STABILITY OF ANISOTROPIC PARABOLIC EQUATIONS WITHOUT BOUNDARY CONDITIONS

HUASHUI ZHAN, ZHAOSHENG FENG

ABSTRACT. In this article, we consider the equation

$$u_t = \sum_{i=1}^{N} \left(a_i(x)|u_{x_i}|^{p_i(x)-2} u_{x_i} \right)_{x_i},$$

with $a_i(x), p_i(x) \in C^1(\overline{\Omega})$ and $p_i(x) > 1$. Where $a_i(x) = 0$ if $x \in \partial\Omega$, and $a_i(x) > 0$ if $x \in \Omega$, without any boundary conditions. We propose an analytical method for studying the stability of weak solutions. We also study the uniqueness of a weak solution, and establish its stability under certain conditions.

1. INTRODUCTION

In past decades, the so-called electrorheological fluid equation [1, 15]:

$$u_t = \operatorname{div}\left(a(x)|\nabla u|^{p(x)-2}\nabla u \right), \quad (x,t) \in Q_T, \tag{1.1}$$

has received a lot of attention from a rather diverse group of scientists such as physicists and mathematicians [3, 4, 6, 7, 11, 13, 16, 19]. In this work, we consider an anisotropic parabolic equation

$$u_t = \sum_{i=1}^{N} \left(a_i(x)|u_{x_i}|^{p_i(x)-2} u_{x_i} \right)_{x_i}, \quad (x,t) \in Q_T, \tag{1.2}$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

but without the boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with the smooth boundary $\partial\Omega$, $Q_T = \Omega \times (0,T)$, and $p_i(x)$ is a $C^1(\overline{\Omega})$ function with $p_i(x) > 1$. Equation (1.2) arises in several scientific fields. For instance, in biology [6, 7] it is suggested as a model to describe the spread of an epidemic disease in heterogeneous environments. In fluid mechanics [2, 5], it is used as the mathematical description for the dynamics of fluids with different conductivities in different directions. For equation (1.1), considerable

---

attention has been devoted to the existence and uniqueness of its solution. One can refer to [8, 9, 10, 12, 14, 17, 18] and the references therein.

When $a(x) \in C^1(\overline{\Omega})$, and

$$a(x) > 0, \ x \in \Omega \text{ and } a(x) = 0, \ x \in \partial\Omega, \tag{1.5}$$

the initial-boundary value problem of equation (1.1) was discussed by means of the parabolic regularized method [19]. In this study, we assume that $a_i(x) \in C^1(\overline{\Omega})$, and

$$a_i(x) > 0, \ x \in \Omega \text{ and } a_i(x) = 0, \ x \in \partial\Omega, \ i = 1, 2, \dots, N, \tag{1.6}$$

and denote

$$p_0 = \min_{x \in \overline{\Omega}} \{p_1(x), p_2(x), \dots, p_{N-1}(x), p_N(x)\}.$$

Throughout this paper, we assume that $p_0 > 1$. Before stating our main results, let us recall two definitions.

Definition 1.1. If $u(x, t)$ satisfies

$$u \in L^\infty(Q_T), \ \frac{\partial u}{\partial t} \in L^2(Q_T), \ u_{x_i} \in L^\infty(0, T; L^{p_i(x)}(a_i, \Omega)), \tag{1.7}$$

and for $\varphi_1 \in C_0^1(Q_T)$, $\varphi_2 \in L^\infty(0, T; W_{\text{loc}}^{1,p_0}(\Omega))$ and $\varphi_{2x_i} \in L^\infty(0, T; L^{p_i(x)}(a_i, \Omega))$, it holds

$$\iint_{Q_T} \left[\frac{\partial u}{\partial t}(\varphi_1 \varphi_2) + \sum_{i=1}^N a_i(x)|u_{x_i}|^{p_i(x)-2} u_{x_i}(\varphi_1 \varphi_2)_{x_i} \right] dx \, dt = 0, \tag{1.8}$$

then we call $u(x, t)$ a weak solution of equation (1.2) with the initial condition (1.3) in the sense of

$$\lim_{t \to 0} \int_\Omega |u(x, t) - u_0(x)| \, dx = 0. \tag{1.9}$$

Here, $L^{p_i(x)}(a_i, \Omega)$ is the weighted variable exponent Lebesgue space. One can refer to [11] for the definition of such a space and the corresponding Hölder inequality.

Recall that the characteristic function χ of Ω is defined by

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \overline{\Omega}, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \overline{\Omega}. \end{cases}$$

Definition 1.2. A nonnegative continuous function χ is said to be a weak characteristic function of Ω, if

$$\chi(x) \begin{cases} > 0, & x \in \Omega, \\ = 0, & x \in \partial\Omega. \end{cases} \tag{1.10}$$

Apparently, the weak characteristic function is not unique for a bounded domain Ω. For examples, the distance function $d(x) = \text{dist}(x, \partial\Omega)$ and the diffusion function $a_i(x)$ in (1.6) both are the weak characteristic functions. Based on Definition 1.2, we propose a new analytical method, currently called the weak characteristic function method, to study the stability of weak solutions to the nonlinear degenerate parabolic equations independent of the boundary condition.

Theorem 1.3. *Let $a_i(x) \in C^1(\overline{\Omega})$ satisfy (1.6), and $u(x,t)$ and $v(x,t)$ be two solutions of equation (1.2) with the initial values $u_0(x)$ and $v_0(x)$ respectively. If for sufficiently large n, there are a weak characteristic function $\chi(x)$ of Ω and a constant c such that*

$$n\Big(\int_{\Omega\backslash\Omega_n} a_i(x)|\chi_{x_i}(x)|^{p_i(x)}dx\Big)^{1/p_i^+} \leq c, \tag{1.11}$$

then

$$\int_{\Omega} |u(x,t) - v(x,t)|dx \leq c\int_{\Omega} |u_0(x) - v_0(x)|dx, \tag{1.12}$$

where $p_i^+ = \max_{x \in \overline{\Omega}} p_i(x)$ and $\Omega_n = \{x \in \Omega : \chi(x) > 1/n\}$.

