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MULTIPLE SOLUTIONS FOR MIXED BOUNDARY VALUE PROBLEMS WITH φ-LAPLACIAN OPERATORS

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Abstract. Using Leray-Schauder degree theory and the method of upper and lower solutions we establish existence and multiplicity of solutions for problems of the form

$$\begin{gathered}(\varphi(u'))' = f(t,u,u')\\ u(0)=u(T)=u'(0),\end{gathered}$$

where φ is an increasing homeomorphism such that $\varphi(0)=0$, and f is a continuous function.

1. Introduction

The purpose of this article is to obtain multiplicity of solutions for problems of the form

$$\begin{gathered}(\varphi(u'))' = f(t,u,u')\\ u(0)=u(T)=u'(0),\end{gathered} \tag{1.1}$$

where $0<T<\infty$, $\varphi:\mathbb{R}\to\mathbb{R}$ is an increasing homeomorphism such that $\varphi(0)=0$, and $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function. We call *solution* of this problem a function $u:[0,T]\to\mathbb{R}$ of class C^1 such that $\varphi(u')$ is continuously differentiable, satisfying the boundary conditions and $(\varphi(u'(t)))'=f(t,u(t),u'(t))$ for all $t\in[0,T]$.

Existence of solutions for boundary value problems can be studied by different methods:fixed point theorems, topological degree, fixed point index theory, lower and upper functions, etc.; for bounded intervals see for example [1, 2, 3, 4, 7, 8, 9] and for unbounded intervals [5, 6] and the reference therein. In particular, using the method of upper and lower solutions and the fixed point index theory the authors in [9] obtained existence and multiplicity results of solutions for the Dirichlet boundary value problem. These results were established under a growth condition of Wintner-Nagumo type of the form:

$$|f(t,x,y)|\le\psi(|y|)(l(t)+c(t)|y|^{(p-1)/p}),$$

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where $l \in L^1([0,T])$, $c \in L^p([0,T])$ with $1 \le p \le \infty$, f is a Carathéodory function, and $\psi : [0,\infty) \to (0,\infty)$ is such that

$$\int_{-\infty}^{\infty} \frac{ds}{\psi(|\varphi^{-1}(s)|)} = \infty. \tag{1.2}$$

Santos [8] proved the existence of at least one solution for (1.1) using the method of upper and lower solutions and the fixed point theorem of Schauder, see Theorem 3.5 below.

Inspired by these results, the main aim of this paper is to study the existence and multiplicity of solutions for (1.1) using the method of upper and lower solutions and topological methods based upon Leray-Schauder degree. In this work, we highlight several aspects of these results. On the one hand, our problem consists of equations for general type of boundary conditions. On the other hand, we generalize the results of [8, Section 4].

Finally, we establish multiplicity results for (1.1) using the method of upper and lower solutions and Leray-Schauder degree theory. For these results, we impose the growth condition of Wintner-Nagumo type

$$|f(t,x,y)| \le \psi(|y|)$$

where f is a continuous function and ψ satisfies (1.2). Which is needed to ensure an a priori bound for the derivatives of the solutions to apply Leray-Schauder degree. These results improve the literature concerning Dirichlet-type equations.

2. Notation and preliminaries

For a fixed T, we denote for $C = C([0,T],\mathbb{R})$ the Banach space of continuous functions $u : [0,T] \to \mathbb{R}$ with the norm $\|u\|_\infty$, $C^1 = C^1([0,T],\mathbb{R})$ denote the Banach space of continuously differentiable functions from $[0,T]$ into $\mathbb{R}$ equipped with the usual norm $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$. We introduce the following operators: the *Nemytskii operator* $N_f : C^1 \to C$,

$$N_f(u)(t) = f(t,u(t),u'(t)),$$

and the *integral operator* $H : C \to C^1$,

$$H(u)(t) = \int_0^t u(s)ds.$$

The following results are taken from [1, 8], respectively. The first one is needed in the construction of the equivalent fixed point problem.

Lemma 2.1. *For each $h \in C$, there exists a unique $Q_\varphi = Q_\varphi(h) \in \operatorname{im}(h)$ (where $\operatorname{im}(h)$ denotes the range of h) such that*

$$\int_0^T \varphi^{-1}(h(t) - Q_\varphi(h))dt = 0.$$

Moreover, the function $Q_\varphi : C \to \mathbb{R}$ is continuous and sends bounded sets into bounded sets.

