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KDV TYPE ASYMPTOTICS FOR SOLUTIONS TO HIGHER-ORDER NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider the Cauchy problem for the higher-order nonlinear Schrödinger equation

$$i\partial_t u - \frac{a}{3}|\partial_x|^3 u - \frac{b}{4}\partial_x^4 u = \lambda i \partial_x(|u|^2 u), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R},$$
$$u(0,x) = u_0(x), \quad x \in \mathbb{R},$$

where $a, b > 0$, $|\partial_x|^\alpha = \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}$ and $\mathcal{F}$ is the Fourier transformation. Our purpose is to study the large time behavior of the solutions under the non-zero mass condition $\int u_0(x)dx \neq 0$.

1. Introduction

We consider the Cauchy problem for the higher-order nonlinear Schrödinger equation

$$\begin{gathered} i\partial_t u - \frac{a}{3}|\partial_x|^3 u - \frac{b}{4}\partial_x^4 u = i\partial_x(|u|^2 u), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{gathered} \tag{1.1}$$

where $a, b > 0$, $|\partial_x|^\alpha = \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}$ and $\mathcal{F}$ is the Fourier transformation defined by $\mathcal{F}\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\phi dx$, and its inverse by $\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\phi(\xi)d\xi$. Note that we have the relation $u(-t,x) = \overline{u}(t,-x)$, so we only consider the case $t > 0$. Note that for the classical solution of (1.1) we have the conservation laws $\int_{\mathbb{R}} u(t,x)dx = \int_{\mathbb{R}} u_0(x)dx$ and $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$.

Equation (1.1) arises in the context of high-speed soliton transmission in long-haul optical communication system [13]. Also it can be considered as a particular form of the higher order nonlinear Schrödinger equation introduced by [42] to describe the nonlinear propagation of pulses through optical fibers. This equation also represents the propagation of pulses by taking higher dispersion effects into account than those given by the Schrödinger equation (see [10, 11, 25, 28, 38, 32, 43, 49]).

Higher order nonlinear Schrödinger equations have been widely studied recently. For the local and global well-posedness of the Cauchy problem we refer to [5, 6, 40] and references cited therein. The dispersive blow-up was obtained in [2]. The existence and uniqueness of solutions to (1.1) were proved in [1, 19, 30, 33, 34, 39, 47, 50], and the smoothing properties of solutions were studied in [3, 8, 12, 30, 33,

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34, 36, 37, 48]. The blow-up effect for a special class of slowly decaying solutions of the Cauchy problem (1.1) was studied in [1].

In this article we are interested in the case of non zero total mass of the initial data

$$\int_{\mathbb{R}} u_0(x)dx \neq 0.$$

Then by (1.1) we obtain the non-zero total mass for the solution $\int_{\mathbb{R}} u(t,x)dx = \int_{\mathbb{R}} u_0(x)dx \neq 0$ for all $t > 0$. We develop the factorization technique originated in our previous papers [22, 21, 26, 20, 44, 45, 46]. The case of zero total mass $\int_{\mathbb{R}} u(t,x)dx = \int_{\mathbb{R}} u_0(x)dx = 0$ is easier and the corresponding results can be obtained following the approach in [24].

We denote the Lebesgue space by $L^p = \{\phi \in \mathbf{S}' : \|\phi\|_{L^p} < \infty\}$, with norms

$$\|\phi\|_{L^p} = \Big(\int |\phi(x)|^p dx\Big)^{1/p}$$

for $1 \le p < \infty$ and $\|\phi\|_{L^\infty} = \sup_{x\in\mathbb{R}} |\phi(x)|$. The weighted Sobolev space is $H_p^{m,s} = \{\varphi \in \mathbf{S}' : \|\phi\|_{H_p^{m,s}} = \|\langle x\rangle^s \langle i\partial_x\rangle^m \phi\|_{L^p} < \infty\}$, where $m, s \in \mathbb{R}$, $1 \le p \le \infty$, $\langle x\rangle = \sqrt{1+x^2}$, $\langle i\partial_x\rangle = \sqrt{1-\partial_x^2}$. We also use the notations $H^{m,s} = H_2^{m,s}$, $H^m = H^{m,0}$ shortly, if it does not cause any confusion. Let $C(I;B)$ be the space of continuous functions from an interval I to a Banach space B. Different positive constants might be denoted by the same letter C.

In [29], it was proved the existence of the self-similar solutions of the form $u = t^{-1/3} f_m(xt^{-1/3})$ to the reduced equation

$$i\partial_t u - \frac{1}{3}|\partial_x|^3 u = i\partial_x(|u|^2 u), \tag{1.2}$$

which is defined by the mean value $m = \int_{\mathbf{R}} f_m(x)dx \neq 0$.

Now we state the main result of this article. We show that the asymptotic behavior of the solutions to (1.1) resembles that of the KdV equation, which was studied intensively (see [15, 18, 23]). We denote by $\mu(x)$ the root of the equation $\Lambda'(\xi) = \xi|\xi|(a + b|\xi|) = x$ for all $x \in \mathbb{R}$.

Theorem 1.1. *Suppose that $\int_{\mathbf{R}} u_0(x)dx \neq 0$, and that the initial data $u_0 \in H^{1,1}$ have a sufficiently small norm $\|u_0\|_{H^{1,1}} \le \varepsilon$. Then there exists a unique global solution $u \in C([0,\infty); H^{1,1})$ to the Cauchy problem (1.1). Furthermore there exists $W_+ \in L^\infty$ such that the large time behavior satisfies*

$$\begin{aligned} u(t,x) &= \frac{M}{\sqrt{it\Lambda''(\mu(\frac{x}{t}))}} W_+\big(\mu(\frac{x}{t})\big) \exp\Big(\frac{i\mu(\frac{x}{t})}{|\Lambda''(\mu(\frac{x}{t}))|}\big|W_+\big(\mu(\frac{x}{t})\big)\big|^2 \log t\Big) \\ &\quad + O(t^{-1/3}\langle t^{1/3}\mu(\frac{x}{t})\rangle^{-3/4}) + O(t^{-\frac{1}{3}-\delta}) \end{aligned}$$

on the domain $|x| \ge t^{\frac{1}{3}+2\gamma}$ and $u(t,x) = t^{-1/3} f_m(xt^{-1/3}) + O(t^{-\frac{1}{3}-\delta})$ on the domain $|x| \le t^{\frac{1}{3}+2\gamma}$, where $\gamma, \delta > 0$ are small and $t^{-1/3} f_m(xt^{-1/3})$ is the self-similar solution to equation (1.2) with $m = \int_{\mathbf{R}} f_m(x)dx = \int_{\mathbf{R}} u_0(x)dx \neq 0$.

We organize the rest of the paper as follows. Section 2 is devoted to the factorization formulas and L^2-boundedness of pseudodifferential operators. Then in Sections 3 and 4 we obtain estimates for the operator $\mathcal{V}$ and $\mathcal{V}^*$, respectively, in the uniform norm and the L^2-estimates of commutators. In Section 5, we prove a priori estimates of the solution $u(t)$ in the norm

$$\|u\|_{X_T} = \sup_{t\in[1,T]} \Big(\|\widehat{\varphi}\|_{L^\infty} + t^{-\gamma}\|\partial_x \mathcal{J} u(t)\|_{L^2} + t^{-\gamma}\|\partial_x^{-1}\mathcal{I}_b u(t)\|_{L^2} + t^{-\gamma}\|\partial_x^{-1}\mathcal{P}_b u(t)\|_{L^2} + t^{-\gamma}\|u(t)\|_{H^1}\Big),$$

where $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$, $\mathcal{U}(t) = \mathcal{F}^{-1}e^{-it\Lambda(\xi)}\mathcal{F}$, $\Lambda(\xi) = \frac{a}{3}|\xi|^3 + \frac{b}{4}\xi^4$, $\mathcal{J} = x - t\Lambda'(-i\partial_x)$, $\mathcal{I}_b = \partial_b + it\partial_b\Lambda(-i\partial_x)$, $\mathcal{P}_b = 3t\partial_t + \partial_x x + b\partial_b$. Section 6 is devoted to the proof of Theorem 1.1.

2. Preliminaries

2.1. **Factorization techniques.** We define the free evolution group $\mathcal{U}(t) = \mathcal{F}^{-1}e^{-it\Lambda(\xi)}\mathcal{F}$, $\Lambda(\xi) = \frac{a}{3}|\xi|^3 + \frac{b}{4}\xi^4$. We write

$$\mathcal{U}(t)\mathcal{F}^{-1}\phi = \mathcal{D}_t\sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{it(x\xi-\Lambda(\xi))}\phi(\xi)d\xi,$$

where $\mathcal{D}_t\phi = |t|-1/2\phi(\frac{x}{t})$ is the dilation operator. There is a unique stationary point $\xi = \mu(x)$ in the integral $\int_{\mathbb{R}} e^{it(x\xi-\Lambda(\xi))}\phi(\xi)d\xi$, which is defined as the root of the equation $\Lambda'(\xi) = \xi|\xi|(a + b|\xi|) = x$ for all $x \in \mathbb{R}$. Thus we obtain $\Lambda'(\mu(x)) = x$ for all $x \in \mathbb{R}$. Also $\Lambda''(\xi) = 2|\xi|(a + \frac{3}{2}b|\xi|) > 0$ for all $\xi \in \mathbb{R}$. We have $\Lambda'(\xi) = O(\xi^2\langle\xi\rangle)$, therefore $\mu(x) = O(\{x\}^{1/2}\langle x\rangle^{1/3})$. Define the scaling operator $(\mathcal{B}\phi)(x) = \phi(\mu(x))$. Hence $\mathcal{U}(t)\mathcal{F}^{-1}\phi = \mathcal{D}_t\mathcal{B}M\mathcal{V}\phi$, where $M = e^{-it(\Lambda(\eta)-\eta\Lambda'(\eta))}$, the operator $\mathcal{V}(t)\phi = \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\phi(\xi)d\xi$ and the phase function $S(\xi,\eta) = \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi-\eta)$. Denote $\mathcal{A}_k = \overline{M}^k\frac{1}{t\Lambda''(\eta)}\partial_\eta M^k$, $k = 0, 1$. We have $\mathcal{A}_1 = \mathcal{A}_0 + i\eta$, also $\mathcal{A}_1\mathcal{V} = \mathcal{V}i\xi$, $[i\eta,\mathcal{V}] = -\mathcal{A}_0\mathcal{V}$, therefore we obtain the commutator $\partial_\eta\mathcal{V} = -t\Lambda''(\eta)[i\eta,\mathcal{V}]$. Since $\partial_\xi S(\xi,\eta) = \Lambda'(\xi) - \Lambda'(\eta)$, we obtain the commutator $it[\Lambda'(\eta),\mathcal{V}]\phi = -\mathcal{V}\partial_\xi\phi$. Also we use the representation for the inverse evolution group $\mathcal{F}\mathcal{U}(-t)\phi = \mathcal{V}^*\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}$, where the inverse dilation operator $\mathcal{D}_t^{-1}\phi = |t|^{1/2}\phi(xt)$ and the inverse scaling operator $(\mathcal{B}^{-1}\phi)(\eta) = \phi(\Lambda'(\eta))$, and the conjugate operator

$$\mathcal{V}^*(t)\phi = \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\phi(\eta)\Lambda''(\eta)d\eta.$$

We have $i\xi\mathcal{V}^*\phi = \mathcal{V}^*\mathcal{A}_1\phi$. Hence $[i\xi,\mathcal{V}^*] = \mathcal{V}^*\mathcal{A}_0$.

Define a new dependent variable $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$. Since $\mathcal{F}\mathcal{U}(-t)\mathcal{L} = \partial_t\mathcal{F}\mathcal{U}(-t)$ with $\mathcal{L} = \partial_t + i\mathbf{\Lambda}$, where $\mathbf{\Lambda} = \Lambda(-i\partial_x) = \mathcal{F}^{-1}\Lambda(\xi)\mathcal{F}$, applying the operator $\mathcal{F}\mathcal{U}(-t)$ to equation (1.1) $\mathcal{L}u = \partial_x(|u|^2u)$, we obtain

$$\begin{aligned} \partial_t\widehat{\varphi} &= \mathcal{F}\mathcal{U}(-t)\mathcal{L}u = \mathcal{F}\mathcal{U}(-t)\partial_x(|\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}|^2\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}) \\ &= i\xi t^{-1}\mathcal{V}^*(|\mathcal{V}\widehat{\varphi}|^2\mathcal{V}\widehat{\varphi}), \end{aligned} \tag{2.1}$$

since the nonlinearity is gauge invariant. Also we mention that the operator $\mathcal{J} = \mathcal{U}(t)x\mathcal{U}(-t) = x - t\mathbf{\Lambda}'$, with $\mathbf{\Lambda}' = \Lambda'(-i\partial_x) = \mathcal{F}^{-1}\Lambda'(\xi)\mathcal{F}$, plays a crucial role in the large time asymptotic estimates. Note that $\mathcal{J}$ commutes with $\mathcal{L}$, i.e. $[\mathcal{J},\mathcal{L}] = 0$. Also we see that the symbol $\Lambda(\xi) = \frac{a}{3}|\xi|^3 + \frac{b}{4}\xi^4$ satisfies the identities $\xi\partial_\xi\Lambda - b\partial_b\Lambda = 3\Lambda$ and $\xi\partial_\xi\Lambda + a\partial_a\Lambda = 4\Lambda$. Hence we have the commutator relations $[\widehat{\mathcal{P}}_a, e^{-it\Lambda(\xi)}] = [\widehat{\mathcal{P}}_b, e^{-it\Lambda(\xi)}] = 0$, with $\widehat{\mathcal{P}}_a = 4t\partial_t - \xi\partial_\xi - a\partial_a$, $\widehat{\mathcal{P}}_b = 3t\partial_t - \xi\partial_\xi + b\partial_b$.

We also use the operators $\mathcal{P}_a = 4t\partial_t + \partial_x x - a\partial_a$, $\mathcal{P}_b = 3t\partial_t + \partial_x x + b\partial_b$ and $\mathcal{I}_a = \partial_a + it\partial_a\Lambda(-i\partial_x)$, $\mathcal{I}_b = \partial_b + it\partial_b\Lambda(-i\partial_x)$, and the commutator relations $[\mathcal{L},\mathcal{I}_a] = [\mathcal{L},\mathcal{I}_b] = 0$ and $[\mathcal{L},\mathcal{P}_a] = 4\mathcal{L}$, $[\mathcal{L},\mathcal{P}_b] = 3\mathcal{L}$ hold. Using the relation $u(t) = \mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi} = \mathcal{F}^{-1}e^{-it\Lambda(\xi)}\widehat{\varphi}$, we obtain

$$\mathcal{P}_a u = \mathcal{F}^{-1}\widehat{\mathcal{P}}_a e^{-it\Lambda(\xi)}\widehat{\varphi} = \mathcal{F}^{-1}[\widehat{\mathcal{P}}_a, e^{-it\Lambda(\xi)}]\widehat{\varphi} + \mathcal{F}^{-1}e^{-it\Lambda(\xi)}\widehat{\mathcal{P}_a}\widehat{\varphi} = \mathcal{U}(t)\mathcal{F}^{-1}\widehat{\mathcal{P}}_a\widehat{\varphi},$$

and $\mathcal{P}_b u = \mathcal{F}^{-1}\widehat{\mathcal{P}}_b e^{-it\Lambda(\xi)}\widehat{\varphi} = \mathcal{F}^{-1}[\widehat{\mathcal{P}}_b, e^{-it\Lambda(\xi)}]\widehat{\varphi} + \mathcal{F}^{-1}e^{-it\Lambda(\xi)}\widehat{\mathcal{P}_b}\widehat{\varphi} = \mathcal{U}(t)\mathcal{F}^{-1}\widehat{\mathcal{P}}_b\widehat{\varphi}$. Note that $\mathcal{I}_a u = \mathcal{I}_a\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi} = \mathcal{F}^{-1}(\partial_a + it\partial_a\Lambda(\xi))E\widehat{\varphi} = \mathcal{F}^{-1}E\partial_a\widehat{\varphi} = \mathcal{U}(t)\mathcal{F}^{-1}\partial_a\widehat{\varphi}$, and $\mathcal{I}_b u = \mathcal{I}_b\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi} = \mathcal{F}^{-1}(\partial_b + it\partial_b\Lambda(\xi))E\widehat{\varphi} = \mathcal{F}^{-1}E\partial_b\widehat{\varphi} = \mathcal{U}(t)\mathcal{F}^{-1}\partial_b\widehat{\varphi}$. Hence $\|\mathcal{I}_a u\|_{L^2} = \|\partial_b\widehat{\varphi}\|_{L^2}$ and $\|\mathcal{I}_b u\|_{L^2} = \|\partial_b\widehat{\varphi}\|_{L^2}$. Also we have the identities $\mathcal{P}_a = 4t\mathcal{L} + \partial_x\mathcal{J} - a\mathcal{I}_a$ and $\mathcal{P}_b = 3t\mathcal{L} + \partial_x\mathcal{J} + b\mathcal{I}_b$.

2.2. **Boundedness of pseudodifferential operators.** Define the pseudodifferential operator

$$a(x,D)\phi \equiv \int_{\mathbb{R}} e^{ix\xi}a(x,\xi)\widehat{\phi}(\xi)d\xi.$$

There are many papers devoted to the L^2-estimates of pseudodifferential operator $a(x,D)$ (see [4, 7, 9, 27]). Below we will use the following results on the L^2-boundedness of pseudodifferential operator $a(x,D)$ (see [27]).

Lemma 2.1. *Let the symbol $a(x,\xi)$ be such that $\sup_{x,\xi\in\mathbb{R}}|\partial_x^k\partial_\xi^l a(x,\xi)| \le C$ for $k,l = 0,1$. Then $\|a(x,D)\phi\|_{L^2_x} \le C\|\phi\|_{L^2}$.*

Lemma 2.2. *Let the symbol $a(x,\xi)$ be such that $\sup_{x\in\mathbb{R}}\|\partial_\xi^l a(x,\xi)\|_{L^2_\xi} \le C$ for $l = 0,1$. Then $\|a(x,D)\phi\|_{L^2_x} \le C\|\phi\|_{L^2}$.*

Next we consider the time dependent pseudodifferential operator

$$a(t,x,D)\phi \equiv \int_{\mathbb{R}} e^{ix\xi}a(t,x,\xi)\widehat{\phi}(\xi)d\xi.$$

Lemma 2.3. *Let the symbol $a(t,x,\xi)$ be such that*

$$\sup_{x,\xi\in\mathbb{R},t\ge1}|\{\xi\}^{-\nu}\langle\xi\rangle^{\nu}(\xi\partial_\xi)^k a(t,x,\xi)| \le C$$

for $k = 0,1,2$, where $\nu\in(0,1)$. Then $\|a(t,x,D)\phi\|_{L^2_x} \le C\|\phi\|_{L^2}$ for all $t\ge1$.

Proof. We define the kernel $K(t,x,y) = \int_{\mathbb{R}} e^{iy\xi}a(t,x,\xi)d\xi$, then we write

$$a(t,x,D)\phi = \int_{\mathbb{R}} e^{ix\xi}a(t,x,\xi)\widehat{\phi}(\xi)d\xi = \int_{\mathbb{R}} K(t,x,y)\phi(x-y)dy.$$

Integrating two times by parts via the identity $e^{iy\xi} = H\partial_\xi(\xi e^{iy\xi})$ with $H = (1+i\xi y)^{-1}$ we obtain $K(t,x,y) = \int_{\mathbb{R}} e^{iy\xi}\xi\partial_\xi(H\xi\partial_\xi(Ha(x,\xi)))d\xi$. By the condition in this lemma, we find $|\xi\partial_\xi(H\xi\partial_\xi(Ha(t,x,\xi)))| \le \frac{C\{\xi\}^{\nu}\langle\xi\rangle^{-\nu}}{\langle\xi y\rangle^2}$ for all $x,y,\xi\in\mathbb{R}$. Hence we obtain the estimate

$$\begin{aligned}
|K(t,x,y)| &\le C\int_0^1 \frac{\xi^{\nu}d\xi}{\langle\xi y\rangle^2} + C\int_1^\infty \frac{\xi^{-\nu}d\xi}{\langle\xi y\rangle^2} \\
&\le C|y|^{-1-\nu}\int_0^{|y|}\eta^{\nu}\langle\eta\rangle^{-2}d\eta + C|y|^{\nu-1}\int_{|y|}^\infty \eta^{-\nu}\langle\eta\rangle^{-2}d\eta \\
&\le C|y|^{\nu-1}\langle y\rangle^{-2\nu}.
\end{aligned}$$

Then by Young's inequality we have

$$
\begin{aligned}
&\|a(t,x,D)\phi\|_{L^2} = \|\int_{\mathbb{R}} K(t,x,y)\phi(x-y)dy\|_{L^2} \\
&\le C\|\int_{\mathbb{R}} |y|^{\nu-1}\langle y\rangle^{-2\nu}|\phi(x-y)|dy\|_{L^2} \le C\big\||x|^{\nu-1}\langle x\rangle^{-2\nu}\big\|_{L^1}\|\phi\|_{L^2} \le C\|\phi\|_{L^2}.
\end{aligned}
$$

The proof is complete. □

Similarly, by considering the conjugate operator.