Theorem 1.4. *Let $a_i(x) \in C^1(\overline{\Omega})$ satisfy (1.6), and $u(x,t)$ and $v(x,t)$ be two weak solutions of (1.2) with the initial values $u_0(x)$ and $v_0(x)$ respectively, If there exists a weak characteristic function χ such that*

$$\int_{\Omega} a_i(x)\Big|\frac{\chi_{x_i}(x)}{\chi(x)}\Big|^{p_i(x)}dx < \infty, \tag{1.13}$$

then the stability (1.12) is true.

Theorem 1.5. *Let $a_i(x) \in C^1(\overline{\Omega})$ satisfy (1.6), and $u(x,t)$ and $v(x,t)$ be two solutions of (1.2) with the different initial values $u_0(x)$ and $v_0(x)$ respectively, but without any boundary condition. If there exist a weak characteristic function $\chi(x)$ and a constant c such that*

$$\frac{a_i(x)|\chi_{x_i}(x)|^{p_i(x)}}{\chi(x)} \leq c, \tag{1.14}$$

then

$$\int_{\Omega} \chi(x)|u(x,t) - v(x,t)|^2dx \leq c\int_{\Omega} \chi(x)|u_0(x) - v_0(x)|^2dx. \tag{1.15}$$

If we choose

$$\chi(x) = \min_{1\leq i \leq N}\{a_i(x)\},$$

then (1.14) holds, and

$$\int_{\Omega} \min_{1\leq i \leq N}\{a_i(x)\}|u(x,t) - v(x,t)|^2dx \leq c\int_{\Omega} \min_{1\leq i \leq N}\{a_i(x)\}|u_0(x) - v_0(x)|^2dx.$$

This inequality implies that the uniqueness of weak solution is always true provided that $a_i(x)$ satisfies conditions (1.5) and (1.6).

Note that by choosing various characteristic functions $\chi(x)$, one may obtain different results. For example, choosing

$$\chi(x) = \prod_{i=1}^{N} a_i(x),$$

then we obtain

$$\chi_{x_i}(x) = \sum_{k=1}^{N}\Big(\prod_{j=1,j\neq k}^{N} a_j(x)\Big)a_{x_i} = \prod_{j=1}^{N} a_j(x)\sum_{k=1}^{N}\frac{a_{kx_i}}{a_k}$$

and

$$n\Big(\int_{\Omega\backslash\Omega_n} a_i(x)|\chi_{x_i}(x)|^{p_i(x)}dx\Big)^{1/p_i^+}$$

$$= n\Big(\int_{\Omega\setminus\Omega_n} a_i(x)\chi^{p_i(x)}(x) \Big| \sum_{k=1}^{N} \frac{a_{kx_i}}{a_k} \Big|^{p_i(x)} dx \Big)^{1/p_i^+}$$

$$\leq n^{1-\frac{p_i^-}{p_i^+}} \Big(\int_{\Omega\setminus\Omega_n} a_i(x) \Big| \sum_{k=1}^{N} \frac{a_{kx_i}}{a_k} \Big|^{p_i(x)} dx \Big)^{1/p_i^+}.$$

From Theorem 1.3 we obtain the following result.

Corollary 1.6. *Let $a_i(x) \in C^1(\overline{\Omega})$ satisfy (1.6), and $u(x,t)$ and $v(x,t)$ be two solutions of equation (1.2) with the initial values $u_0(x)$ and $v_0(x)$ respectively. If for the sufficiently large n, it holds*

$$n^{1-\frac{p_i^-}{p_i^+}} \Big(\int_{\Omega\setminus\Omega_n} a_i(x) \Big| \sum_{k=1}^{N} \frac{a_{kx_i}}{a_k} \Big|^{p_i(x)} dx \Big)^{1/p_i^+} \leq c, \tag{1.16}$$

then the stability (1.12) is true.

Similarly, since

$$\int_{\Omega} a_i(x) \Big| \frac{\chi_{x_i}(x)}{\chi(x)} \Big|^{p_i(x)} dx = \int_{\Omega} a_i(x) \Big| \sum_{k=1}^{N} \frac{a_{kx_i}}{a_k} \Big|^{p_i(x)} dx,$$

by Theorem 1.4, we have the following result.

Corollary 1.7. *Let $a_i(x) \in C^1(\overline{\Omega})$ satisfy (1.6), and $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (1.2) with the initial values $u_0(x)$ and $v_0(x)$ respectively, If there exists a characteristic function $\chi(x)$ such that*

$$\int_{\Omega} a_i(x) \Big| \sum_{k=1}^{N} \frac{a_{kx_i}}{a_k} \Big|^{p_i(x)} dx < \infty, \tag{1.17}$$

then the stability (1.12) is true.

If $a_i(x) \equiv a(x)$, then condition (1.14) holds, i.e. equation (1.2) reduces to

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \Big(a(x)|u_{x_i}|^{p(x)-2} u_{x_i} \Big)_{x_i}. \tag{1.18}$$

From Theorem 1.5, we have the following result.

Corollary 1.8. *Let $a(x) \in C^1(\overline{\Omega})$ satisfy (1.5) and $u(x,t)$ and $v(x,t)$ be two solutions of equation (1.18) with the differential initial values $u_0(x)$ and $v_0(x)$ respectively. Then*

$$\int_{\Omega} (a(x))^N |u(x,t) - v(x,t)|^2 dx \leq c \int_{\Omega} (a(x))^N |u_0(x) - v_0(x)|^2 dx.$$

If $a_i(x) \equiv a(x)$ and $p_i(x) \equiv p$, then condition (1.16) is equivalent to condition (1.17), which is also equivalent to

$$\int_{\Omega} \frac{|a_{x_i}|^p}{a^{p-1}} dx < \infty. \tag{1.19}$$

In this case, equation (1.2) reduces to

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \Big(a(x)|u_{x_i}|^{p_i-2} u_{x_i} \Big). \tag{1.20}$$

If (1.19) is true, then the stability (1.12) is true without any boundary condition. As we can see, equation (1.20) is different from the evolutionary p-Laplacian equation:

$$u_t = \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u). \tag{1.21}$$

It is notable that if we choose appropriate weak characteristic functions, we can obtain nice results on the stability. One can see that the weak characteristic function method can also be generalized to study the stability of weak solutions to a more general degenerate parabolic equation as well as the evolutionary p-Laplacian equations.

