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MONOTONE ITERATIVE METHOD FOR RETARDED EVOLUTION EQUATIONS INVOLVING NONLOCAL AND IMPULSIVE CONDITIONS

XUPING ZHANG, PENGYU CHEN, YONGXIANG LI

Abstract. In this article, we apply the perturbation technique and monotone iterative method in the presence of the lower and the upper solutions to discuss the existence of the minimal and maximal mild solutions to the retarded evolution equations involving nonlocal and impulsive conditions in an ordered Banach space X

$$\begin{aligned} u'(t)+Au(t)&=f(t,u(t),u_t), \quad t\in[0,a],\ t\neq t_k,\\ u(t_k^+)&=u(t_k^-)+I_k(u(t_k)), \quad k=1,2,\dots,m,\\ u(s)&=g(u)(s)+\varphi(s), \quad s\in[-r,0], \end{aligned}$$

where $A:D(A)\subset X\to X$ is a closed linear operator and $-A$ generates a strongly continuous semigroup $T(t)$ ($t\geq 0$) on X, a, $r>0$ are two constants, $f:[0,a]\times X\times C_0\to X$ is Carathéodory continuous, $0<t_1<t_2<\dots<t_m<a$ are pre-fixed numbers, $I_k\in C(X,X)$ for $k=1,2,\dots,m$, $\varphi\in C_0$ is a priori given history, while the function $g:C_a\to C_0$ implicitly defines a complementary history, chosen by the system itself. Under suitable monotonicity conditions and noncompactness measure conditions, we obtain the existence of the minimal and maximal mild solutions, the existence of at least one mild solutions as well as the uniqueness of mild solution between the lower and the upper solutions. An example is given to illustrate the feasibility of our theoretical results.

1. Introduction

Let X be a real Banach space with norm $\|\cdot\|$, and let $a,r>0$ be two constants. For every $t\in[0,a]$, we denote by $C_t:=C([-r,t],X)$ the Banach space of all the continuous functions from $[-r,t]$ into X endowed with the sup-norm $\|u\|_{C_t}=\sup_{-r\leq s\leq t}\|u(s)\|$.

In this article, we use the perturbation technique and monotone iterative method in the presence of the lower and the upper solutions to discuss the existence of the existence of the minimal and the maximal mild solutions, the existence of at least one mild solutions as well as the uniqueness of the mild solution to the retarded evolution equations involving nonlocal and impulsive conditions in an ordered Banach

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space X,

$$\begin{aligned} u'(t)+Au(t)&=f(t,u(t),u_t), \quad t\in[0,a], \quad t\neq t_k,\\ u(t_k^+)&=u(t_k^-)+I_k(u(t_k)), \quad k=1,2,\dots,m,\\ u(s)&=g(u)(s)+\varphi(s), \quad s\in[-r,0], \end{aligned} \tag{1.1}$$

where state $u(\cdot)$ takes values in the Banach space X endowed with norm $\|\cdot\|$; $A: D(A)\subset X\to X$ is a closed linear operator and $-A$ generates a strongly continuous semigroup $T(t)$ $(t\geq 0)$ on X; $f:[0,a]\times X\times C_0\to X$ is a Carathéodory continuous nonlinear function; $0<t_1<t_2<\dots<t_m<a$ are pre-fixed numbers, $I_k\in C(X,X)$ is an impulsive function, $k=1,2,\dots,m$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and the left limits of $u(t)$ at $t=t_k$, respectively; $\varphi\in C_0$ is a priori given history, while the function $g:C_a\to C_0$ implicitly defines a complementary history, chosen by the system itself; u_t denotes the function in C_0 defined as $u_t(\tau)=u(t+\tau)$ for $\tau\in[-r,0]$ and $u_t(\cdot)$ represent the time history of the state from the time $t-r$ up to the present time t.

The study of abstract nonlocal Cauchy problem was initiated by Byszewski [11]. It is demonstrated that the nonlocal problems have better effects in applications than the traditional Cauchy problems, differential equations with nonlocal conditions were studied by many authors and some basic results on nonlocal problems have been obtained, see [3, 12, 16, 24, 26, 37, 45, 48, 49] and the references therein.

It is a well-established idea to model the evolution of some physical, biological and economic systems using delay evolution equations, in which the response of the system depends not only on the current state of the system, but also on the past history of the system. The history function g, which defines the initial nonlocal condition, is in fact a feedback operator which adjusts a part of the past — if the local initial condition φ is present — or even the whole past — if φ is absent — according to some precise future requirements. Namely, if we wish that the system behavior in the future be described by a function u belonging to a certain class, then it is left at the system disposal to choose its past wherefrom to start its evolution, in the nonlocal implicit form $u(t)=g(u)(t)+\varphi(t)$ for each $t\in[-r,0]$. For more details on this topic, see for example, the books of Hale and Verduyn Lunel [30], Kolmanovskii and Myshkis [35] and the papers of Chuong and Ke [18], Kamaljeet and Bahuguna [34] and Travis and webb [43].

The dynamics of evolving processes is often subjected to abrupt changes at certain moments such as shocks, harvesting and natural disasters. Often these short term perturbations are treated as having acted instantaneously or in the form of impulses. One of the emerging branches of the study associated with impulses is the theory of impulsive differential equations, which describes processes which experience a sudden change in their states at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical, biological, population and dynamics, engineering and economics. The theory of impulsive differential equations has emerged as an important area of research in the previous decades. For more details on differential equations with impulses, one can see the monographs of Benchohra, Henderson and Ntouyas [9], Lakshmikantham, Bainov and Simeonov [36] and the papers of Benchohra and Hammouche [1], Ahmed [2], Abada, Barreira and Valls [8], Bonottoa et al. [10], Guo and Liu [29], Li and Liu [39] and Qian, Chen and Sun [42], where numerous properties of their solutions are studied and detailed bibliographies are given. Particularly, there has been a significant development in the theory of impulsive evolution equations with nonlocal conditions in Banach spaces. Liang, Liu and Xiao [37, 38] combined impulsive conditions and nonlocal conditions, and investigated the nonlocal impulsive evolution equation in Banach spaces. Later, Balachandran, Kiruthika and Trujillo [5], Chen, Li and Yang [17], Debbouche and Baleanu [19], Fan and Li [26], Wang and Wei [47], Yan [50] studied the impulsive evolution equation with nonlocal conditions. Moreover, Abada, Benchohra and Hammouche [1], Cardindi and Rubbioni [13], Chang, Anguraj and Karthikeyan [14], Chuong and Ke [18], Fu and Cao [27], Ji and Li [33] studied the impulsive evolution inclusion with nonlocal conditions.

We mention that in 2012, Chuong and Ke [18] studied the retarded evolution inclusions involving nonlocal and impulsive conditions

$$\begin{gathered}
u'(t)+Au(t)\in F(t,u(t),u_t), \quad t\in[0,a],\ \ t\neq t_k,\\
u(t_k^+)=u(t_k^-)+I_k(u(t_k)), \quad k=1,2,\dots,m,\\
u(s)+g(u)(s)=\varphi(s), \quad s\in[-r,0],
\end{gathered}\tag{1.2}$$

where X is a Banach space, $F:[0,a]\times X\times C_0\to P(X)$ is a multivalued map, $P(X)$ stands for the collection of all nonempty subsets of X. A is a closed linear operator on X. By using the fixed point theory for multi-valued maps and the theory of differential inclusions, the authors obtain the existence of mild solutions for nonlocal problem (1.2). Furthermore, by applying corresponding measure of noncompactness estimates, they also proved the continuity of the solution map, which demonstrates that the solution set depends continuously on initial data.

But so far we have not seen relevant papers that study delay evolution equations involving nonlocal and impulsive conditions by applying the iterative method, perturbation technique and the method of lower and upper solutions. It is well known that the monotone iterative technique in the presence of the lower and the upper solutions is an important method for seeking solutions of differential equations in abstract spaces. The most advantage by using the iterative method based on lower and upper solutions is that it not only provides a method to obtain the existence of extremal mild solutions, but also yields iterative sequences of lower and upper approximate solutions that converge to the minimal and maximal mild solutions between the lower and upper solutions. The iterative sequences are very useful in numerical calculation, which provide a computing rule in computer simulation. Early on, Du and Lakshmikantham [21] investigated the existence of extremal solutions to the initial value problem of ordinary differential equations without impulse by using the method of the lower and the upper solutions coupled with the monotone iterative technique. Latter, Guo and Liu [29], Li and Liu [39] developed the iterative method for ordinary differential equations with instantaneous impulses in Banach spaces. Recently, the iterative method has been extended to evolution equations in ordered Banach spaces, we refer to the papers by EI-Gebeily, O'Regan and Nieto [23] and Wang and Wang [46] for evolution equations with classical initial vlaue conditions, and to the paper by Chen and Li [15] and Chen, Li and Yang [17] for evolution equations with impulses in Banach spaces.

Inspired by the above-mentioned aspects, in the present paper we will combine these earlier works and extend the study to the retarded evolution equations involving nonlocal and impulsive conditions (1.1), which is more general than those in many previous publications. By combining the theory of semigroups of linear operators and the method of the lower and the upper solutions coupled with the monotone iterative technique, we construct two monotone iterative sequences, and prove that the sequences monotonically converge to the minimal and the maximal mild solutions of problem (1.1), respectively, under the reasonably weak conditions on the semigroup $T(t)$ $(t \geq 0)$, nonlinear function f, impulsive function I_k for $k = 1, 2, \dots, m$ and nonlocal term g.

The work of this paper has three wedges: firstly, we will extend the study of impulsive Cauchy problems with nonlocal initial conditions to retarded evolution equations involving nonlocal and impulsive conditions. Secondly, we will obtain the existence theorems of minimal and the maximal mild solutions and the uniqueness theorems of mild solution to the nonlocal problem (1.1) under weaker conditions on the semigroup $T(t)$ $(t \geq 0)$, nonlinear function f, impulsive function I_k for $k = 1, 2, \dots, m$ and nonlocal term g. Lastly, the perturbation technique and monotone iterative method are extended to study the retarded evolution equations with nonlocal and impulsive conditions in ordered Banach space. Moreover, even for corresponding retarded impulsive Cauchy problems without nonlocal initial conditions, the results here are new.

