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NONNEGATIVE CONTROLLABILITY FOR A CLASS OF NONLINEAR DEGENERATE PARABOLIC EQUATIONS WITH APPLICATION TO CLIMATE SCIENCE

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Abstract. We consider a nonlinear degenerate reaction-diffusion equation. First we prove that if the initial state is nonnegative, then the solution remains nonnegative for all time. Then we prove the approximate controllability between nonnegative states via multiplicative controls, this is done using the reaction coefficient as control.

1. Introduction

In this article, we study the one-dimensional semilinear reaction-diffusion equation

$$u_t - (a(x)u_x)_x = \alpha(x,t)u + f(x,t,u), \quad (x,t) \in Q_T := (-1,1) \times (0,T),\ T > 0,$$

where $a \in C([-1,1]) \cap C^1(-1,1)$ is strictly positive on $(-1,1)$ and $a(\pm 1) = 0$ (for example $a(x) = (1-x^2)^\eta$ with $\eta > 0$), α is a bounded function on Q_T and $f(\cdot,\cdot,u)$ is a suitable non-linearity that will be defined below. The above semilinear equation is a degenerate parabolic equation since the diffusion coefficient vanishes at the boundary points of $[-1,1]$.

Our interest in this degenerate reaction-diffusion equations is motivated by its applications to the energy balance models in climate science. For example, the Budyko-Sellers model that is obtained from the above equation when $a(x) = 1-x^2$. We devote the entire Section 4 to the presentation of applications of degenerate equations to climate science.

In our mathematical study we need to distinguish two classes of degenerate problems: weakly degenerate problems (WDeg) (see [14, 36]) when the degenerate diffusion coefficient is such that $\frac{1}{a} \in L^1(-1,1)$ (e.g. $a(x) = \sqrt{1-x^2}$), and strongly degenerate problems (SDeg) (see [13, 35]) when $\frac{1}{a} \notin L^1(-1,1)$ (if $a \in C^1([-1,1])$ follows that $\frac{1}{a} \notin L^1(-1,1)$, e.g. $a(x) = 1-x^2$). It is well-known [1] that, in the (WDeg) case, all functions in the domain of the corresponding differential operator

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possess a trace on the boundary, in spite of the fact that the operator degenerates at such points. Thus, in the (WDeg) case we can consider the general Robin type boundary conditions, in a similar way to the uniformly parabolic case. Conversely, in the harder (SDeg) case, one is limited to only the weighted Neumann type boundary conditions. This preamble allows us to justify the following general problem formulation.

1.1. **Problem formulation.** Let us introduce the semilinear degenerate parabolic Cauchy problems:

$$
\begin{gathered}
u_t - (a(x)u_x)_x = \alpha(x,t)u + f(x,t,u) \quad \text{in } Q_T := (-1,1)\times(0,T)\\
\begin{cases}
\begin{cases}
\beta_0 u(-1,t) + \beta_1 a(-1)u_x(-1,t) = 0 & t\in(0,T)\\
\gamma_0\, u(1,t) + \gamma_1\, a(1)\, u_x(1,t) = 0 & t\in(0,T)
\end{cases} & \text{for (WDeg)}\\
a(x)u_x(x,t)|_{x=\pm 1} = 0 & \text{for (SDeg)}
\end{cases}\\
u(x,0) = u_0(x) \in L^2(-1,1),
\end{gathered}
\tag{1.1}
$$

where the reaction coefficient $\alpha(x,t) \in L^\infty(Q_T)$ will represent the *multiplicative control* (that is the variable function through which we can act on the system), and α is chosen, in this article, as a piecewise static function (in the sense of Definition 1.1). We consider problem (1.1) under the following assumptions:

(A1) The function $f : Q_T \times \mathbb{R} \to \mathbb{R}$ satisfies:
- $(x,t,u) \mapsto f(x,t,u)$ is a Carathéodory function on $Q_T \times \mathbb{R}$, that is
  $\star$ $(x,t) \mapsto f(x,t,u)$ is measurable, for every $u \in \mathbb{R}$,
  $\star$ $u \mapsto f(x,t,u)$ is a continuous function, for a.e. $(x,t) \in Q_T$;
- $t \mapsto f(x,t,u)$ is locally absolutely continuous for a.e. $x \in (-1,1)$, for every $u \in \mathbb{R}$, and $f_t(x,t,u)u \geq -\nu u^2$ for a.e. $t \in (0,T)$;
- there exist constants $\delta_* \geq 0$, $\vartheta \in [1, \vartheta_{\sup})$, $\vartheta_{\sup} \in \{3,4\}$, and $\nu \geq 0$ such that for a.e. $(x,t) \in Q_T$ and all $u,v \in \mathbb{R}$, we have

$$
|f(x,t,u)| \leq \delta_* \, |u|^\vartheta, \tag{1.2}
$$

$$
\begin{aligned}
-\nu\big(1 + |u|^{\vartheta-1} + |v|^{\vartheta-1}\big)(u-v)^2 &\leq \big(f(x,t,u) - f(x,t,v)\big)(u-v)\\
&\leq \nu(u-v)^2,
\end{aligned}
\tag{1.3}
$$

(A2) The function $a \in C([-1,1]) \cap C^1(-1,1)$ satisfies

$$
a(x) > 0,\ \forall x \in (-1,1), \quad a(-1) = a(1) = 0\,.
$$

Then, we consider the following two cases:
- case (WDeg): if $\frac{1}{a} \in L^1(-1,1)$, then $\vartheta_{\sup} = 4$. In (1.1) we consider the Robin boundary conditions, where $\beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$, with $\beta_0^2 + \beta_1^2 > 0$ and $\gamma_0^2 + \gamma_1^2 > 0$ satisfy the sign condition:

$$
\beta_0\beta_1 \leq 0 \quad \text{and} \quad \gamma_0\gamma_1 \geq 0;
$$

- case (SDeg): if $\frac{1}{a} \notin L^1(-1,1)$ and $\xi_a(x) := \int_0^x \frac{1}{a(s)}ds \in L^{q_\vartheta}(-1,1)$, where $q_\vartheta = \max\big\{\frac{1+\vartheta}{3-\vartheta}, 2\vartheta - 1\big\}$, then $\vartheta_{\sup} = 3$. In (1.1) we consider the weighted Neumann boundary conditions.

To clarify the kind of multiplicative controls used, we recall the definition of piecewise static function.

Definition 1.1. We say that a function $\alpha \in L^\infty(Q_T)$ is *piecewise static* (or a *simple function* with respect to the variable t), if there exist $m \in \mathbb{N}$, $\alpha_k(x) \in L^\infty(-1,1)$ and $t_k \in [0,T]$, $t_{k-1} < t_k$, $k = 1,\dots,m$ with $t_0 = 0$ and $t_m = T$, such that

$$\alpha(x,t) = \alpha_1(x)\chi_{[t_0,t_1]}(t) + \sum_{k=2}^{m} \alpha_k(x)\chi_{(t_{k-1},t_k]}(t), \tag{1.4}$$

where $\chi_{[t_0,t_1]}$ and $\chi_{(t_{k-1},t_k]}$ are the indicator function of $[t_0,t_1]$ and $(t_{k-1},t_k]$, respectively. Sometimes, for clarity purposes, we will call the function α in (1.4) an m-step piecewise static function

1.2. **Main results.** In this article, we study the controllability of (1.1) using *multiplicative controls*, that is the reaction coefficients $\alpha(x,t)$. First, we find that the following general nonnegative result holds also for the degenerate PDE of system (1.1). That is, if the initial state is nonnegative the corresponding strong solution to (1.1) remains nonnegative for all time. For the notion of strict/strong solutions of the nonlinear degenerate problem (1.1) see Section 2. The following result is classic only for the uniformly parabolic (non degenerate) case.

Proposition 1.2. *Let $u_0 \in L^2(-1,1)$ such that $u_0(x) \geq 0$ a.e. $x \in (-1,1)$. Let u be the corresponding unique strong solution of* (1.1)*. Then*

$$u(x,t) \geq 0, \quad \textit{for a.e. } (x,t) \in Q_T\,.$$

A consequence of this result, the solution to (1.1) cannot be steered from a nonnegative initial state to any target state which is negative on a nonzero measure set in the space domain, regardless of the choice of the reaction coefficient $\alpha(x,t)$ as multiplicative control.

In Theorem 1.4 below, we obtain an optimal goal, that is, we approximately control system (1.1) between nonnegative states via multiplicative controls at any time.

Definition 1.3. System (1.1) is said to be non-negatively globally approximately controllable in $L^2(-1,1)$ at any time $T > 0$, by means of multiplicative controls α, if for any nonnegative $u_0, u^* \in L^2(-1,1)$ with $u_0 \neq 0$, and for every $\varepsilon > 0$ there exists a piecewise static multiplicative control $\alpha = \alpha(\varepsilon, u_0, u^*)$ in $L^\infty(Q_T)$, such that for the corresponding strong solution $u(x,t)$ of (1.1) we have

$$\|u(\cdot,T) - u^*\|_{L^2(-1,1)} < \varepsilon.$$

Now we can state the main controllability result.

Theorem 1.4. *The nonlinear degenerate system* (1.1) *is non-negatively globally approximately controllable in $L^2(-1,1)$ at any time $T > 0$, by means of 2-steps piecewise static multiplicative controls.*

The outline of this article is as follows: The proofs of the main results are given in Section 3. In Section 2 we recall the well-posedness of (1.1), in particular we introduce the notions of strict and strong solutions, and we give some useful estimates and properties for this kind of degenerate PDEs, that we use in Section 3. The proofs of the existence and uniqueness results for strict and strong solutions are contained in Section 5. In Section 4 we present some motivations for studying degenerate parabolic problems with the above structure, in particular we introduce the Budyko-Sellers model, an energy balance model in climate science.

1.3. **State of the art in multiplicative controllability and degenerate parabolic equations.** Control theory appeared in the second part of the previous century in the context of linear ordinary differential equations and was motivated by several engineering, Life sciences and economics applications. Then, it was extended to various linear partial differential equations (PDEs) governed by additive locally distributed controls (see [1, 3, 19, 24, 32, 34, 47]), or by boundary controls. Methodologically-speaking, these kind of controllability results for PDEs are typically obtained using the linear duality pairing technique between the control-to-state mapping at hand and its dual observation map (see the Hilbert Uniqueness Method - HUM - introduced in 1988 by J. L. Lions), sometimes using the Carleman estimates tool (see [1, 15, 17]). If the above map is nonlinear, as it happens in our case for the multiplicative controllability, in general the aforementioned approach does not apply.

From the point of view of applications, the approach based on multiplicative controls seems more realistic than the other kinds of controllability, since additive and boundary controls do not model in a realistic way the problems that involve inputs with high energy levels; such as energy balance models in climate science (see Section 4), chemical reactions controlled by catalysts, nuclear chain reactions, smart materials, social science, ecological population dynamic (see [49]) and biomedical models. An important class of biomedical reaction-diffusion problems consists in the models of tumor growth (see, e.g. Section 7 "Control problems" of the survey paper [7] by Bellomo and Preziosi). As regards degenerate reaction-diffusion equations there are also interesting models in population genetics, in particular we recall the Fleming-Viot model (see Epstein's and Mazzeo's book [30]).