The remainder of this paper is structured as follows. In Sections 2-4, we prove Theorems 1.3-1.5 respectively, by means of the proposed weak characteristic function method. In Section 5, we extend this method to study the stability of solutions of the evolutionary p-Laplacian equation (1.21).

2. Proof of Theorem 1.3

Following [13, 19], we denote the variable exponent Sobolev space by $W^{1,p(x)}(\Omega)$. To prove Theorem 1.4, we need the following technical lemma [13, 19].

Lemma 2.1.

(i) *The spaces* $\big(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)}\big)$, $\big(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)}\big)$ *and* $W_0^{1,p(x)}(\Omega)$ *are reflexive Banach spaces.*

(ii) *($p(x)$-Hölder's inequality) Let* $q_1(x)$ *and* $q_2(x)$ *be real functions with* $\frac{1}{q_1(x)} + \frac{1}{q_2(x)} = 1$ *and* $q_1(x) > 1$. *Then, the conjugate space of* $L^{q_1(x)}(\Omega)$ *is* $L^{q_2(x)}(\Omega)$. *For any* $u \in L^{q_1(x)}(\Omega)$ *and* $v \in L^{q_2(x)}(\Omega)$, *it holds*

$$\big|\int_\Omega uv dx\big| \le 2\|u\|_{L^{q_1(x)}(\Omega)}\|v\|_{L^{q_2(x)}(\Omega)}. \tag{2.1}$$

(iii) *It holds*

$$\text{If } \|u\|_{L^{p(x)}(\Omega)} = 1, \text{ then } \int_\Omega |u|^{p(x)} dx = 1.$$

$$\text{If } \|u\|_{L^{p(x)}(\Omega)} > 1, \text{ then } |u|^{p^-}_{L^{p(x)}(\Omega)} \le \int_\Omega |u|^{p(x)} dx \le |u|^{p^+}_{L^{p(x)}(\Omega)}.$$

$$\text{If } \|u\|_{L^{p(x)}(\Omega)} < 1, \text{ then } |u|^{p^+}_{L^{p(x)}(\Omega)} \le \int_\Omega |u|^{p(x)} dx \le |u|^{p^-}_{L^{p(x)}(\Omega)}.$$

(iv) *If* $p_1(x) \le p_2(x)$, *then* $L^{p_1(x)}(\Omega) \supset L^{p_2(x)}(\Omega)$.

(v) *If* $p_1(x) \le p_2(x)$, *then* $W^{1,p_2(x)}(\Omega) \hookrightarrow W^{1,p_1(x)}(\Omega)$.

(vi) *($p(x)$-Poincaré inequality) If* $p(x) \in C(\Omega)$, *then there is a constant* $C > 0$, *such that*

$$\|u\|_{L^{p(x)}}(\Omega) \le C\|\nabla u\|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

This implies $\|\nabla u\|_{L^{p(x)}}(\Omega)$ *and* $\|u\|_{W^{1,p(x)}(\Omega)}$ *being equivalent to the norms of* $W_0^{1,p(x)}(\Omega)$.

For $n > 0$, let

$$g_n(s) = \int_0^s h_n(\tau)d\tau, \quad h_n(s) = 2n(1 - |ns|)_+.$$

Obviously, $h_n(s) \in C(\mathbb{R})$, and

$$h_n(s) \geq 0, \quad |sh_n(s)| \leq 1, \quad |g_n(s)| \leq 1,$$
$$\lim_{\eta \to 0} g_n(s) = \operatorname{sgn} s, \quad \lim_{\eta \to 0} sg_n'(s) = 0. \tag{2.2}$$

Let $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (1.2) with the initial values $u_0(x)$ and $v_0(x)$ respectively, but without any boundary condition. Let $\chi(x)$ be a weak characteristic function of Ω. We define

$$\phi_n(x) = \begin{cases} 1, & \text{if } x \in \Omega_n, \\ n\chi(x), & \text{if } x \in \Omega \setminus \Omega_n, \end{cases} \tag{2.3}$$

where $\Omega_n = \{x \in \Omega : \chi(x) > \frac{1}{n}\}$. By a process of limit, we choose

$$\varphi_1 = \chi_{[\tau,s]}\phi_n, \quad \varphi_2 = g_n(u-v),$$

and take $\chi_{[\tau,s]}\phi_n g_n(u-v)$ as the test function. Here, $\chi_{[\tau,s]}$ is the characteristic function of $[\tau, s) \subseteq [0, T)$. Then we have

$$\int_\tau^s \int_\Omega \phi_n g_n(u-v) \frac{\partial(u-v)}{\partial t} \, dx \, dt + \sum_{i=1}^N \int_\tau^s \int_\Omega a_i(x) \Big(|u_{x_i}|^{p_i(x)-2}u_{x_i}$$
$$- |v_{x_i}|^{p_i(x)-2}v_{x_i} \Big)(u_{x_i} - v_{x_i})g_n'(u-v)\phi_n(x) \, dx \, dt$$
$$+ \sum_{i=1}^N \int_\tau^s \int_\Omega a_i(x) \Big(|u_{x_i}|^{p_i(x)-2}u_{x_i} - |v_{x_i}|^{p_i(x)-2}v_{x_i} \Big)$$
$$\times (u_{x_i} - v_{x_i})g_n(u-v)\phi_{nx_i} \, dx \, dt = 0. \tag{2.4}$$