Remark 1.1. As the reader can see, the hypotheses on nonlinear function f, impulsive function I_k $(k = 1, 2, \dots, m)$ and nonlocal term g in our theorems are reasonably weak and different from those in many previous papers such as [5, 19, 26, 47, 50]. Furthermore, the techniques in the proofs of our theorems are essentially different from those used in [18].

The outline of this paper is as follows. In section 2, notation and preliminaries are introduced, which are used throughout this paper. In Section 3, we obtained the existence of extremal mild solutions as well as mild solutions for nonlocal problem (1.1) under the situation that the semigroup $T(t)$ $(t \geq 0)$ and the nonlocal function g are compact. The existence of extremal mild solutions and the uniqueness of mild solution for nonlocal problem (1.1) are obtained under the situation that the semigroup $T(t)$ $(t \geq 0)$ and nonlocal function g are not compact in Section 4. Finally, an concrete example is given to illustrate the feasibility of our abstract results.

2. Preliminaries

We begin by giving some notation. Let X be an ordered Banach space with norm $\|\cdot\|$ and partial order " $\leq$ " with positive cone $P = \{u \in X \mid u \geq \theta\}$ (θ is the zero element of X), which defines a partial ordering in X by $u \leq v$ if and only if $v - u \in P$. If $u \leq v$ and $u \neq v$, we say $u < v$. The cone P is called normal if there exists a positive constant N such that $\theta \leq u \leq v$ implies $\|u\| \leq N\|v\|$, in which N is called normal constant. Evidently, for any $t \in [0, a]$, $C_t = ([-r, t], X)$ is also an ordered Banach space whose partial order " $\leq$ " induced by the positive cone $P_{C_t} := \{u \in C_t \mid u(s) \geq \theta, s \in [-r, t]\}$, and P_{C_t} is also a normal cone with the same normal constant N. Denote by

$$
\begin{aligned}
PC([-r, a], X) = \big\{&u : [-r, a] \to X : u \text{ is continuous for } t \neq t_k, \\
&\text{left continuous at } t = t_k \text{ and } u(t_k^+) \text{ exists for } k = 1, 2, \dots, m\big\}
\end{aligned}
$$

be a piecewise continuous function space. It is easy to see that $PC([-r,a],X)$ is a Banach space endowed with the supremum-norm

$$\|u\|_{PC} = \sup_{t\in[-r,a]} \|u(t)\|, \quad \forall u \in PC([-r,a],X). \tag{2.1}$$

Evidently, $PC([-r,a],X)$ is also an ordered Banach space with the partial order " $\leq$ " induced by the positive cone $K_{PC} = \{u \in PC([-r,a],X) \mid u(t) \geq \theta,\ t \in [-r,a]\}$. K_{PC} is also a normal cone with the same normal constant N. For v, $w \in PC([-r,a],X)$ with $v \leq w$, we use $[v,w]$ to denote the order interval

$$\{u \in PC([-r,a],X) \mid v \leq u \leq w\}$$

in $PC([-r,a],X)$, and $[v(t),w(t)]$ to denote the order interval

$$\{u \in PC([-r,a],X) \mid v(t) \leq u(t) \leq w(t),\, t \in [-r,a]\}$$

in X.

In the following, we denote $J_0 = [-r,0]$, $J_1 = [0,t_1]$, $J_k = (t_{k-1},t_k]$, $k = 2,\dots,m+1$, $t_{m+1} = a$, $I' = [-r,a]\backslash\{t_1,t_2,\dots,t_m\}$ and $I'' = [-r,a]\backslash\{0,t_1,t_2,\dots,t_m\}$, and use X_1 to denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\|+\|A\cdot\|$. An abstract function $u \in PC([-r,a],X)\cap C^1(I'',X)\cap C(I',X_1)$ is called a solution of nonlocal problem (1.1) if $u(t)$ satisfies all the equalities in (1.1).

Definition 2.1. If a function $u \in PC([-r,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ satisfies

$$\begin{gathered} u'(t) + Au(t) \leq f(t,u(t),u_t), \quad t \in [0,a], \ \ t \neq t_k, \\ u(t_k^+) \leq u(t_k^-) + I_k(u(t_k)), \quad k = 1,2,\dots,m, \\ u(s) \leq g(u)(s) + \varphi(s), \quad s \in [-r,0], \end{gathered} \tag{2.2}$$

we call it a lower solution of nonlocal problem (1.1); if all the inequalities in (2.2) are reversed, we call it an upper solution of nonlocal problem (1.1).

Definition 2.2. A function $f : [0,a] \times X \times C_0 \to X$ is said to be Carathéodory continuous provided that

(i) for all $(u,v) \in X \times C_0$, $f(\cdot,u,v) : [0,a] \to X$ is measurable,
(ii) for a.e. $t \in [0,a]$, $f(t,\cdot,\cdot) : X \times C_0 \to X$ is continuous.

Throughout this paper, let $A : D(A) \subset X \to X$ be a closed linear operator and let $-A$ generate a strongly continuous semigroup $T(t)$ $(t \geq 0)$ on ordered Banach space X. Then there exist constants $C_1 \geq 1$ and $\delta \in \mathbb{R}$ such that

$$\|T(t)\| \leq C_1 e^{\delta t}, \quad t \geq 0. \tag{2.3}$$

Denote $\mathcal{L}(X)$ be the Banach space of all bounded linear operators from X to X equipped with its natural topology. From (2.3) we know that

$$C := \sup_{t\in[0,a]} \|T(t)\|_{\mathcal{L}(X)} \geq 1 \tag{2.4}$$

is a finite number.

Definition 2.3. A function $u \in PC([-r,a],X)$ is said to be a mild solution of nonlocal problem (1.1) if it satisfies the following equation

$$u(t) = \begin{cases} g(u)(t) + \varphi(t), & t \in [-r,0]; \\ T(t)[g(u)(0) + \varphi(0)] + \int_0^t T(t-s)f(s,u(s),u_s)ds, & \\ + \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k)), & t \in [0,a]. \end{cases} \tag{2.5}$$

Definition 2.4. A strongly continuous semigroup $T(t)$ $(t \geq 0)$ in X is said to be compact, if $T(t)$ is a compact operator in X for every $t > 0$.

Definition 2.5. A strongly continuous semigroup $T(t)$ $(t \geq 0)$ in X is said to be equicontinuous, if $T(t)$ is continuous in the operator norm for every $t > 0$.

Definition 2.6. A strongly continuous semigroup $T(t)$ $(t \geq 0)$ in X is said to be positive, if order inequality $T(t)u \geq \theta$ holds for each $u \geq \theta$, $u \in X$ and $t \geq 0$.

One can easily to see that for any constant $M \geq 0$, $-(A + MI)$ also generates a strongly continuous semigroup $S(t) = e^{-Mt}T(t)$ $(t \geq 0)$ in X, and

$$\sup_{t\in[0,a]} \|S(t)\|_{\mathcal{L}(X)} = \sup_{t\in[0,a]} \|e^{-Mt}T(t)\|_{\mathcal{L}(X)} = C \geq 1. \tag{2.6}$$

Therefore, $S(t)$ $(t \geq 0)$ is a positive strongly continuous semigroup if $T(t)$ $(t \geq 0)$ is a positive strongly continuous semigroup, $S(t)$ $(t \geq 0)$ is a compact semigroup if $T(t)$ $(t \geq 0)$ is a compact semigroup, $S(t)$ $(t \geq 0)$ is an equicontinuous semigroup if $T(t)$ $(t \geq 0)$ is an equicontinuous semigroup. For more details about the properties of the operator semigroups and positive strongly continuous semigroup, we refer to Banasiak and Arlotti [7], Henry [32], Pazy [41] and Vrabie [44].

Next, we recall some basic definitions and properties about Kuratowski measure of noncompactness that will be used in the proof of our main results.

Definition 2.7 ([20, 6])**.** The Kuratowski measure of noncompactness $\alpha(\cdot)$ defined on bounded set S of Banach space X is

$$\alpha(S) := \inf\{\delta > 0 : S = \cup_{i=1}^{m} S_i \text{ with } \operatorname{diam}(S_i) \leq \delta \text{ for } i = 1, 2, \dots, m\}. \tag{2.7}$$

It is easy to know from Definition 2.7 that $0 \leq \alpha(S) < \infty$. The following properties about the Kuratowski measure of noncompactness are well known.

Lemma 2.8 ([6, 20])**.** *Let E be a Banach space and S, $U \subset E$ be bounded. The following properties are satisfied:*

(i) $\alpha(S) = 0$ *if and only if* $\overline{S}$ *is compact, where* $\overline{S}$ *means the closure hull of* S*;*
(ii) $\alpha(S) = \alpha(\overline{S}) = \alpha(\operatorname{conv} S)$*, where* $\operatorname{conv} S$ *means the convex hull of* S*;*
(iii) $\alpha(\lambda S) = |\lambda|\alpha(S)$ *for any* $\lambda \in \mathbb{R}$*;*
(iv) $S \subset U$ *implies* $\alpha(S) \leq \alpha(U)$*;*
(v) $\alpha(S \cup U) = \max\{\alpha(S), \alpha(U)\}$*;*
(vi) $\alpha(S + U) \leq \alpha(S) + \alpha(U)$*, where* $S + U = \{x \mid x = y + z, y \in S, z \in U\}$*;*
(vii) *If the map* $Q : \mathcal{D}(Q) \subset E \to X$ *is Lipschitz continuous with constant* k*, then* $\alpha(Q(V)) \leq k\alpha(V)$ *for any bounded subset* $V \subset \mathcal{D}(Q)$*, where* X *is another Banach space.*

To introduce the useful lemmas which will be used in our argument, we use $\alpha(\cdot)$ and $\alpha_{PC}(\cdot)$ to denote the Kuratowski measure of noncompactness on the bounded set of X and $PC([-r, a], X)$, respectively. For any $D \subset PC([-r, a], X)$ and $t \in [0, a]$, set $D(t) = \{u(t) \mid u \in D\}$ and $D_t = \{u_t \mid u \in D\}$, then $D(t) \subset X$ and $D_t \subset C_0$. If $D \subset PC([-r, a], X)$ is bounded, then for any $t \in [0, a]$, $D(t) \subset X$ and $D_t \subset C_0$ are bounded, and $\alpha(D(t)) \leq \alpha_{PC}(D)$. For details about the definition and properties of the measure of noncompactness, we refer to the monographs by Ayerbe, Domínguez and López [4], Banaś and Goebel [6], Deimling [20].