The second results gives an equivalent formulation of problem (1.1) as a fixed point problem.

Lemma 2.2. *A function u is a solution of* (1.1) *if and only if $u \in C^1$ is a fixed point of the operator M_1 defined on C^1 by*

$$M_f(u) = \varphi^{-1}(-Q_\varphi(H(N_f(u)))) + H(\varphi^{-1}[H(N_f(u)) - Q_\varphi(H(N_f(u)))]). \quad (2.1)$$

Here φ^{-1} is understood as the operator $\varphi^{-1} : C \to C$ defined by $\varphi^{-1}(v)(t) = \varphi^{-1}(v(t))$. It is clear that φ^{-1} is continuous and sends bounded sets into bounded sets. Using the Arzelà-Ascoli theorem it is not difficult to see that M_f is completely continuous.

3. Existence results

In this section we prove the existence of at least one solution for problem (1.1).

3.1. **Upper and lower solutions.**

The functions considered as lower and upper solutions for the initial problem (1.1) are defined as follows.

Definition 3.1. A lower solution α (resp. upper solution β) of (1.1) is a function $\alpha \in C^1$ such that $\varphi(\alpha') \in C^1$, $\alpha'(0) \geq \alpha(0) > \alpha(T)$ (resp. $\beta \in C^1$, $\varphi(\beta') \in C^1$, $\beta'(0) \leq \beta(0) < \beta(T)$) and

$$(\varphi(\alpha'(t)))' \geq f(t, \alpha(t), \alpha'(t)) \quad (\text{resp. } (\varphi(\beta'(t)))' \leq f(t, \beta(t), \beta'(t))) \quad (3.1)$$

for all $t \in [0, T]$. Such a lower or upper solution is called strict if the inequality (3.1) is strict for all for all $t \in [0, T]$.

We will use the following general assumptions.

1. There exist α, β, respectively lower and upper solutions for (1.1) such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, T]$.
2. There exists $\psi : [0, \infty) \to (0, \infty)$ such that
$$\int_{-\infty}^{\infty} \frac{ds}{\psi(|\varphi^{-1}(s)|)} = \infty.$$
and $|f(t, x, y)| \leq \psi(|y|)$ for all $x \in [\alpha(t), \beta(t)]$, $t \in [0, T]$ and $y \in \mathbb{R}$.

We can now prove some existence results for (1.1).

Theorem 3.2. *Let $\alpha \leq \beta$ be respectively a lower and an upper solution of* (1.1)*, let $R > \max\{\|\alpha'\|_\infty, \|\beta'\|_\infty\}$, and let $E = \{(t, x, y) : t \in [0, T], \alpha(t) \leq x \leq \beta(t), |y| \leq R\}$. Suppose that f satisfies*

$$|f(t, x, y)| \leq \psi(|y|) \quad (3.2)$$

over E for some ψ such that

$$\min\Big\{ \int_0^{\varphi(R)} \frac{ds}{\psi(|\varphi^{-1}(s)|)}, \int_{\varphi(-R)}^0 \frac{ds}{\psi(|\varphi^{-1}(s)|)} \Big\} > T. \quad (3.3)$$

Then (1.1) *has a solution u such that $\|u'\|_\infty < R$ and $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0, T]$.*

Proof. Let α, β be, respectively, lower and upper solutions of (1.1). Let $\gamma : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \times \mathbb{R}$ be the continuous functions defined by

$$\gamma(t, x) = \begin{cases} \beta(t), & x \geq \beta(t) \\ x, & \alpha(t) \leq x \leq \beta(t) \\ \alpha(t), & x \leq \alpha(t), \end{cases} \qquad Q(y) = \begin{cases} y, & |y| \leq R \\ R, & y \geq R \\ -R, & y \leq -R, \end{cases}$$

and define $F : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$F(t, x, y) = f(t, \gamma(t, x), Q(y)) + \frac{x - \gamma(t, x)}{1 + |x - \gamma(t, x)|}.$$

Now, we consider the modified problem

$$\begin{gathered} (\varphi(u'))' = F(t, u, u') \\ u(0) = u(T) = u'(0). \end{gathered} \tag{3.4}$$

For clearness, the proof will follow several steps.