The above considerations motivate our investigation of the multiplicative controllability. As regards the topic of multiplicative controllability of PDEs we recall the pioneering paper [5] by Ball, Marsden, and Slemrod, and in the framework of the Schrödinger equation we especially mention [6] by Beauchard and Laurent, and [22] by Coron, Gagnon and Moranceu. As regards parabolic and hyperbolic equations we focus on some results by Khapalov contained in the book [41], and in the references therein.

The main results of this paper deal with approximate multiplicative controllability of semilinear degenerate reaction-diffusion equations. This study is motivated by its applications (see in Section 4 to an energy balance model in climate science: the Budyko-Sellers model) and also by the classical results that hold for the corresponding non-degenerate reaction-diffusion equations, governed via the coefficient of the reaction term (multiplicative control). For the above class of uniformly parabolic equations there are some important obstructions to multiplicative controllability due to the strong maximum principle (see the seminal papers by Diaz [25] and [26], and also the papers [16, 41]), this implies the well-known nonnegative constraint. In Proposition 1.2 we extend the nonnegative constraint to the semilinear degenerate system (1.1). This motivates our investigations regarding the nonnegative controllability for the semilinear degenerate system (1.1) with general weighted Robin/Neumann boundary conditions.

Regarding the nonnegative controllability for reaction-diffusion equations, first, Khapalov in [41] obtained the nonnegative approximate controllability in large time of the one dimensional heat equation via multiplicative controls. Thus, the author and Cannarsa considered the linear degenerate problem associated with (1.1), both in the weakly degenerate (WDeg) case, in [14], and in the strongly degenerate (SDeg) case, in [13]. Then, the author in [35] investigated semilinear strongly degenerate problems. This article can be seen as the final step of the study started in [13, 14, 35], where the global nonnegative approximate controllability was obtained in large time. Indeed, in this article we introduce a new proof, that permits us to obtain the nonnegative controllability in *arbitrary small time* and consequently at any time, instead of *large time*. Moreover, the proof, contained in [35], of the nonnegative controllability in large time for the (SDeg) case has the further obstruction that permitted to treat only superlinear growth, with respect to u, of the nonlinearity function $f(x,t,u)$. While the new proof, adopted in this paper, permits us to control also linear growth of f, with respect to u.

Finally, we mention some recent papers about the approximate multiplicative controllability for reaction-diffusion equations between sign-changing states: in [16] by the author with Cannarsa and Khapalov regarding a semilinear uniformly parabolic system, and [37] by the author with Nitsch and Trombetti, concerning degenerate parabolic equations. Furthermore, some interesting contributions about exact controllability issues for evolution equations via bilinear controls have recently appeared, in particular we mention [2] by Alabau-Boussouira, Cannarsa and Urbani, and [29] by Duprez and Lissy.

To complete the discussion regarding the multiplicative controllability, we note that recently there has been an increasing interest in these topics; many authors are starting to extend the above results from reaction-diffusion equations to other operators. In [50], Vancostenoble proved a nonnegative controllability result in large time for a linear parabolic equation with singular potential, following the approach of [13] and [14]. An interesting work in progress, using the technique of this paper, consists of approaching the problem of the approximate controllability via multiplicative control of nonlocal operators, e.g. the fractional heat equation studied in [10] by Biccari, Warma and Zuazua. Other interesting open problems are suggested in [38, 42].

2. Well-posedness

The well-posedness of case (SDeg) in problem (1.1) was introduced in [35], while the well-posedness of case (WDeg) in problem (1.1) was presented in [36]. To study the well-posedness of (1.1), it is necessary to introduce in the weighted Sobolev spaces $H^1_a(-1,1)$ and $H^2_a(-1,1)$, and their main propertie. Also, in Section 2.2 we introduce the notions of strict and strong solutions semilinear degenerate problems, and we give the corresponding existence and uniqueness results, that are proved in Section 5.

2.1. **Weighted Sobolev spaces.**

Let $a \in C([-1,1]) \cap C^1(-1,1)$ such that the assumption (A2) holds, we define the following spaces:

$$H^1_a(-1,1) = \begin{cases} \{u \in L^2(-1,1) \cap AC([-1,1]) : \sqrt{a}\, u_x \in L^2(-1,1)\} & \text{for (WDeg)} \\ \{u \in L^2(-1,1) \cap AC_{\rm loc}(-1,1) : \sqrt{a}\, u_x \in L^2(-1,1)\} & \text{for (SDeg),} \end{cases}$$

where $AC([-1,1])$ denotes the space of the absolutely continuous functions on $[-1,1]$, and $AC_{\rm loc}(-1,1)$ denotes the space of the locally absolutely continuous functions on $(-1,1)$.

$$H^2_a(-1,1) := \{u \in H^1_a(-1,1) : a u_x \in H^1(-1,1)\}.$$

See [1, 13, 14, 35, 36] for the main properties of the weighted Sobolev spaces. In particular we note that $H^1_a(-1,1)$ and $H^2_a(-1,1)$ are Hilbert spaces with their natural scalar products that induce, respectively, the norms:

$$\|u\|^2_{1,a} := \|u\|^2_{L^2(-1,1)} + |u|^2_{1,a}, \quad \|u\|^2_{2,a} := \|u\|^2_{1,a} + \|(au_x)_x\|^2_{L^2(-1,1)},$$

where $|u|^2_{1,a} := \|\sqrt{a}u_x\|^2_{L^2(-1,1)}$ is a seminorm.

Remark 2.1. The space $H^1_a(-1,1)$ is embedded in $L^\infty(-1,1)$ only in the weakly degenerate case (see [1, 13, 14, 35]).

The following proposition was presented in [18, Proposition 2.1], [35, Appendix], and [12, Lemma 2.5].

Proposition 2.2. *In the* (SDeg) *case, for every* $u \in H^2_a(-1,1)$ *we have*

$$\lim_{x\to\pm1} a(x)u_x(x) = 0 \quad \textit{and} \quad au \in H^1_0(-1,1)\,.$$

Some spectral properties. Let us define the operator $(A_0, D(A_0))$ by

$$D(A_0) = \begin{cases} \left\{ u \in H^2_a(-1,1) : \begin{cases} \beta_0 u(-1) + \beta_1 a(-1)u_x(-1) = 0 \\ \gamma_0 u(1) + \gamma_1 a(1)\, u_x(1) = 0 \end{cases} \right\} & \text{for (WDeg)} \\ H^2_a(-1,1) & \text{for (SDeg)} \end{cases}$$
$$A_0 u = (au_x)_x\,, \quad \forall u \in D(A_0). \tag{2.1}$$

Remark 2.3. We note that in the (SDeg) case, for every $u \in D(A_0)$ Proposition 2.2 guarantees that u satisfies the weighted Neumann boundary conditions.

When $\alpha \in L^\infty(-1,1)$, we define the operator $(A, D(A))$ as

$$\begin{gathered} D(A) = D(A_0) \\ Au = (au_x)_x + \alpha u, \quad \forall u \in D(A). \end{gathered} \tag{2.2}$$

We recall some spectral results obtained from [14] for (WDeg), and from [13] for (SDeg).

Proposition 2.4. $(A, D(A_0))$ *is a closed, self-adjoint, dissipative operator with dense domain in* $L^2(-1,1)$*. Therefore,* A *is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on* $L^2(-1,1)$*.*

Proposition 2.4 allows us to obtain the following result.

Proposition 2.5. *There exists an increasing sequence* $\{\lambda_p\}_{p\in\mathbb{N}}$*, with* $\lambda_p \to +\infty$ *as* $p \to \infty$*, such that* $-\lambda_p$ *are the eigenvalues of* $(A_0, D(A_0))$*, and the corresponding eigenfunctions* $\{\omega_p\}_{p\in\mathbb{N}}$ *form a complete orthonormal system in* $L^2(-1,1)$*.*

Remark 2.6. When $a(x) = 1 - x^2$, that is in the case of the Budyko-Sellers model, the orthonormal eigenfunctions of $(A_0, D(A_0))$ are reduced to Legendre's polynomials (see [45, Section 5.6] and [35, Remark 3.2]).

Spaces involving time: $\mathcal{B}(Q_T)$ and $\mathcal{H}(Q_T)$. Given $T>0$, let us define the Banach spaces:

$$\mathcal{B}(Q_T) := C([0,T];L^2(-1,1)) \cap L^2(0,T;H^1_a(-1,1))$$

with the norm

$$\|u\|^2_{\mathcal{B}(Q_T)} = \sup_{t\in[0,T]} \|u(\cdot,t)\|^2_{L^2(-1,1)} + 2\int_0^T\int_{-1}^1 a(x)u_x^2 dx\, dt\,,$$

and

$$\mathcal{H}(Q_T) := L^2(0,T;D(A_0)) \cap H^1(0,T;L^2(-1,1)) \cap C([0,T];H^1_a(-1,1))$$

with the norm

$$\begin{aligned}\|u\|^2_{\mathcal{H}(Q_T)} &= \sup_{[0,T]} \Big(\|u\|^2_{L^2(-1,1)} + \|\sqrt{a}u_x\|^2_{L^2(-1,1)}\Big) \\ &\quad + \int_0^T \Big(\|u_t\|^2_{L^2(-1,1)} + \|(au_x)_x\|^2_{L^2(-1,1)}\Big)\, dt.\end{aligned}$$

The following embedding lemma was obtained for the space $\mathcal{H}(Q_T)$: in [35] for the (SDeg) case, and in [36] for the (WDeg) case.

Lemma 2.7. *Let $\vartheta \ge 1$. Then $\mathcal{H}(Q_T) \subset L^{2\vartheta}(Q_T)$ and*

$$\|u\|_{L^{2\vartheta}(Q_T)} \le cT^{\frac{1}{2\vartheta}}\|u\|_{\mathcal{H}(Q_T)},$$

where c is a positive constant.

We note that Lemma 2.7 holds in a more general setting than under the assumptions (A1) and (A2), where $\vartheta \in [1,\vartheta_{\sup})$, with $\vartheta_{\sup} \in \{3,4\}$.

2.2. **Existence and uniqueness of solutions of semilinear degenerate problems.** To study the well-posedness, we represent the semilinear problem (1.1) using the following abstract setting in the Hilbert space $L^2(-1,1)$,

$$\begin{gathered} u'(t) = (A_0 + \alpha(t)I)\, u(t) + \phi(u)\,, \quad t>0 \\ u(0) = u_0 \in L^2(-1,1)\,, \end{gathered} \tag{2.3}$$

where A_0 is the operator defined in (2.1), I is the identity operator and, for every $u \in \mathcal{B}(Q_T)$, the Nemytskii operator associated with the problem (1.1) is defined as

$$\phi(u)(x,t) := f(x,t,u(x,t)), \ \ \forall (x,t)\in Q_T. \tag{2.4}$$

The following proposition was proved in [35], for the (SDeg) case, and in [36], for the (WDeg) case.