Now, we give the following lemmas about the measure of noncompactness which are used further in this paper.

Lemma 2.9 ([29])**.** *Let E be a Banach space. Assume that $\Omega \subset E$ is a bounded closed and convex set on E, the operator $Q : \Omega \to \Omega$ is k-set-contractive. Then Q has at least one fixed point in Ω.*

Lemma 2.10 ([31])**.** *Let X be a Banach space, and $D = \{u_n\}_{n=1}^{\infty} \subset PC([0,a], X)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integrable on $[0,a]$, and*

$$\alpha\Big(\Big\{\int_0^a u_n(t)dt \mid n \in \mathbb{N}\Big\}\Big) \le 2\int_0^a \alpha(D(t))dt.$$

Lemma 2.11 ([16])**.** *Let X be a Banach space, and let $D \subset X$ be bounded. Then there exists a countable set $D^* \subset D$, such that*

$$\alpha(D) \le 2\alpha(D^*).$$

Lemma 2.12 ([28])**.** *Let P be a normal cone of the ordered Banach space X and v_0, $w_0 \in X$ with $v_0 \le w_0$. Suppose that $Q : [v_0, w_0] \to X$ is a nondecreasing strict set-contraction operator such that $v_0 \le Qv_0$ and $Qw_0 \le w_0$. Then Q has a minimal fixed point $\underline{u}$ and a maximal fixed point $\overline{u}$ in $[v_0, w_0]$; moreover,*

$$v_n \to \underline{u} \quad \text{and} \quad w_n \to \overline{u} \quad \text{as } n \to \infty,$$

where $v_n = Qv_{n-1}$ and $w_n = Qw_{n-1}\,(n = 1, 2, \dots)$ which satisfy

$$v_0 \le v_1 \le \dots \le v_n \le \dots \le \underline{u} \le \overline{u} \le \dots \le w_n \le \dots \le w_1 \le w_0.$$

3. Case g compact

In this section, we discuss the existence of extremal mild solutions as well as the existence of at least one mild solutions for nonlocal problem (1.1) under the situation that $T(t)$ $(t \ge 0)$ is a compact strongly continuous semigroup and the nonlocal function $g : PC([-r, a], X) \to C_0$ is also compact.

Theorem 3.1. *Let X be an ordered Banach space, whose positive cone P is normal, and let $A : D(A) \subset X \to X$ be a closed linear operator and the positive strongly continuous semigroup $T(t)$ $(t \ge 0)$ generated by $-A$ be compact on X. Assume that the nonlinear function $f : [0, a] \times X \times C_0 \to X$ is Carathéodory continuous, $I_k \in C(X, X)$ for $k = 1, 2, \dots, m$ and the nonlocal function $g : PC([-r, a], X) \to C_0$ is continuous and compact as well as the nonlocal problem (1.1) has a lower solution $v^{(0)} \in PC([-r, a], X) \cap C^1(I'', X) \cap C(I', X_1)$ and an upper solution $w^{(0)} \in PC([-r, a], X) \cap C^1(I'', X) \cap C(I', X_1)$ with $v^{(0)} \le w^{(0)}$. Suppose also that the following conditions are satisfied:*

- (H1) *There exists a constant $M > 0$ such that*
$$f(t, u_2, v_2) - f(t, u_1, v_1) \ge -M(u_2 - u_1),$$
for $\forall\, t \in [0, a]$, $u_1, u_2 \in X$ and v_1, $v_2 \in C_0$ with $v^{(0)}(t) \le u_1 \le u_2 \le w^{(0)}(t)$ and $(v^{(0)})_t \le v_1 \le v_2 \le (w^{(0)})_t$;
- (H2) *The impulsive function $I_k(\cdot)$ satisfies*
$$I_k(u_1) \le I_k(u_2), \ k = 1, 2, \dots, m,$$
for any $t \in [0, a]$, and $v^{(0)}(t) \le u_1 \le u_2 \le w^{(0)}(t)$;
- (H3) *The nonlocal function $g(u)$ is increasing on order interval $[v^{(0)}, w^{(0)}]$.*

Then nonlocal problem (1.1) has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ between $v^{(0)}$ and $w^{(0)}$, which can be obtained by a monotone iterative procedure starting from $v^{(0)}$ and $w^{(0)}$, respectively.

Proof. It is easy to see that nonlocal problem (1.1) is equivalent to the following retarded evolution equations involving nonlocal and impulsive conditions

$$
\begin{gathered}
u'(t)+Au(t)+Mu(t)=f(t,u(t),u_t)+Mu(t), \quad t\in[0,a],\ \ t\neq t_k,\\
u(t_k^+)=u(t_k^-)+I_k(u(t_k)), \quad k=1,2,\dots,m,\\
u(s)=g(u)(s)+\varphi(s), \quad s\in[-r,0],
\end{gathered}
\tag{3.1}
$$

for any constant $M>0$. We consider the operator $\mathcal{Q}:\ [v^{(0)},w^{(0)}]\to PC([-r,a],X)$ defined by

$$
(\mathcal{Q}u)(t)=\begin{cases}
g(u)(t)+\varphi(t), & t\in[-r,0];\\
S(t)[g(u)(0)+\varphi(0)] & \\
+\int_0^t S(t-s)[f(s,u(s),u_s)+Mu(s)]ds & \\
+\sum_{0<t_k<t}S(t-t_k)I_k(u(t_k)), & t\in[0,a],
\end{cases}
\tag{3.2}
$$

where $S(t)=e^{-Mt}T(t)$ $(t\geq 0)$ is the strongly continuous semigroup generated by $-(A+MI)$. By Definition 2.3, we know that the mild solution of nonlocal problem (1.1) is equivalent to the fixed point of operator $\mathcal{Q}$ defined by (3.2).

At first, we prove that the operator $\mathcal{Q}:\ [v^{(0)},w^{(0)}]\to PC([-r,a],X)$ defined by (3.2) is continuous. For this purpose, let $\{u^{(n)}\}_{n=1}^{\infty}\subset[v^{(0)},w^{(0)}]$ be a sequence such that $\lim_{n\to\infty}u^{(n)}=u$ in $[v^{(0)},w^{(0)}]$. Then $\lim_{n\to\infty}(u^{(n)})_t=u_t$. If $t\in[-r,0]$, by (3.2) and the continuity of the nonlocal function g, we have

$$
\|(\mathcal{Q}u^{(n)})(t)-(\mathcal{Q}u)(t)\|=\|g(u^{(n)})(t)-g(u)(t)\|\to 0 \quad textas\ n\to\infty,
\tag{3.3}
$$

and if $t\in[0,a]$, by the Carathéodory continuity of the nonlinear function f, and the continuity of the impulsive function I_k for $k=1,2,\dots,m$, we get that for a.e. $s\in[0,a]$,

$$
\lim_{n\to\infty}\|f(s,u^{(n)}(s),(u^{(n)})_s)+Mu^{(n)}(s)-f(s,u(s),u_s)-Mu(s)\|=0,
\tag{3.4}
$$

$$
\lim_{n\to\infty}\|I_k(u^{(n)}(t_k))-I_k(u(t_k))\|=0 \quad \text{for } k=1,2,\dots,m.
\tag{3.5}
$$

Applying the condition (H1), we know that for any $u\in[v^{(0)},w^{(0)}]$ and $s\in[0,t]$, $t\in[0,a]$,

$$
\begin{aligned}
f(s,v^{(0)}(s),(v^{(0)})_s)+Mv^{(0)}(s)&\leq f(s,u(s),u_s)+Mu(s)\\
&\leq f(s,w^{(0)}(s),(w^{(0)})_s)+Mw^{(0)}(s).
\end{aligned}
$$

The above inequality combined with the normality of the positive cone P, we know that there exists a constant $M_1>0$, such that

$$
\|f(s,u(s),u_s)+Mu(s)\|\leq M_1, \quad s\in[0,t], \quad t\in[0,a].
\tag{3.6}
$$

Combining with (2.6), (3.2), (3.4)-(3.6) and Lebesgue's dominated convergence theorem, we know that for any $t \in [0,a]$,

$$\begin{aligned} &\|(\mathcal{Q}u^{(n)})(t) - (\mathcal{Q}u)(t)\| \\ &\le C\|g(u^{(n)})(0) - g(u)(0)\| \\ &\quad + C\int_0^t \|f(s,u^{(n)}(s),(u^{(n)})_s) + Mu^{(n)}(s) - f(s,u(s),u_s) - Mu(s)\|ds \\ &\quad + C\sum_{0<t_k<t} \|I_k(u^{(n)}(t_k)) - I_k(u(t_k))\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.7}$$

Hence, from (3.3) and (3.7) we obtain

$$\|\mathcal{Q}u^{(n)} - \mathcal{Q}u\|_{PC} \to 0 \quad (n \to \infty),$$

which means that $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to PC([-r,a], X)$ is a continuous operator.