Lemma 2.4. *Let the symbol $a(t,x,\xi)$ be such that*

$$
\sup_{x,\xi\in\mathbb{R},t\ge1} |\{x\}^{-\nu}\langle x\rangle^{\nu}(x\partial_x)^k a(t,x,\xi)| \le C
$$

for $k=0,1,2$, where $\nu\in(0,1)$. Then $\|a(t,x,D)\phi\|_{L^2_x} \le C\|\phi\|_{L^2}$ for all $t\ge1$.

3. Estimates for the operator $\mathcal{V}$

Let $\chi\in C^4(\mathbb{R})$ be such that $\chi(x)=1$ for $|x|\le1$ and $\chi(x)=0$ for $|x|\ge2$. Define the cut off functions $\chi_j(z)\in C^4(\mathbb{R})$, $j=1,2,3$, such that $\chi_1+\chi_2+\chi_3\equiv1$, $\chi_2(z)=0$ for $z\le\frac13$ or $z\ge3$ and $\chi_2(z)=1$ for $\frac23\le z\le\frac32$, $\chi_1(z)=1-\chi_2(z)$ for $-\frac32<z<\frac23$ and $\chi_1(z)=0$ for $z\ge\frac23$ or $z\le-3$ and $\chi_3(z)=1-\chi_2(z)$ for $z>\frac32$, $\chi_3(z)=1-\chi_1(z)$ for $z<-\frac32$ and $\chi_3(z)=0$ for $-\frac32\le z\le\frac32$. Denote $\Psi_1(t,\xi,\eta)=(1-\chi(\eta t^{1/3}))\chi_1(\frac{\xi}{\eta})$, $\Psi_2(t,\xi,\eta)=(1-\chi(\eta t^{1/3}))\chi_2(\frac{\xi}{\eta})$, $\Psi_3(t,\xi,\eta)=(1-\chi(\eta t^{1/3}))\chi_3(\frac{\xi}{\eta})$, $\Psi_4(t,\xi,\eta)=\chi(\eta t^{1/3})\chi(\frac13\xi t^{1/3})$ and $\Psi_5(t,\xi,\eta)=\chi(\eta t^{1/3})(1-\chi(\frac13\xi t^{1/3}))$ and consider the operators

$$
\mathcal{V}_j(t)\phi = \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\Psi_j(t,\xi,\eta)\phi(\xi)d\xi
$$

for $j=1,2,3,4,5$.

3.1. **Estimates for commutators.** We first obtain the L^2-estimate for the operator $\mathcal{V}_1$.

Lemma 3.1. *Let the weight $P\in C^2(\mathbb{R}\setminus0)$ be such that $\partial_\eta^k P(\eta)=O(|\eta|^{\alpha_1-k})$ for $k=0,1,2$, with $\alpha_1\le2$. Assume that the weight $Q\equiv1$ if $\alpha_2=0$, or $Q\in C^2(\mathbb{R}\setminus0)$ satisfies the estimate $\partial_\xi^k Q(\xi)=O(|\xi|^{\alpha_2-k})$, $k=0,1,2$, if $\alpha_2\ge1$. Suppose that $2\le\alpha_1+\alpha_2<5/2$. Then*

$$
\big\||\Lambda''|^{1/2}Pt\mathcal{V}_1Q\phi\big\|_{L^2} \le C\|\partial_\xi\phi\|_{L^2} + C|t|^{\frac13(\frac52-\alpha_1-\alpha_2)}|\phi(0)|
$$

for all $t\ge1$.

Proof. Integrating by parts we obtain

$$
\begin{aligned}
Pt\mathcal{V}_1Q\phi &= C\big(1-\chi(\eta t^{1/3})\big)P(\eta)|t|^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\partial_\xi\Big(\frac{Q(\xi)\chi_1(\frac{\xi}{\eta})}{\partial_\xi S(\xi,\eta)}\phi(\xi)\Big)d\xi \\
&= C|t|^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_1(t,\eta,\xi)\phi_\xi(\xi)d\xi \\
&\quad + C|t|^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_2(t,\eta,\xi)\frac{\phi(\xi)-\phi(0)}{\xi}d\xi \\
&\quad + C|t|^{1/2}\phi(0)\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_3(t,\eta,\xi)d\xi,
\end{aligned}
$$

where $q_k(t,\eta,\xi) = (1-\chi(t^{1/3}\eta))P(\eta)(\xi\partial_\xi)^{k-1}(Q(\xi)\chi_1(\frac{\xi}{\eta})\frac{1}{\partial_\xi S(\xi,\eta)})$, $k=1,2$,

$$q_3(t,\eta,\xi) = (1-\chi(t^{1/3}\eta))P(\eta)\partial_\xi(Q(\xi)\chi_1(\frac{\xi}{\eta})\frac{1}{\partial_\xi S(\xi,\eta)}).$$

Next we change $\eta = \mu(x)$ (recalling that the inverse scaling operator $(\mathcal{B}^{-1}\phi)(\eta) = \phi(\Lambda'(\eta))$ and $\mu(x)$ is the root of the equation $\Lambda'(\eta) = x$), we obtain

$$\begin{aligned} Pt\mathcal{V}_1 Q\phi &= \overline{M}\mathcal{B}^{-1}|t|^{1/2}\int_{\mathbb{R}} e^{itx\xi} q_1(t,\mu(x),\xi)e^{-it\Lambda(\xi)}\phi_\xi(\xi)d\xi \\ &\quad + \overline{M}\mathcal{B}^{-1}|t|^{1/2}\int_{\mathbb{R}} e^{itx\xi} q_2(t,\mu(x),\xi)e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi}d\xi \\ &\quad + \phi(0)\overline{M}\mathcal{B}^{-1}|t|^{1/2}\int_{\mathbb{R}} e^{itx\xi} q_3(t,\mu(x),\xi)e^{-it\Lambda(\xi)}d\xi \end{aligned}$$

Also we change the variable of integration $\xi = t^{-1/3}\xi'$; then

$$\begin{aligned} Pt\mathcal{V}_1 Q\phi &= \overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}\Big(\int_{\mathbb{R}} e^{ix\xi} q_1(t,\mu(xt^{-2/3}),\xi t^{-1/3})\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\phi_\xi(\xi)d\xi \\ &\quad + \int_{\mathbb{R}} e^{ix\xi} q_2(t,\mu(xt^{-2/3}),\xi t^{-1/3})\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi}d\xi \\ &\quad + \phi(0)\int_{\mathbb{R}} e^{ix\xi} q_3(t,\mu(xt^{-2/3}),\xi t^{-1/3})\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}d\xi\Big), \end{aligned}$$

here $\mathcal{D}^{-1}_{t^{2/3}}\phi(x) = |t|^{1/3}\phi(xt^{2/3})$ and $\mathcal{D}_{t^{1/3}}\phi(\xi) = |t|^{-1/6}\phi(\xi t^{-1/3})$. Define the pseudodifferential operators $a_k(t,x,D)\phi \equiv \int_{\mathbb{R}} e^{ix\xi}a_k(t,x,\xi)\widehat{\phi}(\xi)d\xi$ with symbols $a_k(t,x,\xi) = q_k(t,\mu(xt^{-2/3}),\xi t^{-1/3})$. Then we obtain

$$\begin{aligned} Pt\mathcal{V}_1 Q\phi &= \overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}\Big(a_1(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\phi_\xi(\xi) \\ &\quad + a_2(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi} \\ &\quad + \phi(0)a_3(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\Big). \end{aligned}$$

Let us prove the L^2-boundedness of the operators $a_k(t,x,D)$, $k=1,2$. We obtain

$$\begin{aligned} a_k(t,x,\xi) &= \big(1-\chi\big(t^{1/3}\mu(xt^{-2/3})\big)\big)P(\mu(xt^{-2/3})) \\ &\quad \times (\xi\partial_\xi)^{k-1}\Big(\frac{Q(\xi t^{-1/3})\chi_1(\xi t^{-1/3}/\mu(xt^{-2/3}))}{\Lambda'(\xi t^{-\frac{1}{3}}) - xt^{-2/3}}\Big). \end{aligned}$$

Note that $\mu(x) = O(|x|^{1/2})$ for small $|x|$, therefore $(1-\chi(t^{1/3}\mu(xt^{-2/3}))) \neq 0$ for $|x| \geq C > 0$, $t \geq 1$. Also we have $\chi_1(z) \neq 0$ for $-3 < z < 2/3$, hence

$$\begin{aligned} |\Lambda'(\xi t^{-1/3}) - xt^{-2/3}| &= \Big|\int_{\mu(xt^{-\frac{2}{3}})}^{\xi t^{-1/3}} \Lambda''(\eta)d\eta\Big| \geq \Big|\int_{\mu(xt^{-2/3})}^{\frac{2}{3}\mu(xt^{-2/3})} \Lambda''(\eta)d\eta\Big| \\ &\geq \frac{1}{3}|\mu(xt^{-2/3})||\Lambda''(\frac{2}{3}\mu(xt^{-2/3})) \\ &\geq C|\Lambda'(\mu(xt^{-2/3}))| = C|x|t^{-2/3}. \end{aligned}$$

Note that $|\partial_x^k\partial_\xi^l a_k(t,x,\xi)| \leq C$ for all $x,\xi \in \mathbb{R}$, $t \geq 1$,$k,l = 0,1$, if $2 \leq \alpha_1 + \alpha_2 \leq 3$. Therefore by Lemma 2.1 we have $\|a_k(t,x,D)\phi\|_{L^2_x} \leq C\|\phi\|_{L^2}$. Thus the pseudodifferential operators $a_k(t,x,D)$ are L^2-bounded for $k=1,2$. Then we find that

$$\begin{aligned}
&\||\Lambda''|^{\frac12}\overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}a_1(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\phi_\xi(\xi)\|_{L^2_\eta}\\
&\le \|a_1(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\phi_\xi(\xi)\|_{L^2_x}\\
&\le C\|\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\phi_\xi(\xi)\|_{L^2} = C\|\partial_\xi\phi\|_{L^2}
\end{aligned}$$

and by the Hardy inequality

$$\begin{aligned}
&\big\||\Lambda''|^{\frac12}\overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}a_2(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi}\|_{L^2_\eta}\\
&\le \|a_2(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi}\big\|_{L^2_x}\\
&\le C\|\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi}\|_{L^2_\xi} = C\|\frac{\phi(\xi)-\phi(0)}{\xi}\|_{L^2_\xi} \le C\|\partial_\xi\phi\|_{L^2}.
\end{aligned}$$

Now let us prove the L^2 - L^∞ boundedness of the pseudodifferential operator $a_3(t,x,D)$. We have

$$\begin{aligned}
a_3(t,x,\xi) &= q_3\big(t,\mu(xt^{-2/3}),\xi t^{-1/3}\big)\\
&= \Big(1-\chi\big(t^{1/3}\mu(xt^{-2/3})\big)\Big)P\big(\mu(xt^{-2/3})\big)\\
&\quad\times t^{1/3}\partial_\xi\Big(\frac{Q(\xi t^{-1/3})\chi_1(\xi t^{-1/3}/\mu(xt^{-2/3}))}{\Lambda'(\xi t^{-1/3})-xt^{-2/3}}\Big).
\end{aligned}$$

We consider two symbols $a_{3,1}(t,x,\xi)=a_3(t,x,\xi)\chi(\xi)$ and $a_{3,2}(t,x,\xi)=a_3(t,x,\xi)(1-\chi(\xi))$. Note that $|\langle x\rangle^\nu(x\partial_x)^j\langle\xi\rangle^\delta a_{3,1}(t,x,\xi)|\le C|t|^{\frac13(3-\alpha_1-\alpha_2)}$ for all $x,\xi\in\mathbb{R}$, $t\ge 1$, $j=0,1,2$ with some $\nu\in(0,1)$, if $\alpha_2=0$ or $\alpha_2\ge 1$, $\alpha_1\le 2$, $2\le\alpha_1+\alpha_2<5/2$. Therefore by Lemma 2.4 we find that

$$\|a_{3,1}(t,x,D)\phi\|_{L^2_x}\le C|t|^{\frac13(3-\alpha_1-\alpha_2)}\|\langle\xi\rangle^{-\delta}\widehat{\phi}\|_{L^2}\le C|t|^{\frac13(3-\alpha_1-\alpha_2)}\|\widehat{\phi}\|_{L^\infty}.$$

Also we have $|\partial_x^k\partial_\xi^l\langle\xi\rangle^\delta a_{3,2}(t,x,\xi)|\le C|t|^{\frac13(3-\alpha_1-\alpha_2)}$ for all $x,\xi\in\mathbb{R}$, $t\ge1$, $k,l=0,1$, if $2\le\alpha_1+\alpha_2<5/2$, $\delta>1/2$. Therefore by Lemma 2.1 we find that

$$\|a_{3,2}(t,x,D)\phi\|_{L^2_x}\le C|t|^{\frac13(3-\alpha_1-\alpha_2)}\|\langle\xi\rangle^{-\delta}\widehat{\phi}\|_{L^2}\le C|t|^{\frac13(3-\alpha_1-\alpha_2)}\|\widehat{\phi}\|_{L^\infty}.$$

Then

$$\begin{aligned}
&\big\||\Lambda''|^{1/2}\phi(0)\overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{\frac23}}a_3(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{\frac13}}e^{-it\Lambda(\xi)}\big\|_{L^2_\eta}\\
&\le|\phi(0)|\|a_3(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\|_{L^2_x}\\
&\le C|t|^{\frac13(3-\alpha_1-\alpha_2)}|\phi(0)|\|\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\|_{L^\infty_\xi}\\
&\le C|t|^{\frac13(\frac52-\alpha_1-\alpha_2)}|\phi(0)|,
\end{aligned}$$

which yields the result of the lemma. □

In the next lemma we estimate the commutator $[h,\mathcal{V}_2]$.

Lemma 3.2. *Let the weights $P\in C^1(\mathbb{R}\setminus 0)$ and $Q\in C^2(\mathbb{R}\setminus 0)$ be such that $\partial_\eta^kP(\eta)=O(|\eta|^{\alpha_1-k})$, $k=0,1$, and $\partial_\xi^kQ(\xi)=O(|\xi|^{\alpha_2-k})$, $k=0,1,2$. Suppose that*

$h(\xi) \in C^4(\mathbb{R}\setminus 0)$ is such that $|\partial_\xi^k h(\xi)| \le C|\xi|^{\alpha_3 - k}$ for $\xi \in \mathbb{R}\setminus 0$, $0 \le k \le 4$. Assume that $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_3 \ge 1$, $2 \le \alpha_1+\alpha_2+\alpha_3 < 5/2$. Then

$$\||\Lambda''|^{1/2} Pt[h,\mathcal{V}_2]Q\phi\|_{L^2} \le C\|\partial_\xi \phi\|_{L^2} + C|t|^{\frac{1}{3}(\frac{5}{2}-\alpha_1-\alpha_2-\alpha_3)}|\phi(0)|$$

for all $t \ge 1$.

Proof. Integrating by parts we obtain

$$\begin{aligned}
&Pt[h,\mathcal{V}_2]Q\phi \\
&= C\Big(1-\chi\big(\eta t^{1/3}\big)\Big)P(\eta)|t|^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\partial_\xi\Big(\frac{h(\eta)-h(\xi)}{\partial_\xi S(\xi,\eta)}\Big)Q(\xi)\phi(\xi)\chi_2(\frac{\xi}{\eta})d\xi \\
&= C|t|^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_1(\eta,\xi)\phi_\xi(\xi)d\xi + C|t|^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_2(\eta,\xi)\frac{\phi(\xi)-\phi(0)}{\xi}d\xi \\
&\quad + C\phi(0)|t|^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_3(\eta,\xi)d\xi,
\end{aligned}$$

where $q_k(\eta,\xi) = (1-\chi(\eta t^{1/3}))P(\eta)(\xi\partial_\xi)^{k-1}(Q(\xi)\chi_2(\frac{\xi}{\eta})\frac{h(\eta)-h(\xi)}{\partial_\xi S(\xi,\eta)})$, $q_3(\eta,\xi) = (1-\chi(\eta t^{1/3}))P(\eta)\partial_\xi(Q(\xi)\chi_2(\frac{\xi}{\eta})\frac{h(\eta)-h(\xi)}{\partial_\xi S(\xi,\eta)})$. Next we change $\eta = \mu(x)$ and the variable of integration $\xi = t^{-1/3}\xi'$, then we obtain

$$\begin{aligned}
Pt[h,\mathcal{V}_2]Q\phi &= \overline{M}\mathcal{B}^{-1}\mathcal{D}_{t^{2/3}}^{-1}\Big(\int_{\mathbb{R}} e^{ix\xi}q_1(t,\mu(xt^{-2/3}),\xi t^{-1/3})\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\phi_\xi(\xi)d\xi \\
&\quad + \int_{\mathbb{R}} e^{ix\xi}q_2\big(t,\mu(xt^{-2/3}),\xi t^{-1/3}\big)\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi}d\xi \\
&\quad + \phi(0)\int_{\mathbb{R}} e^{ix\xi}q_3\big(t,\mu(xt^{-2/3}),\xi t^{-1/3}\big)\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}d\xi\Big).
\end{aligned}$$

We define the pseudodifferential operators $a_k(t,x,D)\phi \equiv \int_{\mathbb{R}} e^{ix\xi}a_k(t,x,\xi)\widehat{\phi}(\xi)d\xi$ with symbols $a_k(t,x,\xi) = q_k(t,\mu(xt^{-2/3}),\xi t^{-1/3})$. Then we obtain

$$\begin{aligned}
Pt[h,\mathcal{V}_2]Q\phi = \overline{M}\mathcal{B}^{-1}\mathcal{D}_{t^{2/3}}^{-1}\Big(&a_1(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\phi_\xi(\xi) \\
&+ a_2(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi} \\
&+ \phi(0)a_3(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\Big).
\end{aligned}$$

We will prove the L^2-boundedness of $a_k(t,x,D)$. Note that $\chi_2\big(\frac{\xi t^{-1/3}}{\mu(xt^{-2/3})}\big) \ne 0$ for $1/3 < \frac{\xi}{t^{1/3}\mu(xt^{-2/3})} < 3$. Using the identities $\Lambda'(\xi)-x = (\xi-\mu(x))\int_0^1 \Lambda''(\xi+(\mu(x)-\xi)\tau)d\tau$, and $h(\mu(x))-h(\xi) = (\mu(x)-\xi)\int_0^1 h'(\xi+(\mu(x)-\xi)\tau)d\tau$, we obtain

$$\begin{aligned}
a_k(t,x,\xi) = -\Big(1-\chi\big(t^{1/3}\mu(xt^{-2/3})\big)\Big)P\big(\mu(xt^{-2/3})\big)(\xi\partial_\xi)^{k-1}\Big(\chi_2\big(\xi t^{-1/3}/\mu(xt^{-2/3})\big) \\
\times Q(\xi t^{-1/3})\frac{\int_0^1 h'(\xi t^{-1/3}+(\mu(xt^{-2/3})-\xi t^{-1/3})\tau)d\tau}{\int_0^1 \Lambda''(\xi t^{-1/3}+(\mu(xt^{-2/3})-\xi t^{-1/3})\tau)d\tau}\Big)
\end{aligned}$$

for $k = 1,2$. We have $|\partial_x^k\partial_\xi^l a_k(t,x,\xi)| \le C$ for all $x,\xi \in \mathbb{R}$, $t \ge 1$, $k,l = 0,1$, if $\alpha_1,\alpha_2 \in \mathbb{R}$, $\alpha_3 \ge 1$, $2 \le \alpha_1+\alpha_2+\alpha_3 < 5/2$. Therefore by Lemma 2.1 we find that