Proposition 2.8. *Let $1 \le \vartheta < \vartheta_{\sup}$, let $f: Q_T\times\mathbb{R}\to\mathbb{R}$ and assume that* (A1) *and* (A2) *hold. Then $\phi: \mathcal{B}(Q_T) \to L^{1+\frac{1}{\vartheta}}(Q_T)$, defined in* (2.4)*, is a locally Lipschitz continuous map and $\phi(\mathcal{H}(Q_T)) \subseteq L^2(Q_T)$.*

The above proposition justifies the introduction of the notions of *strict solutions* and *strong solutions*. Such notions are classical in PDE theory, see, for instance, the book [8, pp. 62-64] (see also [35, 36]) and the pioneer paper [39] by Friedrichs.

Strict solutions. In this section we give the notion of solutions of (1.1) with initial state in $H^1_a(-1,1)$, introduced in [35] for (SDeg) and in [36] for (WDeg).

Definition 2.9. If $u_0 \in H^1_a(-1,1)$, then u is a *strict solution* of (1.1), if $u \in \mathcal{H}(Q_T)$ and

$$u_t - (a(x)u_x)_x = \alpha(x,t)u + f(x,t,u) \quad \text{a.e. in } Q_T := (-1,1)\times(0,T)$$
$$\begin{cases} \begin{cases} \beta_0 u(-1,t) + \beta_1 a(-1)u_x(-1,t) = 0 & \text{a.e. } t\in(0,T) \\ \gamma_0 u(1,t) + \gamma_1 a(1)u_x(1,t) = 0 & \text{a.e. } t\in(0,T), \end{cases} & \text{for (WDeg)} \\ a(x)u_x(x,t)\big|_{x=\pm1} = 0 \quad \text{a.e. } t\in(0,T) & \text{for (SDeg)} \end{cases}$$
$$u(x,0) = u_0(x) \quad x\in(-1,1)\,,$$

for almost every t in $(0,T)$.

Remark 2.10. Since a strict solution u belongs to $\mathcal{H}(Q_T) \subseteq L^2(0,T;D(A_0))$, we have

$$u(\cdot,t)\in D(A_0), \quad \text{for a.e. } t\in(0,T).$$

Thus, thanks to the definition of the operator $(A,D(A))$ given in (2.2) and Remark 2.3, we deduce that the associated boundary conditions hold, for almost every $t\in(0,T)$.

The following existence and uniqueness result for strict solutions is proved in Section 5 (see also [35, Appendix B] for (SDeg), and [36] for (WDeg)).

Theorem 2.11. *For each $u_0 \in H^1_a(-1,1)$ there exists a unique strict solution $u\in\mathcal{H}(Q_T)$ to* (1.1).

Strong solutions. In this subsection we introduce the notion of solutions when the initial state belongs to $L^2(-1,1)$. These solutions are called *strong solutions* and are defined by approximation sequence of strict solutions.

Definition 2.12. Let $u_0\in L^2(-1,1)$. We say that $u\in\mathcal{B}(Q_T)$ is a *strong solution* of (1.1), if $u(\cdot,0)=u_0$ and there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}(Q_T)$ such that, as $k\to\infty$, $u_k\to u$ in $\mathcal{B}(Q_T)$ and, for every $k\in\mathbb{N}$, u_k is the strict solution of the Cauchy problem

$$u_{kt} - (a(x)u_{kx})_x = \alpha(x,t)u_k + f(x,t,u_k) \quad \text{a.e. in } Q_T := (-1,1)\times(0,T)$$
$$\begin{cases} \begin{cases} \beta_0 u_k(-1,t) + \beta_1 a(-1)u_{kx}(-1,t) = 0 & \text{a.e. } t\in(0,T) \\ \gamma_0 u_k(1,t) + \gamma_1 a(1)\,u_{kx}(1,t) = 0 & \text{a.e. } t\in(0,T) \end{cases} & \text{for (WDeg)} \\ a(x)u_{kx}(x,t)|_{x=\pm1} = 0 \quad \text{a.e. } t\in(0,T) & \text{for (SDeg)} \end{cases}$$
$$u(x,0) = u_0(x) \quad x\in(-1,1)$$

with initial datum $u_k(x,0)$.

Remark 2.13. Let us consider the sequence of strict solutions in Definition 2.12, $\{u_k\}_{k\in\mathbb{N}}\subseteq\mathcal{H}(Q_T)$ such that, as $k\to\infty$, $u_k\to u$ in $\mathcal{B}(Q_T)$. Thus, it follows that $u_k(\cdot,0)\to u_0$ in $L^2(-1,1)$, because of the definition of the $\mathcal{B}(Q_T)-$norm.

The next proposition, obtained in [35] for (SDeg) and in [36] for (WDeg), will be very useful in the proof of the main results.

Proposition 2.14. *Let $\alpha \in L^\infty(Q_T)$ a piecewise static function and let $u_0, v_0 \in L^2(-1,1)$. Let u, v be the corresponding strong solutions of* (1.1)*, with initial data u_0, v_0 respectively. Then,*

$$\|u - v\|_{\mathcal{B}(Q_T)} \le C_T \, \|u_0 - v_0\|_{L^2(-1,1)}, \tag{2.5}$$

where $C_T = e^{(\nu + \|\alpha^+\|_\infty)T}$ and $\alpha^+ := \max\{\alpha, 0\}$.

From Proposition 2.14 trivially we have a corollary.

Corollary 2.15. *Let $u_0 \in L^2(-1,1)$, $\alpha \in L^\infty(Q_T)$, α be a piecewise static function with $\alpha(x,t) \le 0$ in Q_T, and u be the corresponding strong solution of* (1.1)*. If $T \in (0, 1/(4\nu))$, then*

$$\|u\|_{C([0,T],L^2(-1,1))} \le \sqrt{2}\, \|u_0\|_{L^2(-1,1)}. \tag{2.6}$$

The following existence and uniqueness result for strong solutions, was given in [35] for (SDeg) and in [36] for (WDeg)). It will be proved in Section 5.

Theorem 2.16. *For each $u_0 \in L^2(-1,1)$ and each piecewise static function $\alpha \in L^\infty(Q_T)$, there exists a unique strong solution to* (1.1)*.*

Further estimates. First, we recall the following Lemma that was obtained in [35, Lemma B.2] for the (SDeg) case, and in [36], for the (WDeg) case. In the case of static reaction $\alpha \in L^\infty(-1,1)$ (a similar argument to that used in Subsection 5.1.4 to prove Theorem 2.11 which permits to extend the following lemma to the case of $\alpha \in L^\infty(Q_T)$ piecewise static function).

Lemma 2.17. *Let $\alpha \in L^\infty(Q_T)$ be a piecewise static function and $u_0 \in H^1_a(-1,1)$. Then the strict solution $u \in \mathcal{H}(Q_T)$ of system* (1.1)*, under the assumptions* (A1) *and* (A2)*, satisfies*

$$\|u\|_{\mathcal{H}(Q_T)} \le c e^{kT} \|u_0\|_{1,a},$$

where $c = c(\|u_0\|_{1,a})$ and k are positive constants.

Using Lemma 2.7, Proposition 2.8 and Lemma 2.17 we obtain the following Proposition.

Proposition 2.18. *Let $\alpha \in L^\infty(Q_T)$ be a piecewise static function and $u_0 \in H^1_a(-1,1)$. Let $u \in \mathcal{H}(Q_T)$ be the strict solution of* (1.1)*, under the assumptions* (A1) *and* (A2)*. Then, the function $(x,t) \mapsto f(x,t,u(x,t))$ belongs to $L^2(Q_T)$ and*

$$\|f(\cdot,\cdot,u)\|_{L^2(Q_T)} \le C e^{k\vartheta T} \sqrt{T} \|u_0\|^{\vartheta}_{1,a}\,,$$

where $C = C(\|u_0\|_{1,a})$ and k are positive constants.

Proof. Applying (1.2), Lemma 2.7 and Lemma 2.17, we obtain

$$\int_0^T \int_{-1}^1 f^2(x,t,u)\,dx\,dt \le \delta_*^2 \int_0^T \int_{-1}^1 |u|^{2\vartheta}\,dx\,dt \le cT\|u\|^{2\vartheta}_{\mathcal{H}(Q_T)} \le Ce^{2k\vartheta T} T \|u_0\|^{2\vartheta}_{1,a},$$

where c and $C = C(\|u_0\|_{1,a})$ and k are positive constants. □

3. Proof of main results

In Proposition 1.2 we showed that the solution to (1.1) remains nonnegative for all time when the initial state is nonnegative, regardless of the choice of the multiplicative control $\alpha(x,t)$. In Theorem 1.4 we showed that the global approximate multiplicative controllability between nonnegative states at any time.

For brevity of notation, we will use $\|\cdot\|$, $\|\cdot\|_\infty$ and $\langle\cdot,\cdot\rangle$ instead of the norms $\|\cdot\|_{L^2(-1,1)}$ and $\|\cdot\|_{L^\infty(Q_T)}$, and the inner product $\langle\cdot,\cdot\rangle_{L^2(-1,1)}$, respectively.

3.1. **Nonnegative solutions.** Before proving Proposition 1.2 we give a regularity property of the positive and negative part of a function, that will be used in that proof. For $u:(-1,1)\to\mathbb{R}$, we consider the positive and negative part functions, respectively,

$$u^+(x):=\max\{u(x),0\}\,,\quad u^-(x):=\max\{0,-u(x)\}\,,\quad x\in(-1,1)\,.$$

Then $u=u^+-u^-$. We have the following regularity result in weighted Sobolev spaces, obtained as trivial consequence of a classical result for the usual Sobolev spaces, that we can find in [43, Appendix A].

Proposition 3.1. *Let $u\in H^1_a(-1,1)$, then $u^+,u^-\in H^1_a(-1,1)$. Moreover,*

$$(u^+)_x=\begin{cases}u_x(x) & \text{if } u(x)>0\\ 0 & \text{if } u(x)\le 0\end{cases}\quad \text{and}\quad (u^-)_x=\begin{cases}-u_x(x) & \text{if } u(x)<0\\ 0 & \text{if } u(x)\ge 0\,.\end{cases}$$

Proof of Proposition 1.2. **Case 1:** $u_0\in H^1_a(-1,1)$. Firstly, we prove Proposition 1.2 under the further assumption that $u_0\in H^1_a(-1,1)$. Note that the corresponding unique solution $u(x,t)$ is a strict solution, that is

$$u\in\mathcal{H}(Q_T)=L^2(0,T;D(A_0))\cap H^1(0,T;L^2(-1,1))\cap C([0,T];H^1_a(-1,1)).$$

We denote with u^+ and u^- the positive and negative part of u, respectively. Since $u=u^+-u^-$, it sufficies to prove that $u^-(x,t)=0$, a.e. in Q_T. Multiplying by u^- both sides of the equation in (1.1) and integrating on $(-1,1)$ we obtain

$$\int_{-1}^1 u_tu^-dx=\int_{-1}^1[(a(x)u_x)_xu^-+\alpha uu^-+f(x,t,u)u^-]dx. \tag{3.1}$$

We start by estimating of the second term in (3.1). Integrating by parts, recalling that $u^-(\cdot,t)\in H^1_a(-1,1)$ for every $t\in(0,T)$,and using Proposition 3.1 we deduce

$$\begin{aligned}\int_{-1}^1(a(x)u_x)_xu^-\,dx&=[a(x)u_xu^-]_{-1}^1-\int_{-1}^1 a(x)u_x(u^-)_x\,dx\\&=[a(x)u_xu^-]_{-1}^1+\int_{-1}^1 a(x)u_x^2\,dx\,.\end{aligned}\tag{3.2}$$