Secondly, we prove that $\mathcal{Q}$ maps $[v^{(0)}, w^{(0)}]$ to $[v^{(0)}, w^{(0)}]$ is a monotonic increasing operator. Let u, $v \in [v^{(0)}, w^{(0)}]$ with $u \le v$, then $u(t) \le v(t)$ for $t \in [-r,a]$ and $u_t \le v_t$ for $t \in [0,a]$. Since $S(t)$ $(t \ge 0)$ is a positive C_0-semigroup, combining this fact with the assumptions (H1)–(H3), it is easy to prove that

$$\mathcal{Q}u \le \mathcal{Q}v,$$

which means that $\mathcal{Q}$ is an increasing operator in $[v^{(0)}, w^{(0)}]$. Next, we show that $v^{(0)} \le \mathcal{Q}v^{(0)}$ and $\mathcal{Q}w^{(0)} \le w^{(0)}$. By the definition of lower solution, we know that for $t \in [-r,0]$,

$$v^{(0)}(t) \le g(v^{(0)})(t) + \varphi(t) = (\mathcal{Q}v^{(0)})(t). \tag{3.8}$$

Letting $h(t) = (v^{(0)})'(t) + Av^{(0)}(t) + Mv^{(0)}(t)$, $t \in [0,a]$, $t \ne t_k$ for $k = 1,2,\dots,m$. By Definition 2.1, we get that $h \in PC([0,a], X)$ and $h(t) \le f(t, v^{(0)}(t), (v^{(0)})_t) + Mv^{(0)}(t)$ for $t \in [0,a]$. Therefore, by Definitions 2.1 and 2.3 and the positivity of the strongly continuous semigroup $S(t)$ $(t \ge 0)$, we have

$$\begin{aligned} v^{(0)}(t) &= S(t)v^{(0)}(0) + \int_0^t S(t-s)h(s)ds + \sum_{0<t_k<t} S(t-t_k)[v^{(0)}(t_k^+) - v^{(0)}(t_k^-)] \\ &\le S(t)[g(v^{(0)})(0) + \varphi(0)] + \int_0^t S(t-s)[f(s,v^{(0)}(s),(v^{(0)})_s) + Mv^{(0)}(s)]ds \\ &\quad + \sum_{0<t_k<t} S(t-t_k)I_k(v^{(0)}(t_k)) \\ &= (\mathcal{Q}v^{(0)})(t), \quad t \in [0,a]. \end{aligned}$$

The above inequality together with (3.8) imply $v^{(0)} \le \mathcal{Q}v^{(0)}$. Similarly, it can be shown that $\mathcal{Q}w^{(0)} \le w^{(0)}$. Therefore, $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a monotonic increasing operator.

Now, we define two sequences $\{v^{(n)}\}$ and $\{w^{(n)}\}$ in ordered interval $[v^{(0)}, w^{(0)}]$ by the following iterative scheme

$$v^{(n)} = \mathcal{Q}v^{(n-1)}, \quad w^{(n)} = \mathcal{Q}w^{(n-1)}, \quad n = 1,2,\dots. \tag{3.9}$$

Then from the fact that $\mathcal{Q}$ is a monotonic increasing operator, it follows that

$$v^{(0)} \le v^{(1)} \le v^{(2)} \le \dots \le v^{(n)} \le \dots \le w^{(n)} \le \dots \le w^{(2)} \le w^{(1)} \le w^{(0)}. \tag{3.10}$$

In what follows, we prove that $\{v^{(n)}\}$ and $\{w^{(n)}\}$ are convergent on $[-r,a]$. For convenience, let $B=\{v^{(n)} \mid n\in\mathbb{N}\}$ and $B^*=\{v^{(n-1)} \mid n\in\mathbb{N}\}$. Then $B=\mathcal{Q}(B^*)$. From the fact that the nonlocal function $g: PC([-r,a],X)\to C_0$ is a compact map, we know that for $t\in[-r,0]$, the set

$$\{(\mathcal{Q}v^{(n-1)})(t) \mid v^{(n-1)}\in B^*\}=\{g(v^{(n-1)})(t)+\varphi(t) \mid v^{(n-1)}\in B^*\}$$

is precompact in X. For $0<t\le a$ and $v^{(n-1)}\in B_0$, let

$$\begin{aligned}(\mathcal{Q}_1v^{(n-1)})(t)&=S(t)[g(v^{(n-1)})(0)+\varphi(0)]\\&\quad+\int_0^t S(t-s)[f(s,v^{(n-1)}(s),v_s^{(n-1)})+Mv^{(n-1)}(s)]ds,\end{aligned}\tag{3.11}$$

$$(\mathcal{Q}_2v^{(n-1)})(t)=\sum_{0<t_k<t}S(t-t_k)I_k(v^{(n-1)}(t_k)).\tag{3.12}$$

For any $v^{(n-1)}\in B^*$, $s\in[0,t]$, $t\in[0,a]$, by the assumption (H1), we have

$$\begin{aligned}f(s,v^{(0)}(s),(v^{(0)})_s)+Mv^{(0)}(s)&\le f(s,v^{(n-1)}(s),(v^{(n-1)})_s)+Mv^{(n-1)}(s)\\&\le f(s,w^{(0)}(s),(w^{(0)})_s)+Mw^{(0)}(s).\end{aligned}$$

Above inequality together with the normality of cone P, we know that there exists a constant $M_2>0$ such that for any $v^{(n-1)}\in B^*$,

$$\|f(s,v^{(n-1)}(s),(v^{(n-1)})_s)+Mv^{(n-1)}(s)\|\le M_2,\quad s\in[0,t],\ \ t\in[0,a].\tag{3.13}$$

Hence, for $t\in(0,a]$ and $0<\epsilon<t$, the operator

$$\begin{aligned}&(\mathcal{Q}_1^\epsilon v_{n-1})(t)\\&=S(t)[g(v^{(n-1)})(0)+\varphi(0)]\\&\quad+\int_0^{t-\epsilon}S(t-s)[f(s,v^{(n-1)}(s),(v^{(n-1)})_s)+Mv^{(n-1)}(s)]ds\\&=S(t)[g(v^{(n-1)})(0)+\varphi(0)]\\&\quad+S(\epsilon)\int_0^{t-\epsilon}S(t-s-\epsilon)[f(s,v^{(n-1)}(s),(v^{(n-1)})_s)+Mv^{(n-1)}(s)]ds\end{aligned}\tag{3.14}$$

is precompact in X since $S(t)$ is compact for $t>0$. By (2.6), (3.11), (3.13) and (3.14), we obtain

$$\begin{aligned}&\|(\mathcal{Q}_1v^{(n-1)})(t)-\mathcal{Q}_1^\epsilon v^{(n-1)}(t)\|\\&=\int_{t-\epsilon}^t\|S(t-s)[f(s,v^{(n-1)}(s),v_s^{(n-1)})+Mv^{(n-1)}(s)]\|ds\\&\le CM_2\epsilon,\end{aligned}$$

which means that there exists precompact set $\{(\mathcal{Q}_1^\epsilon v^{(n-1)})(t) \mid v^{(n-1)}\in B^*\}$ sufficiently close to the set $\{(\mathcal{Q}_1v^{(n-1)})(t) \mid v^{(n-1)}\in B^*\}$ for every $t\in(0,a]$. Therefore, by the total boundedness we know that for $t\in(0,a]$, the set $\{(\mathcal{Q}_1v^{(n-1)})(t) \mid v^{(n-1)}\in B^*\}$ is precompact in X.

On the other hand, for any $v^{(n-1)}\in B^*$ and $k=1,2,\dots,m$, by the assumption (H2), we have

$$I_k(v^{(0)}(t_k))\le I_k(v^{(n-1)}(t_k))\le I_k(w^{(0)}(t_k)).\tag{3.15}$$

By the normality of the cone P and (3.15), there exists a constant $M_3 > 0$ such that

$$\|I_k(v^{(n-1)}(t_k))\| \le M_3, \quad v^{(n-1)} \in B^*, \quad k = 1, 2, \dots, m.$$

Therefore, $\{(\mathcal{Q}_2 v^{(n-1)})(t) \mid v^{(n-1)} \in B^*\}$ is precompact in X since the operator $S(t)$ is compact for $t > 0$.

Therefore, the set

$$\{v^{(n)}(t)\} = \{(\mathcal{Q}v^{(n-1)})(t) \mid v^{(n-1)} \in B^*\}$$

is precompact in X for any $t \in [-r, a]$. Hence, $\{v^{(n)}(t)\}$ has a convergent subsequence. Combining this with the monotonicity (3.10), we easily prove that $\{v^{(n)}(t)\}$ itself is convergent, i.e.,

$$\lim_{n\to\infty} v^{(n)}(t) = \underline{u}(t) \quad \text{for } t \in [-r, a].$$

Similarly, we can prove

$$\lim_{n\to\infty} w^{(n)}(t) = \overline{u}(t) \quad \text{for } t \in [-r, a].$$

Obviously, $\{v_n(t)\} \subset PC([-r, a], X)$, and $\underline{u}(t)$ is bounded integrable when t belongs to $[-r, 0]$ and $[0, a]$, respectively. For any $t \in [-r, a]$, we know from (3.2) that

$$\begin{aligned} v^{(n)}(t) &= (\mathcal{Q}v^{(n-1)})(t) \\ &= \begin{cases} g(v^{(n-1)})(t) + \varphi(t), & t \in [-r, 0], \\ S(t)[g(v^{(n-1)})(0) + \varphi(0)] \\ + \int_0^t S(t-s)[f(s, v^{(n-1)}(s), (v^{(n-1)})_s) \\ + Mv^{(n-1)}(s)]ds + \sum_{0<t_k<t} S(t-t_k) I_k(v^{(n-1)}(t_k)), & t \in [0, a]. \end{cases} \end{aligned}$$

Letting $n \to \infty$ in the above inequality, by the Lebesgue's dominated convergence theorem, we have that

$$\underline{u}(t) = \begin{cases} g(\underline{u})(t) + \varphi(t), & t \in [-r, 0], \\ S(t)[g(\underline{u})(0) + \varphi(0)] + \int_0^t S(t-s)[f(s, \underline{u}(s), \underline{u}_s) + M\underline{u}(s)]ds \\ + \sum_{0<t_k<t} S(t-t_k) I_k(\underline{u}(t_k)), & t \in [0, a]. \end{cases}$$

Therefore, $\underline{u} \in PC([-r, a], X)$ and $\underline{u} = \mathcal{Q}\underline{u}$. Similarly, we can prove that $\overline{u}(t)$ belongs to $PC([-r, a], X)$ and $\overline{u} = \mathcal{Q}\overline{u}$. Combining this fact with the monotonicity (3.10), we can prove that $\underline{u}$, $\overline{u} \in [v^{(0)}, w^{(0)}]$ are fixed points of operator $\mathcal{Q}$ and $\underline{u} \le \overline{u}$.

Next, we show that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of $\mathcal{Q}$ in $[v^{(0)}, w^{(0)}]$, respectively. In fact, for any $u \in [v^{(0)}, w^{(0)}]$, $\mathcal{Q}u = u$, we have $v^{(0)} \le u \le w^{(0)}$, and $v^{(1)} = \mathcal{Q}v^{(0)} \le \mathcal{Q}u = u \le \mathcal{Q}w^{(0)} = w^{(1)}$. Continuing such a progress, we get $v^{(n)} \le u \le w^{(n)}$. Letting $n \to \infty$, we get $\underline{u} \le u \le \overline{u}$. Therefore, $\underline{u}$ and $\overline{u}$ are minimal and maximal mild solutions of nonlocal problem (1.1) in $[v^{(0)}, w^{(0)}]$, and $\underline{u}$ and $\overline{u}$ can be obtained by the iterative scheme (3.9) starting from $v^{(0)}$ and $w^{(0)}$, respectively. This completes the proof of Theorem 3.1. □

From the proof of Theorem 3.1, we can easily obtain the following result.