$\|a_k(t,x,D)\phi\|_{L^2_x} \le C\|\phi\|_{L^2}$. Thus the pseudodifferential operators $a_k(t,x,D)$ are L^2-bounded for $k=1,2$. Then as above we have

$$\big\||\Lambda''|^{\frac12}\overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}a_1(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\phi_\xi(\xi)\big\|_{L^2_\eta} \le C\|\partial_\xi\phi\|_{L^2}$$

and

$$\big\||\Lambda''|^{\frac12}\overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}a_2(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi}\big\|_{L^2_\eta} \le C\|\partial_\xi\phi\|_{L^2}.$$

Now let us prove the L^2 - L^∞ boundedness of the pseudodifferential operator $a_3(t,x,D)$. We have

$$\begin{aligned} a_3(t,x,\xi) = &-\Big(1-\chi\big(t^{1/3}\mu(xt^{-2/3})\big)\Big)P\big(\mu(xt^{-2/3})\big)t^{1/3}\partial_\xi\Big(Q(\xi t^{-1/3})\\ &\times\chi_2\Big(\frac{\xi t^{-1/3}}{\mu(xt^{-2/3})}\Big)\frac{\int_0^1 h'(\xi t^{-1/3}+(\mu(xt^{-2/3})-\xi t^{-1/3})\tau)d\tau}{\int_0^1\Lambda''(\xi t^{-1/3}+(\mu(xt^{-2/3})-\xi t^{-1/3})\tau)d\tau}\Big). \end{aligned}$$

Note that $|\partial_x^k\partial_\xi^l\langle\xi\rangle^\delta a_3(t,x,\xi)| \le C|t|^{\frac13(3-\alpha_1-\alpha_2)}$ for all $x,\xi\in\mathbb{R}$, $t\ge1$, $k,l=0,1$, if $\alpha_1,\alpha_2\in\mathbb{R}$, $\alpha_3\ge1$, $2\le\alpha_1+\alpha_2+\alpha_3<5/2$, $\delta>1/2$. Therefore by Lemma 2.1, we have

$$\|a_3(t,x,D)\phi\|_{L^2_x} \le C|t|^{\frac13(3-\alpha_1-\alpha_2-\alpha_3)}\|\langle\xi\rangle^{-\delta}\widehat{\phi}\|_{L^2} \le C|t|^{\frac13(3-\alpha_1-\alpha_2-\alpha_3)}\|\widehat{\phi}\|_{L^\infty}.$$

Then

$$\begin{aligned} &\big\||\Lambda''|^{1/2}\phi(0)\overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{\frac23}}a_3(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{\frac13}}e^{-it\Lambda(\xi)}\big\|_{L^2_\eta}\\ &\le|\phi(0)|\|a_3(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\|_{L^2_x}\\ &\le C|t|^{\frac13(3-\alpha_1-\alpha_2-\alpha_3)}|\phi(0)|\|\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\|_{L^\infty_\xi}\\ &\le C|t|^{\frac13(\frac52-\alpha_1-\alpha_2-\alpha_3)}|\phi(0)|, \end{aligned}$$

which yields the result of the lemma. □

In the next lemma we estimate the operator $\mathcal{V}_3$.

Lemma 3.3. *Let the weights $P\in C^1(\mathbb{R}\setminus0)$ and $Q\in C^2(\mathbb{R}\setminus0)$ be such that $\partial_\eta^kP(\eta)=O(|\eta|^{\alpha_1-k})$, $k=0,1$, and $\partial_\xi^kQ(\xi)=O(|\xi|^{\alpha_2-k})$, $k=0,1,2$. Assume that $\alpha_1\ge0$, $\alpha_2\in\mathbb{R}$, $2\le\alpha_1+\alpha_2\le5/2$. Then $\big\||\Lambda''|^{1/2}Pt\mathcal{V}_3Q\phi\big\|_{L^2}\le C\|\partial_\xi\phi\|_{L^2}+C|t|^{\frac13(\frac52-\alpha_1-\alpha_2)}|\phi(0)|$ for all $t\ge1$.*

Proof. Integrating by parts we obtain

$$\begin{aligned} Pt\mathcal{V}_3Q\phi &= CP(\eta)|t|^{1/2}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}\partial_\xi\frac{1}{\partial_\xi S(\xi,\eta)}Q(\xi)\phi(\xi)\Psi_3(t,\xi,\eta)d\xi\\ &= C|t|^{1/2}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}q_1(t,\eta,\xi)\phi_\xi(\xi)d\xi\\ &\quad+C|t|^{1/2}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}q_2(t,\eta,\xi)\frac{\phi(\xi)-\phi(0)}{\xi}d\xi\\ &\quad+C\phi(0)|t|^{1/2}\int_{\mathbb{R}}e^{-itS(\xi,\eta)}q_3(t,\eta,\xi)d\xi, \end{aligned}$$

where $q_k(t,\eta,\xi)=P(\eta)\big(\xi\partial_\xi\big)^{k-1}(Q(\xi)\Psi_3(t,\xi,\eta)\frac{1}{\partial_\xi S(\xi,\eta)})$, $q_3(t,\eta,\xi)=P(\eta)\partial_\xi(Q(\xi)\Psi_3(t,\xi,\eta)\frac{1}{\partial_\xi S(\xi,\eta)})$. Next we change $\eta=\mu(x)$, and the variable of integration $\xi=t^{-1/3}\xi'$, then we obtain

$$\begin{aligned}
Pt\mathcal{V}_3Q\phi &= \overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}\Big(\int_{\mathbb{R}}e^{ix\xi}q_1(t,\mu(xt^{-2/3}),\xi t^{-1/3})\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\phi_\xi(\xi)d\xi\\
&\quad+\int_{\mathbb{R}}e^{ix\xi}q_2(t,\mu(xt^{-\frac{2}{3}}),\xi t^{-1/3})\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi}d\xi\\
&\quad+\phi(0)\int_{\mathbb{R}}e^{ix\xi}q_3\big(t,\mu(xt^{-2/3}),\xi t^{-1/3}\big)\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}d\xi\Big).
\end{aligned}$$

Define the pseudodifferential operators $a_k(t,x,D)\phi\equiv\int_{\mathbb{R}}e^{ix\xi}a_k(t,x,\xi)\widehat{\phi}(\xi)d\xi$ with symbols $a_k(t,x,\xi)=q_k\big(t,\mu(xt^{-2/3}),\xi t^{-\frac{1}{3}}\big)$. Then we obtain

$$\begin{aligned}
Pt\mathcal{V}_3Q\phi &= \overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}\Big(a_1(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\phi_\xi(\xi)\\
&\quad+a_2(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi}\\
&\quad+\phi(0)a_3(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\Big).
\end{aligned}$$

Let us prove the L^2-boundedness of the operators $a_k(t,x,D)$, $k=1,2$. We have

$$\begin{aligned}
a_k(t,x,\xi) &= \Big(1-\chi\big(t^{1/3}\mu(xt^{-2/3})\big)\Big)P(\mu(xt^{-2/3}))\\
&\quad\times(\xi\partial_\xi)^{k-1}\Big(\frac{Q(\xi t^{-1/3})\chi_3(\xi/(t^{1/3}\mu(xt^{-2/3})))}{\Lambda'(\xi t^{-1/3})-xt^{-2/3}}\Big).
\end{aligned}$$

Note that $\mu(x)=O(|x|^{1/2})$ for small $|x|$, therefore $\chi(t^{1/3}\mu(xt^{-2/3}))\neq 0$ for $|x|\leq C$, $t\geq 1$. Also $\chi_3(\xi t^{-1/3}/\mu(xt^{-2/3}))\neq 0$ for $\frac{\xi}{t^{1/3}\mu(xt^{-2/3})}>3/2$, hence

$$\begin{aligned}
|\Lambda'(\xi t^{-1/3})-xt^{-2/3}| &= |\int_{\mu(xt^{-2/3})}^{\xi t^{-1/3}}\Lambda''(\eta)d\eta|\geq|\int_{\frac{2}{3}\xi t^{-1/3}}^{\xi t^{-1/3}}\Lambda''(\eta)d\eta|\\
&\geq\frac{1}{3}|\xi t^{-1/3}|\Lambda''\big(\frac{2}{3}\xi t^{-1/3}\big)\geq C|\Lambda'(\xi t^{-1/3})|.
\end{aligned}$$

Note that $|\partial_x^k\partial_\xi^l a_k(t,x,\xi)|\leq C$ for all $x,\xi\in\mathbb{R}$, $t\geq 1$, $k,l=0,1$, if $2\leq\alpha_1+\alpha_2\leq 3$. Therefore by Lemma 2.1 we find that $\|a_k(t,x,D)\phi\|_{L^2_x}\leq C|t|\|\phi\|_{L^2}$. Thus the pseudodifferential operators $a_k(t,x,D)$ are L^2-bounded for $k=1,2$. Then as above we have

$$\big\||\Lambda''|^{\frac{1}{2}}\overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}a_1(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\phi_\xi(\xi)\big\|_{L^2_\eta}\leq C\|\partial_\xi\phi\|_{L^2}\,,$$

$$\big\||\Lambda''|^{\frac{1}{2}}\overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}a_2(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi}\big\|_{L^2_\eta}\leq C\|\partial_\xi\phi\|_{L^2}.$$

Now let us prove the L^2 - L^∞ boundedness of the pseudodifferential operator $a_3(t,x,D)$. We have

$$\begin{aligned}
a_3(t,x,\xi) &= \Big(1-\chi\big(t^{1/3}\mu(xt^{-2/3})\big)\Big)P\big(\mu(xt^{-2/3})\big)t^{1/3}\\
&\quad\times\partial_\xi\Big(\frac{Q(\xi t^{-1/3})\chi_3(\xi/(t^{1/3}\mu(xt^{-2/3})))}{\Lambda'(\xi t^{-1/3})-xt^{-2/3}}\Big).
\end{aligned}$$

Note that $|\partial_x^k \partial_\xi^l \langle\xi\rangle^\delta a_3(t,x,\xi)| \le C|t|^{\frac{1}{3}(3-\alpha_1-\alpha_2)}$ for all $x,\xi \in \mathbb{R}$, $t \ge 1$, $k,l = 0,1$, if $2 \le \alpha_1+\alpha_2+\delta \le 3$, $\delta > 1/2$. Therefore by Lemma 2.1 we have

$$\|a_3(t,x,D)\phi\|_{L^2_x} \le C|t|^{\frac{1}{3}(3-\alpha_1-\alpha_2)}\|\langle\xi\rangle^{-\delta}\widehat{\phi}\|_{L^2} \le C|t|^{\frac{1}{3}(3-\alpha_1-\alpha_2)}\|\widehat{\phi}\|_{L^\infty}.$$

Thus as above we obtain

$$\||\Lambda''|^{1/2}\phi(0)\overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{\frac{2}{3}}}a_3(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{\frac{1}{3}}}e^{-it\Lambda(\xi)}\|_{L^2_\eta} \le C|t|^{\frac{1}{3}(\frac{5}{2}-\alpha_1-\alpha_2)}|\phi(0)|.$$

The proof is complete. □

Next we obtain the L^2-estimate for the operator $\mathcal{V}_4$.

Lemma 3.4. *Let the weight P be such that $P(\eta) = O(|\eta|^{\alpha_1})$, with $\alpha_1 > -1$. Assume that the weight Q satisfies the estimate $Q(\xi) = O(|\xi|^{\alpha_2})$, if $\alpha_2 > -1$. Then*

$$\big\||\Lambda''|^{1/2}P\mathcal{V}_4 Q\phi\big\|_{L^p} \le C|t|^{-\frac{1}{3}(\frac{1}{2}+\alpha_1+\alpha_2)-\frac{1}{3p}}(\|\partial_\xi\phi\|_{L^2} + |t|^{\frac{1}{6}}|\phi(0)|)$$

for all $t \ge 1$, where $1 \le p \le \infty$.

Proof. Using the estimate $|\phi(\xi)-\phi(0)| \le C|\xi|^{1/2}\|\partial_\xi\phi\|_{L^2}$, we have

$$\begin{aligned}
&\||\Lambda''|^{1/2}P\mathcal{V}_4Q\phi\|_{L^p} \\
&\le C|t|^{1/2}\||\Lambda''|^{1/2}P(\eta)\chi(\eta t^{1/3})\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\chi(\frac{1}{3}\xi t^{1/3})Q(\xi)\phi(\xi)d\xi\|_{L^p} \\
&\le C|t|^{1/2}\||\Lambda''|^{1/2}P(\eta)\chi(\eta t^{1/3})\|_{L^p}\|\chi(\frac{1}{3}\xi t^{1/3})Q(\xi)\phi(\xi)\|_{L^1} \\
&\le C|t|^{1/2}\|\partial_\xi\phi\|_{L^2}\||\eta|^{\frac{1}{2}+\alpha_1}\|_{L^p(|\eta|\le 2t^{-1/3})}\||\xi|^{\frac{1}{2}+\alpha_2}\|_{L^1(|\xi|\le 6t^{-1/3})} \\
&+ C|t|^{1/2}|\phi(0)|\||\eta|^{\frac{1}{2}+\alpha_1}\|_{L^p(|\eta|\le 2t^{-1/3})}\||\xi|^{\alpha_2}\|_{L^1(|\xi|\le 6t^{-1/3})} \\
&\le C|t|^{-\frac{1}{3}(\frac{1}{2}+\alpha_1+\alpha_2)-\frac{1}{3p}}(\|\partial_\xi\phi\|_{L^2} + |t|^{1/6}|\phi(0)|).
\end{aligned}$$

The proof is complete. □

In the next lemma we estimate the operator $\mathcal{V}_5$.

Lemma 3.5. *Let the weights P and $Q \in C^3(\mathbb{R}\setminus 0)$ be such that $P(\eta) = O(|\eta|^{\alpha_1})$, and $\partial_\xi^k Q(\xi) = O(|\xi|^{\alpha_2-k})$, $k = 0,1,2,3$. Assume that $\alpha_1 \ge 0$, $\alpha_2 \in \mathbb{R}$, $-1 \le \alpha_1+\alpha_2 \le 2$. Then*

$$\big\||\Lambda''|^{1/2}P\mathcal{V}_5Q\phi\big\|_{L^2} \le Ct^{-\frac{1+\alpha_1+\alpha_2}{3}}(\|\partial_\xi\phi\|_{L^2} + t^{\frac{1}{6}}|\phi(0)|)$$

for all $t \ge 1$.

Proof. Integrating by parts we obtain

$$\begin{aligned}
P\mathcal{V}_5Q\phi &= C\chi(\eta t^{1/3})P(\eta)|t|^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\partial_\xi\frac{1}{\partial_\xi S(\xi,\eta)}Q(\xi)\Big(1-\chi(\frac{1}{3}\xi t^{1/3})\Big)\phi(\xi)d\xi \\
&= C|t|^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_1(t,\eta,\xi)\phi_\xi(\xi)d\xi \\
&\quad + C|t|^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_2(t,\eta,\xi)\frac{\phi(\xi)-\phi(0)}{\xi}d\xi \\
&\quad + C\phi(0)|t|^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}q_3(t,\eta,\xi)d\xi,
\end{aligned}$$

where $q_k(t,\eta,\xi) = t^{-1}\chi(\eta t^{1/3})P(\eta)(\xi\partial_\xi)^{k-1}(Q(\xi)(1-\chi(\frac{1}{3}\xi t^{1/3}))\frac{1}{\partial_\xi S(\xi,\eta)})$, $q_3(t,\eta,\xi) = t^{-1}\chi(\eta t^{1/3})P(\eta)\partial_\xi(Q(\xi)(1-\chi(\frac{1}{3}\xi t^{1/3}))\frac{1}{\partial_\xi S(\xi,\eta)})$. Next we change $\eta = \mu(x)$, and the variable of integration $\xi = t^{-1/3}\xi'$, then we obtain

$$\begin{aligned} P\mathcal{V}_5 Q\phi &= \overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}\Big(\int_{\mathbb{R}} e^{ix\xi} q_1(t,\mu(xt^{-2/3}),\xi t^{-1/3})\mathcal{D}_{t^{1/3}} e^{-it\Lambda(\xi)}\phi_\xi(\xi)d\xi \\ &\quad + \int_{\mathbb{R}} e^{ix\xi} q_2(t,\mu(xt^{-2/3}),\xi t^{-1/3})\mathcal{D}_{t^{1/3}} e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi}d\xi \\ &\quad + \phi(0)\int_{\mathbb{R}} e^{ix\xi} q_3(t,\mu(xt^{-2/3}),\xi t^{-1/3})\mathcal{D}_{t^{1/3}} e^{-it\Lambda(\xi)}d\xi\Big). \end{aligned}$$

Define the pseudodifferential operators $a_k(t,x,D)\phi \equiv \int_{\mathbb{R}} e^{ix\xi} a_k(t,x,\xi)\widehat{\phi}(\xi)d\xi$ with symbols $a_k(t,x,\xi) = q_k(t,\mu(xt^{-2/3}),\xi t^{-\frac{1}{3}})$. Then we obtain

$$\begin{aligned} P\mathcal{V}_5 Q\phi &= \overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}\Big(a_1(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}} e^{-it\Lambda(\xi)}\phi_\xi(\xi) \\ &\quad + a_2(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}} e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi} \\ &\quad + \phi(0)a_3(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}} e^{-it\Lambda(\xi)}\Big). \end{aligned}$$

Let us prove the L^2-boundedness of the operators $a_k(t,x,D)$, $k=1,2$. We have

$$a_k(t,x,\xi) = t^{-1}\chi\big(t^{1/3}\mu(xt^{-2/3})\big)P(\mu(xt^{-2/3}))(\xi\partial_\xi)^{k-1}\Big(\frac{Q(\xi t^{-1/3})(1-\chi(\frac{1}{3}\xi))}{\Lambda'(\xi t^{-1/3}) - xt^{-2/3}}\Big),$$

Note that $\mu(x) = O(|x|^{1/2})$ for small $|x|$, therefore $\chi(t^{1/3}\mu(xt^{-2/3})) \neq 0$ for $|x| \leq C$, $t \geq 1$. Also $\chi_3(\xi t^{-1/3}/\mu(xt^{-2/3})) \neq 0$ for $\frac{\xi}{t^{1/3}\mu(xt^{-2/3})} > 3/2$, hence

$$\begin{aligned} |\Lambda'(\xi t^{-1/3}) - xt^{-2/3}| &= |\int_{\mu(xt^{-2/3})}^{\xi t^{-1/3}} \Lambda''(\eta)d\eta| \geq |\int_{\frac{2}{3}\xi t^{-1/3}}^{\xi t^{-1/3}} \Lambda''(\eta)d\eta| \\ &\geq \frac{1}{3}|\xi t^{-1/3}|\Lambda''(\frac{2}{3}\xi t^{-1/3}) \geq C|\Lambda'(\xi t^{-1/3})|. \end{aligned}$$