Step 1. If u is a solution of (3.4), then $\alpha(t) \le u(t) \le \beta(t)$ for all $t \in [0, T]$). Let u be a solution of the modified problem (3.4) and suppose by contradiction that there is some $t_0 \in [0, T]$ such that

$$\max_{[0,T]}(\alpha(t) - u(t)) = \alpha(t_0) - u(t_0) > 0. \tag{3.5}$$

If $t_0 \in (0, T)$, there are sequences (t_k) in $[t_0 - \epsilon, t_0)$ and (t'_k) in $(t_0, t_0 + \epsilon]$ converging to t_0 such that $\alpha'(t_k) - u'(t_k) \ge 0$ and $\alpha'(t'_k) - u'(t'_k) \le 0$. Therefore $\alpha'(t_0) = u'(t_0)$. Since $R > \|\alpha'\|_\infty$ we deduce that $Q(u'(t_0)) = \alpha'(t_0)$. Using that φ is an increasing homeomorphism, this implies $(\varphi(\alpha'(t_0)))' \le (\varphi(u'(t_0)))'$. By (3.1) we obtain the contradiction

$$\begin{aligned} (\varphi(\alpha'(t_0)))' \le (\varphi(u'(t_0)))' &= F(t_0, u(t_0), u'(t_0)) \\ &\le f(t_0, \alpha(t_0), \alpha'(t_0))) + \frac{u(t_0) - \alpha(t_0)}{1 + |u(t_0) - \alpha(t_0)|} \\ &< f(t_0, \alpha(t_0), \alpha'(t_0))) \le (\varphi(\alpha'(t_0)))'. \end{aligned}$$

So $\alpha(t) \le u(t)$ for all $t \in (0, T)$. If the maximum is attained at $t_0 = 0$ then

$$\max_{[0,T]}(\alpha(t) - u(t)) = \alpha(0) - u(0) > 0.$$

Using that $u(0) = u'(0)$ and $\alpha'(0) \le u'(0)$, we obtain the contradiction

$$\alpha(0) \le \alpha'(0) \le u'(0) = u(0) < \alpha(0).$$

If

$$\max_{[0,T]}(\alpha(t) - u(t)) = \alpha(T) - u(T) > 0,$$

then $\alpha(0) = \alpha(T)$. Using that $u(0) = u(T)$ we obtain again a contradiction. In consequence we have that $\alpha(t) \le u(t)$ for all $t \in [0, T]$. In a similar way we can prove that $u(t) \le \beta(t)$ for all $t \in [0, T]$.

Step 2. If u is a solution of (3.4), then $|u'\|_\infty < R$. Let u be a solution of the modified problem (3.4) and suppose by contradiction that u' is such that $\|u'\|_\infty \ge R$. If $\max\{u'(t) : t \in [0, T]\} \ge R$, then there exist t_0, t_1 such that $u'(t_0) = 0, u'(t_1) = R$ and $0 < u'(t) < R$ for all t between t_0 and t_1 (without loss of generality we assume that $t_0 < t_1$). Then $\varphi(u'(t_0)) = 0, \varphi(u'(t_1)) = \varphi(R)$ and $0 < \varphi(u'(t)) < \varphi(R)$. Using the substitution $s = \varphi(u'(t))$ we obtain

$$\int_0^{\varphi(R)} \frac{ds}{\psi(|\varphi^{-1}(s)|)} = \int_{t_0}^{t_1} \frac{(\varphi(u'(t)))'dt}{\psi(|u'(t)|)} = \int_{t_0}^{t_1} \frac{f(t, u(t), Q(u'(t)))dt}{\psi(|u'(t)|)}.$$

Since $(t, u(t), Q(u'(t))) = (t, u(t), u'(t)) \in E$ and $u'(t) > 0$, we conclude by (3.2) that

$$\int_0^{\varphi(R)} \frac{ds}{\psi(|\varphi^{-1}(s)|)} \leq |\int_{t_0}^{t_1} dt| = |t_1 - t_0| \leq T.$$

This contradicts (3.3). Similarly, if $\min\{u'(t) : t \in [0,T]\} \leq -R$, then there exist t_0, t_1 such that $\varphi(u'(t_0)) = 0$, $\varphi(u'(t_1)) = \varphi(-R)$, $\varphi(-R) < \varphi(u'(t)) < 0$ for all t between t_0 and t_1. Arguing as above leads to a contradiction.