Theorem 3.2. *Let X be an ordered Banach space, whose positive cone P is normal, and let $A : D(A) \subset X \to X$ be a closed linear operator and the positive strongly continuous semigroup $T(t)$ $(t \geq 0)$ generated by $-A$ be compact on X. Assume that the nonlinear function $f : [0,a] \times X \times C_0 \to X$ is Carathéodory continuous, $I_k \in C(X,X)$ for $k = 1,2,\dots,m$ and the nonlocal function $g : PC([-r,a],X) \to C_0$ is continuous and map a monotonic set into a precompact set. Assume that the nonlocal problem (1.1) has a lower solution $v^{(0)} \in PC([-r,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ and an upper solution $w^{(0)} \in PC([-r,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ with $v^{(0)} \leq w^{(0)}$, and the conditions (H1)–(H3) are satisfied. Then nonlocal problem (1.1) has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ between $v^{(0)}$ and $w^{(0)}$, which can be obtained by a monotone iterative procedure starting from $v^{(0)}$ and $w^{(0)}$, respectively.*

Applying the famous Schauder's fixed point theorem, we can also obtain the following existence result.

Theorem 3.3. *Let X be an ordered Banach space, whose positive cone P is normal, and let $A : D(A) \subset X \to X$ be a closed linear operator and the positive strongly continuous semigroup $T(t)$ $(t \geq 0)$ generated by $-A$ be compact on X. Assume that the nonlinear function $f : [0,a] \times X \times C_0 \to X$ is Carathéodory continuous, $I_k \in C(X,X)$ for $k = 1,2,\dots,m$ and the nonlocal function $g : PC([-r,a],X) \to C_0$ is continuous and compact. If nonlocal problem* (1.1) *has a lower solution $v^{(0)} \in PC([-r,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ and an upper solution $w^{(0)} \in PC([-r,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ with $v^{(0)} \leq w^{(0)}$, and the conditions* (H1)–(H3) *are satisfied. Then nonlocal problem* (1.1) *exists at least one mild solution in ordered interval $[v^{(0)}, w^{(0)}]$.*

Proof. In accordance with the proof of Theorem 3.1, we know that $\mathcal{Q}$ defined by (3.2) is a continuous mapping from $[v^{(0)}, w^{(0)}]$ to $[v^{(0)}, w^{(0)}]$. Therefore, in order to apply Schauder's fixed point theorem to obtain a fixed point, we need to prove that $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a compact operator. For this purpose, let

$$
\begin{aligned}
\Pi_1 &= \{g(u)(\cdot) + \varphi(\cdot) :\, \cdot \in [-r,0],\, u \in [v^{(0)}, w^{(0)}]\},\\
\Pi_2 &= \Big\{S(\cdot)[g(u)(0) + \varphi(0)] + \int_0^{\cdot} S(\cdot - s)[f(s,u(s),u_s) + Mu(s)]ds :\\
&\qquad \cdot \in [0,a],\, u \in [v^{(0)}, w^{(0)}]\Big\},\\
\Pi_3 &= \Big\{\sum_{0<t_k<\cdot} S(\cdot - t_k) I_k(u(t_k)) :\, \cdot \in [0,a],\, u \in [v^{(0)}, w^{(0)}]\Big\}.
\end{aligned}
$$

By the compactness of the nonlocal function g, we know that the set Π_1 is precompact in C_0. In what follows, we prove that Π_2 is a precompact set. For $t \in [0,a]$, by the fact that the semigroup $T(t)$ $(t \geq 0)$ is compact and therefore $S(t)$ $(t \geq 0)$ is also compact for every $t > 0$ as well as the compactness of nonlocal function g, we get that the set

$$
\{S(t)[g(u)(0) + \varphi(0)] :\, u \in [v^{(0)}, w^{(0)}]\}
$$

is precompact in X. For $t \in (0,a]$ and $0<\epsilon<t$, the set

$$\begin{aligned}&\Big\{\int_0^{t-\epsilon} S(t-s)[f(s,u(s),u_s)+Mu(s)]ds : u\in[v^{(0)},w^{(0)}]\Big\}\\ &=\Big\{S(\epsilon)\int_0^{t-\epsilon} S(t-s-\epsilon)[f(s,u(s),u_s)+Mu(s)]ds : u\in[v^{(0)},w^{(0)}]\Big\}\end{aligned}\tag{3.16}$$

is precompact in X since $S(t)$ is compact for $t>0$. Furthermore, we know that for every $u\in[v^{(0)},w^{(0)}]$,

$$\begin{aligned}&\int_0^{t-\epsilon} S(t-s)[f(s,u(s),u_s)+Mu(s)]ds\\ &\to \int_0^{t} S(t-s)[f(s,u(s),u_s)+Mu(s)]ds \quad \text{as } \epsilon\to 0.\end{aligned}\tag{3.17}$$

Combining (3.16) and (3.17) with the total boundedness, we know that the set

$$\Big\{\int_0^{t} S(t-s)[f(s,u(s),u_s)+Mu(s)]ds :\ u\in[v^{(0)},w^{(0)}]\Big\}$$

is precompact in X. Therefore, for each $t\in[0,a]$, $\Pi_2(t)$ is precompact in X.

Next, we prove the equicontinuity of Π_2. For any $u\in[v^{(0)},w^{(0)}]$ and $s\in[0,t]$, $t\in[0,a]$, by the assumption (H1), we have

$$\begin{aligned}f(s,v^{(0)}(s),(v^{(0)})_s)+Mv^{(0)}(s) &\le f(s,u(s),u_s)+Mu(s)\\ &\le f(s,w^{(0)}(s),(w^{(0)})_s)+Mw^{(0)}(s).\end{aligned}$$

By the normality of the cone P, there exists $M_4>0$ such that

$$\|f(s,u(s),u_s)+Mu(s)\|\le M_4,\quad s\in[0,t],\ t\in[0,a],\ u\in[v^{(0)},w^{(0)}].\tag{3.18}$$

Therefore, by (2.6), (3.18) and the definition of the set Π_2, we obtain that for $0\le t'<t''\le a$ and $u\in[v^{(0)},w^{(0)}]$,

$$\begin{aligned}&\Big\|S(t'')[g(u)(0)+\varphi(0)]-S(t')[g(u)(0)+\varphi(0)]\\ &+\int_0^{t'}[S(t''-s)-S(t'-s)]\cdot[f(s,u(s),u_s)+Mu(s)]ds\\ &+\int_{t'}^{t''} S(t''-s)[f(s,u(s),u_s)+Mu(s)]ds\Big\|\\ &\le C\|S(t''-t')[g(u)(0)+\varphi(0)]-[g(u)(0)+\varphi(0)]\|\\ &\quad+M_4\int_0^{t'}\|S(t''-s)-S(t'-s)\|ds+CM_4(t''-t')\\ &\le C\|S(t''-t')[g(u)(0)+\varphi(0)]-[g(u)(0)+\varphi(0)]\|\\ &\quad+M_4\int_0^{t'}\|S(t''-t'+s)-S(s)\|ds+CM_4(t''-t').\end{aligned}\tag{3.19}$$

Since the semigroup $S(t)$ $(t\ge 0)$ is strongly continuous for $t\ge 0$ and is continuous in the uniform operator topology for $t>0$, then it is easy to see that the right hand of (3.19) tends to zero independently of $u\in[v^{(0)},w^{(0)}]$ as $t''-t'\to 0$, which means that the functions in Π_2 are equicontinuous. Therefore, by the Arzela-Ascoli theorem one can easily to justify that the set Π_2 is precompact.

Now, we are in the position to prove the precompactness of Π_3. From the definition of the set Π_3 combined with the fact that the interval $[0,a]$ is divided into finite subintervals by t_k, $k=1,2,\dots,m$, we only need to prove that the set

$$\{S(\cdot - t_1)I_1(u(t_1)) : \cdot \in [t_1,t_2],\, u \in [v^{(0)}, w^{(0)}]\}$$

is precompact in $C([t_1,t_2],X)$, as the cases for other subintervals are the same. By the fact that the semigroup $S(t)$ $(t \geq 0)$ is compact for every $t>0$ we know that for each $t \in [t_1,t_2]$, the set

$$\{S(t - t_1)I_1(u(t_1)) : t \in [t_1,t_2],\, u \in [v^{(0)}, w^{(0)}]\}$$

is precompact in X. For $t_1 \leq t' < t'' \leq t_2$ and $u \in [v^{(0)}, w^{(0)}]$, by (2.6) we obtain

$$\begin{aligned}
&\|S(t''-t_1)I_1(u(t_1)) - S(t'-t_1)I_1(u(t_1))\| \\
&= \|S(t'-t_1)[S(t''-t') - S(0)]I_1(u(t_1))\| \\
&\leq C\|[S(t''-t') - S(0)]I_1(u(t_1))\|.
\end{aligned}$$

The above inequality combined with the strong continuity of the operator $S(t)$ for $t>0$ imply that the functions in

$$\{S(\cdot - t_1)I_1(u(t_1)) : \cdot \in [t_1,t_2],\, u \in [v^{(0)}, w^{(0)}]\}$$

are equicontinuous. Therefore, an application of the Arzela-Ascoli theorem justifies the precompactness of the set

$$\{S(\cdot - t_1)I_1(u(t_1)) : \cdot \in [t_1,t_2],\, u \in [v^{(0)}, w^{(0)}]\}$$

in $C([t_1,t_2],X)$. Hence, using a completely similar method for the case $k = 2,3,\dots,m$, one can prove that the set Π_3 is precompact.

Hence, by the above discuss we know that $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a compact operator, and therefore a completely continuous operator. Therefore, the famous Schauder's fixed point theorem implies that the operator $\mathcal{Q}$ has at least one fixed point in ordered interval $[v^{(0)}, w^{(0)}]$, which gives rise to a mild solution of problem (1.1). This completes the proof. □

Remark 3.4. In Theorems 3.1 and 3.2, we obtained the existence of minimal mild solution and maximal mild solution (two mild solutions) for nonlocal problem (1.1) by using the monotone iterative method. In Theorems 3.3, we obtained the existence of at least one mild solution by utilizing Schauder's fixed point theorem. The results as well as the proof method used are all different.