Note that $|\langle\xi\rangle^\nu(\xi\partial_\xi)^j a_k(t,x,\xi)| \leq Ct^{-\frac{1+\alpha_1+\alpha_2}{3}}$ for all $x,\xi \in \mathbb{R}$, $t \geq 1$, $j=0,1,2$ with some $\nu \in (0,1)$, if $\alpha_1 \geq 0$, $\alpha_2 \in \mathbb{R}$, $-1 \leq \alpha_1+\alpha_2 < 2$. Therefore by Lemma 2.3 we find $\|a_k(t,x,D)\phi\|_{L^2_x} \leq Ct^{-\frac{1+\alpha_1+\alpha_2}{3}}\|\phi\|_{L^2}$. Then as above we have

$$\big\||\Lambda''|^{1/2}\overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}a_1(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\phi_\xi(\xi)\big\|_{L^2_\eta} \leq Ct^{-\frac{1+\alpha_1+\alpha_2}{3}}\|\partial_\xi\phi\|_{L^2}$$

and

$$\begin{aligned} &\big\||\Lambda''|^{1/2}\overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}a_2(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\frac{\phi(\xi)-\phi(0)}{\xi}\big\|_{L^2_\eta} \\ &\leq Ct^{-\frac{1+\alpha_1+\alpha_2}{3}}\|\partial_\xi\phi\|_{L^2}. \end{aligned}$$

Now let us prove the L^2 - L^∞ boundedness of the pseudodifferential operator $a_3(t,x,D)$. We have

$$a_3(t,x,\xi) = t^{-1}\chi\big(t^{1/3}\mu(xt^{-2/3})\big)P(\mu(xt^{-2/3}))t^{1/3}\partial_\xi\Big(\frac{Q(\xi t^{-1/3})(1-\chi(\frac{1}{3}\xi))}{\Lambda'(\xi t^{-1/3}) - xt^{-2/3}}\Big),$$

Note that $|\langle\xi\rangle^\nu(\xi\partial_\xi)^j\langle\xi\rangle^\delta a_3(t,x,\xi)| \leq C|t|^{-\frac{1}{3}(\alpha_1+\alpha_2)}$ for all $x,\xi \in \mathbb{R}$, $t \geq 1$, $j = 0,1,2$ with some small $\nu \in (0,1)$, $\delta > 1/2$. Therefore by Lemma 2.3 we find

$\|a_3(t,x,D)\phi\|_{L^2_x} \le C|t|^{-\frac{1}{3}(\alpha_1+\alpha_2)}\|\langle\xi\rangle^{-\delta}\widehat{\phi}\|_{L^2} \le Ct^{-\frac{1+\alpha_1+\alpha_2}{3}}t^{\frac{1}{6}}\|\widehat{\phi}\|_{L^\infty}$. Thus we obtain

$$\big\||\Lambda''|^{1/2}\phi(0)\overline{M}\mathcal{B}^{-1}\mathcal{D}^{-1}_{t^{2/3}}a_3(t,x,D)\mathcal{F}^{-1}\mathcal{D}_{t^{1/3}}e^{-it\Lambda(\xi)}\big\|_{L^2_\eta} \le Ct^{-\frac{1+\alpha_1+\alpha_2}{3}}t^{\frac{1}{6}}|\phi(0)|.$$

The proof is complete. □

Applying the above lemmas we obtain estimates for the derivatives, $\partial_\eta\mathcal{V}_k = |\Lambda''(\eta)|[\eta,\mathcal{V}_k] + \mathcal{V}_k\partial_\eta\big((1-\chi(\eta t^{1/3}))\chi_k(\xi/\eta)\big)$, $k=1,2$, $\partial_\eta\mathcal{V}_4 = |\Lambda''(\eta)|[\eta,\mathcal{V}_4] + \mathcal{V}_4\partial_\eta(\chi(\eta t^{1/3})\chi(\frac{1}{3}\xi t^{1/3}))$. Note that $\partial_\eta((1-\chi(\eta t^{1/3}))\chi_k(\xi/\eta)) \le C\eta^{-1} \le Ct\eta^2$ since $\eta t^{1/3} \ge 1$ and $\partial_\eta(\chi(\eta t^{1/3})\chi(\frac{1}{3}\xi t^{1/3})) \le C\eta^{-1} \le Ct\eta^2$. We choose in Lemma 3.1 and 3.4 $P(\eta) = \eta^{1-j}|\Lambda''(\eta)|$, $Q(\xi) = \xi^j$, or $P(\eta) = \eta^{-j}|\Lambda''(\eta)|$ and $Q(\xi) = \xi^{1+j}$. Also we choose in Lemma 3.2 $h(\xi) = \xi$, $P(\eta) = \eta^{-j}|\Lambda''(\eta)|$, $Q(\xi) = \xi^j$. Then we obtain

Corollary 3.6. *The following estimates hold*

$$\big\||\Lambda''|^{1/2}\eta^{-j}\partial_\eta\mathcal{V}_k\xi^j\phi\big\|_{L^2} \le C\Big(\|\partial_\xi\phi\|_{L^2} + t^{\frac{1}{6}}|\phi(0)|\Big)$$

for all $t \ge 1$, $j \ge 0$, $k = 1,4$; and

$$\big\||\Lambda''|^{1/2}\eta^{-j}\partial_\eta\mathcal{V}_2\xi^j\phi\big\|_{L^2} \le C\Big(\|\partial_\xi\phi\|_{L^2} + t^{\frac{1}{6}}|\phi(0)|\Big)$$

for all $t \ge 1$, $j \in \mathbb{Z}$.

Choosing in Lemma 3.3 $P(\eta) = \eta^{1-j}|\Lambda''(\eta)|$ and $Q(\xi) = \xi^j$, or $P(\eta) = \eta^{-j}|\Lambda''(\eta)|$ and $Q(\xi) = \xi^{1+j}$, and since $\partial_\eta\mathcal{V}_3 = |\Lambda''(\eta)|[\eta,\mathcal{V}_3] + \mathcal{V}_3(\partial_\eta(\chi(\eta t^{1/3})(1-\chi(\frac{1}{3}\xi t^{1/3}))))$, we note that

$$\partial_\eta\big((1-\chi(\eta t^{1/3}))\chi_3(\xi/\eta)\big) \le C\eta^{-1} \le Ct\eta^2$$

since $\eta t^{1/3} \ge 1$.

Corollary 3.7. *It holds*

$$\big\||\Lambda''|-1/2\eta^{1-j}\partial_\eta\mathcal{V}_3\xi^j\phi\big\|_{L^2} \le C\Big(\|\partial_\xi\phi\|_{L^2} + t^{1/6}|\phi(0)|\Big)$$

for all $t \ge 1$, $j = 0,1$.

Choosing in Lemma 3.5 $P(\eta) = \eta^{1-j}|\Lambda''(\eta)|$ and $Q(\xi) = \xi^j$, or $P(\eta) = \eta^{-j}|\Lambda''(\eta)|$ and $Q(\xi) = \xi^{1+j}$, since $\eta t^{1/3} \le 2$ in $\mathcal{V}_5$ and $\partial_\eta\mathcal{V}_5 = |\Lambda''(\eta)|[\eta,\mathcal{V}_5] + \mathcal{V}_5(\partial_\eta((1-\chi(\eta t^{1/3}))\chi_3(\xi/\eta)))$, we note that $\partial_\eta(\chi(\eta t^{1/3})(1-\chi(\frac{1}{3}\xi t^{1/3}))) \le C\eta^{-1} \le Ct\eta^2$.

Corollary 3.8. *It holds*

$$\||\Lambda''|-1/2\partial_\eta\mathcal{V}_5\xi^j\phi\|_{L^2} \le Ct^{\frac{1-j}{3}}(\|\partial_\xi\phi\|_{L^2} + t^{\frac{1}{6}}|\phi(0)|)$$

for all $t \ge 1$, $j = 0,1$.

3.2. **Asymptotic behavior.** Define the kernel

$$A_j(t,\eta) = \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\Psi_2(t,\xi,\eta)\xi^j d\xi.$$

With the changing of variables $\xi = \eta y$, we obtain

$$A_j(t,\eta) = (1-\chi(\eta t^{1/3}))|\eta|\eta^j\sqrt{\frac{|t|}{2\pi}}\int_{1/3}^3 e^{-it|\eta|^3G(y,\eta)}\chi_2(y)y^j dy,$$

where $S(\eta y,\eta)=\Lambda(\eta y)-\Lambda(\eta)-\eta\Lambda'(\eta)(y-1)=|\eta|^3G(y,\eta)$ and $G(y,\eta)=(\frac{1}{3}(y+2)+\frac{b}{4}|\eta|(y^3+2y^2+3))(y-1)^2$, $y>0$. To study the asymptotic behavior of the kernel $A_j(t,\eta)$ for large t we apply the stationary phase method (see [14, p. 110])

$$\int_{\mathbb{R}} e^{izg(y)}f(y)dy=e^{izg(y_0)}f(y_0)\sqrt{\frac{2\pi}{z|g''(y_0)|}}e^{i\frac{\pi}{4}\operatorname{sgn} g''(y_0)}+O\big(z^{-3/2}\big) \tag{3.1}$$

for $z\to+\infty$, where the stationary point y_0 is defined by the equation $g'(y_0)=0$. By (3.1) with $g(y)=-G(y,\eta)$, $f(y)=\chi_2(y)y^j$ and $y_0=1$, we obtain

$$A_j(t,\eta)=\frac{t^{1/2}|\eta|\eta^j}{\sqrt{i\langle t\eta^3\rangle\frac{\Lambda''(\eta)}{|\eta|}}}+O\Big(t^{1/2}\eta^{1+j}\langle\eta\rangle^{-1}\langle t\eta^3\rangle^{-1}\Big)$$

for $t^{1/3}|\eta|\to\infty$. Also since $\Lambda''(\eta)=O(|\eta|\langle\eta\rangle)$ we have the estimate $|A_j(t,\eta)|\leq Ct^{\frac{1}{2}}|\eta|^{j+1}\langle t\eta^3\rangle-1/2\langle\eta\rangle-1/2$. By the Sobolev embedding theorem and Lemma 3.1 we have $\||\Lambda''|^{1/2}\langle t\eta^3\rangle\eta^{-1-j}\mathcal{V}_1\xi^j\phi\|_{L^2}\leq C(t^{\frac{1}{6}}|\phi(0)|+\|\partial_\xi\phi\|_{L^2})$ and using Corollary 3.6,

$$\||\Lambda''|^{1/2}\eta^{-j}\partial_\eta\mathcal{V}_1\xi^j\phi\|_{L^2}\leq C(t^{\frac{1}{6}}|\phi(0)|+\|\partial_\xi\phi\|_{L^2})$$

we obtain

$$\begin{aligned}\|\langle t\eta^3\rangle^{1/2}\eta^{-j}\mathcal{V}_1\xi^j\phi\|_{L^\infty}&\leq C\||\Lambda''|^{1/2}\langle t\eta^3\rangle\eta^{-1-j}\mathcal{V}_1\xi^j\phi\|_{L^2}^{1/2}\||\Lambda''|^{1/2}\eta^{-j}\partial_\eta\mathcal{V}_1\xi^j\phi\|_{L^2}^{1/2}\\&\quad+C\||\Lambda''|^{1/2}\langle t\eta^3\rangle\eta^{-1-j}\mathcal{V}_1\xi^j\phi\|_{L^2}\\&\leq C\Big(t^{1/6}|\phi(0)|+\|\partial_\xi\phi\|_{L^2}\Big).\end{aligned}$$

for $j\geq0$, Similarly by the Sobolev embedding theorem and Lemma 3.3 we have

$$\||\Lambda''|^{1/2}\langle t\eta^3\rangle|\eta|^{-1-j}\mathcal{V}_3\xi^j\phi\|_{L^2}\leq C\|\partial_\xi\phi\|_{L^2}+Ct^{\frac{1}{6}}|\phi(0)|.$$

Then by Corollary 3.7, $\||\Lambda''|^{1/2}\eta^{-j}\partial_\eta\mathcal{V}_3\xi^j\phi\|_{L^2}\leq C\|\partial_\xi\phi\|_{L^2}+Ct^{\frac{1}{6}}|\phi(0)|$, we have

$$\begin{aligned}&\|\langle t\eta^3\rangle^{\frac{1}{2}-\frac{j}{3}}\mathcal{V}_3\xi^j\phi\|_{L^\infty}\\&\leq Ct^{-j/3}\||\Lambda''|^{1/2}\langle t\eta^3\rangle|\eta|^{-1-j}\mathcal{V}_3\xi^j\phi\|_{L^2}^{1/2}\||\Lambda''|^{1/2}\eta^{-j}\partial_\eta\mathcal{V}_3\xi^j\phi\|_{L^2}^{1/2}\\&\quad+Ct^{-j/3}\||\Lambda''|^{1/2}\langle t\eta^3\rangle|\eta|^{-1-j}\mathcal{V}_3\xi^j\phi\|_{L^2}\\&\leq Ct^{-j/3}(t^{\frac{1}{6}}|\phi(0)|+\|\partial_\xi\phi\|_{L^2})\end{aligned}$$

for $j=0,1$. Similarly by the Sobolev embedding theorem and Lemma 3.5 we have $\||\Lambda''|^{1/2}\mathcal{V}_5\xi^j\phi\|_{L^2}\leq Ct^{-\frac{1+j}{3}}(\|\partial_\xi\phi\|_{L^2}+Ct^{\frac{1}{6}}|\phi(0)|)$. Then by Corollary 3.8,

$$\||\Lambda''|-1/2\partial_\eta\mathcal{V}_5\xi^j\phi\|_{L^2}\leq Ct^{\frac{1-j}{3}}(\|\partial_\xi\phi\|_{L^2}+t^{1/6}|\phi(0)|)$$

for $j=0,1$, and

$$\begin{aligned}\|\mathcal{V}_5\xi^j\phi\|_{L^\infty}&\leq C\||\Lambda''|^{1/2}\mathcal{V}_5\xi^j\phi\|_{L^2}^{1/2}\||\Lambda''|^{-1/2}\partial_\eta\mathcal{V}_5\xi^j\phi\|_{L^2}^{1/2}\\&\leq Ct^{-j/3}(t^{1/6}|\phi(0)|+\|\partial_\xi\phi\|_{L^2})\end{aligned}$$

for $j=0,1$. In the next lemma we estimate the operators $\mathcal{V}_2$ in the uniform norm. We denote $\{\eta\}=|\eta|\langle\eta\rangle^{-1}$.

Lemma 3.9. *The estimate $\langle t^{1/3}\eta\rangle^{3/4}|\mathcal{V}_2\xi^j\phi-A_j\phi|\leq C\|\partial_\xi\phi\|_{L^2}$ holds for all $t\geq1$ if $j\geq0$.*

Proof. We integrate by parts via identity $e^{-itS(\xi,\eta)} = H_1\partial_\xi((\xi-\eta)e^{-itS(\xi,\eta)})$ with $H_1 = (1-it(\xi-\eta)\partial_\xi S(\xi,\eta))^{-1}$, to obtain

$$\begin{aligned}\mathcal{V}_2\xi^j\phi - A_j\phi &= Ct^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}(\phi(\xi)-\phi(\eta))(\xi-\eta)\partial_\xi(H_1\Psi_2(t,\xi,\eta)\xi^j)d\xi\\ &\quad + Ct^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}(\xi-\eta)H_1\Psi_2(t,\xi,\eta)\xi^j\partial_\xi\phi(\xi)d\xi.\end{aligned}$$

Note that $\Lambda''(\xi) > 0$ for all $\xi > 0$, and $\partial_\xi^k\Lambda(\xi) = O(\{\xi\}^{3-k}\langle\xi\rangle^{4-k})$, $k = 0,1,2$. Also $\partial_\xi S(\xi,\eta) = \Lambda'(\xi)-\Lambda'(\eta)$. Hence we have

$$|H_1\Psi_2(t,\xi,\eta)\xi^j| + |(\xi-\eta)\partial_\xi(H_1\Psi_2(t,\xi,\eta)\xi^j)| \le \frac{C|\eta|^j}{1+t|\eta|\langle\eta\rangle(\xi-\eta)^2}$$

on the domain $1/3 < \frac{\xi}{\eta} < 3$. Therefore we obtain

$$\begin{aligned}|\mathcal{V}_2\xi^j\phi - A_j\phi| &\le Ct^{1/2}|\eta|^j\int_{1/3<\xi/\eta<3}\frac{|\phi(\xi)-\phi(\eta)|}{|\xi-\eta|}\frac{|\xi-\eta|d\xi}{1+t|\eta|\langle\eta\rangle(\xi-\eta)^2}\\ &\quad + Ct^{1/2}|\eta|^j\int_{1/3<\xi/\eta<3}\frac{|\xi-\eta||\partial_\xi\phi(\xi)|d\xi}{1+t|\eta|\langle\eta\rangle(\xi-\eta)^2}.\end{aligned}$$

By the Hardy inequality we have $\int_{1/3<\xi/\eta<3}\frac{|\phi(\xi)-\phi(\eta)|^2}{|\xi-\eta|^2}d\xi \le C\|\partial_\xi\phi\|_{L^2}^2$, then by the Cauchy-Schwarz inequality, $|\mathcal{V}_2\xi^j\phi - A_j\phi| \le Ct^{1/2}|\eta|^j\|\partial_\xi\phi\|_{L^2}I^{1/2}$, where $I = \int_{1/3<\xi/\eta<3}\frac{(\xi-\eta)^2d\xi}{(1+t|\eta|\langle\eta\rangle(\xi-\eta)^2)^2}$. Changing $\xi = \eta y$ we have

$$I \le C|\eta|^3\int_{1/3}^3\frac{(y-1)^2dy}{(1+|t||\eta|^3\langle\eta\rangle(y-1)^2)^2} \le C|\eta|^3\langle t|\eta|^3\langle\eta\rangle\rangle^{-3/2}.$$

Thus we obtain $|\mathcal{V}_2\xi^j\phi - A_j\phi| \le Ct^{1/2}\|\partial_\xi\phi\|_{L^2}|\eta|^{j+\frac{3}{2}}\langle t|\eta|^3\langle\eta\rangle\rangle^{-3/4}$. The proof is complete. □

4. Estimates for the operator $\mathcal{V}^*$

4.1. **Asymptotic behavior.** Define the kernel

$$A^*(t,\xi) = \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\chi_2(\eta\xi^{-1})|\Lambda''(\eta)|d\eta$$

for $\xi \ne 0$. Changing variables, $\eta = \xi y$, we obtain

$$A^*(t,\xi) = |\xi|\sqrt{\frac{|t|}{2\pi}}\int_{1/3}^3 e^{it|\xi|^3\widetilde{G}(y,\xi)}\chi_2(y)|\Lambda''(\xi y)|dy,$$

where $S(\xi,\xi y) = \Lambda(\xi)-\Lambda(\xi y)-\xi\Lambda'(\xi y)(1-y) = |\xi|^3\widetilde{G}(y,\xi)$ with $\widetilde{G}(y,\xi) = (\frac{a}{3}(2y+1)+\frac{b}{4}|\xi|(3y^2+2y+1))(y-1)^2$, $y > 0$. Then by (3.1) with $g(y) = \widetilde{G}(y,\xi)$, $f(y) = \chi_2(y)|\Lambda''(\xi y)|$, $y_0 = 1$, we obtain

$$A^*(t,\xi) = t^{1/2}\xi^2\sqrt{\frac{i\Lambda''(\xi)}{\langle t\xi^3\rangle|\xi|}} + O\Big(t^{1/2}\{\xi\}^2\langle\xi\rangle^3\langle t\xi^3\rangle^{-1}\Big)$$

for $t|\xi|^3 \to \infty$. Since $\Lambda''(\xi) = O(\xi\langle\xi\rangle)$ we have $|A^*(t,\xi)| \le Ct^{1/2}\xi^2\langle t\xi^3\rangle{-1/2}\langle\xi\rangle^{1/2}$.

In the next lemma we study the asymptotic behavior of $\mathcal{V}^*$. We denote $\widehat{\xi} = \xi t^{1/3}$ and define

$$\|\phi\|_{I_{\alpha,\beta}} = \|\{\eta\}^\alpha\langle\widehat{\eta}\rangle^{-\beta}\langle\eta\rangle^\sigma\partial_\eta\phi\|_{L^2} + \|\{\eta\}^{\alpha-1}\langle\widehat{\eta}\rangle^{-\beta}\langle\eta\rangle^\sigma\phi\|_{L^2},$$

where $\sigma = \frac{3}{4} + 2\beta$.