If $\beta_1\gamma_1\neq 0$, keeping in mind the boundary conditions, for $t\in(0,T)$ we have

$$\begin{aligned}&[a(x)u_xu^-]_{-1}^1\\&=a(1)u_x(1,t)u^-(1,t)-a(-1)u_x(-1,t)u^-(-1,t)\\&=-\frac{\gamma_0}{\gamma_1}(u^+(1,t)-u^-(1,t))u^-(1,t)+\frac{\beta_0}{\beta_1}(u^+(-1,t)-u^-(-1,t))u^-(-1,t)\\&=\frac{\gamma_0}{\gamma_1}(u^-(1,t))^2-\frac{\beta_0}{\beta_1}(u^-(-1,t))^2\ge 0.\end{aligned}\tag{3.3}$$

Thus, including the simple case $\beta_1\gamma_1 = 0$, from (3.2) and (3.3) we obtain

$$\int_{-1}^{1} (a(x)u_x)_x u^- \, dx \geq 0. \tag{3.4}$$

We also have the equality

$$\int_{-1}^{1} \alpha u u^- dx = -\int_{-1}^{1} \alpha (u^-)^2 dx, \tag{3.5}$$

moreover, using (1.3) we have

$$\begin{aligned}
\int_{-1}^{1} f(x,t,u)u^- \, dx &= \int_{-1}^{1} f(x,t,u^+ - u^-)u^- \, dx = \int_{-1}^{1} f(x,t,-u^-)u^- \, dx \\
&= -\int_{-1}^{1} f(x,t,-u^-)(-u^-) \, dx \\
&\geq -\int_{-1}^{1} \nu(-u^-)^2 \, dx = -\int_{-1}^{1} \nu(u^-)^2 \, dx \,.
\end{aligned} \tag{3.6}$$

We can compute the first term in (3.1) the following way

$$\int_{-1}^{1} u_t u^- dx = \int_{-1}^{1} (u^+ - u^-)_t u^- dx = -\int_{-1}^{1} (u^-)_t u^- dx = -\frac{1}{2}\frac{d}{dt}\int (u^-)^2 dx \,.$$

Applying to (3.1) the above equality and (3.4)-(3.6), we have

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} (u^-)^2 dx \leq \int_{-1}^{1} (\alpha(x,t) + \nu)\,(u^-)^2 dx \leq (\|\alpha\|_\infty + \nu)\int_{-1}^{1} (u^-)^2 dx,$$

so by Gronwall's Lemma, since $u_0^-(x) \equiv 0$, we obtain

$$\int_{-1}^{1} (u^-(x,t))^2 dx \leq e^{2(\nu + \|\alpha\|_\infty)T} \int_{-1}^{1} (u^-(x,0))^2 dx = 0, \quad \forall t \in (0,T).$$

Therefore,

$$u^-(x,t) = 0 \quad \forall (x,t) \in Q_T, \tag{3.7}$$

that proves Proposition 1.2 in the case $u_0 \in H_a^1(-1,1)$.

Case 2: $u_0 \in L^2(-1,1)$. If $u_0 \in L^2(-1,1)$, $u_0 \geq 0$ a.e. $x \in (-1,1)$, then there exists $\{u_{0k}\}_{k\in\mathbb{N}} \subseteq C^\infty([-1,1])$, such that $u_{0k} \geq 0$ on $(-1,1)$ for every $k \in \mathbb{N}$, and $u_{0k} \to u_0$ in $L^2(-1,1)$, as $k \to \infty$. For every $k \in \mathbb{N}$, we consider $u_k \in \mathcal{H}(Q_T)$ the strict solution to (1.1) with initial state u_{0k}. For the well-posedness there exists $u \in \mathcal{B}(Q_T)$ such that $u_k \to u$ in $\mathcal{B}(Q_T)$, as $k \to \infty$. This convergence implies that there exists $\{u_{k_p}\}_{p\in\mathbb{N}} \subseteq \{u_k\}_{k\in\mathbb{N}}$ such that, as $p \to \infty$,

$$u_{k_p}(x,t) \to u(x,t), \quad \text{a.e. } (x,t) \in Q_T. \tag{3.8}$$

For every $p \in \mathbb{N}$, we can apply the Case 1 to the system (1.1) with initial datum u_{0k_p}, so by (3.7) we deduce

$$u_{k_p}(x,t) \geq 0, \quad \text{a.e. } (x,t) \in Q_T,$$

thus from the convergence (3.8) it follows that

$$u(x,t) \geq 0, \quad \text{a.e. } (x,t) \in Q_T.$$

□

3.2. **Nonnegative controllability.**

Proof of Theorem 1.4. Let us fix $\varepsilon > 0$. Since $u_0, u^* \in L^2(-1,1)$, there exist $u_0^\varepsilon, u_\varepsilon^* \in C^1([-1,1])$ such that

$$u_0^\varepsilon,\ u_\varepsilon^* > 0 \text{ on } [-1,1], \quad \|u_\varepsilon^* - u^*\| < \frac{\varepsilon}{4}, \quad \|u_0^\varepsilon - u_0\| < \frac{\sqrt{2}}{36 S_\varepsilon e^\nu}\varepsilon\,, \tag{3.9}$$

where ν is the nonnegative constant in assumption (A1) and

$$S_\varepsilon := \max_{x\in[-1,1]}\Big\{\frac{u_\varepsilon^*(x)}{u_0^\varepsilon(x)}\Big\} + 1\,. \tag{3.10}$$

From (3.9) and (3.10) it follows that there exists $\eta^* > 0$ such that

$$\eta^* \le \frac{u_\varepsilon^*(x)}{S_\varepsilon u_0^\varepsilon(x)} \le 1, \quad \forall x \in [-1,1]\,. \tag{3.11}$$

The strategy of the proof consists of using two control actions: in the first step we steer the system from the initial state u_0 to the intermediate state $S_\varepsilon u_0^\varepsilon$, then in the second step we drive the system from this to u_ε^*. In the second step, condition (3.11) will be crucial, that is justify the choice of the intermediate state $S_\varepsilon u_0^\varepsilon$.

Step 1: Steering the system from u_0 to $S_\varepsilon u_0^\varepsilon$. Let us choose the positive constant bilinear control

$$\alpha(x,t) = \alpha_1 := \frac{\log S_\varepsilon}{T_1} > 0, \quad (x,t) \in (-1,1)\times(0,T_1), \quad \text{for some } T_1 > 0.$$

Let us denote by $u^\varepsilon(x,t)$ and $u(x,t)$ the strict and strong solution of (1.1) with initial state u_0^ε and u_0, respectively. So, keeping in mind the abstract formulation (2.3) for problem (1.1), the Duhamel's principle and the Proposition (2.8), the strict solution $u^\varepsilon(x,t)$ is given in terms of a Fourier series approach, by

$$u^\varepsilon(x,T_1) = e^{\alpha_1 T_1}\sum_{p=1}^\infty e^{-\lambda_p T_1}\langle u_0^\varepsilon, \omega_p\rangle \omega_p(x) + R_\varepsilon(x,T_1)\,, \tag{3.12}$$

with

$$R_\varepsilon(x,T_1) := \sum_{p=1}^\infty \Big[\int_0^{T_1} e^{(\alpha_1-\lambda_p)(T_1-t)}\langle f(\cdot,t,u^\varepsilon(\cdot,t)),\omega_p\rangle dt\Big]\omega_p(x),$$

where $\{-\lambda_p\}_{p\in\mathbb{N}}$ are the eigenvalues of the operator $(A_0, D(A_0))$, defined in (2.1), and $\{\omega_p\}_{p\in\mathbb{N}}$ are the corresponding eigenfunctions, that form a complete orthonormal system in $L^2(-1,1)$, see Proposition 2.5. We recall that the eigenvalues of the operator $(A, D(A))$, with $Au = A_0u + \alpha_1 u$ (defined in (2.2)) are obtained from the eigenvalues of the operator $(A_0, D(A_0))$ by shift, that is we have $\{-\lambda_p + \alpha_1\}_{p\in\mathbb{N}}$, and the corresponding orthonormal system in $L^2(-1,1)$ of eigenfunctions is the same as $(A_0, D(A_0))$, that is $\{\omega_p\}_{p\in\mathbb{N}}$.

By the strong continuity of the semigroup, see Proposition 2.4, we have that

$$\sum_{p=1}^\infty e^{-\lambda_p T_1}\langle u_0^\varepsilon,\omega_p\rangle\omega_p(x) \to u_0^\varepsilon \quad \text{in } L^2(-1,1) \text{ as } T_1 \to 0.$$

So, there exists a small time $\overline{T}_1 \in (0,1)$ such that

$$\big\|S_\varepsilon \sum_{p=1}^\infty e^{-\lambda_p T_1}\langle u_0^\varepsilon,\omega_p\rangle\omega_p(\cdot) - S_\varepsilon u_0^\varepsilon(\cdot)\big\| < \frac{\varepsilon}{8}, \quad \forall T_1 \in (0,\overline{T}_1]. \tag{3.13}$$

Since u^ε is a strict solution, by Proposition 2.18 we have $f(\cdot,\cdot,u^\varepsilon(\cdot,\cdot)) \in L^2(Q_T)$, then using also Hölder's inequality and Parseval's identity we deduce that

$$
\begin{aligned}
\|R_\varepsilon(x,T_1)\|^2 &= \sum_{p=1}^{\infty} \Big| \int_0^{T_1} e^{(\alpha_1-\lambda_p)(T_1-t)} \langle f(\cdot,t,u^\varepsilon(\cdot,t)),\omega_p\rangle dt \Big|^2 \\
&\le \sum_{p=1}^{\infty} \Big(\int_0^{T_1} e^{2(\alpha_1-\lambda_p)(T_1-t)} dt \Big) \int_0^{T_1} |\langle f(\cdot,t,u^\varepsilon(\cdot,t)),\omega_p\rangle|^2 dt \\
&\le e^{2\alpha_1 T_1} T_1 \int_0^{T_1} \sum_{p=1}^{\infty} |\langle f(\cdot,t,u^\varepsilon(\cdot,t)),\omega_p\rangle|^2 dt \\
&= S_\varepsilon^2 T_1 \int_0^{T_1} \|f(\cdot,t,u^\varepsilon(\cdot,t))\|^2 dt \le C S_\varepsilon^2 e^{2k\vartheta T_1} T_1^2 \|u_0^\varepsilon\|_{1,a}^{2\vartheta},
\end{aligned}
\tag{3.14}
$$

where $C = C(\|u_0^\varepsilon\|_{1,a})$ and k are the positive constants introduced in the statement of Proposition 2.18. Then there exists $T_1^* \in (0,\overline{T}_1]$ such that

$$
\sqrt{C} S_\varepsilon e^{k\vartheta T_1} T_1 \|u_0^\varepsilon\|_{1,a}^{\vartheta} < \frac{\sqrt{2}}{36}\varepsilon, \quad \forall T_1 \in (0,T_1^*]. \tag{3.15}
$$