4. Case g not compact

In this section, we discuss the existence of extremal mild solutions and the uniqueness of mild solution for nonlocal problem (1.1) under the situation that the nonlocal function $g : PC([-r,a],X) \to C_0$ is not compact. Furthermore, the strongly continuous semigroup $T(t)$ $(t \geq 0)$ generated by $-A$ does not have to be compact.

From the definition of regular cone in ordered Banach spaces and the proof of Theorem 3.1, we can easily obtain the following result in a general ordered Banach space whose positive cone is regular.

Theorem 4.1. *Let X be an ordered Banach space, whose positive cone P is regular, and let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generates a positive strongly continuous semigroup $T(t)$ $(t \geq 0)$ on X. Assume that the nonlinear function $f : [0,a] \times X \times C_0 \to X$ is Carathéodory continuous, $I_k \in C(X,X)$ for $k = 1,2,\dots,m$ and the nonlocal function $g : PC([-r,a],X) \to C_0$ is continuous. If the nonlocal problem* (1.1) *has a lower solution $v^{(0)} \in PC([-r,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ and an upper solution $w^{(0)} \in PC([-r,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ with $v^{(0)} \leq w^{(0)}$, and the conditions* (H1)–(H3) *are satisfied. Then nonlocal problem* (1.1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ between $v^{(0)}$ and $w^{(0)}$, which can be obtained by a monotone iterative procedure starting from $v^{(0)}$ and $w^{(0)}$, respectively.*

In the application of differential equations, we often choose Hilbert space, reflexive space and $L^p(\Omega)$ space for $1 \leq p < \infty$ as working spaces, which are all weakly sequentially complete spaces. Therefore, it is interesting to discuss the existence of extremal mild solutions for nonlocal problem (1.1) in weakly sequentially complete space.

Theorem 4.2. *Let X be an ordered Banach space, whose positive cone P is regular, and let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generates a positive strongly continuous semigroup $T(t)$ $(t \geq 0)$ on X. Assume that the nonlinear function $f : [0,a] \times X \times C_0 \to X$ is Carathéodory continuous, $I_k \in C(X,X)$ for $k = 1,2,\dots,m$ and the nonlocal function $g : PC([-r,a],X) \to C_0$ is continuous. If the nonlocal problem* (1.1) *has a lower solution $v^{(0)} \in PC([-r,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ and an upper solution $w^{(0)} \in PC([-r,a],X) \cap C^1(I'',X) \cap C(I',X_1)$ with $v^{(0)} \leq w^{(0)}$, and the conditions* (H1)–(H3) *are satisfied. Then nonlocal problem* (1.1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ between $v^{(0)}$ and $w^{(0)}$, which can be obtained by a monotone iterative procedure starting from $v^{(0)}$ and $w^{(0)}$, respectively.*

Proof. By the proof of Theorem 3.1, we know that the operator $\mathcal{Q}$ defined by (3.2) maps ordered interval $[v^{(0)}, w^{(0)}]$ to $[v^{(0)}, w^{(0)}]$ is continuous and monotone increasing. Furthermore, if the conditions (H1)–(H3) are satisfied, then the sequences $\{v^{(n)}\}$ and $\{w^{(n)}\}$ defined by (3.9) satisfying the monotonicity (3.10). Therefore, for any $t \in [-r, 0]$, the sequences $\{v^{(n)}(t)\}$ and $\{w^{(n)}(t)\}$ are monotone and order-bounded sequences in X. Noticing that X is a weakly sequentially complete Banach space, from [22, Theorem 2.2] we know that $\{v^{(n)}(t)\}$ and $\{w^{(n)}(t)\}$ are precompact in X. Combining this fact with the monotonicity (3.10), it follows that $\{v^{(n)}(t)\}$ and $\{w^{(n)}(t)\}$ are convergent in X. Denote

$$\lim_{n\to\infty} v^{(n)}(t) = \underline{u}(t) \quad \text{for } t \in [-r, a].$$

Similarly, we can prove that

$$\lim_{n\to\infty} w^{(n)}(t) = \overline{u}(t) \quad \text{for } t \in [-r, a].$$

Using a completely similar method to the one we used to prove Theorem 3.1 we can easily to prove that $\underline{u}$ and $\overline{u}$ are minimal and maximal mild solutions of nonlocal problem (1.1) in $[v^{(0)}, w^{(0)}]$, and $\underline{u}$ and $\overline{u}$ can be obtained by the iterative scheme (3.10) starting from $v^{(0)}$ and $w^{(0)}$, respectively. This completes the proof. $\square$

Remark 4.3. In Theorems 4.1 and 4.2, we only assume that positive semigroup $T(t)$ $(t \geq 0)$ is strongly continuous, the nonlinear function f, the impulsive function I_k and the nonlocal term g are continuous and satisfy some monotonicity conditions, which are more easily to satisfied. Therefore, Theorem 4.1 and Theorem 4.2 in this paper essentially extends the main results of the previous research in several ways, as far as the mild solution of the retarded evolution equations involving nonlocal and impulsive conditions (1.1) is concerned, by dropping the compactness and Lipschitz continuity of the impulsive item from the hypotheses. This distinguishes the present paper from earlier works on impulsive Cauchy problems.

Theorem 4.4. *Let X be an ordered Banach space, whose positive cone P is normal, and let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generates a positive and equicontinuous strongly continuous semigroup $T(t)$ $(t \geq 0)$ on X. Assume that the nonlinear function $f : [0, a] \times X \times C_0 \to X$ is Carathéodory continuous, $I_k \in C(X, X)$ for $k = 1, 2, \dots, m$ and the nonlocal function $g : PC([-r, a], X) \to C_0$ is continuous. If the nonlocal problem* (1.1) *has a lower solution $v^{(0)} \in PC([-r, a], X) \cap C^1(I'', X) \cap C(I', X_1)$ and an upper solution $w^{(0)} \in PC([-r, a], X) \cap C^1(I'', X) \cap C(I', X_1)$ with $v^{(0)} \leq w^{(0)}$, and the conditions* (H1)–(H2) *and the condition*

(H4) *There exist positive constants L_f, L_g and L_k $(k = 1, 2, \dots, m)$ satisfying*

$$4aC(2L_f + M) + C\Big(2\sum_{k=1}^{M} L_k + L_g\Big) < 1$$

such that

$$\begin{gathered}
\|g(u) - g(v)\|_{C_0} \leq L_g \|u - v\|_{PC}, \quad u, v \in [v^{(0)}, w^{(0)}], \\
\alpha\Big(\Big\{f(t, u^{(n)}(t), (u^{(n)})_t)\Big\}\Big) \\
\leq L_f\Big[\alpha\Big(\Big\{u^{(n)}(t)\Big\}\Big) + \sup_{-r\leq\tau\leq 0} \alpha\Big(\Big\{u^{(n)}(t+\tau)\Big\}\Big)\Big], \quad \forall t \in [0, a] \\
\alpha\Big(\Big\{I_k(u^{(n)}(t_k))\Big\}\Big) \leq L_k \alpha\Big(\Big\{u^{(n)}(t_k)\Big\}\Big), \quad k = 1, 2, \dots, m,
\end{gathered}$$

where $\{u^{(n)}\} \subset [v^{(0)}, w^{(0)}]$ is countable and increasing or decreasing monotonic set and $\{(u^{(n)})_t\} \subset C_0$,

are satisfied. Then nonlocal problem (1.1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ between $v^{(0)}$ and $w^{(0)}$; moreover,*

$$v^{(n)}(t) \to \underline{u}(t), \quad w^{(n)}(t) \to \overline{u}(t) \quad \text{uniformly for } t \in [-r, a] \text{ as } n \to \infty,$$

where

$$v^{(n)}(t) = (\mathcal{Q}v^{(n-1)})(t), \quad w^{(n)}(t) = (\mathcal{Q}w^{(n-1)})(t),$$

which satisfy

$$\begin{aligned}
v^{(0)}(t) &\leq v^{(1)}(t) \leq \dots \leq v^{(n)}(t) \leq \dots \leq \underline{u}(t) \leq \overline{u}(t) \leq \dots \\
&\leq w^{(n)}(t) \leq \dots \leq w^{(1)}(t) \leq w^{(0)}(t), \quad \forall t \in [-r, a].
\end{aligned}$$

Proof. By the proof of Theorem 3.1, we know that the operator $\mathcal{Q}$ defined by (3.2) maps $[v^{(0)}, w^{(0)}]$ to $[v^{(0)}, w^{(0)}]$ is continuous and monotone increasing. In the following, we will prove that the operator $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is strict set-contraction. For this purpose, denote by

$$(\mathcal{Q}_1 u)(t) = \begin{cases} g(u)(t) + \varphi(t), & t \in [-r, 0]; \\ S(t)[g(u)(0) + \varphi(0)], & t \in [0, a], \end{cases}$$

$$(\mathcal{Q}_2 u)(t) = \begin{cases} 0, & t \in [-r, 0]; \\ \int_0^t S(t-s)[f(s, u(s), u_s) + Mu(s)]ds \\ + \sum_{0<t_k<t} S(t-t_k) I_k(u(t_k)), & t \in [0, a]. \end{cases}$$

Then it is easily to see that

$$(\mathcal{Q}u)(t) = (\mathcal{Q}_1 u)(t) + (\mathcal{Q}_2 u)(t), \quad t \in [-r, a].$$

Firstly, we prove that the operator $\mathcal{Q}_1 : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is Lipschitz continuous. For u, $v \in [v^{(0)}, w^{(0)}]$, by (2.6), the definition of operator $\mathcal{Q}_1$ and the condition (H4), we have

$$\begin{aligned} \|(\mathcal{Q}_1 u)(t) - (\mathcal{Q}_1 v)(t)\| &= \|g(u)(t) - g(v)(t)\| \le \|g(u) - g(v)\|_{C_0} \\ &\le L_g \|u - v\|_{PC} \quad \text{for } t \in [-r, 0] \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \|(\mathcal{Q}_1 u)(t) - (\mathcal{Q}_1 v)(t)\| &= \|S(t)\| \cdot \|g(u)(0) - g(v)(0)\| \\ &\le C\|g(u) - g(v)\|_{C_0} \\ &\le CL_g \|u - v\|_{PC} \quad \text{for} \quad t \in [0, a]. \end{aligned} \tag{4.2}$$

