\frac{h(G^{-1}(u_n))}{g(G^{-1}(u_n))} - \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}\Big| \\
&= \frac{|h(G^{-1}(u_n))g(G^{-1}(v_n)) - h(G^{-1}(v_n))g(G^{-1}(u_n))|}{g(G^{-1}(u_n))g(G^{-1}(v_n))} \\
&\le \frac{|h(G^{-1}(u_n)) - h(G^{-1}(v_n))|g(G^{-1}(v_n))}{g(G^{-1}(u_n))g(G^{-1}(v_n))} \\
&\quad + \frac{|h(G^{-1}(v_n))||g(G^{-1}(v_n)) - g(G^{-1}(u_n))|}{g(G^{-1}(u_n))g(G^{-1}(v_n))} \\
&\le \sqrt{1 + \frac{\kappa}{2}} C|G^{-1}(u_n) - G^{-1}(v_n)| \\
&\quad + C|G^{-1}(v_n) - 0||g'(v_n + \theta_1(u_n - v_n))||G^{-1}(v_n) - G^{-1}(u_n)| \\
&\le \sqrt{1 + \frac{\kappa}{2}} C|(G^{-1}(v_n + \theta_2(u_n - v_n)))'||u_n - v_n|
\end{aligned}$$

$$
\begin{aligned}
&+ C|v_n|\tilde{C}|(G^{-1}(v_n+\theta_2(u_n-v_n)))'||u_n-v_n| \\
&= \Big(\sqrt{1+\frac{\kappa}{2}}C + C\tilde{C}|v_n|\Big)\frac{1}{g(G^{-1}(v_n+\theta_2(u_n-v_n)))}|u_n-v_n| \\
&\le \Big(\sqrt{1+\frac{\kappa}{2}}C + C\tilde{C}|v_n|\Big)|u_n-v_n|,
\end{aligned}
$$

where

$$
g'(t) = \frac{\kappa t}{\sqrt{1+\frac{\kappa t^2}{2(1+t^2)}}(1+t^2)^2} \le \tilde{C}, \theta_1, \theta_2 \in (0,1).
$$

By the Hölder's inequality,

$$
\int_{\mathbb{R}^N} |(u_n-v_n)\varphi| dx \le \Big(\int_{\mathbb{R}^N} |u_n-v_n|^2 dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N} |\varphi|^2 dx\Big)^{1/2} \to 0
$$

and

$$
\int_{\mathbb{R}^N} |v_n||(u_n-v_n)\varphi| dx \le \Big(\int_{\mathbb{R}^N} |u_n-v_n|^2 dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N} |v_n\varphi|^2 dx\Big)^{1/2} \to 0
$$

as $n\to\infty$. Therefore,

$$
\int_{\mathbb{R}^N} \Big|\Big[\frac{h(G^{-1}(u_n))}{g(G^{-1}(u_n))} - \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}\Big]\varphi\Big| dx \to 0 \quad \text{as } n\to\infty.
$$

Then $J'(u_n)\to 0$ as $n\to\infty$. □

Now, we define

$$
b := \inf\{J(v) : v\in\mathcal{P}, \beta(v)=0\}.
$$

From Lemma 4.10, similarly to the proof of [22, Lemma 4.11], we have the following result.

Lemma 4.11. $b > c_\infty$.

Let us consider a positive, radially symmetric, ground state solution $w\in H^1(\mathbb{R}^N)$ to the autonomous problem at infinity. We define the operator $\Pi : \mathbb{R}^N \to \mathcal{P}$ by

$$
\Pi[y](x) = w\big(\frac{x-y}{\theta_y}\big),
$$

where θ_y projects $w(\cdot - y)$ onto $\mathcal{P}$. Π is continuous as θ_y is unique and $\theta_y(w(\cdot-y))$ is a continuous function of $w(\cdot-y)$. The following lemma describes some properties of Π, its proof can be founded in [10], [11].

Lemma 4.12. *It holds that* $\beta(\Pi[y](x)) = y$ *and* $J(\Pi[y]) \to c_\infty, |y|\to\infty$.

Lemma 4.13. *Assume that*

(H9) $\|V_\infty - V\|_\infty < \frac{2(\min\{c_\sharp, 2c_\infty\}-c_\infty)}{\bar\theta^N\|w\|_2^2}$, $\bar\theta = \sup_{y\in\mathbb{R}^N}\theta_y$.

Then $J(\Pi[y]) < \min\{c_\sharp, 2c_\infty\}$.

Proof. Since J_∞ is translation invariant, the maximum of $\theta\mapsto J_\sharp(w(\cdot/\theta))$ is attained at $\theta=1$ and $\theta_y>1$. It follows from (H9) and Lemma 2.1 (4) that

$$
\begin{aligned}
J(\Pi[y]) &= J_\infty(\Pi[y]) + J(\Pi[y]) - J_\infty(\Pi[y]) \\
&= J_\infty(\Pi[y]) + \frac{1}{2}\int_{\mathbb{R}^N} (V(x)-V_\infty)|G^{-1}(\Pi[y])|^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&< c_\infty + \frac{\min\{c_\sharp, 2c_\infty\} - c_\infty}{\bar\theta^N \|w\|_2^2} \int_{\mathbb{R}^N} |\Pi[y]|^2 dx \\
&= c_\infty + \frac{\min\{c_\sharp, 2c_\infty\} - c_\infty}{\bar\theta^N \|w\|_2^2} \int_{\mathbb{R}^N} \big|w\big(\frac{x-y}{\theta_y}\big)\big|^2 dx \\
&= c_\infty + \frac{\min\{c_\sharp, 2c_\infty\} - c_\infty}{\bar\theta^N \|w\|_2^2} \theta_y^N \|w\|_2^2 \\
&\le \min\{c_\sharp, 2c_\infty\}.
\end{aligned}
$$

□

Remark 4.14. Replacing (H9) with

$$\|V_\infty - V\|_\infty < \frac{2c_\infty}{\bar\theta^N \|w\|_2^2}$$

yields $J(\Pi[y]) < 2c_\infty$.