Step 3. Problem (3.4) has at least one solution. For $\lambda \in [0,1]$, we consider the family of boundary value problems

$$\begin{gathered}(\varphi(u'))' = \lambda F(t, u, u') \\ u(0) = u(T) = u'(0).\end{gathered} \tag{3.6}$$

Notice that (3.6) coincides with (3.4) for $\lambda = 1$. So, for each $\lambda \in [0,1]$, the operator associated to (3.6) by Lemma 2.2 is the operator $M(\lambda, \cdot)$, where M is defined on $[0,1] \times C^1$ by

$$\begin{aligned}M(\lambda, u) = \varphi^{-1}&(-Q_\varphi(\lambda H(N_F(u)))) \\ &+ H(\varphi^{-1}[\lambda H(N_F(u)) - Q_\varphi(\lambda H(N_F(u)))]).\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}M(1, u) &= M_F(u) \\ &= \varphi^{-1}(-Q_\varphi(H(N_F(u)))) + H(\varphi^{-1}[H(N_F(u)) - Q_\varphi(H(N_F(u)))]).\end{aligned}$$

On the other hand, we let $(\lambda, u) \in [0, T] \times C^1$ be such that $u = M(\lambda, u)$. Then

$$\varphi(u') = [\lambda H(N_F(u)) - Q_\varphi(\lambda H(N_F(u)))], \tag{3.8}$$

where

$$\begin{aligned}|\lambda H(N_F(u))(t)| &\leq \int_0^T \big|f(s, \gamma(s, u(s)), Q(u'(s))) + \frac{u(s) - \gamma(s, u(s))}{1 + |u(s) - \gamma(s, u(s))|}\big|ds \\ &\leq \int_0^T |f(s, \gamma(s, u(s)), Q(u'(s)))|ds + T \\ &\leq \int_0^T |f(s, \gamma(s, u(s), Q(u'(s)))|ds + T \\ &\leq \sigma T + T,\end{aligned}$$

with $\sigma := \sup_{s\in[0,T]} |f(s, \gamma(s, u(s), Q(u'(s))))|$. Using (3.8), we have

$$|\varphi(u'(t))| \leq 2(\sigma T + T) := \delta \quad (t \in [0,T]), \tag{3.9}$$

and hence

$$\|u'\|_\infty \leq \omega, \tag{3.10}$$

where $\omega = \max\{|\varphi^{-1}(\delta)|, |\varphi^{-1}(-\delta)|\}$. Because $u \in C^1$ is such that $u(0) = u'(0)$, we have

$$|u(t)| \leq |u(0)| + \int_0^T |u'(s)|ds \leq \omega + T\omega \quad (t \in [0,T]),$$

and hence

$$\|u\|_1 = \|u\|_\infty + \|u'\|_\infty \leq \omega + T\omega + \omega = \omega(2 + T).$$

Let M be the operator given by (3.7) and let $\rho > \omega(2+T)$. Then, for each $\lambda \in [0,T]$, the Leray-Schauder degree $\deg_{LS}(I - M(\lambda,\cdot), B_\rho(0), 0)$ is well defined, and by the homotopy invariance, one has

$$\deg_{LS}(I - M(0,\cdot), B_\rho(0), 0) = \deg_{LS}(I - M(1,\cdot), B_\rho(0), 0).$$

On the other hand,

$$\deg_{LS}(I - M(0,\cdot), B_\rho(0), 0) = \deg_{LS}(I, B_\rho(0), 0) = 1.$$

Hence, there exists $u \in B_\rho(0)$ such that $M_F(u) = u$, which is a solution of (3.4). $\square$

Remark 3.3. If α and β in Theorem 3.2 are strict, then $\alpha(t) < u(t) < \beta(t)$ for all for all $t \in [0,T]$. If ρ is large enough, then, using that $\deg_{LS}(I - M_F, B_\rho(0), 0) = 1$ and the additivity-excision property of the Leray-Schauder degree, we obtain that

$$\deg_{LS}(I - M_F, B_\rho(0), 0) = \deg_{LS}(I - M_F, \Omega_{\alpha,\beta}, 0) = 1,$$

where $\Omega_{\alpha,\beta} := \{u \in C^1 : \alpha < u < \beta\}$.