From (4.1), (4.2), (2.1) and the fact that $C \ge 1$, we know that

$$\|\mathcal{Q}_1 u - \mathcal{Q}_1 v\|_{PC} \le CL_g \|u - v\|_{PC}. \tag{4.3}$$

Therefore, by Lemma 2.8 (vii) and (4.3) we get that for any bounded set $D \subset [v^{(0)}, w^{(0)}]$,

$$\alpha_{PC}(\mathcal{Q}_1(D)) \le CL_g \alpha_{PC}(D). \tag{4.4}$$

Secondly, we estimate the measure of noncompactness to the operator $\mathcal{Q}_2$. From the proof of Theorem 3.3, one can easily to prove that the operator $\mathcal{Q}_2 : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is equicontinuous on each J_k $(k = 1, 2, \dots, m+1)$. For any bounded set $D \subset [v^{(0)}, w^{(0)}]$, by Lemma 2.11, there exists a countable set $D^* = \{u^{(n)}\} \subset D$, such that

$$\alpha_{PC}(\mathcal{Q}_2(D)) \le 2\alpha_{PC}(\mathcal{Q}_2(D^*)). \tag{4.5}$$

Since $\mathcal{Q}_2(D^*) \subset \mathcal{Q}_2(D)$ is equicontinuous on each J_k $(k = 1, 2, \dots, m+1)$, we have by the definition of operator Q_2 and Lemma 2.9

$$\alpha_{PC}(\mathcal{Q}_2(D^*)) = \sup_{t \in [-r,a]} \alpha(\mathcal{Q}_2(D^*)(t)) = \sup_{t \in [0,a]} \alpha(\mathcal{Q}_2(D^*)(t)). \tag{4.6}$$

For every $t \in [0,a]$, by (2.6), the definition of operator $\mathcal{Q}_2$, the condition (H4) and Lemma 2.10, we arrive at

$$\begin{aligned}
&\alpha((\mathcal{Q}_2 D^*)(t)) \\
&= \alpha\Big(\Big\{ \int_0^t S(t-s)[f(s,u^{(n)}(s),(u^{(n)})_s) + Mu^{(n)}(s)] \\
&\quad + \sum_{0<t_k<t} S(t-t_k) I_k(u^{(n)}(t_k)) ds \Big\}\Big) \\
&\le 2C \int_0^t \alpha\Big(\Big\{ f(s,u^{(n)}(s),u_s^{(n)}) + Mu^{(n)}(s)\Big\}\Big) ds \\
&\quad + C \sum_{0<t_k<t} \alpha\Big(\Big\{ I_k(u^{(n)}(t_k))\Big\}\Big) \\
&\le 2C \int_0^t \Big[L_f \alpha(D^*(s)) + L_f \sup_{-r\le\tau\le 0} \alpha(D^*(s+\tau)) + M\alpha(D^*(s))\Big] ds \\
&\quad + C \sum_{0<t_k<t} L_k \alpha(D^*(t_k)) \\
&\le \Big[2aC(2L_f + M) + C\sum_{k=1}^m L_k\Big]\alpha_{PC}(D).
\end{aligned} \tag{4.7}$$

Combining this with (4.5), (4.6) and (4.7), we have

$$\alpha_{PC}(\mathcal{Q}_2(D)) \le 2\Big[2aC(2L_f + M) + C\sum_{k=1}^m L_k\Big]\alpha_{PC}(D). \tag{4.8}$$

Therefore, by (4.4), (4.8), Lemma 2.8 (vi) and the assumption (H4), we know that

$$\alpha_{PC}(\mathcal{Q}(D)) \le \alpha_{PC}(\mathcal{Q}_1(D)) + \alpha(\mathcal{Q}_2(D))_{PC} \le \mu_{PC}\alpha(D), \tag{4.9}$$

where

$$\mu = 4aC(2L_f + M) + C\Big(2\sum_{k=1}^m L_k + L_g\Big) < 1. \tag{4.10}$$

Hence, from (4.9) and (4.10) we know that $\mathcal{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a strict set-contraction operator. Therefore, our conclusion follows from Lemma 2.12. This completes the proof. □

Remark 4.5. Analytic semigroup and differentiable semigroup are equicontinuous semigroup [41]. In the application of partial differential equations, such as parabolic and strongly damped wave equations, the corresponding solution semigroup are analytic semigroup. Therefore, Theorem 4.4 in this paper has broad applicability.

Now we discuss the uniqueness of the mild solution to nonlocal problem (1.1). If we replace the assumption (H4) by the following assumption:

(H5) There exist positive constants $\overline{L}_f$, $\overline{M}_f$, $\overline{L}_g$ and $\overline{L}_k$ $(k = 1, 2, \dots, m)$ satisfying

$$NC\Big(\overline{L}_g + \sum_{k=1}^m \overline{L}_k + a(M + \overline{M}_f + \overline{L}_f)\Big) < 1$$

such that for any $u, v \in [v^{(0)}, w^{(0)}]$,

$$g(u)(t) - g(v)(t) \le \overline{L}_g(u(t) - v(t)), \quad t \in [-r, 0],$$

$$\begin{aligned} &f(t, u(t), u_t) - f(t, v(t), v_t) \\ &\le \overline{M}_f(v(t) - v(t)) + \overline{L}_f(u(t+\tau) - v(t+\tau)), \quad \forall t \in [0, a],\ \tau \in [-r, 0] \end{aligned}$$

$$I_k(u(t_k)) - I_k(v(t_k)) \le \overline{L}_k(u(t_k) - v(t_k)), \quad k = 1, 2, \dots, m,$$

we have the following unique existence results:

Theorem 4.6. *Let X be an ordered Banach space, whose positive cone P is normal, and let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generates a positive and equicontinuous strongly continuous semigroup $T(t)$ $(t \ge 0)$ on X. Assume that the nonlinear function $f : [0, a] \times X \times C_0 \to X$ is Carathéodory continuous, $I_k \in C(X, X)$ for $k = 1, 2, \dots, m$ and the nonlocal function $g : PC([-r, a], X) \to C_0$ is continuous. If the nonlocal problem* (1.1) *has a lower solution $v^{(0)} \in PC([-r, a], X) \cap C^1(I'', X) \cap C(I', X_1)$ and an upper solution $w^{(0)} \in PC([-r, a], X) \cap C^1(I'', X) \cap C(I', X_1)$ with $v^{(0)} \le w^{(0)}$, such that* (H1)–(H3), (H5) *hold, then nonlocal problem* (1.1) *has a unique mild solution between $v^{(0)}$ and $w^{(0)}$, which can be obtained by a monotone iterative procedure starting from $v^{(0)}$ or $w^{(0)}$.*

Proof. We first prove that (H1)–(H3) and (H5) imply (H4). For $t \in [-r, 0]$, let $u, v \in [v^{(0)}, w^{(0)}]$ such that $u > v$. In accordance with (H3) and (H5), we arrive at

$$\theta \le g(u)(t) - g(v)(t) \le \overline{L}_g(u(t) - v(t)), \quad t \in [-r, 0].$$

By this and the normality of cone P, we have

$$\|g(u)(t) - g(v)(t)\| \le N\overline{L}_g\|u(t) - v(t)\| \le N\overline{L}_g\|u - v\|_{PC}, \quad t \in [-r, 0].$$

Hence

$$\|g(u) - g(v)\|_{C_0} \le L_g\|u - v\|_{PC},$$

where $L_g = N\overline{L}_g$.

On the other hand, for $t \in [0, a]$, let $\{u^{(n)}\} \subset [v^{(0)}, w^{(0)}]$ be an increasing sequences. For $m, n \in \mathbb{N}$ with $m > n$, from the conditions (H1), (H2) and (H5), we obtain

$$\begin{aligned} \theta &\le f(t, u^{(m)}(t), (u^{(m)})_t) - f(t, u^{(n)}(t), (u^{(n)})_t) \\ &\quad + M[u^{(m)}(t) - u^{(n)}(t)] \\ &\le (M + \overline{M}_f)[u^{(m)}(t) - u^{(n)}(t)] \\ &\quad + \overline{L}_f[u^{(m)}(t+\tau) - u^{(n)}(t+\tau)], \ \ t \in [0, a],\ \tau \in [-r, 0] \end{aligned}$$

and

$$\theta \le I_k(u^{(m)}(t_k)) - I_k(u^{(n)}(t_k)) \le \overline{L}_k(u^{(m)}(t_k) - u^{(n)}(t_k)), \quad k = 1, 2, \dots, m.$$

From the above inequalities and the normality of cone P, it follows that for any $t \in [0, a]$, $\tau \in [-r, 0]$,

$$\begin{aligned} &\|f(t, u^{(m)}(t), (u^{(m)})_t) - f(t, u^{(n)}(t), (u^{(n)})_t)\| \\ &\le N(M + \overline{M}_f)\|u^{(m)}(t) - u^{(n)}(t)\| \\ &\quad + N\overline{L}_f\|u^{(m)}(t+\tau) - u^{(n)}(t+\tau)\| + M\|u^{(m)}(t) - u^{(n)}(t)\| \end{aligned}$$

$$\begin{aligned}&\le [N(M+\overline{M}_f)+M]\|u^{(m)}(t)-u^{(n)}(t)\|\\&\quad + N\overline{L}_f\sup_{-r\le\tau\le 0}\|u^{(m)}(t+\tau)-u^{(n)}(t+\tau)\|\end{aligned}$$

and

$$\|I_k(u^{(m)}(t_k))-I_k(u^{(n)}(t_k))\|\le N\overline{L}_k\|u^{(m)}(t_k)-u^{(n)}(t_k)\|,\quad k=1,2,\dots,m.$$

From these inequalities and the definition of the measure of noncompactness, we have

$$\begin{aligned}&\alpha\Big(\Big\{f(t,u^{(n)}(t),(u^{(n)})_t)\Big\}\Big)\\&\le L_f\Big[\alpha\Big(\Big\{u^{(n)}(t)\Big\}\Big)+\sup_{-r\le\tau\le 0}\alpha\Big(\Big\{u^{(n)}(t+\tau)\Big\}\Big)\Big],\quad \forall t\in[0,a]\end{aligned}$$

and

$$\alpha\Big(\Big\{I_k(u^{(n)}(t_k))\Big\}\Big)\le L_k\alpha\Big(\Big\{u^{(n)}(t_k)\Big\}\Big),\quad k=1,2,\dots,m,$$

where

$$L_f=\max\big\{N(M+\overline{M}_f)+M, N\overline{L}_f\big\},\quad L_k=N\overline{L}_k,\ \ k=1,2,\dots,m.$$

If $\{u^{(n)}\}\subset[v^{(0)},w^{(0)}]$ is a decreasing sequence, then above inequality is also valid. Therefore, the condition (H4) holds.