Using Proposition 2.14, by (3.12)-(3.15) and keeping in mind (3.9), for every $T_1 \in (0,T_1^*]$, we have

$$
\begin{aligned}
&\|u(\cdot,T_1) - S_\varepsilon u_0^\varepsilon(\cdot)\| \\
&\le \|u(\cdot,T_1) - u^\varepsilon(\cdot,T_1)\| + \|u^\varepsilon(\cdot,T_1) - S_\varepsilon u_0^\varepsilon(\cdot)\| \\
&\le e^{(\nu+\|\alpha_1^+\|_\infty)T_1} \|u_0 - u_0^\varepsilon\| \\
&\quad + \Big\| S_\varepsilon \sum_{p=1}^{\infty} e^{-\lambda_p T_1} \langle u_0^\varepsilon,\omega_p\rangle \omega_p(\cdot) - S_\varepsilon u_0^\varepsilon(\cdot)\Big\| + \|R_\varepsilon(\cdot,T_1)\| \\
&\le e^{(\nu+\|\alpha_1^+\|_\infty)} \frac{\sqrt{2}}{36 S_\varepsilon e^{\nu}}\varepsilon + \frac{\sqrt{2}}{36}\varepsilon + \sqrt{C} S_\varepsilon e^{k\vartheta T_1} T_1 \|u_0^\varepsilon\|_{1,a}^{\vartheta} \\
&< \frac{\sqrt{2}}{12}\varepsilon,
\end{aligned}
\tag{3.16}
$$

where $\nu \ge 0$ is given in assumption (A1). Let us set

$$
\sigma_0^\varepsilon(x) := u(x,T_1) - S_\varepsilon u_0^\varepsilon(x), \tag{3.17}
$$

we note that by (3.16), we have

$$
\|\sigma_0^\varepsilon\| < \frac{\sqrt{2}}{12}\varepsilon\,. \tag{3.18}
$$

Step 2: Steering the system from $S_\varepsilon u_0^\varepsilon + \sigma_0^\varepsilon$ to u^* at $T \in (0,T^*]$, for some $T^* > 0$. In this step let us restart at time T_1 from the initial state $S_\varepsilon u_0^\varepsilon + \sigma_0^\varepsilon$ and our goal is to steer the system arbitrarily close to u^*. Let us consider

$$
\alpha_\varepsilon(x) := \begin{cases} \log\big(\frac{u_\varepsilon^*(x)}{S_\varepsilon u_0^\varepsilon(x)}\big) & \text{for } x \ne \pm 1, \\ 0 & \text{for } x = \pm 1; \end{cases} \tag{3.19}
$$

thus by (3.11) we deduce that $\alpha_\varepsilon \in L^\infty(-1,1)$ and $\alpha_\varepsilon(x) \le 0$ for a.e. $x \in [-1,1]$. So, there exists a sequence $\{\alpha_{\varepsilon j}\}_{j\in\mathbb{N}} \subset C^2([-1,1])$ such that

$$\begin{gathered} \alpha_{\varepsilon j}(x) \le 0 \ \ \forall x \in [-1,1], \quad \alpha_{\varepsilon j}(\pm 1) = 0, \\ \alpha_{\varepsilon j} \to \alpha_\varepsilon \text{ in } L^2(-1,1) \text{ as } j \to \infty, \end{gathered} \tag{3.20}$$

and

$$\lim_{x\to\pm1} \frac{\alpha'_{\varepsilon j}(x)}{a(x)} = 0, \quad \lim_{x\to\pm1} \alpha'_{\varepsilon j}(x) a'(x) = 0. \tag{3.21}$$

From (3.20) we deduce

$$e^{\alpha_{\varepsilon j}(x)} S_\varepsilon u_0^\varepsilon(x) \to e^{\alpha_\varepsilon(x)} S_\varepsilon u_0^\varepsilon(x) = u_\varepsilon^*(x) \text{ in } L^2(-1,1) \text{ as } j \to \infty;$$

then there exists $j^* \in \mathbb{N}$ such that

$$\|e^{\alpha_{\varepsilon j}} S_\varepsilon u_0^\varepsilon - u_\varepsilon^*\| < \frac{\varepsilon}{12}, \quad \forall j \in \mathbb{N} \quad \text{with } j \ge j^*\,. \tag{3.22}$$

Let us fix an arbitrary $j \in \mathbb{N}$ with $j \ge j^*$, and let us choose as control the static multiplicative function

$$\alpha(x,t) := \frac{1}{T - T_1} \alpha_{\varepsilon j}(x) \le 0 \quad \forall (x,t) \in \widetilde{Q}_T := (-1,1) \times (T_1, T), \tag{3.23}$$

and call $u^\sigma(x,t)$ the unique strong solution that solves problem (1.1) with the following changes:

- time interval (T_1, T) instead of $(0,T)$;
- multiplicative control given by (3.23);
- initial condition $u^\sigma(x, T_1) = S_\varepsilon u_0^\varepsilon(x) + \sigma_0^\varepsilon(x)$.

Let us also denote by $u(x,t)$ the unique strict solution of the problem

$$\begin{gathered} u_t - (a(x)u_x)_x = \frac{\alpha_{\varepsilon j}(x)}{T - T_1} u + f(x,t,u) \quad \text{in } \widetilde{Q}_T := (-1,1) \times (T_1, T) \\ \text{B. C.} \\ u(x, T_1) = S_\varepsilon u_0^\varepsilon(x), \quad x \in (-1,1)\,. \end{gathered} \tag{3.24}$$

For a.e. $x \in (-1,1)$, from the equation

$$u_t(\cdot, t) = \frac{\alpha_{\varepsilon j}(\cdot)}{T - T_1} u(\cdot, t) + \big((a(\cdot) u_x(\cdot,t))_x + f(\cdot, t, u)\big) \qquad t \in (T_1, T),$$

by the classical variation constants technique, we obtain a representation formula of the solution $u(x,t)$ of (3.24), that computed at time T, for $x \in (-1,1)$, becomes

$$u(x,T) = e^{\alpha_{\varepsilon j}(x)} S_\varepsilon u_0^\varepsilon(x) + \int_{T_1}^T e^{\alpha_{\varepsilon j}(x)\frac{(T-\tau)}{T}} \big((a(x)u_x)_x(x,\tau) + f(x,\tau,u(x,\tau))\big) d\tau\,.$$

Let us show that $u(\cdot, T) \to u_\varepsilon^*$ in $L^2(-1,1)$, as $T \to {T_1}^+$. Since $\alpha_{\varepsilon j}(x) \le 0$ from that by the above formula, using Hölder's inequality and (3.22), we deduce that

$$\begin{aligned} &\|u(\cdot,T) - u_\varepsilon^*(\cdot)\|^2 \\ &\le 2\|e^{\alpha_{\varepsilon j}(x)} S_\varepsilon u_0^\varepsilon(x) - u_\varepsilon^*\|^2 \\ &\quad + 2\int_{-1}^1 \Big(\int_{T_1}^T e^{\alpha_{\varepsilon j}(x)\frac{(T-\tau)}{T}} ((a(x)u_x)_x(x,\tau) + f(x,\tau,u(x,\tau)))d\tau \Big)^2 dx \\ &\le \frac{\varepsilon^2}{72} + (T - T_1) \|\big(a(\cdot)u_x\big)_x + f(\cdot,\cdot,u)\|^2_{L^2(\widetilde{Q}_T)}. \end{aligned} \tag{3.25}$$

In Step 3 below, as an appendix to this proof, we will prove that the norm at right-hand side of (3.25) is bounded as $T \to {T_1}^+$. More precisely, we will find that

$$\|\big(a(\cdot)u_x\big)_x + f(\cdot,\cdot,u)\|_{L^2(\widetilde{Q}_T)} \le K, \quad \text{for a.e. } T \in (T_1, T_1+1), \tag{3.26}$$

where $K = K(u_0^\varepsilon, u_\varepsilon^*, \|u_0^\varepsilon\|_{1,a})$ is a positive constant.

Thus, from (3.25) and (3.26) there exists $T_2 \in (T_1, T_1+1)$ such that for every $T \in (T_1, T_2)$ we have

$$\|u(\cdot,T) - u_\varepsilon^*(\cdot)\| < \frac{\varepsilon}{6}. \tag{3.27}$$

Then, using Corollary 2.15, from (3.9), (3.27) and (3.18), there exists T^* in the interval $(T_1, \min\{T_2, T_1 + \frac{1}{4\nu}\})$ such that for every $T \in (T_1, T^*]$ we have

$$\begin{aligned}
\|u^\sigma(\cdot,T) - u^*(\cdot)\| &\le \|u^\sigma(\cdot,T) - u(\cdot,T)\| + \|u(\cdot,T) - u_\varepsilon^*(\cdot)\| + \|u_\varepsilon^* - u^*\| \\
&\le \sqrt{2}\|S_\varepsilon u_0^\varepsilon + \sigma_0^\varepsilon - S_\varepsilon u_0^\varepsilon\| + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} < \frac{\varepsilon}{2},
\end{aligned}$$

from which it follows the approximate controllability at any $T \in (0, T^*]$, since $T_1 > 0$ was arbitrarily small. Moreover, if $T > T^*$ using the above argument we first obtain the approximate controllability at time T^*. Then, we restart at time T^* close to u^*, and we stabilize the system into the neighborhood of u^*, applying the above strategy n times, for some $n \in \mathbb{N}$, on n small time interval by measure $\frac{T-T^*}{n}$, steering the system in every interval from a suitable approximation of u^* to u^*.

Step 3: Evaluation of $\|\big(a(\cdot)u_x\big)_x + f(\cdot,\cdot,u)\|^2_{L^2(\widetilde{Q}_T)}$: Proof of the inequality (3.26). Multiplying by $\big(a(x)u_x\big)_x$ the equation in (3.24), integrating over $\widetilde{Q}_T = (-1,1) \times (T_1, T)$ and applying Young's inequality we have

$$\begin{aligned}
&\|\big(a(\cdot)u_x\big)_x\|^2_{L^2(\widetilde{Q}_T)} \\
&\le \int_{T_1}^T \int_{-1}^1 u_t \big(a(x)u_x\big)_x \,dx\,dt - \frac{1}{T-T_1} \int_{T_1}^T \int_{-1}^1 \alpha_{\varepsilon j}(x) u \big(a(x)u_x\big)_x \,dx\,dt \\
&\quad + \frac{1}{2} \int_{T_1}^T \int_{-1}^1 f^2(x,t,u)\,dx\,dt + \frac{1}{2} \int_{T_1}^T \int_{-1}^1 \big|\big(a(x)u_x\big)_x\big|^2 \,dx\,dt\,.
\end{aligned} \tag{3.28}$$