In the third term of the left-hand side of (2.4), we note that

$$\int_\Omega a_i(x) \Big(|u_{x_i}|^{p_i(x)-2}u_{x_i} - |v_{x_i}|^{p_i(x)-2}v_{x_i} \Big)(u_{x_i} - v_{x_i})g_n'(u-v)\phi_n(x) dx \geq 0. \tag{2.5}$$

For the first term of the left hand side of (2.4), in view of $u_t \in L^2(Q_T)$, it follows the Lebesgue dominated convergence theorem that

$$\lim_{n \to \infty} \int_\tau^s \int_\Omega \phi_n(x)g_n(u-v) \frac{\partial(u-v)}{\partial t} \, dx \, dt$$
$$= \int_\Omega |u-v|(x,s)dx - \int_\Omega |u-v|(x,\tau)dx. \tag{2.6}$$

Since $\phi_{nx_i} = n\chi_{x_i}$ when $x \in \Omega \setminus \Omega_n$, by (iii) of Lemma 2.1 we deduce that

$$\left| \int_\Omega a_i(x) \Big(|u_{x_i}|^{p_i(x)-2}u_{x_i} - |v_{x_i}|^{p_i(x)-2}v_{x_i} \Big) \phi_{nx_i}g_n(u-v)dx \right|$$
$$= \left| \int_{\Omega \setminus \Omega_n} a_i(x) \Big(|u_{x_i}|^{p_i(x)-2}u_{x_i} - |v_{x_i}|^{p_i(x)-2}v_{x_i} \Big) \phi_{nx_i}g_n(u-v)dx \right|$$
$$\leq n \int_{\Omega \setminus \Omega_n} a_i(x) \Big(|u_{x_i}|^{p_i(x)-1} + |v_{x_i}|^{p_i(x)-1} \Big) \chi_{x_i}g_n(u-v)dx$$
$$\leq cn \Big(\int_{\Omega \setminus \Omega_n} a_i(x) \Big(|u_{x_i}|^{p_i(x)} + |v_{x_i}|^{p_i(x)} \Big) dx \Big)^{1/q_i^+} \Big(\int_{\Omega \setminus \Omega_n} a_i(x)|\chi_{x_i}|^{p_i(x)}dx \Big)^{1/p_i^+}$$
$$\leq c \Big[\Big(\int_{\Omega \setminus \Omega_n} a_i(x)|u_{x_i}|^{p_i(x)}dx \Big)^{1/q_i^+} + \Big(\int_{\Omega \setminus \Omega_n} a_i(x)|v_{x_i}|^{p_i(x)}dx \Big)^{1/q_i^+} \Big]$$

$$\times \Big[n\Big(\int_{\Omega \backslash \Omega_n} a_i(x) |\chi_{x_i}|^{p_i(x)} dx \Big)^{1/p_i^+} \Big]$$

$$\leq c\Big(\int_{\Omega \backslash \Omega_n} a_i(x) |u_{x_i}|^{p_i(x)} dx \Big)^{1/q_i^+} + c\Big(\int_{\Omega \backslash \Omega_n} a_i(x) |v_{x_i}|^{p_i(x)} dx \Big)^{1/q_i^+},$$

where $q_i(x) = \frac{p_i(x)}{p_i(x)-1}$ and $q_i^+ = \max_{x \in \overline{\Omega}} q_i(x)$.

Therefore,

$$\lim_{n \to \infty} \Big| \int_\tau^s \int_\Omega a_i(x) \Big(|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i} \Big) \phi_{nx_i} g_n(u-v) \, dx \, dt \Big|$$

$$\leq c \lim_{n \to \infty} \Big[\Big(\int_{\Omega \backslash \Omega_n} a_i(x) |u_{x_i}|^{p_i(x)} dx \Big)^{1/q_i^+} + \Big(\int_{\Omega \backslash \Omega_n} a_i(x) |v_{x_i}|^{p_i(x)} dx \Big)^{1/q_i^+} \Big] \quad (2.7)$$

$$= 0.$$

Let $\eta \to 0$ in (2.4). Then we have

$$\int_\Omega |u(x,s) - v(x,s)| dx \leq \int_\Omega |u(x,\tau) - v(x,\tau)| dx, \quad (2.8)$$

Because of the arbitrariness of τ, we obtain

$$\int_\Omega |u(x,s) - v(x,s)| \, dx \leq c \int_\Omega |u_0(x) - v_0(x)| \, dx.$$

3. Proof of Theorem 1.4

Making a minor modification, we can generalize Definition 1.1 to the following version.

Definition 3.1. Suppose that $u(x,t)$ satisfies (1.7). If for any function $g(s) \in C^1(\mathbb{R})$ with $g(0) = 0$, $\varphi_1 \in C_0^1(\Omega)$ and $\varphi_{2x_i} \in L^2(0,T; L^{p_i(x)}(a_i, \Omega))$ it holds

$$\iint_{Q_T} \Big[\frac{\partial u}{\partial t} g(\varphi_1 \varphi_2) + \sum_{i=1}^N a_i(x) |u_{x_i}|^{p_i(x)-2} u_{x_i} g_{x_i}(\varphi_1 \varphi_2) \Big] dx \, dt = 0, \quad (3.1)$$

and the initial value condition (1.3) is satisfied in the sense of (1.9), then $u(x,t)$ is said to be a weak solution of equation (1.2) with initial condition (1.3).