Lemma 4.1. *Let* $\frac{1}{4} + 2\beta \le \alpha < \frac{5}{2} - 2\beta$, $0 \le \beta \le \frac{11}{16}$. *Then*

$$\|\langle\widehat{\xi}\rangle^{\beta}(\mathcal{V}^*\phi - A^*\phi)\|_{L^\infty} \le C\max(t^{\frac{2\beta}{3}-\frac{1}{2}}, t^{-\frac{1}{4}}, t^{\frac{\alpha-1}{3}})\|\phi\|_{I_{\alpha,\beta}}$$

for all $t \ge 1$.

Proof. We write

$$\begin{aligned}\mathcal{V}^*\phi - A^*\phi &= \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}(\phi(\eta) - \phi(\xi))\Lambda''(\eta)\chi_2(\eta\xi^{-1})d\eta \\ &\quad + \sqrt{\frac{|t|}{2\pi}}\int_{\mathbb{R}} e^{itS(\xi,\eta)}\phi(\eta)\Lambda''(\eta)(1 - \chi_2(\eta\xi^{-1}))d\eta \\ &= I_1 + I_2\end{aligned}$$

for $\xi \ne 0$. In the integral I_1 we use the identity $e^{itS(\xi,\eta)} = H_3\partial_\eta((\eta-\xi)e^{itS(\xi,\eta)})$ with $H_3 = (1 + it(\eta-\xi)\partial_\eta S(\xi,\eta))^{-1}$, $\partial_\eta S(\xi,\eta) = -\Lambda''(\eta)(\xi-\eta)$, and integrate by parts

$$\begin{aligned}I_1 = Ct^{1/2}&\int_{\mathbb{R}} e^{itS(\xi,\eta)}\frac{\phi(\eta)-\phi(\xi)}{\eta-\xi}(\eta-\xi)^2\partial_\eta(H_3\Lambda''(\eta)\chi_2(\eta\xi^{-1}))d\eta \\ &+ Ct^{1/2}\int_{\mathbb{R}} e^{itS(\xi,\eta)}(\eta-\xi)H_3\Lambda''(\eta)\chi_2(\eta\xi^{-1})\partial_\eta\phi(\eta)d\eta.\end{aligned}$$

Then applying the estimates $\Lambda''(\eta) = O(\{\eta\}\langle\eta\rangle^2)$, we obtain

$$\begin{aligned}&|(\eta-\xi)H_3|\Lambda''(\eta)|\{\eta\}^{-\alpha}\langle\widehat{\eta}\rangle^{\beta}\langle\eta\rangle^{-\sigma}\chi_2(\eta\xi^{-1})| \\ &+ |(\eta-\xi)^2\langle\widehat{\eta}\rangle^{\beta}\langle\eta\rangle^{-\sigma}\partial_\eta\Big(H_3|\Lambda''(\eta)|\{\eta\}^{-\alpha}\chi_2(\eta\xi^{-1})\Big)| \\ &\le \frac{C\{\xi\}^{1-\alpha}\langle\xi\rangle^{2-\sigma}\langle\widehat{\xi}\rangle^{\beta}|\eta-\xi|}{1+t\{\xi\}\langle\xi\rangle^2(\eta-\xi)^2}\end{aligned}$$

in the domain $1/3 \le \frac{\eta}{\xi} \le 3$. If $|\kappa(x)| \le C|\kappa(yx)|$ for all $y \in (0,1)$ then we find the Hardy inequality

$$\begin{aligned}\|\frac{\kappa(x)}{x}(\phi(x)-\phi(0))\|_{L^2} &= \|\frac{\kappa(x)}{x}\int_0^x \phi'(y)dty\|_{L^2} = \|\kappa(x)\int_0^1\phi'(xy)dy\|_{L^2} \\ &\le C\|\int_0^1\kappa(xy)\phi'(xy)dy\|_{L^2} \le \int_0^1\|\kappa(xy)\phi'(xy)\|_{L^2}dt \\ &\le \int_0^1 y{-}1/2dy\|\kappa\phi'\|_{L^2} \le 2\|\kappa\phi'\|_{L^2}.\end{aligned}$$

Hence $\int_{1/3\le\eta/\xi\le3}\frac{|\phi(\eta)-\phi(\xi)|^2}{(\eta-\xi)^2}\{\eta\}^{2\alpha}\langle\widehat{\eta}\rangle^{-2\beta}\langle\eta\rangle^{2\sigma}d\eta \le C\|\phi\|_{I_{\alpha,\beta}}^2$. Therefore

$$\langle\widehat{\xi}\rangle^{\beta}|I_1| \le Ct^{1/2}\{\xi\}^{1-\alpha}\langle\xi\rangle^{2-\sigma}\langle\widehat{\xi}\rangle^{2\beta}\|\phi\|_{I_{\alpha,\beta}}I_3^{1/2},$$

where $I_3 = \int_{1/3\le\eta/\xi\le3}\frac{(\eta-\xi)^2d\eta}{(1+t\{\xi\}\langle\xi\rangle^2(\eta-\xi)^2)^2}$. Changing $\eta = y\xi$, we have

$$I_3 \le C|\xi|^3\int_{1/3}^3\frac{(1-y)^2dy}{\big(1+t\{\xi\}^3\langle\xi\rangle^4(1-y)^2\big)^2} \le C|\xi|^3\langle t|\xi|^3\langle\xi\rangle\rangle^{-3/2}.$$

Hence

$$\langle\widehat{\xi}\rangle^{\beta}|I_1| \le Ct^{1/2}\{\xi\}^{\frac{5}{2}-\alpha}\langle\xi\rangle^{2+\frac{3}{2}-\sigma}\langle\xi t^{1/3}\rangle^{2\beta}\langle t|\xi|^3\langle\xi\rangle\rangle^{-3/4}\|\phi\|_{I_{\alpha,\beta}}$$

$$\le C\max(t^{-\frac14}, t^{\frac{\alpha-1}{3}})\|\phi\|_{I_{\alpha,\beta}}$$

if $\frac14+2\beta\le\alpha\le 5/2$. In the integral I_2, using the identity $e^{itS(\xi,\eta)}=H_4\partial_\eta(\eta e^{itS(\xi,\eta)})$ with $H_4=(1+it\eta\partial_\eta S(\xi,\eta))^{-1}$, $\partial_\eta S(\xi,\eta)=-\Lambda''(\eta)(\xi-\eta)$, we integrate by parts

$$\begin{aligned}I_2&=Ct^{1/2}\int_{-\infty}^{\infty}e^{itS(\xi,\eta)}\frac{\{\eta\}^\alpha\langle\eta\rangle^\sigma\phi(\eta)}{\eta\langle\widehat{\eta}\rangle^\beta}\eta^2\partial_\eta\Big(H_4(1-\chi_2(\eta\xi^{-1}))\frac{\Lambda''(\eta)\langle\widehat{\eta}\rangle^\beta}{\{\eta\}^\alpha\langle\eta\rangle^\sigma}\Big)d\eta\\&\quad+Ct^{1/2}\int_{-\infty}^{\infty}e^{itS(\xi,\eta)}\eta H_4(1-\chi_2(\eta\xi^{-1}))\frac{\Lambda''(\eta)\langle\widehat{\eta}\rangle^\beta}{\{\eta\}^\alpha\langle\eta\rangle^\sigma}\partial_\eta\Big(\frac{\{\eta\}^\alpha\langle\eta\rangle^\sigma}{\langle\widehat{\eta}\rangle^\beta}\phi(\eta)\Big)d\eta.\end{aligned}$$

Then using the estimates $\partial_\eta S(\xi,\eta)=O(|\eta|\langle\eta\rangle(\xi+|\eta|))$ in the domains $\frac{\eta}{\xi}<\frac23$ and $\frac{\eta}{\xi}\ge\frac32$, we obtain

$$\begin{aligned}&\big|\eta^2\partial_\eta\Big(H_4(1-\chi_2(\eta\xi^{-1}))\frac{\Lambda''(\eta)\langle\widehat{\eta}\rangle^\beta}{\{\eta\}^\alpha\langle\eta\rangle^\sigma}\Big)\big|+\big|\eta H_4(1-\chi_2(\eta\xi^{-1}))\frac{\Lambda''(\eta)\langle\widehat{\eta}\rangle^\beta}{\{\eta\}^\alpha\langle\eta\rangle^\sigma}\big|\\&\le\frac{C\{\eta\}^{2-\alpha}\langle\widehat{\eta}\rangle^\beta\langle\eta\rangle^{3-\sigma}}{1+t\eta^2\langle\eta\rangle(|\xi|+|\eta|)}.\end{aligned}$$

Therefore by Hardy's inequality

$$\langle\widehat{\xi}\rangle^\beta|I_2|\le C^\beta t^{1/2}\|\partial_\eta(\{\eta\}^\alpha\langle\widehat{\eta}\rangle^{-\beta}\langle\eta\rangle^\sigma\phi(\eta))\|_{L^2}I_3^{1/2}\le Ct^{1/2}\|\phi\|_{I_{\alpha,\beta}}I_3^{1/2},$$

where $I_3=\int_{\mathbb{R}}\frac{\langle\widehat{\xi}\rangle^{2\beta}\langle\widehat{\eta}\rangle^{2\beta}\{\eta\}^{4-2\alpha}\langle\eta\rangle^{6-2\sigma}d\eta}{(1+t\eta^2\langle\eta\rangle(|\xi|+|\eta|))^2}$. We have

$$\begin{aligned}I_3&\le C\int_0^1\frac{\langle\widehat{\xi}\rangle^{2\beta}\eta^{4-2\alpha}\langle\widehat{\eta}\rangle^{2\beta}d\eta}{(1+t\eta^2(|\xi|+|\eta|))^2}+Ct^{\frac{2\beta}{3}-2}\int_1^\infty\langle\widehat{\xi}\rangle^{2\beta}\eta^{2\beta-2\sigma}(|\xi|+|\eta|)^{-2}d\eta\\&\le C\int_0^1\frac{\eta^{4-2\alpha}(1+t^{\frac{2\beta}{3}}\eta^{2\beta})d\eta}{(1+t\eta^3)^2}+Ct^{\frac{2\beta}{3}}\int_0^1\frac{|\xi|^{2\beta}(1+t^{\frac{2\beta}{3}}\eta^{2\beta})\eta^{4-2\alpha}d\eta}{(t\eta^2|\xi|)^{2\beta}(1+t\eta^3)^{2-2\beta}}\\&\quad+Ct^{\frac{4\beta}{3}-2}\int_1^\infty\eta^{4\beta-2\sigma-2}d\eta\\&\le Ct^{\frac{2\alpha-5}{3}}\int_0^{t^{1/3}}\eta^{4+2\beta-2\alpha}\langle\eta\rangle^{-6}d\eta+Ct^{\frac{2\alpha-5}{3}}\int_0^{t^{1/3}}\eta^{4-2\beta-2\alpha}\langle\eta\rangle^{6\beta-6}d\eta+Ct^{\frac{4\beta}{3}-2}\\&\le C\max(t^{\frac{4\beta}{3}-2},t^{\frac{2\alpha-5}{3}})\end{aligned}$$

if $-\frac12+2\beta<\alpha<\frac52-2\beta$ with $0\le\beta\le1$.

Then $\langle\widehat{\xi}\rangle^\beta|I_2|\le C\max(t^{\frac{2\beta}{3}-\frac12},t^{\frac{\alpha-1}{3}})\|\phi\|_{I_{\alpha,\beta}}$, and the proof is complete. □

4.2. **Estimates for commutators.** In the next lemma we estimate the commutator $[|\xi|,\mathcal{V}^*]$.

Lemma 4.2. *$t\|[|\xi|,\mathcal{V}^*]\phi\|_{L^2}\le C\||\Lambda''|-1/2\partial_\eta\phi\|_{L^2}+C\||\Lambda''|-1/2\eta^{-1}\phi\|_{L^2}$ holds for all $t\ge1$.*

Proof. Integrating by parts and using $\partial_\eta S(\xi,\eta)=-|\Lambda''(\eta)|(\xi-\eta)$, we obtain

$$\begin{aligned}t[|\xi|,\mathcal{V}^*]\phi&=Ct|t|^{1/2}\int_{\mathbb{R}}e^{itS(\xi,\eta)}(|\xi|-|\eta|)\phi(\eta)|\Lambda''(\eta)|d\eta\\&=C|t|^{1/2}\int_{\mathbb{R}}e^{itS(\xi,\eta)}\partial_\eta\Big(\frac{|\xi|-|\eta|}{\xi-\eta}\phi(\eta)\Big)d\eta\\&=C|t|^{1/2}\int_{\mathbb{R}}e^{itS(\xi,\eta)}q_1(\eta,\xi)\partial_\eta\phi(\eta)d\eta+C|t|^{1/2}\int_{\mathbb{R}}e^{itS(\xi,\eta)}q_2(\eta,\xi)\frac{\phi(\eta)}{\eta}d\eta,\end{aligned}$$

where $q_1(\eta,\xi)=\frac{|\xi|-|\eta|}{\xi-\eta}=\frac{\xi+\eta}{|\xi|+|\eta|}$ and $q_2(\eta,\xi)=\eta\partial_\eta\big(\frac{|\xi|-|\eta|}{\xi-\eta}\big)=\frac{|\xi|\eta}{(|\xi|+|\eta|)^2}$. Next we define the pseudodifferential operators $a_k^*(\xi,D)\phi=C\int_{\mathbb{R}}e^{-ix\xi}a_k(x,\xi)\widehat{\phi}(x)dx$, with symbols $a_k(x,\xi)=q_k(\mu(x),\xi)$, then changing the variable of integration $\eta=\mu(x)$, we obtain

$$t[|\xi|,\mathcal{V}^*]\phi=a_1^*(\xi,D)\mathcal{F}^{-1}\mathcal{D}_t\mathcal{B}\Big(\frac{M}{|\Lambda''|}\partial_\eta\phi\Big)+a_2^*(\xi,D)\mathcal{F}^{-1}\mathcal{D}_t\mathcal{B}\Big(\frac{M}{|\Lambda''|}\frac{\phi}{\eta}\Big).$$

We prove the L^2-boundedness of the pseudodifferential operators $a_k^*(\xi,D)$ by considering the adjoint operators

$$a_k(x,D)\phi=\int_{\mathbb{R}}e^{ix\xi}\overline{a_k(x,\xi)}\widehat{\phi}(\xi)d\xi.$$

Define the symbols $a_{1,1}(x,\xi)=\frac{\xi}{|\xi|}\chi(x)(1-\chi(\xi))$, $a_{2,1}(x,\xi)=0$, $a_{k,2}(x,\xi)=a_k(x,\xi)\chi(x)(1-\chi(\xi))-a_{k,1}(x,\xi)$, $a_{k,3}(x,\xi)=a_k(x,\xi)(1-\chi(x))$, and $a_{k,4}(x,\xi)=a_k(x,\xi)\chi(x)\chi(\xi)$. Applying the Plancherel theorem we see that $\|a_{k,1}(x,D)\phi\|_{L^2_x}\le C\|\phi\|_{L^2}$. Note that the symbols

$$\begin{aligned}a_{1,2}(x,\xi)&=\chi(x)(1-\chi(\xi))\frac{\mu(x)-|\mu(x)|}{|\xi|+|\mu(x)|},\\ a_{2,2}(x,\xi)&=\chi(x)(1-\chi(\xi))\frac{|\xi|\mu(x)}{(|\xi|+|\mu(x)|)^2}\end{aligned}$$

are such that $|\{\xi\}^{-1}\langle\xi\rangle(\xi\partial_\xi)^l a_{k,2}(x,\xi)|\le C$ for all $x,\xi\in\mathbb{R}$, $l=0,1,2$. Then by Lemma 2.3 we have the estimate $\|a_{k,2}(x,D)\phi\|_{L^2_x}\le C\|\phi\|_{L^2}$ for all $t\ge1$. The symbol $a_{k,3}(x,\xi)$ satisfies the estimate $\sup_{x,\xi\in\mathbb{R}}|\partial_x^j\partial_\xi^l a_{k,3}(x,\xi)|\le C$ for $j,l=0,1$, then by Lemma 2.1 we have $\|a_{k,3}(x,D)\phi\|_{L^2_x}\le C\|\phi\|_{L^2}$. By the Cauchy-Schwarz inequality we find that

$$|\int_{\mathbb{R}}e^{ix\xi}\overline{a_{k,4}(x,\xi)}\widehat{\phi}(\xi)d\xi|\le|\chi(x)|\|\chi(\xi)\|_{L^2}\|\widehat{\phi}\|_{L^2}=C|\chi(x)|\|\phi\|_{L^2}.$$

Then we obtain

$$\|a_{k,4}(x,D)\phi\|_{L^2_x}=\|\int_{\mathbb{R}}e^{ix\xi}\overline{a_{k,4}(x,\xi)}\widehat{\phi}(\xi)d\xi\|_{L^2}\le C\|\chi(x)\|_{L^2}\|\phi\|_{L^2}\le C\|\phi\|_{L^2}.$$

Therefore the pseudodifferential operator $a_k(x,D)=\sum_{j=1}^4 a_{k,j}(x,D)$ is L^2-bounded. Hence

$$\begin{aligned}&t\|[|\xi|,\mathcal{V}^*]\phi\|_{L^2}\\ &\le\big\|a_1^*(\xi,D)\mathcal{F}^{-1}\mathcal{D}_t\mathcal{B}\big(\frac{M}{|\Lambda''|}\partial_\eta\phi\big)\big\|_{L^2}+\big\|a_2^*(\xi,D)\mathcal{F}^{-1}\mathcal{D}_t\mathcal{B}\big(\frac{M}{|\Lambda''|}\frac{\phi}{\eta}\big)\big\|_{L^2}\\ &\le C\|\mathcal{B}(\frac{M}{|\Lambda''|}\partial_\eta\phi)\|_{L^2}+C\|\mathcal{B}(\frac{M}{|\Lambda''|}\frac{\phi}{\eta})\|_{L^2}\\ &\le C\||\Lambda''|^{-12}\partial_\eta\phi\|_{L^2}+C\||\Lambda''|-1/2\eta^{-1}\phi\|_{L^2}.\end{aligned}$$

The proof is complete. □

In the next lemma we estimate the operator $\mathcal{V}_{K_j}^*$,

$$\mathcal{V}_{K_j}^*\phi=\sqrt{\frac{|t|}{2\pi}}\int_{-\infty}^{\infty}e^{itS(\xi,\eta)}K_j(\xi,\eta)\phi(\eta)\Lambda''(\eta)d\eta,$$

where the symbols $K_3(\xi,\eta)=\frac{((i\xi)^2-(i\eta)^2)\eta^2}{Z(\xi,\eta)}$ and $K_4(\xi,\eta)=\frac{(|\xi|(i\xi)-|\eta|(i\eta))\eta^2}{Z(\xi,\eta)}$ with $Z(\xi,\eta)=(\Lambda(\xi)-2\Lambda(\eta))\big(1-\chi(\frac{\xi}{4\eta})\big)+2(\frac{\Lambda(\xi)}{\xi}-\frac{\Lambda(\eta)}{\eta})\eta\chi(\frac{\xi}{4\eta})$.

Lemma 4.3. *The estimate* $\|\mathcal{V}_{K_j}^*\phi\|_{L^2}\le C\||\Lambda''|^{1/2}\eta\phi\|_{L^2}$ *holds for all* $t\ge 1$ *and* $j=3,4$.

Proof. We define the operators $a_j^*(\xi,D)\phi=C\int_{\mathbb{R}}e^{-ix\xi}a_j(x,\xi)\phi(x)dx$, with symbols $a_j(x,\xi)=\frac{1}{\mu(x)}K_j(\xi,\mu(x))$, then changing the variable of integration $\eta=\mu(x)$, we obtain $\mathcal{V}_{K_j}^*\phi=a_j^*(\xi,D)\mathcal{F}^{-1}\mathcal{D}_t\mathcal{B}M\eta\phi$. We prove the L^2-boundedness of the pseudodifferential operators $a_j^*(\xi,D)$ by considering the adjoint operators $a_j(x,D)\phi=\int_{\mathbb{R}}e^{ix\xi}\overline{a_j(x,\xi)}\phi(\xi)d\xi$.