Now let us give an application of Theorem 3.2.

Example 3.4. Consider the problem

$$\begin{gathered}(\varphi(u'))' = \frac{(u'^3+1)\sin(\pi u' + (t+T) - u)}{1+u^2u'^2}\\ u(0) = u(T) = u'(0),\end{gathered}\tag{3.11}$$

where $\varphi(s) = s^3$. It is not difficult to verify that φ is an increasing homeomorphism. For $T > 1$ we consider the functions $\alpha(t) = -t - T$ and $\beta(t) = t + T$ as lower and upper solutions for (3.11), respectively,

$$f(t,x,y) = \frac{(y^3+1)\sin(\pi y + (t+T) - x)}{1+x^2y^2}$$

is a continuous function such that

$$|f(t,x,y)| \le |y|^3 + 1, \quad (t,x,y) \in [0,T]\times\mathbb{R}\times\mathbb{R}.$$

Let $R > 0$, and let $\psi(s) = |s|^3 + 1$. One has

$$\begin{gathered}\int_{-\infty}^{\infty} \frac{ds}{\psi(|\varphi^{-1}(s)|)} = \int_{-\infty}^{\infty} \frac{ds}{1+|s|} = \infty,\\ \int_0^{\varphi(R)} \frac{ds}{\psi(|\varphi^{-1}(s)|)} = \int_{\varphi(-R)}^{0} \frac{ds}{\psi(|\varphi^{-1}(s)|)} = \ln(1+R^3).\end{gathered}$$

So, we can choose $R > 0$ and $T < \ln(1+R^3)$ to see Theorem 3.2. Thus, we obtain that (3.11) has at least one solution.

The proof of the following existence theorem can be found in [8].

Theorem 3.5. *Suppose that* (1.1) *has a lower solution α and an upper solution β such that $\alpha(t) \le \beta(t)$ for all $t \in [0,T]$. If there exists a continuous function $g(t,x)$ on $[0,T]\times\mathbb{R}$ such that*

$$|f(t,x,y)| \le |g(t,x)|, \quad \text{for all } (t,x,y) \in [0,T]\times\mathbb{R}\times\mathbb{R}, \tag{3.12}$$

then (1.1) *has a solution u such that $\alpha(t) \le u(t) \le \beta(t)$ for all $t \in [0,T]$.*

Proof. The proof is based on two steps which are analogous to the proof of the Theorem 3.2.

Step 1. We show that if u is a solution of (3.4) with $F(t,x,y) = f(t,\gamma(t,x),y) + \frac{x-\gamma(t,x)}{1+|x-\gamma(t,x)|}$, then $\alpha(t) \le u(t) \le \beta(t)$ for all $t \in [0,T]$ and hence u is a solution of (1.1).

Step 2. We show that the problem (3.4) has at least one solution. □

Corollary 3.6. *Let $f(t,x,y) = f(t,x)$ be a continuous function. If (1.1) has a lower solution α and a upper solution β such that $\alpha(t) \le \beta(t)$ for all $t \in [0,T]$, then problem (1.1) has a solution such that $\alpha(t) \le u(t) \le \beta(t)$ for all $t \in [0,T]$.*

4. Multiplicity result

In this section, we establish the existence of at least three solutions to problem (1.1).