Let $u(x,t)$ and $v(x,t)$ be two weak solutions of (1.2) with the initial values $u_0(x)$ and $v_0(x)$ respectively, and χ be a weak characteristic function. We choose $g_n(\chi(u-v))$ as the test function in Definition 3.1. Then we have

$$\int_\Omega g_n(\chi(u-v)) \frac{\partial(u-v)}{\partial t} dx$$

$$+ \sum_{i=1}^N \int_\Omega \chi(x) a_i(x) \Big(|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i} \Big) (u-v)_{x_i} g_n'(\chi(u-v)) dx$$

$$+ \sum_{i=1}^N \int_\Omega a_i(x) \Big(|u_{x_i}|^{p_i(x)-2} u_{x_i} - |v_{x_i}|^{p_i(x)-2} v_{x_i} \Big) \chi_{x_i}(u-v) g_n'(\chi(u-v)) dx$$

$$= 0. \quad (3.2)$$

Let us evaluate each term in the left hand side of (3.2). For the first two terms, we find that

$$\lim_{n\to\infty}\int_\Omega g_n(\chi(u-v))\frac{\partial(u-v)}{\partial t}dx = \frac{d}{dt}\int_\Omega |u(x,t)-v(x,t)|dx, \qquad (3.3)$$

$$\int_\Omega \chi(x)a_i(x)\left(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i}\right)(u-v)_{x_i}g_n'(\chi(u-v))dx \geq 0,$$

and

$$\left|\int_\Omega a_i(x)(u-v)g_n'(\chi(u-v))\left(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i}\right)\chi_{x_i}dx\right|$$

$$= \left|\int_{\{\Omega:\chi|u-v|<1/n\}} a_i^{-\frac{p_i(x)-1}{p_i(x)}}a_i(x)(u-v)g_n'(\chi(u-v))a_i^{\frac{p_i(x)-1}{p_i(x)}}\right.$$

$$\left.\times\left(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i}\right)\chi_{x_i}dx\right| \qquad (3.4)$$

$$\leq \left(\int_{\{\Omega:\chi|u-v|<\frac{1}{n}\}}|a_i^{\frac{1}{p_i(x)}}(u-v)g_n'(\chi(u-v))\chi_{x_i}|^{p_i(x)}dx\right)^{1/p_{i1}}$$

$$\times\left(\int_{\{\Omega:\chi|u-v|<1/n\}}a_i(x)\left(|u_{x_i}|^{p_i(x)}+|v_{x_i}|^{p_i(x)}\right)dx\right)^{1/q_{i1}},$$

where $p_{i1}=p_i^+$ or p_i^- based on (iii) of Lemma 2.1, and similar for q_{i1}.

If $\{x\in\Omega:|u-v|=0\}$ has zero measure, since

$$\int_\Omega a_i(x)\left|\frac{\chi_{x_i}}{\chi}\right|^{p_i(x)}dx < \infty,$$

we derive that

$$\int_{\{\Omega:\chi|u-v|<1/n\}}\left|a_i^{\frac{1}{p_i(x)}}\frac{\chi_{x_i}}{\chi}\chi(u-v)g_n'(\chi(u-v))\right|^{p_i(x)}dx \leq c, \qquad (3.5)$$

and

$$\lim_{n\to\infty}\left(\int_{\{\Omega:\chi|u-v|<\frac{1}{n}\}}a_i(x)\left(|u_{x_i}|^{p_i(x)}+|v_{x_i}|^{p_i(x)}\right)dx\right)^{1/q_{i1}}$$

$$= \left(\int_{\{\Omega:|u-v|=0\}}a_i(x)\left(|u_{x_i}|^{p_i(x)}+|v|^{p_i(x)}\right)dx\right)^{1/q_{i1}} = 0. \qquad (3.6)$$

If $\{x\in\Omega:u-v=0\}$ has a positive measure, then

$$\lim_{n\to\infty}\left(\int_{\{\Omega:\chi|u-v|<\frac{1}{n}\}}|a_i^{\frac{1}{p_i(x)}}\frac{\chi_{x_i}}{\chi}(u-v)g_n'(\chi(u-v))|^{p_i(x)}dx\right)^{1/p_{i1}}$$

$$= \left(\int_{\{\Omega:|u-v|=0\}}a_i(x)\left|\frac{\chi_{x_i}}{\chi}\right|^{p_i(x)}\lim_{n\to\infty}|(u-v)g_n'((u-v)\chi)|^{p_i(x)}dx\right)^{1/p_{i1}} \qquad (3.7)$$

$$= 0.$$

In view of (2.2) and condition (1.13), it follows the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}\left|\int_\Omega a_i(x)(u-v)g_n'(\chi(u-v))\left(|u_{x_i}|^{p_i(x)-2}u_{x_i}-|v_{x_i}|^{p_i(x)-2}v_{x_i}\right)\chi_{x_i}dx\right| = 0.$$

We now letting $\eta\to 0$ in (3.2), we have

$$\frac{d}{dt}\int_\Omega |u(x,t)-v(x,t)|dx \leq \int_\Omega |u(x,t)-v(x,t)|dx.$$

By Gronwall's inequality, we obtain

$$\int_\Omega |u(x,t) - v(x,t)|\, dx \le c \int_\Omega |u_0(x) - v_0(x)|\, dx, \quad \forall t \in [0, T).$$

4. Proof of Theorem 1.5

Let $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (1.2) with the initial values $u_0(x)$ and $v_0(x)$ respectively. Then we have

$$\iint_{Q_T} \Big[\big(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}\big)\varphi + \sum_{i=1}^N a_i(x)\big(|u_{x_i}|^{p_i(x)-2}u_{x_i} - |v_{x_i}|^{p_i(x)-2}v_{x_i}\big)\varphi_{x_i} \Big] dx\, dt \tag{4.1}$$
$$= 0.$$

Let

$$\varphi = \chi_{[\tau,s]}(u-v)\chi(x),$$

where $\chi_{[\tau,s]}$ is the characteristic function on $[\tau, s]$ and $\chi(x)$ is a weak characteristic function of Ω. Denote $Q_{\tau s} = \Omega \times [\tau, s]$. Then we have

$$\iint_{Q_{\tau s}} a_i(x)\Big(|u_{x_i}|^{p_i(x)-2}u_{x_i} - |v_{x_i}|^{p_i(x)-2}v_{x_i}\Big)[(u-v)\chi]_{x_i}\, dx\, dt$$
$$= \iint_{Q_{\tau s}} a_i(x)\chi(x)\Big(|u_{x_i}|^{p_i(x)-2}u_{x_i} - |v_{x_i}|^{p_i(x)-2}v_{x_i}\Big)(u-v)_{x_i}\, dx\, dt \tag{4.2}$$
$$+ \iint_{Q_{\tau s}} a_i(x)\Big(|u_{x_i}|^{p_i(x)-2}u_{x_i} - |v_{x_i}|^{p_i(x)-2}v_{x_i}\Big)(u-v)\chi_{x_i}\, dx\, dt.$$