As above we define the symbols $a_{j,1}(x,\xi)=a_j(x,\xi)\chi(x)(1-\chi(\xi))$, $a_{j,2}(x,\xi)=a_j(x,\xi)(1-\chi(x))$, and $a_{j,3}(x,\xi)=a_j(x,\xi)\chi(x)\chi(\xi)$. We have

$$a_j(x,\xi)=\frac{((i\xi)^2-(i\mu(x))^2)\mu(x)}{Z(\xi,\mu(x))},$$
$$a_j(x,\xi)=\frac{(|\xi|(i\xi)-|\mu(x)|(i\mu(x)))\mu(x)}{Z(\xi,\mu(x))},$$

where

$$Z(\xi,\mu(x))=(\Lambda(\xi)-2\Lambda(\mu(x)))\big(1-\chi(\frac{\xi}{4\mu(x)})\big)+2\big(\frac{\Lambda(\xi)}{\xi}-\frac{\Lambda(\mu(x))}{\mu(x)}\big)\mu(x)\chi(\frac{\xi}{4\mu(x)}).$$

Note that $|\{\xi\}^{-1}\langle\xi\rangle(\xi\partial_\xi)^l a_{j,1}(x,\xi)|\le C$ for all $x,\xi\in\mathbb{R}$, $l=0,1,2$. Then by Lemma 2.3 we have the estimate $\|a_{j,1}(x,D)\phi\|_{L^2_x}\le C\|\phi\|_{L^2}$ for all $t\ge 1$. The symbol $a_{j,2}(x,\xi)$ satisfies the estimate $\sup_{x,\xi\in\mathbb{R}}|\partial_x^j\partial_\xi^l a_{j,2}(x,\xi)|\le C$ for $j,l=0,1$, then by Lemma 2.1 we have $\|a_{j,2}(x,D)\phi\|_{L^2_x}\le C\|\phi\|_{L^2}$. By the Cauchy-Schwarz inequality we find that $|\int_{\mathbb{R}}e^{ix\xi}\overline{a_{j,3}(x,\xi)}\widehat{\phi}(\xi)d\xi|\le|\chi(x)|\|\chi(\xi)\|_{L^2}\|\widehat{\phi}\|_{L^2}=C|\chi(x)|\|\phi\|_{L^2}$. Then we obtain

$$\|a_{j,3}(x,D)\phi\|_{L^2_x}=\|\int_{\mathbb{R}}e^{ix\xi}\overline{a_{j,3}(x,\xi)}\widehat{\phi}(\xi)d\xi\|_{L^2}\le C\|\chi(x)\|_{L^2}\|\phi\|_{L^2}\le C\|\phi\|_{L^2}.$$

Therefore the pseudodifferential operator $a_j(x,D)=\sum_{l=1}^3 a_{j,l}(x,D)$ is L^2-bounded. Hence the estimate follows

$$\|\mathcal{V}_{K_j}^*\phi\|_{L^2}\le\|a_j^*(\xi,D)\mathcal{F}^{-1}\mathcal{D}_t\mathcal{B}M\eta\phi\|_{L^2}\le C\|\mathcal{B}M\eta\phi\|_{L^2}\le C\||\Lambda''|^{1/2}\eta\phi\|_{L^2}.$$

The proof is complete. ☐

5. A priori estimates

5.1. **Uniform norm.** In the next lemma we estimate the large time behavior of $\mathcal{F}\mathcal{U}(-t)\partial_x(|u|^2u)$. We denote $\widehat{\xi}=\xi t^{1/3}$, and define the norm $\|\phi\|_Y=\|\phi\|_{L^\infty}+t^{-1/6}\|\partial_\xi\phi\|_{L^2}$.

Lemma 5.1. *The asymptotic equality*

$$t\mathcal{F}\mathcal{U}(-t)\partial_x(|u|^2u)=|\widehat{\xi}|^6\langle\widehat{\xi}\rangle^{-6}\frac{i\xi}{\Lambda''(\xi)}|\widehat{\varphi}|^2\widehat{\varphi}+O(|\widehat{\xi}|\langle\widehat{\xi}\rangle^{--1-\nu}\|\widehat{\varphi}\|_Y^3)$$

holds for all $t\ge 1$ *and* $\xi\in\mathbb{R}$, *where* $\widehat{\varphi}(t)=\mathcal{F}\mathcal{U}(-t)u(t)$, $\nu>0$ *is small.*

Proof. In view of (2.1), it follows that $t\mathcal{FU}(-t)\partial_x(|u|^2u) = \mathcal{V}^*(t)\mathcal{A}_1(t)|\psi|^2\psi$ and $t\mathcal{FU}(-t)\partial_x(|u|^2u) = i\xi\mathcal{V}^*(t)|\psi|^2\psi$, where $\psi = \mathcal{V}\widehat{\varphi}$. Then by Lemma 4.1 with $\alpha = \frac{1}{2}+\nu$, $\beta = 2\nu$, $\nu \in (0, \frac{11}{32}]$, we obtain

$$\begin{aligned} t\mathcal{FU}(-t)\partial_x(|u|^2u) &= i\xi\mathcal{V}^*(t)|\psi|^2\psi \\ &= i\xi A^*|\psi|^2\psi + O\Big(t^{\frac{\nu}{3}-\frac{1}{2}}\widehat{\xi}\|\{\eta\}^{\frac{1}{2}+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}\partial_\eta|\psi|^2\psi\|_{L^2}\Big) \\ &\quad + O\Big(t^{\frac{\nu}{3}-\frac{1}{2}}\widehat{\xi}\|\{\eta\}^{\nu-\frac{1}{2}}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}|\psi|^2\psi\|_{L^2}\Big) \end{aligned}$$

when $|\xi| < t^{-1/3}$, and

$$\begin{aligned} &t\mathcal{FU}(-t)\partial_x(|u|^2u) \\ &= \mathcal{V}^*\mathcal{A}_1|\psi|^2\psi \\ &= A^*\mathcal{A}_1|\psi|^2\psi + O\Big(t^{\frac{\nu}{3}-\frac{1}{6}}\langle\widehat{\xi}\rangle^{-2\nu}\|\{\eta\}^{\frac{1}{2}+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}\partial_\eta\mathcal{A}_1|\psi|^2\psi\|_{L^2}\Big) \\ &\quad + O\Big(t^{\frac{\nu}{3}-\frac{1}{6}}\langle\widehat{\xi}\rangle^{-2\nu}\|\{\eta\}^{\nu-\frac{1}{2}}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}\mathcal{A}_1|\psi|^2\psi\|_{L^2}\Big) \end{aligned}$$

when $|\xi| > t^{-1/3}$. Now we consider the remainder terms. Using that $\{\eta\} \le t^{-1/3}\{\widehat{\eta}\}\langle\widehat{\eta}\rangle$, we have

$$\begin{aligned} &\|\{\eta\}^{\frac{1}{2}+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}\partial_\eta|\psi|^2\psi\|_{L^2} \\ &\le C\|\{\eta\}^{\frac{1}{2}+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}|\psi|^2\partial_\eta\psi\|_{L^2} \\ &\le C\||\Lambda''|^{-1/2}\{\eta\}^{\frac{1}{2}+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}\psi^2\|_{L^\infty}\||\Lambda''|^{1/2}\partial_\eta\psi\|_{L^2} \\ &\le Ct^{\frac{1}{6}-\frac{\nu}{3}}\|\phi\|_Y\|\psi\|_{L^\infty}^2 \end{aligned}$$

and denoting $\psi_1 = \mathcal{V}(i\xi)\widehat{\varphi}$, we have

$$\begin{aligned} &\|\{\eta\}^{\frac{1}{2}+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}\partial_\eta\mathcal{A}_1|\psi|^2\psi\|_{L^2} \\ &\le C\|\{\eta\}^{\frac{1}{2}+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}\psi^2\partial_\eta\psi_1\|_{L^2} + C\|\{\eta\}^{\frac{1}{2}+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}\psi\psi_1\partial_\eta\psi\|_{L^2} \\ &\le C\||\Lambda''|^{-1/2}\{\eta\}^{\frac{1}{2}+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}\eta\psi^2\|_{L^\infty}\||\Lambda''|^{1/2}\eta^{-1}\partial_\eta\psi_1\|_{L^2} \\ &\quad + C\||\Lambda''|^{-1/2}\{\eta\}^{\frac{1}{2}+\nu}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}\psi\psi_1\|_{L^\infty}\||\Lambda''|^{1/2}\partial_\eta\psi\|_{L^2} \\ &\le Ct^{\frac{1}{6}-\frac{\nu}{3}}\|\phi\|_Y(\|\eta\psi^2\|_{L^\infty} + \|\psi\psi_1\|_{L^\infty}), \end{aligned}$$

since by Corollary 3.6, 3.7, 3.8 we have

$$\||\Lambda''|^{1/2}\eta^{-j}\partial_\eta\psi_j\|_{L^2} \le C(\|\partial_\xi\phi\|_{L^2} + t^{1/6}|\phi(0)|) = Ct^{1/6}\|\phi\|_Y$$

for $j = 0, 1$. Next using Lemma 3.9, we have $\|\psi^2\|_{L^\infty} \le Ct^{1/3}\|\phi\|_Y^2$, $\|\eta\psi^2\|_{L^\infty} \le C\|\phi\|_Y^2$ and $\|\psi\psi_1\|_{L^\infty} \le C\|\phi\|_Y^2$. Also

$$\begin{aligned} &\|\{\eta\}^{\nu-\frac{1}{2}}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}|\psi|^2\psi\|_{L^2} + \|\{\eta\}^{\nu-\frac{1}{2}}\langle\widehat{\eta}\rangle^{-2\nu}\langle\eta\rangle^{\frac{3}{4}+4\nu}\mathcal{A}_1|\psi|^2\psi\|_{L^2} \\ &\le Ct^{\frac{1}{2}-\frac{\nu}{3}}\|\phi\|_Y^3. \end{aligned}$$

Therefore,

$$t\mathcal{FU}(-t)\partial_x(|u|^2u) = i\xi A^*|\psi|^2\psi + O\Big(\{\widehat{\xi}\}\langle\widehat{\xi}\rangle^{-2\nu}\|\phi\|_Y^3\Big)$$

for $|\xi| < t^{-1/3}$, and

$$t\mathcal{FU}(-t)\partial_x(|u|^2u) = A^*\mathcal{A}_1|\psi|^2\psi + O\Big(\{\widehat{\xi}\}\langle\widehat{\xi}\rangle^{-2\nu}\|\phi\|_Y^3\Big)$$

for $|\xi| > t^{-1/3}$. Next by Lemma 3.9 we have

$$\psi_j(t,\xi) = \frac{t^{1/2}|\xi|(i\xi)^j}{\sqrt{i\langle t\xi^3\rangle \frac{\Lambda''(\xi)}{|\xi|}}}\widehat{\varphi}(\xi) + O\Big(t^{\frac{1}{6}-\frac{j}{3}}\langle t\xi^3\rangle^{\frac{j}{3}-\frac{1}{2}}\|\widehat{\varphi}\|_Y\Big)$$

for $j = 0, 1$. Then we obtain

$$\begin{aligned} i\xi A^*|\psi|^2\psi &= i\xi A^*\Big|\frac{t^{1/2}|\xi|}{\sqrt{\langle t\xi^3\rangle \frac{\Lambda''(\xi)}{|\xi|}}}\widehat{\varphi}(\xi)\Big|^2 \frac{t^{1/2}|\xi|}{\sqrt{\langle t\xi^3\rangle \frac{i\Lambda''(\xi)}{|\xi|}}}\widehat{\varphi}(\xi) + O\big(t\xi^3\langle t\xi^3\rangle^{-2}\|\widehat{\varphi}\|_Y^3\big) \\ &= \frac{t^2\xi^6}{\langle t\xi^3\rangle^2}\frac{i\xi}{\Lambda''(\xi)}|\widehat{\varphi}|^2\widehat{\varphi} + O\big(|\widehat{\xi}|\langle\widehat{\xi}\rangle^{-1-\nu}\|\widehat{\varphi}\|_Y^3\big) \end{aligned}$$

and

$$\begin{aligned} A^*\mathcal{A}_1|\psi|^2\psi &= A^*\big(2\psi\psi_1\overline{\psi} + \psi^2\overline{\psi_1}\big) \\ &= A^*\Big(2\Big|\frac{t^{1/2}|\xi|}{\sqrt{\langle t\xi^3\rangle \frac{\Lambda''(\xi)}{|\xi|}}}\Big|^2 \frac{t^{1/2}|\xi| i\xi}{\sqrt{i\langle t\xi^3\rangle \frac{\Lambda''(\xi)}{|\xi|}}} \\ &\quad + \Big(\frac{t^{1/2}|\xi|}{\sqrt{\langle t\xi^3\rangle \frac{\Lambda''(\xi)}{|\xi|}}}\Big)^2 \overline{\frac{t^{1/2}|\xi| i\xi}{\sqrt{i\langle t\xi^3\rangle \frac{\Lambda''(\xi)}{|\xi|}}}}\Big)|\widehat{\varphi}|^2\widehat{\varphi} + O\Big(t^{1/3}\xi\langle t\xi^3\rangle^{-4/3}\|\widehat{\varphi}\|_Y^3\Big) \\ &= \frac{t^2\xi^6}{\langle t\xi^3\rangle^2}\frac{i\xi}{\Lambda''(\xi)}|\widehat{\varphi}|^2\widehat{\varphi} + O\Big(|\widehat{\xi}|\langle\widehat{\xi}\rangle^{-1-\nu}\|\widehat{\varphi}\|_Y^3\Big). \end{aligned}$$

The proof is complete. □

We next prove a priori estimate in the L^∞-norm for $\widehat{\varphi}$.

Lemma 5.2. *Assume that* $\sup_{t\in[1,T]}\|\widehat{\varphi}\|_Y \le C\varepsilon$. *Then there exists an* ε *such that the estimate* $\sup_{t\in[1,T]}\|\widehat{\varphi}\|_{L^\infty} < C\varepsilon$ *holds for any* $T > 1$.

Proof. On the contrary we assume that there exists a first time $T > 0$ such that $\sup_{t\in[1,T]}\|\widehat{\varphi}\|_{L^\infty} = C\varepsilon$. We use (2.1) for a new dependent variable $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$. In view of Lemma 5.1, we obtain

$$\partial_t\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)\partial_x(|u|^2u) = \frac{|\widehat{\xi}|^6 i\xi}{\langle\widehat{\xi}\rangle^6 t\Lambda''(\xi)}|\widehat{\varphi}|^2\widehat{\varphi} + O\Big(\varepsilon^3 t^{-1}|\widehat{\xi}|\langle\widehat{\xi}\rangle^{-1-\nu}\Big). \tag{5.1}$$

For the case of $|\xi| < t^{-1/3}$, we integrate in time directly

$$|\widehat{\varphi}(t,\xi)| \le |\widehat{\varphi}(1,\xi)| + C\varepsilon^3|\xi|\int_1^t \tau^{-2/3}d\tau \le \varepsilon + C\varepsilon^3|\xi|t^{1/3} \le \varepsilon + C\varepsilon^3.$$

For the case of $|\xi| \ge t^{-1/3}$ multiplying equation (5.1) by $\overline{\widehat{\varphi}}$ and taking the real part of the result we obtain $\partial_t|\widehat{\varphi}(t,\xi)| = O\big(\varepsilon^3 t^{-1}|\widehat{\xi}|\langle\widehat{\xi}\rangle^{-1-\nu}\big)$. Integration in time yields

$$\begin{aligned} |\widehat{\varphi}(t,\xi)| &\le |\widehat{\varphi}(|\xi|^{-3},\xi)| + C\varepsilon^3\int_{|\xi|^{-3}}^t |\xi\tau^{1/3}|\langle\xi\tau^{1/3}\rangle^{-1-\nu}\frac{d\tau}{\tau} \\ &\le \varepsilon + C\varepsilon^3\int_1^{|\xi|t^{1/3}}\langle y\rangle^{-1-\nu}dy \le \varepsilon + C\varepsilon^3. \end{aligned}$$

Therefore $\|\widehat{\varphi}\|_{L^\infty} < C\varepsilon$. The proof is complete. □

5.2. **L^2-norms of $\mathcal{P}_a$ and $\mathcal{P}_b$.** There are many papers devoted to Kato-Ponce type estimates of the commutators of the form $[\langle i\partial_x\rangle^\alpha, u]v$ (see, e.g. [17, 7, 31, 35]). Below we will use the following estimates (see [41]).

Lemma 5.3. *The estimate $\|[|\partial_x|, u]w\|_{L^2} \le C\|\partial_x u\|_{L^\infty}\|w\|_{L^2}$ is true.*

Denote $\mathcal{K}_1 = \frac{\partial_x^4}{\Lambda(-i\partial_x)}$, $\mathcal{K}_2\phi = \frac{|\partial_x|^3\partial_x}{\Lambda(-i\partial_x)}\phi$.

Lemma 5.4. *The estimate $\|[\mathcal{K}_j, u]v\|_{L^2} \le C\|\partial_x u\|_{L^\infty}\|v\|_{H^1}$ holds for $j = 1, 2$.*

Next we consider a-priori estimates of solutions in the norm $\|\partial_x^{-1}\mathcal{P}_b u(t)\|_{L^2} + \|\mathcal{P}_a u(t)\|_{L^2} + \|u(t)\|_{H^1}$ uniformly in time.

Lemma 5.5. *Let $\sup_{t\in[1,T]}\|\widehat{\varphi}\|_Y \le C\varepsilon$. Then the estimate*

$$\sup_{t\in[1,T]} t^{-\gamma}(\|\partial_x^{-1}\mathcal{P}_b u(t)\|_{L^2} + \|\mathcal{P}_a u(t)\|_{L^2} + \|u(t)\|_{H^1}) < C\varepsilon$$

holds for all $T > 1$, where $\gamma > 0$ is small.