We recall a version of the Linking Theorem with Cerami condition by [2, Theorem 2.3], which we state here for the sake of completeness.

Definition 4.15. Let S be a closed subset of a Banach space X and Q be a submanifold of X with relative boundary ∂Q. We say that S and ∂Q link if the following facts hold

1. $S \cap \partial Q = \emptyset$;
2. for any $f \in C^0(X, X)$ with $f|_{\partial Q} = id$, then $f(Q) \cap S \neq \emptyset$.

Moreover, if S and Q are as above and B is a subset of $C^0(X, X)$, then S and ∂Q link with respect to B if (1) and (2) hold for any $f \in B$.

Lemma 4.16. *Suppose that $J \in C^1(X, \mathbb{R})$ is a functional satisfying $(Ce)_c$ condition. Consider a closed subset $S \subset X$ and a submanifold $Q \subset X$ with relative boundary ∂Q are such that*

1. *S and ∂Q link;*
2. *$\alpha = \inf_{v\in S} J(v) > \sup_{v\in\partial Q} J(v) = \alpha_0$;*
3. *$\sup_{v\in Q} J(v) < +\infty$.*

If $B = \{f \in C^0(X, X) : f|_{\partial Q} = id\}$, then $\tau = \inf_{f\in B} \sup_{v\in Q} J(f(v)) \ge \alpha$ is a critical value of J.

Proof of Theorem 1.2. By Lemmas 4.11 and 4.12, we have $b > c_\infty$ and $J(\Pi[y]) \to c_\infty, |y| \to \infty$, there is $\bar\rho > 0$ such that

$$c_\infty < \max_{|y|=\bar\rho} J(\Pi[y]) < b. \tag{4.1}$$

To apply the Linking Theorem 4.16, we take

$$Q := \Pi\left(\overline{B_{\bar\rho}(0)}\right), \quad S := \{v \in H^1(\mathbb{R}^N) : v \in \mathcal{P}, \beta(v) = 0\},$$

and we show that ∂Q and S link with respect to $\mathcal{H} = \{f \in C(Q, \mathcal{P}) : f|_{\partial Q} = id\}$. Since $\beta(\Pi[y](x)) = y$ from Lemma 4.12, we have that $\partial Q \cap S = \emptyset$, as if $v \in S$, then $\beta(v) = 0$, and if $v \in \partial Q, v = \Pi[y]$ for some $y \in \mathbb{R}^N$ with $|y| = \bar\rho$ and then $\beta(v) = y \neq 0$. Now we show that $f(Q) \cap S \neq \emptyset$ for any $f \in \mathcal{H}$. Given $f \in \mathcal{H}$, let $T : \overline{B_{\bar\rho}(0)} \to \mathbb{R}^N$ is defined by $T(y) = \beta \circ f \circ \Pi[y]$. Then the function T is continuous. Moreover, for $|y| = \bar\rho$, we have that $\Pi[y] \in \partial Q$, thus $f \circ \Pi[y] = \Pi[y]$ as $f|_{\partial Q} = id$, and hence $T(y) = y$ by Lemma 4.12. By Brouwer Fixed Point Theorem there is

$\tilde{y} \in B_{\bar{\rho}}(0)$ with $T(\tilde{y}) = 0$, which implies that $f(\Pi[\tilde{y}]) \in S$. Then $f(Q) \cap S \neq \emptyset$ and S and ∂Q link. Now, from (4.1), we may write

$$b = \inf_S J > \max_{\partial Q} J.$$

Let us define

$$k = \inf_{f \in \mathcal{H}} \max_{v \in Q} J(f(v)).$$

Then $k \geq b$. In fact, if $f \in \mathcal{H}$, there exists $w \in S$ with $w = f(u)$ for some $u \in \Pi\big(\overline{B_{\bar{\rho}}(0)}\big)$. Therefore,

$$\max_{v \in Q} J(f(v)) \geq J(f(u)) = J(w) \geq \inf_{v \in S} J(v) = b,$$

and hence $k \geq b$, which implies that $k > c_\infty$. Furthermore, if $f = id$, by Lemma 4.13, we have

$$k = \inf_{f \in \mathcal{H}} \max_{v \in Q} J(f(v)) < \max_{v \in Q} J(v) < \min\{c_\sharp, 2c_\infty\}.$$

Then $k \in (c_\infty, \min\{c_\sharp, 2c_\infty\})$ and it deduces from Lemma 4.7 that the $(\mathrm{Ce})_c$ condition at level k is satisfied. Then, by the linking theorem, k is a critical level of J. □

Remark 4.17. Theorems 1.1 and 1.2 hold for (1.4) with $a(x) = 1$ under assumptions (H1')–(H3') and (H4)–(H9).

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