Theorem 4.1. *Assume that the following conditions are satisfied:*

(i) *For $i = 1,2$, there exist α_i, β_i, respectively strict lower and upper solutions of (1.1), such that $\alpha_i < \beta_i$, $\alpha_1(t) \le \alpha_2(t)$, $\beta_1(t) \le \beta_2(t)$ for all $t \in [0,T]$, and $\{t \in [0,T] : \alpha_2(t) > \beta_1(t)\} \neq \emptyset$.*

(ii) *There exists $\psi : [0,\infty) \to (0,\infty)$ such that*
$$\int_{-\infty}^{\infty} \frac{ds}{\psi(|\varphi^{-1}(s)|)} = \infty.$$

(iii) *Let $R > \max\{\|\alpha_i'\|_\infty, \|\beta_i'\|_\infty\}$, and let*
$$E = \{(t,x,y) : t \in [0,T], \alpha_1(t) \le x \le \beta_2(t), |y| \le R\}.$$
Suppose that $f(t,x,y)$ satisfies
$$|f(t,x,y)| \le \psi(|y|) \tag{4.1}$$
over E, and ψ is such that
$$\min\Big\{ \int_0^{\varphi(R)} \frac{ds}{\psi(|\varphi^{-1}(s)|)}, \int_{\varphi(-R)}^{0} \frac{ds}{\psi(|\varphi^{-1}(s)|)} \Big\} > T. \tag{4.2}$$

Then (3.4) has at least three solutions u_1, u_2, u_3 such that
$$\alpha_1 < u_3 < \beta_2, \quad \alpha_i < u_i < \beta_i, \quad i = 1,2,$$
$$\|u_i'\|_\infty < R \quad i = 1,2,3.$$

Proof. Let γ_1, γ_2, and γ_3 be the functions associated to the pairs of lower and upper solutions (α_1,β_1), (α_2,β_2), (α_1,β_2), respectively. Consider $M_{F_1}, M_{F_2}, M_{F_3}$, the operators associated to the pairs (α_1,β_1), (α_2,β_2), (α_1,β_2), respectively. Using Theorem 3.2, we deduce that there exist $B_{\rho_1}(0), B_{\rho_2}(0)$, and $B_{\rho_3}(0)$, respectively, such that M_{F_i} has no fixed points in $B_{\rho_i}(0) \setminus \overline{\Omega_i}$, with
$$\begin{aligned}
\Omega_1 = \Omega_{\alpha_1,\beta_1} &:= \{u \in C^1 : \alpha_1 < u < \beta_1\},\\
\Omega_2 = \Omega_{\alpha_2,\beta_2} &:= \{u \in C^1 : \alpha_2 < u < \beta_2\},\\
\Omega_3 = \Omega_{\alpha_1,\beta_2} &:= \{u \in C^1 : \alpha_1 < u < \beta_2\}.
\end{aligned}$$
Hence, by Remark 3.3, we have
$$\deg_{LS}(I - M_{F_1}, \Omega_1, 0) = 1,$$

$$\begin{aligned}
&\deg_{LS}(I - M_{F_2}, \Omega_2, 0) = 1, \\
&\deg_{LS}(I - M_{F_3}, \Omega_3, 0) = 1.
\end{aligned}$$

Since $\alpha_1(t) \leq \beta_1(t) \leq \beta_2(t)$, $\alpha_1(t) \leq \alpha_2(t) \leq \beta_2(t)$ for all $t \in [0,T]$, and $\{t \in [0,T] : \alpha_2(t) > \beta_1(t)\} \neq \emptyset$, one has

$$\begin{aligned}
&\Omega_1 \cup \Omega_2 \subset \Omega_3, \\
&\Omega_3 \setminus \overline{\Omega_1 \cup \Omega_2} \neq \emptyset.
\end{aligned}$$

Moreover, $M_{F_i}(u) = M_{F_3}(u)$ for all $u \in \Omega_i$ and $i = 1, 2$. Thus, using the additivity property of Leray-Schauder degree implies that

$$\begin{aligned}
&\deg_{LS}(I - M_{F_3}, \Omega_3 \setminus \overline{\Omega_1 \cup \Omega_2}, 0) \\
&= \deg_{LS}(I - M_{F_3}, \Omega_3, 0) - \deg_{LS}(I - M_{F_2}, \Omega_2, 0) - \deg_{LS}(I - M_{F_1}, \Omega_1, 0) = -1.
\end{aligned}$$

Then problem (3.4) has at least three solutions u_1, u_2, u_3 such that

$$\alpha_1(t) < u_3(t) < \beta_2(t), \quad \alpha_i(t) < u_i(t) < \beta_i(t),$$

for all $t \in [0,T]$ and $i = 1, 2$. Moreover, $\|u_i'\|_\infty < R$ $i = 1, 2, 3$. $\square$

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