Proof. We apply operators $\mathcal{P}_a = 4t\partial_t + \partial_x x - a\partial_a$ and $\mathcal{P}_b = 3t\partial_t + \partial_x x + b\partial_b$ to equation (1.1). Using the commutators $[\mathcal{L}, \mathcal{P}_a] = 4\mathcal{L}$ and $[\mathcal{L}, \mathcal{P}_b] = 3\mathcal{L}$, we obtain

$$\mathcal{L}\mathcal{P}_a u = (\mathcal{P}_a + 4)\mathcal{L}u = \partial_x(\mathcal{P}_a + 3)(|u|^2 u) = 2\partial_x|u|^2\mathcal{P}_a u + \partial_x u^2\overline{\mathcal{P}_a u} + 3\partial_x(|u|^2 u)$$

and

$$\mathcal{L}\partial_x^{-1}\mathcal{P}_b u = (\partial_x^{-1}\mathcal{P}_b + 3\partial_x^{-1})\mathcal{L}u = (\mathcal{P}_b + 2)(|u|^2 u) = 2|u|^2\mathcal{P}_b u + u^2\overline{\mathcal{P}_b u}.$$

Then we have

$$\begin{aligned} \frac{d}{dt}\|\partial_x^{-1}\mathcal{P}_b u\|_{L^2}^2 &= 4\operatorname{Re}(\partial_x^{-1}\mathcal{P}_b u, |u|^2\mathcal{P}_b u) + 2\operatorname{Re}(\partial_x^{-1}\mathcal{P}_b u, u^2\overline{\mathcal{P}_b u}) \\ &\le C\||u|u_x\|_{L^\infty}\|\partial_x^{-1}\mathcal{P}_b u\|_{L^2}^2 \end{aligned}$$

and

$$\frac{d}{dt}\|\mathcal{P}_a u\|_{L^2} \le C\||u|u_x\|_{L^\infty}(\|\mathcal{P}_a u\|_{L^2} + \|u\|_{L^2}).$$

In the same manner we obtain $\frac{d}{dt}\|u\|_{H^1} \le C\||u|u_x\|_{L^\infty}\|u\|_{H^1}$. We denote $\partial_x^k u = v_k + w_k$, $v_k = \mathcal{D}_t\mathcal{B}M(\mathcal{V}_1(i\xi)^k\widehat{\varphi} + \mathcal{V}_2(i\xi)^k\widehat{\varphi} + \mathcal{V}_4(i\xi)^k\widehat{\varphi})$, and $w_k = \mathcal{D}_t\mathcal{B}M(\mathcal{V}_3(i\xi)^k\widehat{\varphi} + \mathcal{V}_5(i\xi)^k\widehat{\varphi})$. Via Lemmas 3.4, 3.9 we have $||\eta|^{\frac{1}{2}-j}\mathcal{V}_k\xi^j\widehat{\varphi}| \le C\varepsilon$, for $j \ge 0$, $k = 1, 2, 4$, therefore denoting $\widetilde{\mu} = \mu(xt^{-1})$ we obtain

$$\begin{aligned} \|\widetilde{\mu}^{\frac{1}{2}-j}v_j\|_{L^\infty} &= \sum_{k=1,2,4}\|\widetilde{\mu}^{\frac{1}{2}-j}\mathcal{D}_t M\mathcal{B}\mathcal{V}_k\xi^j\widehat{\varphi}\|_{L^\infty} \\ &\le Ct{-}1/2\sum_{k=1,2,4}\||\eta|^{\frac{1}{2}-j}\mathcal{V}_k\xi^j\widehat{\varphi}\|_{L^\infty} \le C\varepsilon t{-}1/2 \end{aligned}$$

for $j \ge 0$. Also via Lemma 3.9 we have

$$\begin{aligned} \|\widetilde{\mu}^{1/2}w\|_{L^\infty} &= \sum_{k=3,5}\|\widetilde{\mu}^{1/2}\mathcal{D}_t M\mathcal{B}\mathcal{V}_k\widehat{\varphi}\|_{L^\infty} \\ &\le Ct{-}1/2\sum_{k=3,5}\||\eta|^{1/2}\mathcal{V}_k\widehat{\varphi}\|_{L^\infty} \le C\varepsilon t{-}1/2. \end{aligned}$$

Similarly $\|w_1\|_{L^\infty} \le C\varepsilon t^{-2/3}$. Hence

$$\begin{aligned} \||u|\partial_x u\|_{L^\infty} &= \|\widetilde{\mu}^{1/2}|v|\widetilde{\mu}{-}1/2v_1\|_{L^\infty} + \|\widetilde{\mu}^{1/2}|w|\widetilde{\mu}{-}1/2v_1\|_{L^\infty} + \|(|v| + |w|)w_1\|_{L^\infty} \\ &\le C\varepsilon^2 t^{-1}. \end{aligned}$$

Therefore

$$\frac{d}{dt}(\|\partial_x^{-1}\mathcal{P}_b u\|_{L^2}+\|\mathcal{P}_a u\|_{L^2}+\|u\|_{H^1})\le C\varepsilon^2 t^{-1}(\|\partial_x^{-1}\mathcal{P}_b u\|_{L^2}+\|\mathcal{P}_a u\|_{L^2}+\|u\|_{H^1}).$$

Integration in time of the above inequality yields $\|\partial_x^{-1}\mathcal{P}_b u\|_{L^2}+\|\mathcal{P}_a u\|_{L^2}+\|u\|_{H^1}\le \varepsilon + C\varepsilon^3 t^{\gamma}$. The proof is complete. □

5.3. **L^2-norms of $\mathcal{I}_a$ and $\mathcal{I}_b$.** In this subsection we prove L^2-estimates for $\mathcal{I}_a u(t)$ and $\partial_x^{-1}\mathcal{I}_b u(t)$. We define the norm

$$\begin{aligned}\|u\|_{X_T}=\sup_{t\in[1,T]}\Big(&\|\widehat{\varphi}\|_{L^\infty}+t^{-\gamma}\|\partial_x\mathcal{J}u(t)\|_{L^2}+t^{-\gamma}\|\partial_x^{-1}\mathcal{I}_b u(t)\|_{L^2}\\&+t^{-\gamma}\|\partial_x^{-1}\mathcal{P}_b u(t)\|_{L^2}+t^{-\gamma}\|u(t)\|_{H^1}\Big).\end{aligned}$$

Lemma 5.6. *Let $\|u\|_{X_T}\le C\varepsilon$. Then the estimate*

$$\sup_{t\in[1,T]} t^{-\gamma}(\|\mathcal{I}_a u(t)\|_{L^2}+\|\partial_x^{-1}\mathcal{I}_b u(t)\|_{L^2})<C\varepsilon$$

holds for any $T>1$.

Proof. Arguing by the contradiction we assume that there exists a first time $T>1$ such that $\sup_{t\in[1,T]} t^{-\gamma}(\|\mathcal{I}_a u(t)\|_{L^2}+\|\partial_x^{-1}\mathcal{I}_b u(t)\|_{L^2})=C\varepsilon$. We apply $\mathcal{I}_a$ and $\partial_x^{-1}\mathcal{I}_b$ to (1.1). Via the commutator relations $[\mathcal{L},\mathcal{I}_a]=[\mathcal{L},\mathcal{I}_b]=0$ and by the definitions $\mathcal{I}_a=\partial_b+it\frac{1}{3}|\partial_x|^3$ and $\mathcal{I}_b=\partial_b+it\frac{1}{4}\partial_x^4$ we find

$$\begin{aligned}\mathcal{L}\mathcal{I}_a u&=\mathcal{I}_a\partial_x(|u|^2u)\\&=\partial_a\partial_x(|u|^2u)+\frac{it}{3}|\partial_x|^3\partial_x(|u|^2u)\\&=2\partial_x(|u|^2\mathcal{I}_a u)+\partial_x(u^2\overline{\mathcal{I}_a u})+\frac{it}{3}N_a\end{aligned}\tag{5.2}$$

and

$$\begin{aligned}\mathcal{L}\partial_x^{-1}\mathcal{I}_b u&=\mathcal{I}_b(|u|^2u)\\&=\partial_b(|u|^2u)+\frac{it}{4}\partial_x^4(|u|^2u)\\&=2|u|^2\mathcal{I}_b u+u^2\overline{\mathcal{I}_b u}+\frac{it}{4}N_b,\end{aligned}\tag{5.3}$$

where

$$\begin{aligned}N_a&=|\partial_x|^3\partial_x(|u|^2u)-2\partial_x|u|^2|\partial_x|^3u+\partial_x u^2\overline{|\partial_x|^3u}\\&=-2(|\partial_x|(|u|^2u_{xxx})-|u|^2|\partial_x|u_{xxx})-(|\partial_x|(u^2\overline{u_{xxx}})-u^2\overline{|\partial_x|u_{xxx}})\\&\quad-2(\partial_x|u|^2)|\partial_x|^3u+2uu_x\overline{|\partial_x|^3u}\\&\quad-6|\partial_x|(uu_x\overline{u_{xx}}+|u_x|^2u_x+(\partial_x|u|^2)u_{xx})-2u^2\overline{|\partial_x|u_{xxx}}\end{aligned}$$

and

$$\begin{aligned}N_b&=\partial_x^4(|u|^2u)-2|u|^2\partial_x^4u+u^2\overline{\partial_x^4u}\\&=8uu_x\overline{\partial_x^3u}+8(\partial_x|u|^2)\partial_x^3u+12u_x^2\overline{u_{xx}}+24|u_x|^2u_{xx}\\&\quad+12u|u_{xx}|^2+6\overline{u}u_{xx}^2+2u^2\overline{\partial_x^4u}.\end{aligned}$$

We write $\partial_x^k u=v_k+w_k$, $v_k=\mathcal{D}_t\mathcal{B}M(\mathcal{V}_1(i\xi)^k\widehat{\varphi}+\mathcal{V}_2(i\xi)^k\widehat{\varphi}+\mathcal{V}_4(i\xi)^k\widehat{\varphi})$, and $w_k=\mathcal{D}_t\mathcal{B}M(\mathcal{V}_3(i\xi)^k\widehat{\varphi}+\mathcal{V}_5(i\xi)^k\widehat{\varphi})$, $v_{|k|}=\mathcal{D}_t\mathcal{B}M(\mathcal{V}_1|\xi|^k\widehat{\varphi}+\mathcal{V}_2|\xi|^k\widehat{\varphi}+\mathcal{V}_4|\xi|^k\widehat{\varphi})$, $w_{|k|}=$

$\mathcal{D}_t\mathcal{B}M(\mathcal{V}_3|\xi|^k\widehat{\varphi}+\mathcal{V}_5|\xi|^k\widehat{\varphi})$. Also $\rho_k=\mathcal{D}_t\mathcal{B}M\mathcal{V}_2(i\xi)^k\widehat{\varphi}$, $\omega_k=\mathcal{D}_t\mathcal{B}M(\mathcal{V}_1(i\xi)^k\widehat{\varphi}+\mathcal{V}_4(i\xi)^k\widehat{\varphi})$, i.e. $v_k=\rho_k+\omega_k$. We denote $f\approx g$ if $f-g=O_{L^2}(\varepsilon t^{\gamma-1})$, i.e. $\|f-g\|_{L^2}\le C\varepsilon t^{\gamma-1}$.

Substituting $u_{xxx}=v_3+w_3$, we obtain

$$t(|\partial_x|(|u|^2u_{xxx})-|u|^2|\partial_x|u_{xxx})\approx t|\partial_x|(|u|^2(i\widetilde{\mu})^2\rho_1)-t|u|^2|\partial_x|(i\widetilde{\mu})^2\rho_1\,.$$

By Lemma 5.3 we find

$$\begin{aligned}&\||\partial_x|(|u|^2w_3)-|u|^2|\partial_x|w_3\|_{L^2}+\||\partial_x|(|u|^2\omega_3)-|u|^2|\partial_x|\omega_3\|_{L^2}\\&\le\|\partial_x|u|^2\|_{L^\infty}(\|w_3\|_{L^2}+\|\omega_3\|_{L^2})\le C\varepsilon^3t^{\gamma-1},\end{aligned}$$

by Lemma 3.3, 3.5 we have

$$\|w_3\|_{L^2}\le\sum_{k=3,5}\||\Lambda''|^{1/2}\mathcal{V}_k\xi^2(\xi\widehat{\varphi})\|_{L^2}\le Ct^{-1}\|\partial_\xi\xi\widehat{\varphi}\|_{L^2}\le C\varepsilon t^{\gamma-1},$$

by Lemmas 3.1 and 3.4 we have

$$\|\omega_3\|_{L^2}\le\sum_{k=1,4}\||\Lambda''|^{1/2}\mathcal{V}_k\xi^2(\xi\widehat{\varphi})\|_{L^2}\le Ct^{-1}\|\partial_\xi\xi\widehat{\varphi}\|_{L^2}\le C\varepsilon t^{\gamma-1},$$

and by Lemma 3.2 we find

$$\begin{aligned}\|\rho_3-(i\widetilde{\mu})^2\rho_1\|_{L^2}&\le\sum_{j=1}^{2}\||\Lambda''|^{1/2}\eta^{2-j}[i\eta,\mathcal{V}_2]\xi^{j-1}(\xi\widehat{\varphi})\|_{L^2}\\&\le Ct^{-1}\|\partial_\xi\xi\widehat{\varphi}\|_{L^2}\le C\varepsilon t^{\gamma-1}.\end{aligned}$$

Next we represent

$$\begin{aligned}t|\partial_x|(|u|^2(i\widetilde{\mu})^2\rho_1)&\approx t\frac{|\partial_x|}{\partial_x}((\partial_x|u|^2)(i\widetilde{\mu})^2\rho_1+|u|^2\partial_x((i\widetilde{\mu})^2\rho_1))\\&\approx t\frac{|\partial_x|}{\partial_x}(u\overline{u_x}(i\widetilde{\mu})^3\rho+u_x\overline{u}(i\widetilde{\mu})^3\rho+|u|^2(i\widetilde{\mu})^4\rho)\\&\approx t\frac{|\partial_x|}{\partial_x}(-\widetilde{\mu}^4u|\rho|^2+\widetilde{\mu}^4\overline{u}\rho^2+\widetilde{\mu}^4|u|^2\rho)\\&\approx t\frac{|\partial_x|}{\partial_x}(-\widetilde{\mu}^4(\omega+\rho)|\rho|^2+\widetilde{\mu}^4(\overline{\omega}+\overline{\rho})\rho^2+\widetilde{\mu}^4(\overline{\omega}\rho+\overline{\rho}\omega+|\rho|^2)\rho)\\&\approx t\frac{|\partial_x|}{\partial_x}(\widetilde{\mu}^4|\rho|^2\rho+2\widetilde{\mu}^4\overline{\omega}\rho^2).\end{aligned}$$

Next we represent

$$|u|^2|\partial_x|(i\widetilde{\mu})^2\rho_1\approx(i\widetilde{\mu})^2|u|^2|\partial_x|\rho_1\approx(i\widetilde{\mu})^3|\widetilde{\mu}||u|^2\rho,$$

since taking $p=2+\gamma$, $q=\frac{2p}{\gamma}$ we obtain

$$\begin{aligned}\|t|u|^2[|\partial_x|,\widetilde{\mu}^2]\rho_1\|_{L^2}&\le Ct\|u\|_{L^{2p}}^2\|[|\partial_x|,\widetilde{\mu}^2]\rho_1\|_{L^q}\\&\le Ct\|u\|_{L^{2p}}^2\|\partial_x\widetilde{\mu}^2\|_{L^\infty}\|\rho_1\|_{L^q}\\&\le C\varepsilon^3t^{-\frac{2}{3}(1-\frac{1}{2p})-\frac{1}{2}(1-\frac{2}{q})}\|\rho_1\|_{L^q}\le C\varepsilon^3t^{\gamma-1}\end{aligned}$$

and denoting $\rho_{|1|1}=\mathcal{D}_t\mathcal{B}M\mathcal{V}_2|\xi|(i\xi)\widehat{\varphi}$, and similarly $\omega_{|1|1}$, $w_{|1|1}$ we obtain

$$\begin{aligned}\widetilde{\mu}^2|u|^2|\partial_x|\rho_1&=\widetilde{\mu}^2|u|^2|\partial_x|u_x-|u|^2[|\partial_x|,\widetilde{\mu}^2](\omega_1+w_1)\\&\quad-[|\partial_x|,|u|^2]\widetilde{\mu}^2(\omega_1+w_1)-\frac{|\partial_x|}{\partial_x}((\partial_x|u|^2)\widetilde{\mu}^2(\omega_1+w_1))\end{aligned}$$

$$\begin{aligned}
&- \frac{|\partial_x|}{\partial_x}(|u|^2(\partial_x\widetilde{\mu}^2)(\omega_1 + w_1)) - \frac{|\partial_x|}{\partial_x}(|u|^2\widetilde{\mu}^2(\partial_x\omega_1 + \partial_x w_1))\\
&\approx \widetilde{\mu}^2|u|^2|\partial_x|u_x \approx \widetilde{\mu}^2|u|^2(\omega_{|1|1} + \rho_{|1|1} + w_{|1|1}) \approx \widetilde{\mu}^2|u|^2\rho_{|1|1}\\
&\approx i\widetilde{\mu}^3|\widetilde{\mu}||u|^2\rho \approx i\widetilde{\mu}^3|\widetilde{\mu}|(\omega\overline{\rho} + \overline{\omega}\rho + |\rho|^2)\rho.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&t(|\partial_x|(|u|^2u_{xxx}) - |u|^2|\partial_x|u_{xxx})\\
&\approx t\frac{|\partial_x|}{\partial_x}(\widetilde{\mu}^4|\rho|^2\rho + 2\widetilde{\mu}^4\overline{\omega}\rho^2) + it\widetilde{\mu}^3|\widetilde{\mu}|(\omega\overline{\rho} + \overline{\omega}\rho + |\rho|^2)\rho.
\end{aligned}$$

In the same manner we obtain

$$\begin{aligned}
t(|\partial_x|(u^2\overline{u_{xxx}}) - u^2\overline{|\partial_x|u_{xxx}}) &\approx t|\partial_x|(u^2\overline{(i\widetilde{\mu})^2\rho_1}) - tu^2\overline{|\partial_x|(i\widetilde{\mu})^2\rho_1}\\
&\approx -t\frac{|\partial_x|}{\partial_x}(\widetilde{\mu}^4|\rho|^2\rho) - it|\widetilde{\mu}|^3\widetilde{\mu}(2\omega|\rho|^2 + |\rho|^2\rho)
\end{aligned}$$

and

$$\begin{aligned}
-2t(\partial_x|u|^2)|\partial_x|^3u + 2tuu_x\overline{|\partial_x|^3u} &\approx -2t(\partial_x|u|^2)\rho_{|3|} + 2tuu_x\overline{\rho_{|3|}}\\
&\approx 4it|\widetilde{\mu}|^3\widetilde{\mu}\omega|\rho|^2 - 2it|\widetilde{\mu}|^3\widetilde{\mu}\overline{\omega}\rho^2 + 2it|\widetilde{\mu}|^3\widetilde{\mu}|\rho|^2\rho.
\end{aligned}$$

Next we obtain

$$6t|\partial_x|(uu_x\overline{u_{xx}} + |u_x|^2u_x + (\partial_x|u|^2)u_{xx}) \approx 12t\frac{|\partial_x|}{\partial_x}(\widetilde{\mu}^4\overline{\omega}\rho^2),$$

and writing $w_{|1|3} = \mathcal{D}_t\mathcal{B}M\mathcal{V}_3|\xi|(i\xi)^3\widehat{\varphi}$, we have $tu^2\overline{|\partial_x|u_{xxx}} \approx tu^2\overline{w_{|1|3}} + it|\widetilde{\mu}|\widetilde{\mu}^3|\rho|^2\rho + 2it|\widetilde{\mu}|\widetilde{\mu}^3|\rho|^2\omega$. Therefore

$$tN_a \approx -t\frac{|\partial_x|}{\partial_x}\widetilde{\mu}^4(|\rho|^2\rho + 16\widetilde{\mu}^4\overline{\omega}\rho^2) - it|\widetilde{\mu}|\widetilde{\mu}^3(|\rho|^2\rho + 4\overline{\omega}\rho^2) - 2tu^2\overline{w_{|1|3}}.$$

Next using the factorization formula (2.1), we represent

$$\begin{aligned}
&\mathcal{F}\mathcal{U}(-t)\big(t\frac{|\partial_x|}{\partial_x}(\widetilde{\mu}^4|\rho|^2\rho) + it|\widetilde{\mu}|\widetilde{\mu}^3|\rho|^2\rho\big)\\
&\approx -it\mathcal{F}\mathcal{U}(-t)(|\partial_x|(\widetilde{\mu}^3|\rho|^2\rho) - |\widetilde{\mu}|\widetilde{\mu}^3|\rho|^2\rho)\\
&\approx -i[|\xi|, \mathcal{V}^*](\eta^3|\mathcal{V}_2\widehat{\varphi}|^2\mathcal{V}_2\widehat{\varphi}) \approx 0,
\end{aligned}$$

since by Corollary 3.6

$$\||\Lambda''|{-1/2}\eta^2\partial_\eta\mathcal{V}_2\widehat{\varphi}\|_{L^2} \le \||\Lambda''|^{1/2}\eta\partial_\eta\mathcal{V}_2\xi^{-1}(\xi\widehat{\varphi})\|_{L^2} \le C\varepsilon t^{-1}\|\partial_\xi(\xi\widehat{\varphi})\|_{L^2} \le C\varepsilon t^{\gamma-1}$$

and by Lemma 4.2

$$\begin{aligned}
&\|[|\xi|, \mathcal{V}^*](\eta^3|\mathcal{V}_2\widehat{\varphi}|^2\mathcal{V}_2\widehat{\varphi})\|_{L^2}\\
&\le Ct^{-1}\||\Lambda''|{-1/2}\eta^3|\mathcal{V}_2\widehat{\varphi}|^2\partial_\eta\mathcal{V}_2\widehat{\varphi}\|_{L^2} + Ct^{-1}\||\Lambda''|^{-\frac{1}{2}}\eta^2|\mathcal{V}_2\widehat{\varphi}|^2\mathcal{V}_2\widehat{\varphi}\|_{L^2}\\
&\le C\varepsilon^3t^{\gamma-1}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
tN_a &\approx -16t\frac{|\partial_x|}{\partial_x}(\widetilde{\mu}^4\overline{\omega}\rho^2) - 4it|\widetilde{\mu}|^3\widetilde{\mu}\overline{\omega}\rho^2 - 2tu^2\overline{w_{|1|3}}\\
&\approx 4t(|\partial_x|\partial_x(\widetilde{\mu}^2\overline{\omega}\rho^2) - i|\widetilde{\mu}|\widetilde{\mu}^3\overline{\omega}\rho^2) - 2tu^2\overline{w_{|1|3}}.
\end{aligned}$$