Hence, by Theorem 4.4, the nonlocal problem (1.1) has a minimal mild solution $\underline{u}$ and a maximal mild solution $\overline{u}$ between $v^{(0)}$ and $w^{(0)}$. By the proof Theorem 4.4, (3.2), (3.10) are valid. In what follows, we show that $\underline{u}=\overline{u}$. For $t\in[-r,0]$, by (3.2) and (3.10) and the assumption (H5), we have

$$\theta\le\overline{u}(t)-\underline{u}(t)=A\overline{u}(t)-A\underline{u}(t)=g(\overline{u})(t)-g(\underline{u})(t)\le\overline{L}_g(\overline{u}(t)-\underline{u}(t)).$$

From this and the normality of cone P it follows that

$$\|\overline{u}(t)-\underline{u}(t)\|\le N\overline{L}_g\|\overline{u}(t)-\underline{u}(t)\|\le N\overline{L}_g\|\overline{u}-\underline{u}\|_{PC},\quad t\in[-r,0]. \tag{4.11}$$

For $t\in[0,a]$, by (3.2) and (3.10) and the assumption (H5), we obtain

$$\begin{aligned}\theta&\le\overline{u}(t)-\underline{u}(t)=A\overline{u}(t)-A\underline{u}(t)\\&=S(t)[g(\overline{u})(0)-g(\underline{u})(0)]+\sum_{0<t_k<t}S(t-t_k)[I_k(\overline{u}(t_k))-I_k(\underline{u}(t_k))]\\&\quad+\int_0^t S(t-s)[f(s,\overline{u}(s),\overline{u}_s)-f(s,\underline{u}(s),\underline{u}_s)+M(\overline{u}(s)-\underline{u}(s))]ds\\&\le\overline{L}_gS(t)(\overline{u}(t)-\underline{u}(t))+\sum_{0<t_k<t}\overline{L}_kS(t-t_k)(\overline{u}(t_k)-\underline{u}(t_k))\\&\quad+\int_0^t S(t-s)[(M+\overline{M}_f)(\overline{u}(s)-\underline{u}(s))+\overline{L}_f(\overline{u}(s+\tau)-\underline{u}(s+\tau))]ds,\end{aligned}$$

where $\tau \in [-r, 0]$. In accordance with (2.6), this inequality and the normality of cone P, we get that for any $t \in [0, a]$ and $\tau \in [-r, 0]$,

$$
\begin{aligned}
&\|\overline{u}(t) - \underline{u}(t)\| \\
&\le NC\overline{L}_g\|\overline{u}(t) - \underline{u}(t)\| + NC\sum_{k=1}^{m}\overline{L}_k\|\overline{u}(t_k) - \underline{u}(t_k)\| \\
&\quad + NC\int_0^t [(M + \overline{M}_f)\|\overline{u}(s) - \underline{u}(s)\| + \overline{L}_f\|\overline{u}(s+\tau) - \underline{u}(s+\tau)\|]ds \\
&\le NC\Big(\overline{L}_g + \sum_{k=1}^{m}\overline{L}_k + a(M + \overline{M}_f + \overline{L}_f)\Big)\|\overline{u} - \underline{u}\|_{PC}.
\end{aligned}
\tag{4.12}
$$

Since $C \ge 1$, by (4.11) and (4.12), we know that

$$
\|\overline{u} - \underline{u}\|_{PC} \le \rho\|\overline{u} - \underline{u}\|_{PC},
$$

where

$$
\rho = NC\Big(\overline{L}_g + \sum_{k=1}^{m}\overline{L}_k + a(M + \overline{M}_f + \overline{L}_f)\Big).
$$

Noting that $\rho < 1$ yields $\overline{u} = \underline{u}$. Hence, $\widetilde{u} := \overline{u} = \underline{u}$ is unique mild solution of nonlocal problem (1.1) in $[v^{(0)}, w^{(0)}]$, which can be obtained by the monotone iterative procedure (3.9) starting from $v^{(0)}$ or $w^{(0)}$. This completes the proof. $\square$

5. An example

In this section, we give an example to illustrate the feasibility of our abstract results. We consider the retarded parabolic partial differential equation involving nonlocal and impulsive conditions

$$
\begin{gathered}
\frac{\partial}{\partial t}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = \sin\frac{u(x,t)}{\sqrt{2}} + \int_{-r}^{0}\gamma(s)u_t(x,s)ds, \\
x \in [0,1],\ t \in [0,a],\ t \ne t_k, \\
u(x,t_k^+) = u(x,t_k^-) + \frac{\sqrt{|u(x,t_k)|}}{1 + |u(x,t_k)|}, \quad x \in [0,1],\ k = 1,2,\dots,m, \\
u(0,t) = u(1,t) = 0, \quad t \in [0,a], \\
u(x,s) = \int_0^a \rho(s,t)\lg(1 + |u(x,t)|)dt + \varphi(x,s), \quad x \in [0,1],\ s \in [-r,0],
\end{gathered}
\tag{5.1}
$$

where a, $r > 0$ are two constants, $0 < t_1 < t_2 < \dots < t_m < a$, $u_t(x,s) = u(t+s,x)$ for $s \in [-r,0]$ and $x \in [0,1]$, $\gamma \in L([-r,0],\mathbb{R}^+)$, $\rho(s,t)$ is a continuous function from $[-r,0] \times [0,a]$ to $\mathbb{R}^+$, $\varphi \in C([0,1] \times [-r,0],\mathbb{R}^+)$.

Let $X = L^2([0,1],\mathbb{R})$ with the norm $\|\cdot\|_2$, and let $P = \{u \in L^2([0,1],\mathbb{R}) \mid u(x) \ge 0$a.e. $x \in [0,1]\}$. Then X is a Banach space and P is a regular cone of X. Define an operator $A : D(A) \subset X \to X$ by

$$
Au = -\frac{\partial^2}{\partial x^2}u, \quad u \in D(A),
$$

with domain

$$
D(A) = \{u \in L^2([0,1],\mathbb{R}) : u', u'' \in L^2([0,1],\mathbb{R}),\ u(0) = u(1) = 0\}.
$$

It is well known that $-A$ generates a uniformly bounded strongly continuous semigroup $T(t)$ $(t \geq 0)$ which is positive, compact, and analytic. Furthermore, A has discrete spectrum with eigenvalues $\lambda_n = n^2\pi^2$ for $n \in \mathbb{N}$ associated normalized eigenvectors $e_n(x) = \sqrt{2}\sin(n\pi x)$, the set $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.

For $t \in [0, a]$ and $s \in [-r, 0]$, we denote

$$
\begin{gathered}
u(t) = u(\cdot, t), \quad f(t, u(t), u_t) = \sin\frac{u(\cdot, t)}{\sqrt{2}} + \int_{-r}^{0} \gamma(s) u_t(\cdot, s)\, ds, \\
I_k(u(t_k)) = \frac{\sqrt{|u(\cdot, t_k)|}}{1 + |u(\cdot, t_k)|}, \quad k = 1, 2, \dots, m, \\
g(u)(s) = \int_0^a \rho(s, t) \lg(1 + |u(\cdot, t)|) dt, \quad \varphi(s) = \varphi(\cdot, s).
\end{gathered}
$$

Then the retarded parabolic partial differential equation involving nonlocal and impulsive conditions (5.1) can be transformed into the abstract form of nonlocal problem (1.1).

Theorem 5.1. *Assume that there exists a function $w = w(x, t) \in PC([0, 1] \times [-r, a], \mathbb{R}) \cap C^1([0, 1] \times I'', \mathbb{R})$ such that*

$$
\begin{gathered}
\frac{\partial}{\partial t} w(x, t) - \frac{\partial^2}{\partial x^2} w(x, t) \geq \sin\frac{w(x, t)}{\sqrt{2}} + \int_{-r}^{0} \gamma(s) w_t(x, s) ds, \\
x \in [0, 1],\ t \in [0, a],\ t \neq t_k, \\
w(x, t_k^+) \geq w(x, t_k^-) + \frac{\sqrt{|w(x, t_k)|}}{1 + |w(x, t_k)|}, \quad x \in [0, 1],\ k = 1, 2, \dots, m, \\
w(0, t) = w(1, t) = 0, \quad t \in [0, a], \\
w(x, s) \geq \int_0^a \rho(s, t) \lg(1 + |w(x, t)|) dt + \varphi(x, s), \quad x \in [0, 1],\ s \in [-r, 0].
\end{gathered}
$$

Then the retarded parabolic partial differential equation involving nonlocal and impulsive conditions (5.1) *exist a minimal mild solution and a maximal mild solution between* 0 *and $w(x, t)$, which can be obtained by a monotone iterative procedure starting from* 0 *and $w(x, t)$, respectively.*

Proof. From the assumption and the definition of nonlinear term f, impulsive function I_k for $k = 1, 2, \dots, m$ and nonlocal function g, we can verify that that $v^{(0)} = 0$ and $w^{(0)} = w(x, t)$ are the lower and the upper solutions of the the retarded parabolic partial differential equation involving nonlocal and impulsive conditions (5.1) respectively, nonlocal term g is continuous and compact as well as the conditions (H1)–(H3) are satisfied with $M = \frac{\sqrt{2}}{2}$. Therefore, our conclusion follows from Theorem 3.1. This completes the proof. □

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Xuping Zhang
Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
Email address: `lanyu9986@126.com`

Pengyu Chen (corresponding author)
Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
Email address: chpengyu123@163.com

Yongxiang Li
Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
Email address: liyx@nwnu.edu.cn