Clearly, it has

$$\iint_{Q_{\tau s}} a_i(x)\chi(x)\Big(|u_{x_i}|^{p_i(x)-2}u_{x_i} - |v_{x_i}|^{p_i(x)-2}v_{x_i}\Big)(u-v)_{x_i}\, dx\, dt \ge 0. \tag{4.3}$$

Evaluating the second term on the right hand side of (4.2) yields

$$\Big| \iint_{Q_{\tau s}} (u-v)a_i(x)\Big(|u_{x_i}|^{p_i(x)-2}u_{x_i} - |v_{x_i}|^{p_i(x)-2}v_{x_i}\Big)\chi_{x_i}\, dx\, dt \Big|$$
$$\le \iint_{Q_{\tau s}} |u-v|a_i(x)\Big(|u_{x_i}|^{p_i(x)-1} + |v_{x_i}|^{p_i(x)-1}\Big)|\chi_{x_i}|\, dx\, dt$$
$$\le c\Big(\int_\tau^s \int_\Omega a_i(x)\Big(|u_{x_i}|^{p_i(x)} + |v|^{p_i(x)}\Big)\, dx\, dt \Big)^{1/q_{i1}} \tag{4.4}$$
$$\times \Big(\int_\tau^s \int_\Omega a_i(x)|\chi_{x_i}|^{p_i(x)}|u-v|^{p_i(x)}\, dx\, dt \Big)^{1/p_{i1}}$$
$$\le c\Big(\int_\tau^s \int_\Omega a_i(x)|\chi_{x_i}|^{p_i(x)}|u-v|^{p_i(x)}\, dx\, dt \Big)^{1/p_{i1}}.$$

Since $\frac{a_i(x)|\chi_{x_i}|^{p_i(x)}}{\chi} \le c$, by (4.4) we have

$$\Big| \iint_{Q_{\tau s}} (u-v)a_i(x)\Big(|u_{x_i}|^{p_i(x)-2}u_{x_i} - |v_{x_i}|^{p_i(x)-2}v_{x_i}\Big)\chi_{x_i}\, dx\, dt \Big|$$
$$\le c\Big(\int_\tau^s \int_\Omega \chi|u-v|^{p_i(x)}\, dx\, dt \Big)^{1/p_{i1}}. \tag{4.5}$$

If $p_i(x) \geq 2$, then

$$\Big(\int_\tau^s \int_\Omega \chi(x)|u-v|^{p_i(x)} \, dx \, dt \Big)^{1/p_{i1}} \leq c \Big(\int_\tau^s \int_\Omega \chi(x)|u-v|^2 \, dx \, dt \Big)^{1/p_{i1}}.$$

If $1 < p_i(x) < 2$, by the Hölder inequality we have

$$\int_\tau^s \int_\Omega \chi(x)|u-v|^{p_i(x)} \, dx \, dt \leq c \Big(\int_\tau^s \int_\Omega \chi(x)|u-v|^2 \, dx \, dt \Big)^{\frac{1}{p_{i2}}},$$

where p_{i2} is $\max_{x\in\overline{\Omega}} \frac{2}{p_i(x)}$ or $\min_{x\in\overline{\Omega}} \frac{2}{p_i(x)}$, depending on $\int_\tau^s \int_\Omega \chi|u-v|^{p_i(x)} \, dx \, dt \geq 1$ or $\int_\tau^s \int_\Omega \chi|u-v|^{p_i(x)} \, dx \, dt < 1$. Thus, we obtain

$$\Big(\int_\tau^s \int_\Omega \chi(x)|u-v|^{p_i(x)} \, dx \, dt \Big)^{1/p_{i1}} \leq c \Big(\int_\tau^s \int_\Omega \chi(x)|u-v|^2 \, dx \, dt \Big)^{\frac{1}{p_{i1}} \frac{1}{p_{i2}}} \qquad (4.6)$$

and

$$\iint_{Q_{\tau s}} (u-v)\chi(x)\frac{\partial(u-v)}{\partial t} \, dx \, dt$$
$$= \int_\Omega \chi(x)[u(x,s)-v(x,s)]^2 dx - \int_\Omega \chi(x)[u(x,\tau)-v(x,\tau)]^2 dx. \qquad (4.7)$$

In view of (4.2)-(4.7), letting $\lambda \to 0$ in (4.1) leads to

$$\int_\Omega \chi(x)[u(x,s)-v(x,s)]^2 dx - \int_\Omega \chi(x)[u(x,\tau)-v(x,\tau)]^2 dx$$
$$\leq c \Big(\int_0^s \int_\Omega \chi(x)|u(x,t)-v(x,t)|^2 \, dx \, dt \Big)^q, \qquad (4.8)$$

where $q < 1$. By (4.8), it is easy to see that

$$\int_\Omega \chi(x)|u(x,s)-v(x,s)|^2 dx \leq \int_\Omega \chi(x)|u(x,\tau)-v(x,\tau)|^2 dx. \qquad (4.9)$$

Due to the arbitrariness of τ, we obtain

$$\int_\Omega \chi(x)|u(x,s)-v(x,s)|^2 dx \leq \int_\Omega \chi(x)|u_0(x)-v_0(x)|^2 dx.$$

5. STABILITY OF p-LAPLACIAN EQUATION

In the preceding two sections, we use the weak characteristic function method to prove Theorems 1.3-1.5. In this section, we consider equation (1.21) with the initial value condition (1.3), but without any boundary condition. We apply the proposed weak characteristic function method to prove the stability of solutions of equation (1.21).