Thus, by (3.28) using Proposition (2.18) we obtain

$$\begin{aligned}
&\|\big(a(\cdot)u_x\big)_x + f(\cdot,\cdot,u)\|^2_{L^2(\widetilde{Q}_T)} \\
&\le 2\big(\|\big(a(\cdot)u_x\big)_x\|^2_{L^2(\widetilde{Q}_T)} + \|f(\cdot,\cdot,u)\|^2_{L^2(\widetilde{Q}_T)}\big) \\
&\le 4 \int_{T_1}^T \int_{-1}^1 u_t \big(a(x)u_x\big)_x \,dx\,dt - \frac{4}{T-T_1} \int_{T_1}^T \int_{-1}^1 \alpha_{\varepsilon j}(x) u \big(a(x)u_x\big)_x \,dx\,dt \\
&\quad + 4C^2 S_\varepsilon^{2\vartheta} e^{2k\vartheta(T-T_1)}(T-T_1)\|u_0^\varepsilon\|^{2\vartheta}_{1,a},
\end{aligned} \tag{3.29}$$

where $C = C(\|u_0^\varepsilon\|_{1,a})$ and k are the positive constants given by Proposition 2.18. Let us estimate the first two terms of the right-hand side of (3.28). Without loss of generality, let us consider the (WDeg) problem with $\beta_0\gamma_0 \ne 0$. Integrating by parts and using the sign condition $\beta_0\beta_1 \le 0$ and $\gamma_0\gamma_1 \ge 0$ we have

$$\begin{aligned} &\int_{T_1}^{T}\int_{-1}^{1} u_t\big(a(x)u_x\big)_x\,dx\,dt \\ &= \int_{T_1}^{T} [u_t\big(a(x)u_x\big)]_{-1}^{1} dt - \frac{1}{2}\int_{T_1}^{T}\int_{-1}^{1} a(x)\left(u_x^2\right)_t\,dx\,dt \\ &\le \frac{S_\varepsilon}{2}\frac{\gamma_1}{\gamma_0}a^2(1)(u_{0x}^\varepsilon(1))^2 - \frac{S_\varepsilon}{2}\frac{\beta_1}{\beta_0}a^2(-1)(u_{0x}^\varepsilon(-1))^2 \\ &\quad + \frac{S_\varepsilon^2}{2}\int_{-1}^{1} a(x)(u_{0x}^\varepsilon)^2\,dx\,dt \\ &= c_1(S_\varepsilon, u_0^\varepsilon) + c_2(S_\varepsilon)|u_0^\varepsilon|_{1,a}^2, \end{aligned} \tag{3.30}$$

where $c_1(S_\varepsilon, u_0^\varepsilon) \ge 0$ and $c_2(S_\varepsilon) > 0$ are two constants. Let us note that in the (SDeg) case or in the (WDeg) problem with $\beta_0\gamma_0 = 0$, we obtain a similar estimate, but in the third line of (3.30) at least one of the two boundary contributions is zero. Furthermore, using (3.21) and Proposition 2.14 we obtain

$$\begin{aligned} &\int_{T_1}^{T}\int_{-1}^{1} \alpha_{\varepsilon j}(x)u\big(a(x)u_x\big)_x\,dx\,dt \\ &= -\int_{T_1}^{T}\int_{-1}^{1} \alpha_{\varepsilon j}(x)a(x)u_x^2\,dx\,dt - \frac{1}{2}\int_{T_1}^{T}\int_{-1}^{1} \alpha_{\varepsilon j}'(x)a(x)\left(u^2\right)_x\,dx\,dt \\ &\ge -\frac{1}{2}\int_{T_1}^{T}\left[\alpha_{\varepsilon j}'(x)a(x)u^2\right]_{-1}^{1} dt \\ &\quad + \frac{1}{2}\int_{T_1}^{T}\int_{-1}^{1}\big(\alpha_{\varepsilon j}''(x)a(x) + \alpha_{\varepsilon j}'(x)a'(x)\big)\,u^2\,dx\,dt \\ &\ge -\frac{1}{2}\sup_{x\in[-1,1]}\left|\alpha_{\varepsilon j}''(x)a(x) + \alpha_{\varepsilon j}'(x)a'(x)\right| \int_{T_1}^{T}\int_{-1}^{1} u^2\,dx\,dt \\ &\ge -(T-T_1)c(\alpha_{\varepsilon j}', \alpha_{\varepsilon j}'')e^{\nu(T-T_1)}S_\varepsilon^2\|u_0^\varepsilon\|^2\,. \end{aligned} \tag{3.31}$$

Finally, using (3.29)-(3.31), we prove (3.26), that is for almost every $T \in (T_1, T_1+1)$ we have

$$\begin{aligned} &\|\big(a(\cdot)u_x\big)_x + f(\cdot,\cdot,u)\|_{L^2(\widetilde{Q}_T)}^2 \\ &\le k_1(S_\varepsilon, u_0^\varepsilon) + k_2(S_\varepsilon)|u_0^\varepsilon|_{1,a}^2 + k_3(\nu, S_\varepsilon, \alpha_{\varepsilon j}', \alpha_{\varepsilon j}'')\|u_0^\varepsilon\|^2 + k_4(S_\varepsilon, \|u_0^\varepsilon\|_{1,a})\|u_0^\varepsilon\|_{1,a}^{2\vartheta} \\ &\le k_1(S_\varepsilon, u_0^\varepsilon) + K_2(\nu, S_\varepsilon, \alpha_{\varepsilon j}', \alpha_{\varepsilon j}'', \|u_0^\varepsilon\|_{1,a})\|u_0^\varepsilon\|_{1,a}^2, \end{aligned}$$

where $k_1 \ge 0$ and $k_2, k_3, k_4, K_2 > 0$ are constants. □

4. Energy balance models in climate science

Climate depends on several variables and parameters such as temperature, humidity, wind intensity, the effect of greenhouse gases, and so on. It is also affected by a complex set of interactions in the atmosphere, oceans and continents, that involve physical, chemical, geological and biological processes. One of the first attempts to model the effects of the interaction between large ice masses and solar radiation on climate is the one done, independently, by Budyko [11] and Sellers [48]. A complete treatment of the mathematical formulation of the Budyko-Sellers model has been obtained by Diaz and collaborators in [23]–[28] and [9]; see also the interesting recent monograph [45] on "Energy Balance Climate Models" by North and Kim, and some papers by Cannarsa and coauthors [13, 19]. The Budyko-Sellers model is an *energy balance model*, which studies the role played by continental and oceanic areas of ice on the evolution of the climate. The effect of solar radiation on climate can be summarized in Figure 1.

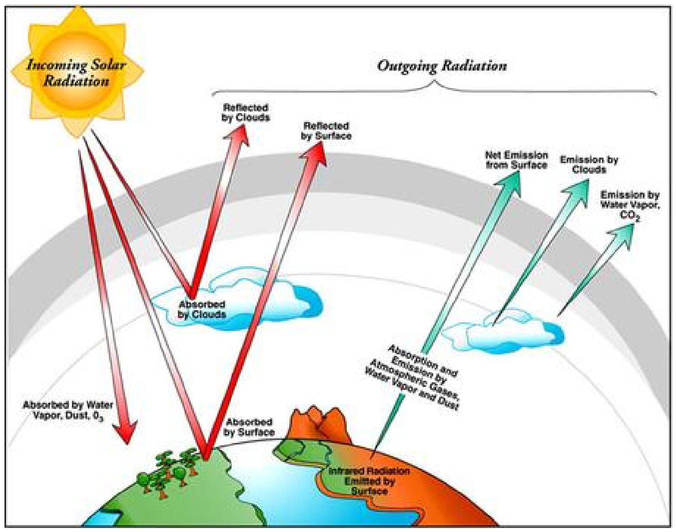

FIGURE 1. Copyrighted by ASR

We have the *energy balance*:

$$\text{Heat variation} = R_a - R_e + D,$$

where R_a is the *absorbed energy*, R_e is the *emitted energy* and D is the *diffusion part*. If we represent the Earth by a compact two-dimensional manifold without boundary $\mathcal{M}$, the general formulation of the Budyko-Sellers model is as follows

$$c(X,t)u_t(X,t) - \Delta_{\mathcal{M}} u(X,t) = R_a(X,t,u) - R_e(u), \tag{4.1}$$

where $c(X,t)$ is a positive function (the heat capacity of the Earth), $u(X,t)$ is the annually (or seasonally) averaged Earth surface temperature, and $\Delta_{\mathcal{M}}$ is the classical Laplace-Beltrami operator. To simplify (4.1), we assume that the thermal capacity is $c \equiv 1$. $R_e(u)$ denotes the Earth radiation, that is, the mean emitted energy flux, that depends on the amount of greenhouse gases, clouds and water vapor in the atmosphere and may be affected by anthropo-generated changes. In the literature there are different empiric expressions of $R_e(u)$. In [48], Sellers proposes a Stefan-Boltzman type radiation law:

$$R_e(u) = \varepsilon(u)u^4,$$

where u is measured in Kelvin degrees (and thus $u > 0$), the positive function $\varepsilon(u) = \sigma\big(1 - m\tanh(\frac{19u^6}{10^6})\big)$ represents the emissivity, σ is the emissivity constant and $m > 0$ is the atmospheric opacity. In its place, in [11] Budyko considers a Newtonian linear type radiation, that is, $R_e(u) = A + Bu$, with suitable $A \in \mathbb{R}$, $B > 0$, which is a linear approximation of the above law near the actual mean temperature of the Earth, $u = 288.15°$ K (15° C). $R_a(X,t,u)$ denotes the fraction of the solar energy absorbed by the Earth and is assumed to be of the form

$$R_a(X,t,u) = QS(X,t)\beta(u),$$

in both the models. In the above relation, Q is the *Solar constant*, $S(X,t)$ is the distribution of solar radiation over the Earth, in seasonal models (when the time scale is smaller) S is a positive "almost periodic" function in time (in particular, it is constant in time, $S = S(X)$, in annually averaged models, that is, when the time scale is long enough), and $\beta(u)$ is the *planetary coalbedo* representing the fraction absorbed according the average temperature ($\beta(u) \in [0,1]$) The *coalbedo* function is equal to 1-*albedo function*. In climate science the albedo (see Figure 2) is more used and well-known than the coalbedo, and is the reflecting power of a surface. It is defined as the ratio of reflected radiation from the surface to incident radiation upon it. It may also be expressed as a percentage, and is measured on a scale from 0, for no reflecting power of a perfectly black surface, to 1, for perfect reflection of a white surface. The coalbedo is assumed to be a non-decreasing function of u, that is, over ice-free zones (like oceans) the coalbedo is greater than over ice-covered regions. Denoted with $u_s = 263.15°$ K ($-10°$ C) the critical value of the temperature at which ice becomes white (the "snow line"), given two experimental values a_i and a_f, such that $0 < a_i < a_f < 1$. Budyko [11] proposed the following coalbedo function, discontinuous at u_s,

$$\beta(u) = \begin{cases} a_i, & \text{over ice-covered } \{X \in \mathcal{M} : u(X,t) < u_s\}, \\ a_f, & \text{over ice-free } \{X \in \mathcal{M} : u(X,t) > u_s\}. \end{cases}$$

Sellers[48] proposed a more regular (at most Lipschitz continuous) function of u. Indeed, Sellers represents $\beta(u)$ as a continuous piecewise linear function (between a_i and a_f) with greatly increasing rate near $u = u_s$, such that $\beta(u) = a_i$, if $u(X,t) < u_s - \eta$ and $\beta(u) = a_f$, if $u(X,t) > u_s + \eta$, for some small $\eta > 0$. If we assume that $\mathcal{M}$ is the unit sphere of $\mathbb{R}^3$, the Laplace-Beltrami operator becomes

$$\Delta_{\mathcal{M}} u = \frac{1}{\sin\phi}\Big\{\frac{\partial}{\partial\phi}\Big(\sin\phi\frac{\partial u}{\partial\phi}\Big) + \frac{1}{\sin\phi}\,\frac{\partial^2 u}{\partial\lambda^2}\Big\},$$

where ϕ is the *colatitude* and λ is the *longitude*.

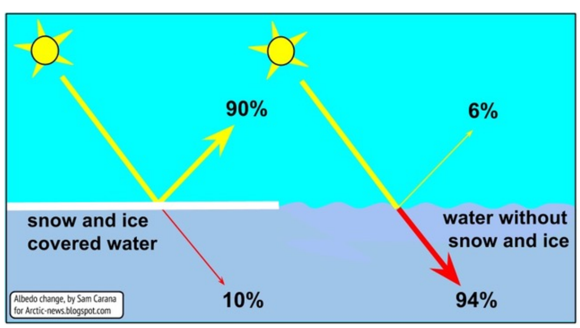

Figure 2. Copyrighted by ABC Columbia

Thus, if we take the average of the temperature at $x = \cos\phi$ (see in Figure 3, that the distribution of the temperature at the same colatitude can be considered approximately uniform). In such a model, the sea level mean zonally averaged temperature $u(x,t)$ on the Earth, where t still denotes time, satisfies a Cauchy-Neumann strongly degenerate problem, in the bounded domain $(-1,1)$, of the type

$$
\begin{gathered}
u_t - \big((1-x^2)u_x\big)_x = \alpha(x,t)\,\beta(u) + f(x,t,u), \quad x \in (-1,1),\\
\lim_{x\to\pm 1}(1-x^2)u_x(x,t) = 0, \quad t \in (0,T)\,.
\end{gathered}
$$

Then, the uniformly parabolic equation (4.1) has been transformed into a 1-D degenerate parabolic equation. So, we have showed that our degenerate reaction-diffusion system (1.1) reduces to the 1-D Budyko-Sellers model when $a(x) = 1 - x^2$.

Environmental aspects. We remark that the Budyko-Sellers model studies the effect of solar radiation on climate, so it takes into consideration the influence of "greenhouse gases" on climate. These cause "global warming" which, consequently, provokes the increase in the average temperature of the Earth's atmosphere and of oceans. This process consists of a warming of the Planet Earth by the action of greenhouse gases, compounds present in the air in a relatively low concentration (carbon dioxide, water vapour, methane, etc.). Greenhouse gases allow solar radiation to pass through the atmosphere while obstructing the passage towards space of a part of the infrared radiation from the Earth's surface and from the lower atmosphere. The majority of climatologists believe that Earth's climate is destined to change, because human activities are altering atmosphere's chemical composition. In fact, the enormous anthropogenic emissions of greenhouse gases are causing an increase in the Earth's temperature, consequently, provoking profound changes in the Planetary climate. One of the aims of this kind of research is to estimate the possibility of controlling the variation of the temperature over decades and centuries and it proposes to provide a study of the possibility of slowing down global warming.

Related open problems. Keeping in mind the meaning of the multiplicative control α in the climate framework, since in the main control result of this paper, Theorem 1.4, the action must be realized over any latitude x in $[-1,1]$, it would be more realistic follow up in future papers the formulation that was already proposed by Von Neumann in 1955 (See Diaz [27] for a comprehensive presentation), that is, by using a localized control defined merely for some set of latitudes. From the

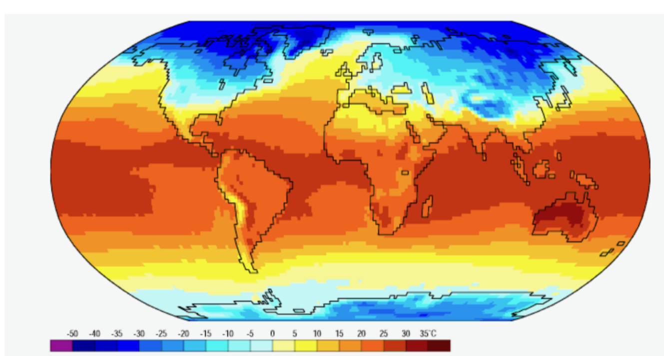

FIGURE 3. Copyrighted by Edu-Arctic.eu

multiplicative controllability point of view, that problem is hard but it is under research by Diaz and the author; one possible approach consists in following some ideas introduced in [33], in the case of uniformly parabolic equations.

5. Appendix: Proofs of existence and uniqueness results

In the first subsection we prove Theorem 2.11, which shows that for all $\alpha \in L^\infty(Q_T)$ piecewise static functions, there exists a unique strict solution $u \in \mathcal{H}(Q_T)$ to (1.1), for all initial state $u_0 \in H^1_a(-1,1)$. Then in subsection 5.2 we can prove Theorem 2.12, that is by an approximation argument we obtain the existence and uniqueness of the strong solution to (1.1), for all initial state $u_0 \in L^2(-1,1)$.

5.1. **Existence and uniqueness of the strict solution to** (1.1)**.** The aim is to prove Theorem 2.11. For this purpose, we will follow the following strategy:

- in Subsection 5.1.1 we present a *maximal regularity* result for abstract non-homogeneous linear evolution equations in Hilbert spaces;
- in Subsection 5.1.2 we introduce the notion of *mild solutions* and we give an existence and uniqueness result for mild solutions;
- in Subsection 5.1.3 we prove the existence and uniqueness of strict solutions for static coefficient $\alpha \in L^\infty(-1,1)$;
- in Subsection 5.1.4, finally we prove that if $u_0 \in H^1_a(-1,1)$ then the mild solution is also a strict solution, for all $\alpha \in L^\infty(Q_T)$ piecewise static function.

5.1.1. *A maximal regularity result for linear problems.* Let us consider the following linear problem in the Hilbert space $L^2(-1,1)$,

$$\begin{gathered} u'(t) = A\,u(t) + g(t), \quad t > 0 \\ u(0) = u_0\,, \end{gathered} \tag{5.1}$$

where A is the operator in (2.2), $g \in L^1(0,T;L^2(-1,1))$, $u_0 \in L^2(-1,1)$.

First, let us recall the notion of "weak solution" introduced by Ball [4] for the linear problem (5.1) (see also [12, 1, 35, 36]).

Definition 5.1. A*weak solution* of (5.1) is a function $u \in C([0,T];L^2(-1,1))$ such that for every $v \in D(A^*)$ (A^* denotes the adjoint of A) the function $\langle u(t), v\rangle$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\langle u(t), v\rangle = \langle u(t), A^* v\rangle + \langle g(t), v\rangle,$$

for almost all $t \in [0,T]$.

Then, we recall the following existence and uniqueness result obtained by Ball [4] (see also [13, 14, 35, 36]).

Proposition 5.2. *For every $u_0 \in L^2(-1,1)$ there exists a unique weak solution u of* (5.1)*, which is given by the following representation formula*

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} g(s)\,ds, \quad t \in [0,T].$$

Now, we are able to present Proposition 5.3, a *maximal regularity* result that holds in the Hilbert space $L^2(-1,1)$. Before giving the statement of Proposition 5.3 we recall that by *maximal regularity* for (5.1) we mean that u' and Au have the same regularity of g.

Proposition 5.3. *Let $T > 0$ and $g \in L^2(0,T;L^2(-1,1))$. For each $u_0 \in H^1_a(-1,1)$, there exists a unique solution of* (5.1),

$$u \in \mathcal{H}(Q_T) = L^2(0,T;D(A_0)) \cap H^1(0,T;L^2(-1,1)) \cap C([0,T];H^1_a(-1,1)).$$

Moreover, there exists a positive constant $C_0(T)$ (nondecreasing in T), such that

$$\|u\|_{\mathcal{H}(Q_T)} \le C_0(T)[\|u_0\|_{1,a} + \|g\|_{L^2(Q_T)}].$$

Proof. It is a direct consequence of [8, Theorem 3.1, Section 3.6.3] (see also [21, 35]), keeping in mind the following three crucial issues related to the abstract setting in [8, Section 3.6.3]:

- $g \in L^2(0,T;X)$, where X is the Hilbert space $L^2(-1,1)$;
- A is the infinitesimal generator of an *analytic* semigroup (see [20]);
- $u_0 \in H^1_a(-1,1)$, where $H^1_a(-1,1)$ is an interpolation space between the domain $D(A_0)$ and $L^2(-1,1)$.

□

5.1.2. *Existence and uniqueness of the mild solution to* (1.1). Before introducing the notion of *mild solutions*, we consider the following abstract representation of the semilinear problem (1.1) in the Hilbert space $L^2(-1,1)$,

$$\begin{gathered} u'(t) = A_0\, u(t) + \psi(t,u(t)), \quad t > 0 \\ u(0) = u_0 \in L^2(-1,1), \end{gathered} \tag{5.2}$$

where A_0 is the operator defined in (2.1) and, for every $u \in \mathcal{B}(Q_T)$,

$$\psi(t,u) := \psi(x,t,u(x,t)) = \alpha(x,t)u(x,t) + f(x,t,u(x,t)), \quad \forall (x,t) \in Q_T, \tag{5.3}$$

with $\alpha \in L^\infty(Q_T)$, piecewise static, given in (1.1).

We note that, since from (1.3), for a.e. $(x,t) \in Q_T$, $\forall u,v \in \mathbb{R}$, it follows that

$$|f(x,t,u) - f(x,t,v)| \le \nu \left(1 + |u|^{\vartheta-1} + |v|^{\vartheta-1}\right)|u-v|,$$

we deduce that

$$\begin{aligned} |\psi(t,u) - \psi(t,v)| &\le |\alpha(x,t)u - \alpha(x,t)v| + |f(x,t,u) - f(x,t,v)| \\ &\le \|\alpha\|_\infty |u-v| + \nu\left(1 + |u|^{\vartheta-1} + |v|^{\vartheta-1}\right)|u-v|. \end{aligned}$$

Therefore,

$$|\psi(t,u) - \psi(t,v)| \le L|u-v|, \quad \text{for a.e. } (x,t) \in Q_T,\ \forall u,v \in L^2(-1,1), \tag{5.4}$$

where $L = L(u,v) = \|\alpha\|_\infty + \nu\left(1 + |u|^{\vartheta-1} + |v|^{\vartheta-1}\right)$ which does not depend on t.