Proposition 5.1. *Let $a(x) \in C^1(\overline{\Omega})$ satisfy (1.5), and $u(x,t)$ and $v(x,t)$ be two weak solutions of equation (1.21) with the initial values $u_0(x)$ and $v_0(x)$ respectively. When $p > 1$, for the sufficiently large n, it holds*

$$n^{1-\frac{(N-1)p+1}{Np}} \Big(\int_{\Omega\backslash\Omega_n} |\nabla a|^p dx \Big)^{1/p} \leq c, \qquad (5.1)$$

where c is a constant. Then the stability (1.12) is true.

Proof. Let $\chi(x) = [a(x)]^N$. We can choose $\phi_n g_n(u - v)$ as the test function, then

$$
\begin{aligned}
&\int_\Omega \phi_n(x) g_n(u - v) \frac{\partial(u - v)}{\partial t} dx \\
&+ \int_\Omega a(x) \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla(u - v) g'_n(u - v) \phi_n(x) dx \\
&+ \int_\Omega a(x) \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla(u - v) g_n(u - v) \nabla \phi_n dx \\
&= 0.
\end{aligned}
\tag{5.2}
$$

Clearly, we see that

$$
\int_\Omega a(x) \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla(u - v) g'_n(u - v) \phi_n(x) dx \geq 0.
\tag{5.3}
$$

By a straightforward computations, we derive that

$$
\begin{aligned}
&\left| \int_\Omega a(x) \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla \phi_n g_n(u - v) dx \right| \\
&= \left| \int_{\Omega \backslash \Omega_n} a(x) \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot \nabla \phi_n g_n(u - v) dx \right| \\
&= \left| \int_{\Omega \backslash \Omega_n} a(x) \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \cdot n g_n(u - v) [a(x)]^{N-1} \nabla a \, dx \right| \\
&\leq c \left(\int_{\Omega \backslash \Omega_n} a(x) (|\nabla u|^p + |\nabla u|^p) \right)^{\frac{p-1}{p}} n \left(\int_{\Omega \backslash \Omega_n} a(x) [a^{N-1} |\nabla a|]^p dx \right)^{1/p} \\
&\leq c \left(\int_{\Omega \backslash \Omega_n} a(x) (|\nabla u|^p + |\nabla u|^p) \right)^{\frac{p-1}{p}} n^{1 - \frac{(N-1)p+1}{Np}} \left(\int_{\Omega \backslash \Omega_n} |\nabla a|^p dx \right)^{1/p} \\
&\leq c \left(\int_{\Omega \backslash \Omega_n} a(x) (|\nabla u|^p + |\nabla u|^p) \right)^{\frac{p-1}{p}} n^{1 - \frac{(N-1)p+1}{Np}} \left(\int_{\Omega \backslash \Omega_n} |\nabla a|^p dx \right)^{1/p},
\end{aligned}
\tag{5.4}
$$

which approaches 0 as $n \to \infty$. Hence, by (5.2)-(5.4), the desired result is obtained. $\square$

If $a(x) = d^\alpha(x)$, then

$$
n^{1 - \frac{(N-1)p+1}{Np}} \left(\int_{\Omega \backslash \Omega_n} |\nabla a|^p dx \right)^{1/p} \leq c n^{1 - \frac{(N-1)p+1}{Np} - \frac{1+p(\alpha-1)}{N\alpha}}.
\tag{5.5}
$$

Let $\alpha \to \infty$. It is easy to see that

$$
\lim_{\alpha \to \infty} \left(1 - \frac{(N-1)p+1}{Np} - \frac{1+p(\alpha-1)}{N\alpha} \right) = \frac{p - 1 - p^2}{Np} < 0.
$$

So we can choose an α such that

$$
\lim_{n \to \infty} n^{1 - \frac{(N-1)p+1}{Np} - \frac{1+p(\alpha-1)}{N\alpha}} = 0.
\tag{5.6}
$$

Proposition 5.2. *Let $a(x) \in C^1(\overline{\Omega})$ satisfy (1.5), and $u(x, t)$ and $v(x, t)$ be two weak solutions of the equation*

$$
u_t = \mathrm{div}(d^\alpha |\nabla u|^{p-2} \nabla u)
\tag{5.7}
$$

with the initial values $u_0(x)$ and $v_0(x)$ respectively. If $p > 1$, for the sufficiently large α, then the stability (1.12) is true.

Next, we give further discussions on the constant α in Proposition 5.2.

Proposition 5.3. *Let $a(x) \in C^1(\overline{\Omega})$ satisfy (1.5), and $u(x,t)$ and $v(x,t)$ be two solutions of equation (5.7) with the initial values $u_0(x)$ and $v_0(x)$ respectively. When $p > 1$, we have*

$$\int_\Omega \frac{|\nabla a|^p}{a^{p-1}} dx \le c, \tag{5.8}$$

where c is a constant. Then the stability (1.12) is true.

Proof. Let $\chi(x) = [a(x)]^N$. We can choose $g_n(\chi(u-v)) = g_n(a^N(u-v))$ as the test function. Then

$$\int_\Omega g_n(a^N(u-v)) \frac{\partial(u-v)}{\partial t} dx$$
$$+ \int_\Omega a(x) \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right) \cdot a^N \nabla(u-v) g'_n(a^N(u-v))\phi_n(x) dx \tag{5.9}$$
$$+ \int_\Omega a(x) \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right) \cdot \nabla a^N(u-v) g'_n(a^N(u-v)) dx$$
$$= 0.$$

Clearly,

$$\int_\Omega a(x) \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right) \cdot \nabla(u-v) g'_n(a^N(u-v)) a^N dx \ge 0. \tag{5.10}$$