Definition 5.4. Let $u_0 \in L^2(-1,1)$. We say that $u \in C([0,T];L^2(-1,1))$ is a *mild solution* of (1.1), if u is a solution of the integral equation

$$u(t) = e^{tA_0}u_0 + \int_0^t e^{A_0(t-s)}\psi(s,u(s))\,ds, \quad t \in [0,T].$$

The existence and the uniqueness of the mild solution of (1.1) follows from Proposition 5.5 below, that is a consequence of a result by Li and Yong [44, Proposition 5.3 in Chapt. 2]. For the next proposition the following assumptions are introduced:

(A3) for each $\bar{u} \in L^2(-1,1)$, $\psi(\cdot,\bar{u}) : [0,T] \to L^2(-1,1)$ is *strongly measurable*, that is there exists a sequence of *simple functions* (piecewise static functions) $\psi_k(\cdot,\bar{u}) : [0,T] \to L^2(-1,1)$ such that
$$\lim_{k\to\infty} |\psi_k(t,\bar{u}) - \psi(t,\bar{u})| = 0, \quad \text{a.e. } t \in [0,T];$$

(A4) there exists a function $L(t) \in L^1(0,T)$ such that
$$|\psi(t,u) - \psi(t,v)| \le L(t)|u-v|, \quad \text{a.e. } t \in [0,T],\ \forall u,v \in L^2(-1,1),$$
$$|\psi(t,0)| \le L(t), \quad \text{a.e. } t \in [0,T].$$

Proposition 5.5. *Under assumptions* (A3) *and* (A4)*, There exists a unique mild solution to* (5.2)*.*

5.1.3. *Existence and uniqueness of strict solutions for static coefficient $\alpha \in L^\infty(-1,1)$.* In this subsection we prove Theorem 2.11 by showing that if the initial state belongs to $H^1_a(-1,1)$, then the *mild solution* is also a *strict* solution.

Lemma 5.6. *For every $M > 0$, there exists $T_M > 0$ such that for all $\alpha \in L^\infty(-1,1)$, and all $u_0 \in H^1_a(-1,1)$ with $\|u_0\|_{1,a} \le M$ there is a unique strict solution $u \in \mathcal{H}(Q_{T_M})$ to* (1.1)*.*

Proof. Let us fix $M > 0$, $u_0 \in H^1_a(-1,1)$ such that $\|u_0\|_{1,a} \le M$. Let $0 < T \le 1$, we define
$$\mathcal{H}_M(Q_T) := \{u \in \mathcal{H}(Q_T) : \|u\|_{\mathcal{H}(Q_T)} \le 2C_0(1)M\},$$
where $C_0(1)$ is the constant $C_0(T)$ (nondecreasing in T) defined in Proposition 5.3 and valued in $T = 1$. Then, let us consider the map $\Phi : \mathcal{H}_M(Q_T) \to \mathcal{H}_M(Q_T)$, defined by
$$\Phi(u)(t) := e^{tA_0}u_0 + \int_0^t e^{(t-s)A_0}\left(\alpha u(s) + f(s,u(s))\right) ds \quad \forall u \in \mathcal{H}(Q_T).$$

Step 1. We prove that the map Φ is well defined for some T. Fix $u \in \mathcal{H}_M(Q_T)$, and consider $y(t) := \Phi(u)(t)$, then, keeping in mind Proposition 5.2, we can see the function y as the solution of the problem
$$\begin{gathered} y'(t) = A_0 y(t) + (\alpha u(t) + f(t,u(t)))\,, \quad t \in [0,T] \\ y(0) = u_0 \in H^1_a(-1,1) \end{gathered} \tag{5.5}$$
By Proposition 2.18 we deduce that $f(\cdot,\cdot,u) \in L^2(Q_T)$, thus
$$g(t) := \psi(t,u(t)) \in L^2(0,T;L^2(-1,1)).$$
Then, applying Proposition 5.3 there exists a unique solution $y \in \mathcal{H}(Q_T)$ of (5.5) such that
$$\|y\|_{\mathcal{H}(Q_T)} \le C_0(T)\left(\|u_0\|_{1,a} + \|\psi(\cdot,u(\cdot))\|_{L^2(Q_T)}\right).$$
Thus, keeping in mind Proposition 5.3, we have $C_0(T) \le C_0(1)$ since $T \le 1$. Applying Lemma 2.7 and Proposition 2.18 there exists $T_0(M) \in (0,1)$ such that
$$\begin{aligned} \|\Phi(u)\|_{\mathcal{H}(Q_T)} &= \|y\|_{\mathcal{H}(Q_T)} \\ &\le C_0(1)\left(\|\alpha\|_\infty \|u\|_{L^2(Q_T)} + \|f(\cdot,\cdot,u)\|_{L^2(Q_T)} + \|u_0\|_{1,a}\right) \\ &\le 2C_0(1)M, \quad \forall T \in [0,T_0(M)]. \end{aligned}$$
Then $\Phi u \in \mathcal{H}_M(Q_T)$ for all $T \in [0,T_0(M)]$.

Step 2.; We prove that there exists T_M such that the map Φ is a contraction. Let $T \in (0, T_0(M)]$. Fix $u, v \in \mathcal{H}_M(Q_T)$ and set

$$w := \Phi(u) - \Phi(v) = \int_0^t e^{(t-s)A_0} \left[\alpha\left(u(s) - v(s)\right) + \left(f(s, u(s)) - f(s, v(s))\right)\right] ds\,.$$

Then, keeping in mind Proposition 5.2, w is solution of the problem

$$\begin{gathered} w_t - (a\, w_x)_x = \psi(t,u) - \psi(t,v) \quad \text{in } Q_T \\ \text{B. C.} \\ w(x,0) = 0\,. \end{gathered} \tag{5.6}$$

By Lemma 2.18 $f(\cdot, u) \in L^2(Q_T)$, and applying Proposition 5.3 we deduce that there exists a unique solution $w \in \mathcal{H}(Q_T)$ of (5.6), and

$$\|\Phi(u) - \Phi(v)\|_{\mathcal{H}(Q_T)} = \|w\|_{\mathcal{H}(Q_T)} \le C_0(1)\|\psi(\cdot, u) - \psi(\cdot, v)\|_{L^2(Q_T)}. \tag{5.7}$$

From (5.7), using (5.4), Lemma 2.7 and Proposition 2.18 (for similar estimates see [35, Lemma B.1]), there exists $T_M \in (0, T_0(M))$ such that Φ is a contraction map. Therefore, Φ has a unique fix point in $\mathcal{H}_M(Q_{T_M})$, from which the conclusion follows. □

Remark 5.7. Using a classical "semigroup" result (see [46]), applying Lemma 5.6 and the "a priori estimate" contained in Lemma 2.17, we directly obtain the existence of the global strict solution to (1.1). That is, following the proof of Lemma 5.6 the unique mild solution, given by Proposition 5.5, is also a strict solution. Thus, we obtain the proof of Theorem 2.11 in the case of a *static* reaction coefficient $\alpha \in L^\infty(-1, 1)$.

5.1.4. *Regularity of the mild solution to* (1.1) *with initial data in* $H^1_a(-1,1)$. Now, we can give the complete proof of Theorem 2.11 in the general case when $\alpha \in L^\infty(Q_T)$ is a piecewise static function.

Proof of Theorem 2.11. Let us consider (1.1) under assumptions (A1) and (A2). Let us assume that $u_0 \in H^1_a(-1,1)$ and $\alpha \in L^\infty(Q_T)$ is a piecewise static function (or a *simple function* with respect to the variable t), in the sense of Definition 1.1, that is, there exist $m \in \mathbb{N}$, $\alpha_k(x) \in L^\infty(-1,1)$ and $t_k \in [0,T]$, $t_{k-1} < t_k$, $k = 1, \dots, m$ with $t_0 = 0$ and $t_m = T$, such that

$$\alpha(x,t) = \alpha_1(x)\chi_{[t_0,t_1]}(t) + \sum_{k=2}^m \alpha_k(x)\chi_{(t_{k-1},t_k]}(t),$$

where $\chi_{[t_0,t_1]}$ and $\chi_{(t_{k-1},t_k]}$ are the indicator function of $[t_0,t_1]$ and $(t_{k-1},t_k]$, respectively. Let $u \in C([0,T]; L^2(-1,1))$ the unique mild solution of (1.1) with initial state $u_0 \in H^1_a(-1,1)$, given by Proposition 5.5,

$$u(t) := e^{tA_0}u_0 + \int_0^t e^{(t-s)A_0}\left(\alpha(s)u(s) + f(s,u(s))\right) ds\,, \quad \forall t \in [0,T].$$

Then u, for $k = 1, \cdots, m$, is the solution of the following m problems:

$$\begin{gathered} U'(t) = A_0\, U(t) + \alpha_k U(t) + f(t, U(t))\,, \quad t \in [t_{k-1}, t_k] \\ U(t_k) = u(t_k) \quad k = 1, \dots, m\,. \end{gathered} \tag{5.8}$$

Since $u_0 \in H^1_a(-1,1)$ and $\alpha_k \in L^\infty(-1,1)$ ($k \in \{1, \cdots, m\}$) is static on $[t_{k-1}, t_k]$, then applying m times Remark 5.7 we obtain that the unique mild solution u is also strict on $[t_{k-1}, t_k]$, then the "new" initial condition $u(t_k)$ will belong to $H^1_a(-1,1)$. Thus, by iteration from $[0, t_1]$ to $[t_{m-1}, t_m]$ we can complete the proof and we obtain that the mild solution $u \in \mathcal{H}(Q_T)$ and it is also a strict solution on $[0,T]$. □

Remark 5.8. Keeping in mind Proposition 5.5 it follows that we can extend Theorem 2.11 to the general case $\alpha \in L^\infty(Q_T)$. Namely, in that case α is *strongly measurable*, in the sense of the condition (A4), moreover we can generalize the proof of Theorem 2.11 from α piecewise static function to α strongly measurable.

Remark 5.9. To prove Theorem 2.11 one can also follow the approach developed by Kato [40] and Evans [31] (see also Pazy's book [46]). This approach considers in (5.2) directly as operator $A(t) := A_0 + \alpha(t)I$, in which the dependence on t is discontinuous, instead of the simple operator A_0 that is constant in t.

5.2. **Existence and uniqueness of the strong solution to** (1.1)**.** In this section we recall the proof of Theorem 2.16, given in [35] for the (SDeg) case, and in [36] for the (WDeg).

Proof of Theorem 2.16. Since $u_0 \in L^2(-1,1)$, there exists $\{u^0_k\}_{k\in\mathbb{N}} \subseteq H^1_a(-1,1)$ such that $\lim_{k\to\infty} u^0_k = u_0$ in $L^2(-1,1)$. For every $k \in \mathbb{N}$, we consider the problem

$$\begin{gathered} u_{kt} - (a(x)u_{kx})_x = \alpha(x,t)u_k + f(x,t,u_k) \quad \text{a.e. in } Q_T \\ \text{B. C.} \\ u_k(x,0) = u^0_k(x) \quad x \in (-1,1)\,. \end{gathered} \tag{5.9}$$

For every $k \in \mathbb{N}$, by Theorem 2.11 there exists a unique $u_k \in \mathcal{H}(Q_T)$ strict solution to (5.9). Then, we consider the sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq \mathcal{H}(Q_T)$ and by direct application of Proposition 2.14 we prove that $\{u_k\}_{k\in\mathbb{N}}$ is a *Cauchy* sequence in the Banach space $\mathcal{B}(Q_T)$. Then, there exists $u \in \mathcal{B}(Q_T)$ such that, as $k \to \infty$, $u_k \to u$ in $\mathcal{B}(Q_T)$ and $u(\cdot,0) = \lim_{k\to\infty} u_k(\cdot,0) = u_0$ in the L^2 sense. So, $u \in \mathcal{B}(Q_T)$ is a strong solution. The uniqueness of the strong solution to (1.1) is a direct consequence of Proposition 2.14. □

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