By a direct calculation, we deduce that

$$\left| \int_\Omega a(x) \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right) \right.$$
$$\left. \cdot \nabla a^N(u-v) g'_n(a^N(u-v)) dx \right|$$
$$= \left| \int_{\{\Omega: a^N|u-v|<1/n\}} a(x) \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right) \right.$$
$$\left. \cdot \nabla a^N(u-v) g'_n(a^N(u-v)) dx \right|$$
$$= N \left| \int_{\{\Omega: a^N|u-v|<1/n\}} a(x) \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right) \right. \tag{5.11}$$
$$\left. \cdot \frac{\nabla a}{a} a^N(u-v) g'_n(a^N(u-v)) dx \right|$$
$$\le c \left(\int_{\{\Omega: a^N|u-v|<\frac{1}{n}\}} a(x) \left(|\nabla u|^p + |\nabla u|^p \right) \right)^{\frac{p-1}{p}}$$
$$\cdot \left(\int_{\{\Omega: a^N|u-v|<\frac{1}{n}\}} a(x) \left| \frac{\nabla a}{a} \right|^p a^N |(u-v) g'_n(a^N(u-v))|^p dx \right)^{1/p}.$$

As for (3.5)-(3.7), we can derive that

$$\lim_{n\to\infty} \left| \int_\Omega a(x) \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right) \cdot \nabla a^N(u-v) g'_n(a^N(u-v)) dx \right| = 0. \tag{5.12}$$

Consequently, using (5.9)-(5.12), we arrive at the desire result. $\square$

If $a(x) = d^\alpha(x)$, then

$$\frac{|\nabla a|^p}{a^{p-1}} = \frac{\alpha^p d^{(\alpha-1)p}}{d^{\alpha(p-1)}} = \alpha^p d^{\alpha-p}. \tag{5.13}$$

Therefore, we can obtain the following proposition which is identical to the corresponding result of [18].

Proposition 5.4. *Let $a(x) \in C^1(\overline{\Omega})$ satisfy* (1.5)*, and $u(x,t)$ and $v(x,t)$ be two solutions of equation* (5.7) *with the initial values $u_0(x)$ and $v_0(x)$ respectively. If $p > 1$ and $\alpha > p - 1$, then the stability* (1.12) *is true.*

References

[1] E. Acerbi, G. Mingione; Regularity results for stationary electrorheological fluids, *Arch. Ration. Mech. Anal.*, **164** (2002), 213-259.

[2] S. Antontsev, J. I. Diaz, S. Shmarev; *Energy methods for free boundary problems: applications to nonlinear PDEs and fluid mechanics.* In: Progress in Nonlinear Differential Equations and theif Applications, Vol.48. Birkäuser, Boston, 2002.

[3] S. Antontsev, S. Shmarev; Parabolic equations with double variable nonlinearlities, *Math. Comput. Simulation*, **81**(2011), 2018-2032.

[4] J. Aramaki; Hölder continuity with exponent $(1 + \alpha)/2$ in the time variable for solutions of parabolic equations, *Electron. J. Differential Equations*, **2015** No. 96 (2015), 1-6.

[5] J. Bear; *Dynamics of Fluids in Porous Media*, American Elsevier, New York, 1972.

[6] M. Bendahmane, K. H. Karlsen; Renormalized solutions of an anisotropic reaction-diffusion-adverction system with L^1 data, *Commu. Pure Appl. Anal.*, **5** (2006), 733-762.

[7] M. Bendahmane, M. Langlais, M. Saad, On some anisotropic reaction-diffusion system with L^1 data modeling the propagation of an epidemic disease, *Nonlinear Anal.*, **54** (2003), 617-636.

[8] A. Di Castro; Anisotropic elliptic problems with natural growth terms, *Manuscripta Math.*, **135** (2011), 521-543.

[9] A. El Hamidi, J. Vétois; Sharp Sobolev asymptotics for critical anisotropic equations, *Arch. Rational Mech. AnaL.*, **192** (2009), 1-26.

[10] I. Fragalà, F. Gazzola, B. Kawohl; Existence and nonexistence results for anisotropic quasilinear equations, *Ann. Inst. H. Poincare Anal. Nonlin.*, **21** (2004), 715-734.

[11] K. Ho, I. Sim; On degenerate $p(x)$−Laplacian equations involving critical growth with two parameters, *Nonlinear Anal.*, **132** (2016), 95-114.

[12] I. Konaté, S. Ouaro; Good Radon measure for anisotropic problems with variable exponent, *Electron. J. Differential Equations*, **2016** No. 221 (2016), 1-19.

[13] S. Lian, W. Gao, H. Yuan, C. Cao; Existence of solutions to an initial Dirichlet problem of evolutional $p(x)$-Laplace equations, *Ann. Inst. H. Poincare Anal. Nonlin.*, **29** (2012), 377-399.

[14] R. D. Nardo, F. Feo; Existence and uniqueness for nonlinear anisotropic elliptic equations, *Arch. Math.*, **102** (2004), 141-153.

[15] M. Ruzicka; *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Math., vol.1748, Springer, Berlin, 2000.

[16] S. Tersenov Alkis, S. Tersenov Aris; Existence of Lipschitz continuous solutions to the Cauchy-Dirichlet problem for anisotropic parabolic equations, *J. Funct. Anal.*, **272** (2017), 3965-3986.

[17] H. Tian, S. Z. Zhen; Orlicz estimates for general parabolic obstacle problems with $p(t,x)$-growth in Reifenberg domains, *Electron. J. Differential Equations*,**2020** No. 13 (2020), 1-25.

[18] J. Yin, C. Wang; Properties of the boundary flux of a singular diffusion process, *Chin. Ann. Math.*, Ser. B, **25** (2004), 175-182.

[19] H. Zhan, J. Wen; Evolutionary $p(x)$-Laplacian equation free from the limitation of the boundary value, *Electron. J. Differential Equations*, **2016** No. 143 (2016), 1-13.

Huashui Zhan

School of Applied Mathematics, Xiamen University of Technology, Xiamen, Fujian 361024, China

Email address: 2012111007@xmut.edu.cn

Zhaosheng Feng

School of Mathematical and Statistical Sciences, University of Texas Rio Grande Valley, Edinburg, TX 78539, USA

Email address: zhaosheng.feng@utrgv.edu