In the same manner we transform $uu_x\overline{\partial_x^3 u} \approx -\widetilde{\mu}^4(\rho+\omega)|\rho|^2$, $(\partial_x|u|^2)\partial_x^3 u \approx -\widetilde{\mu}^4\omega|\rho|^2 + \widetilde{\mu}^4\overline{\omega}\rho^2$, $u_x^2\overline{u_{xx}} \approx \widetilde{\mu}^4\rho|\rho|^2$, $|u_x|^2 u_{xx} \approx -\widetilde{\mu}^4\rho|\rho|^2$, $u|u_{xx}|^2 \approx \widetilde{\mu}^4(\rho+\omega)|\rho|^2$, $\overline{u}u_{xx}^2 \approx \widetilde{\mu}^4(\overline{\rho}+\overline{\omega})\rho^2$, $u^2\overline{\partial_x^4 u} \approx \widetilde{\mu}^4\rho|\rho|^2 + 2\widetilde{\mu}^4\omega|\rho|^2 + u^2\overline{w_4}$. Therefore

$$\begin{aligned} tN_b &\approx 14t\widetilde{\mu}^4\overline{\omega}\rho^2 + 2tu^2\overline{w_4} \\ &\approx -\frac{14}{3}t(\partial_x^2(\widetilde{\mu}^2\overline{\omega}\rho^2) - (i\widetilde{\mu})^2(\widetilde{\mu}^2\overline{\omega}\rho^2)) + 2tu^2\overline{w_4}. \end{aligned}$$

Now we need to transform the terms $2tu^2\overline{w_{|1|3}}$ and $2tu^2\overline{w_4}$. We have $\partial_t(\mathcal{V}^*\overline{M}\phi) = \frac{1}{2t}\mathcal{V}^*\overline{M}\phi + i\Lambda\mathcal{V}^*\overline{M}\phi - i\xi\mathcal{V}^*\Lambda'\overline{M}\phi + \mathcal{V}^*\overline{M}\partial_t\phi$ and $\partial_t(M\mathcal{V}_l\phi) = M\mathcal{V}\partial_t\Psi_l\phi + \frac{1}{2t}M\mathcal{V}_l\phi - M\mathcal{V}_l i\Lambda\phi + M\Lambda'\mathcal{V}_l i\xi\phi + M\mathcal{V}_l\partial_t\phi$. We denote $\mathcal{K}_1 = \frac{\partial_x^4}{\Lambda(-i\partial_x)}$, and $\mathcal{K}_2 = \frac{|\partial_x|^3\partial_x}{\Lambda(-i\partial_x)}$ and the symbols $K_1(\xi) = \frac{\xi^4}{\Lambda(\xi)}$ and $K_2(\xi) = \frac{|\xi|^3 i\xi}{\Lambda(\xi)}$. Then by a direct calculation we obtain

$$\begin{aligned} &\partial_t(\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}}) \\ &= \partial_t(\mathcal{V}^*\overline{M}(M\mathcal{V}\widehat{\varphi})^2\overline{M\mathcal{V}_l K_j\widehat{\varphi}}) \\ &= \frac{1}{2t}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}} + i\Lambda\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}} - i\xi\mathcal{V}^*\Lambda'(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}} \\ &\quad + 2\mathcal{V}^*(\mathcal{V}\widehat{\varphi})(\overline{\mathcal{V}_l K_j\widehat{\varphi}})(\frac{1}{2t}\mathcal{V}\widehat{\varphi} - \mathcal{V}i\Lambda\widehat{\varphi} + \Lambda'\mathcal{V}i\xi\widehat{\varphi} + \mathcal{V}\partial_t\widehat{\varphi}) \\ &\quad + \mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2(\overline{\mathcal{V}\partial_t\Psi_l K_j\widehat{\varphi}} + \frac{1}{2t}\overline{\mathcal{V}_l K_j\widehat{\varphi}} - \overline{\mathcal{V}_l i\Lambda K_j\widehat{\varphi}} + \Lambda'\overline{\mathcal{V}_l i\xi K_j\widehat{\varphi}} + \overline{\mathcal{V}_l K_j\partial_t\widehat{\varphi}}). \end{aligned}$$

Hence

$$\begin{aligned} &\partial_t(\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}}) \\ &= \frac{2}{t}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}} + \mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}\partial_t\Psi_l\widehat{\varphi}} \\ &+ 2\mathcal{V}^*(\mathcal{V}\widehat{\varphi})(\overline{\mathcal{V}_l K_j\widehat{\varphi}})(\mathcal{V}\partial_t\widehat{\varphi}) + \mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\partial_t\widehat{\varphi}} \\ &- i\xi\mathcal{V}^*\Lambda'(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}} + 2\mathcal{V}^*\Lambda'(\mathcal{V}\widehat{\varphi})(\mathcal{V}i\xi\widehat{\varphi})\overline{\mathcal{V}_l K_j\widehat{\varphi}} + \mathcal{V}^*\Lambda'(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l i\xi K_j\widehat{\varphi}} \\ &+ i\Lambda\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}} - 2\mathcal{V}^*(\mathcal{V}\widehat{\varphi})(\mathcal{V}i\Lambda\widehat{\varphi})(\overline{\mathcal{V}_l K_j\widehat{\varphi}}) - \mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l i\Lambda K_j\widehat{\varphi}}. \end{aligned}$$

Next we use the relations

$$\begin{aligned} i\xi\mathcal{V}^*\Lambda'(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}} &= 2\mathcal{V}^*\Lambda'(\mathcal{V}\widehat{\varphi})(\mathcal{V}i\xi\widehat{\varphi})\overline{\mathcal{V}_l K_j\widehat{\varphi}} + \mathcal{V}^*\Lambda'(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j i\xi\widehat{\varphi}} \\ &\quad + \frac{1}{t}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}} + \mathcal{V}^*\Lambda'(\mathcal{V}\widehat{\varphi})^2\overline{\frac{1}{t\Lambda''}\mathcal{V}(\partial_\eta\Psi_l)K_j\widehat{\varphi}} \\ &\approx 2\mathcal{V}^*\Lambda'(\mathcal{V}\widehat{\varphi})(\mathcal{V}i\xi\widehat{\varphi})\overline{\mathcal{V}_l K_j\widehat{\varphi}} + \mathcal{V}^*\Lambda'(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j i\xi\widehat{\varphi}} \end{aligned}$$

and

$$\begin{aligned} i\Lambda\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}} &= \frac{i\Lambda}{i\xi}i\xi\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}} \\ &= 2\frac{i\Lambda}{i\xi}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})(\mathcal{V}i\xi\widehat{\varphi})\overline{\mathcal{V}_l K_j\widehat{\varphi}} + \frac{i\Lambda}{i\xi}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j i\xi\widehat{\varphi}} \\ &\quad + \frac{i\Lambda}{i\xi}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\frac{1}{t\Lambda''}\mathcal{V}(\partial_\eta\Psi_l)K_j\widehat{\varphi}} \\ &\approx \frac{i\Lambda}{i\xi}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j i\xi\widehat{\varphi}}. \end{aligned}$$

Therefore

$$\begin{aligned}
&\partial_t(\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}})\\
&\approx \frac{2}{t}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}}+\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}\partial_t\Psi_l\widehat{\varphi}}+2\mathcal{V}^*(\mathcal{V}\widehat{\varphi})(\overline{\mathcal{V}_l K_j\widehat{\varphi}})(\mathcal{V}\partial_t\widehat{\varphi})\\
&\quad+\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\partial_t\widehat{\varphi}}-2\mathcal{V}^*(\mathcal{V}\widehat{\varphi})(\mathcal{V}i\Lambda\widehat{\varphi})(\overline{\mathcal{V}_l K_j\widehat{\varphi}})\\
&\quad+\frac{i\Lambda}{i\xi}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j i\xi\widehat{\varphi}}+i\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l\Lambda K_j\widehat{\varphi}}.
\end{aligned}$$

We have $\frac{2}{t}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}}+\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}\partial_t\Psi_l\widehat{\varphi}}\approx 0$ and

$$2\mathcal{V}^*(\mathcal{V}\widehat{\varphi})(\overline{\mathcal{V}_l K_j\widehat{\varphi}})(\mathcal{V}\partial_t\widehat{\varphi})-2\mathcal{V}^*(\mathcal{V}\widehat{\varphi})(\mathcal{V}i\Lambda\widehat{\varphi})(\overline{\mathcal{V}_l K_j\widehat{\varphi}})\approx 0.$$

Also we represent

$$\begin{aligned}
&\sum_{l=3,5}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\partial_t\widehat{\varphi}}\\
&=t^{-1}\sum_{l=3,5}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j i\xi\mathcal{V}^*(|\mathcal{V}\widehat{\varphi}|^2\mathcal{V}\widehat{\varphi})}\\
&=t^{-1}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}K_j i\xi\mathcal{V}^*(|\mathcal{V}\widehat{\varphi}|^2\mathcal{V}\widehat{\varphi})}-t^{-1}\sum_{l=1,2,4}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j i\xi\mathcal{V}^*(|\mathcal{V}\widehat{\varphi}|^2\mathcal{V}\widehat{\varphi})}.
\end{aligned}$$

Then by Lemma 5.4,

$$\begin{aligned}
\mathcal{U}(t)\mathcal{F}t^{-1}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}K_j i\xi\mathcal{V}^*(|\mathcal{V}\widehat{\varphi}|^2\mathcal{V}\widehat{\varphi})}&=u^2K_j\partial_x(|u|^2\overline{u})\\
&=[\mathcal{K}_j,u^2]\partial_x(|u|^2\overline{u})+\mathcal{K}_j u^2\partial_x(|u|^2\overline{u})\\
&\approx\mathcal{K}_j u^2\partial_x(|u|^2\overline{u})\approx 0,
\end{aligned}$$

since $t\|[\mathcal{K}_j,u^2]\partial_x(|u|^2\overline{u})\|_{L^2}\le Ct\|\partial_x u^2\|_{L^\infty}\|\partial_x(|u|^2\overline{u})\|_{H^1}\le C\varepsilon^5t^{\gamma-1}$, and

$$\begin{aligned}
&t\|\mathcal{K}_j u^2\partial_x(|u|^2\overline{u})\|_{L^2}\\
&\le Ct\|(\partial_x u^2)\partial_x(|u|^2\overline{u})\|_{L^2}+Ct\|u^2\partial_x^2(|u|^2\overline{u})\|_{L^2}\\
&\le Ct\|\partial_x u^2\|_{L^\infty}^2\|u\|_{L^2}+Ct\|u\|_{L^\infty}^4\|\omega_2\|_{L^2}+Ct\||\widetilde{\mu}|^2|u|^4|\widetilde{\mu}|^{-2}\rho_2\|_{L^2}\\
&\quad+Ct\|u\|_{L^\infty}^4\|w_2\|_{L^2}\le C\varepsilon^5t^{\gamma-1},
\end{aligned}$$

and by Lemmas 3.1, 3.2 and 3.4, for $l=1,2,4$, $j=1,2$, we have

$$\begin{aligned}
&\|\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j i\xi\mathcal{V}^*(|\mathcal{V}\widehat{\varphi}|^2\mathcal{V}\widehat{\varphi})}\|_{L^2}\\
&\le C\||\eta|^{1/2}\mathcal{V}\widehat{\varphi}\|_{L^\infty}^2\||\Lambda''|^{1/2}\eta^{-1}\overline{\mathcal{V}_l K_j i\xi\mathcal{V}^*(|\mathcal{V}\widehat{\varphi}|^2\mathcal{V}\widehat{\varphi})}\|_{L^2}\\
&\le C\varepsilon^2\||\Lambda''|^{1/2}\xi\mathcal{V}^*(|\mathcal{V}\widehat{\varphi}|^2\mathcal{V}\widehat{\varphi})\|_{L^2}\\
&\le C\varepsilon^2\|\partial_x u^2\|_{L^\infty}\|u\|_{L^2}\le C\varepsilon^5t^{\gamma-1}.
\end{aligned}$$

Therefore using $\frac{\Lambda}{i\xi}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l i\xi K_j\widehat{\varphi}}-\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l\frac{\Lambda}{i\xi}i\xi K_j\widehat{\varphi}}\approx 0$, we obtain

$$\begin{aligned}
&\partial_t\Big(\sum_{l=3,5}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}}\Big)\\
&\approx\frac{i\Lambda}{i\xi}\sum_{l=3,5}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j i\xi\widehat{\varphi}}+i\sum_{l=3,5}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l\Lambda K_j\widehat{\varphi}}
\end{aligned}$$

$$\approx 2i\sum_{l=3,5}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l\Lambda K_j\widehat{\varphi}}=\begin{cases}2i\sum_{l=3,5}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l\xi^4\widehat{\varphi}}, & j=1,\\ 2i\sum_{l=3,5}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l|\xi|i\xi^3\widehat{\varphi}}, & j=2,\end{cases}$$

if we choose $K_1=\frac{\xi^4}{\Lambda(\xi)}$ and $K_2=\frac{|\xi|^3 i\xi}{\Lambda(\xi)}$.

Since $\mathcal{L}(tu^2\overline{w_{K_j}})=\mathcal{U}(t)\mathcal{F}^{-1}\partial_t(\sum_{l=3,5}\mathcal{V}^*(\mathcal{V}\widehat{\varphi})^2\overline{\mathcal{V}_l K_j\widehat{\varphi}})$, we have

$$\mathcal{L}(tu^2\overline{w_{K_1}})\approx 2itu^2\overline{w_4}\quad\text{and}\quad \mathcal{L}(tu^2\overline{w_{K_2}})\approx -2tu^2\overline{w_{|1|3}}.$$

Next we transform the terms $t\partial_x^2(\widetilde{\mu}^2\overline{\varpi}\rho^2)-t(i\widetilde{\mu})^2(\widetilde{\mu}^2\overline{\varpi}\rho^2)$ and $t|\partial_x|\partial_x(\widetilde{\mu}^2\overline{\varpi}\rho^2)-it|\widetilde{\mu}|\widetilde{\mu}^3\overline{\varpi}\rho^2$. Using the factorization formula (2.1), we write $\mathcal{F}\mathcal{U}(-t)\partial_x(|u|^2u)=i\xi t^{-1}\mathcal{V}^*(|\mathcal{V}\widehat{\varphi}|^2\mathcal{V}\widehat{\varphi})$. Hence

$$\begin{aligned}\mathcal{F}\mathcal{U}(-t)(t\partial_x^2(\widetilde{\mu}^2\overline{\varpi}\rho^2)-t(i\widetilde{\mu})^2(\widetilde{\mu}^2\overline{\varpi}\rho^2))&\approx[(i\xi)^2,\mathcal{V}^*]\eta^2\big((\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}\big),\\ \mathcal{F}\mathcal{U}(-t)(t|\partial_x|\partial_x(\widetilde{\mu}^2\overline{\varpi}\rho^2)-it|\widetilde{\mu}|\widetilde{\mu}^3\overline{\varpi}\rho^2)&\approx[|\xi|(i\xi),\mathcal{V}^*]\eta^2\big((\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}\big).\end{aligned}$$

Using $\partial_t(\mathcal{V}_K^*\overline{M}\phi)=\frac{1}{2t}\mathcal{V}_K^*\overline{M}\phi+i\Lambda\mathcal{V}_K^*\overline{M}\phi-i\xi\mathcal{V}_K^*\Lambda'\overline{M}\phi+\mathcal{V}_K^*\overline{M}\partial_t\phi$ and $\partial_t(M\mathcal{V}_k\phi)=M\mathcal{V}\partial_t\Psi_k\phi+\frac{1}{2t}M\mathcal{V}_k\phi-M\mathcal{V}_k i\Lambda\phi+M\Lambda'\mathcal{V}_k i\xi\phi+M\mathcal{V}_k\partial_t\phi$, as above by a direct computation we obtain

$$\begin{aligned}&\partial_t(\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}})\\ &=\partial_t(\mathcal{V}_K^*\overline{M}(M\mathcal{V}_2\widehat{\varphi})^2\overline{M\mathcal{V}_1\widehat{\varphi}})\\ &=\frac{1}{2t}\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}+i\Lambda\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}-i\xi\mathcal{V}_K^*\Lambda'(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}\\ &\quad+2\mathcal{V}_K^*\Big((\mathcal{V}_2\widehat{\varphi})\overline{\mathcal{V}_1\widehat{\varphi}}\Big)\Big(\mathcal{V}\partial_t\Psi_2\widehat{\varphi}+\frac{1}{2t}\mathcal{V}_2\widehat{\varphi}-\mathcal{V}_2 i\Lambda\widehat{\varphi}+\Lambda'\mathcal{V}_2 i\xi\widehat{\varphi}+\mathcal{V}_2\partial_t\widehat{\varphi}\Big)\\ &\quad+\mathcal{V}_K^*\Big((\mathcal{V}_2\widehat{\varphi})^2\Big)\Big(\overline{\mathcal{V}\partial_t\Psi_1\widehat{\varphi}}+\frac{1}{2t}\overline{\mathcal{V}_1\widehat{\varphi}}-\overline{\mathcal{V}_1 i\Lambda\widehat{\varphi}}+\overline{\Lambda'\mathcal{V}_1 i\xi\widehat{\varphi}}+\overline{\mathcal{V}_1\partial_t\widehat{\varphi}}\Big).\end{aligned}$$

Hence

$$\begin{aligned}&\partial_t\big(\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}\big)\\ &=\frac{2}{t}\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}+2\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})(\mathcal{V}\partial_t\Psi_2\widehat{\varphi})\overline{\mathcal{V}_1\widehat{\varphi}}+\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2(\overline{\mathcal{V}\partial_t\Psi_1\widehat{\varphi}})\\ &\quad+2\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})\overline{\mathcal{V}_1\widehat{\varphi}}(\mathcal{V}_2\partial_t\widehat{\varphi})+\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\partial_t\widehat{\varphi}}-i\xi\mathcal{V}_K^*\Lambda'(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}\\ &\quad+2\mathcal{V}_K^*\Lambda'(\mathcal{V}_2\widehat{\varphi})(\mathcal{V}_2 i\xi\widehat{\varphi})\overline{\mathcal{V}_1\widehat{\varphi}}+\mathcal{V}_K^*\Lambda'(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1 i\xi\widehat{\varphi}}+i\Lambda\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}\\ &\quad-2\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})(\mathcal{V}_2 i\Lambda\widehat{\varphi})\overline{\mathcal{V}_1\widehat{\varphi}}-\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1 i\Lambda\widehat{\varphi}}.\end{aligned}$$

Then we use the identity

$$\begin{aligned}&i\xi\mathcal{V}_K^*\Lambda'(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}\\ &=\mathcal{V}^*\mathcal{A}_1 K\Lambda'(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}\\ &=2\mathcal{V}_K^*\Lambda'(\mathcal{V}_2 i\xi\widehat{\varphi})(\mathcal{V}_2\widehat{\varphi})\overline{\mathcal{V}_1\widehat{\varphi}}+\mathcal{V}_K^*\Lambda'(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1 i\xi\widehat{\varphi}}+\mathcal{V}^*\mathcal{A}_0(K\Lambda')(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}\\ &\quad+2\mathcal{V}_K^*\Lambda'(\mathcal{V}_2\widehat{\varphi})\overline{\mathcal{V}_1\widehat{\varphi}}(\mathcal{V}\mathcal{A}_0\Psi_2\widehat{\varphi})+\mathcal{V}_K^*\Lambda'\mathcal{A}_0(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}\mathcal{A}_0\Psi_1\widehat{\varphi}}\\ &\approx 2\mathcal{V}_K^*\Lambda'(\mathcal{V}_2 i\xi\widehat{\varphi})(\mathcal{V}_2\widehat{\varphi})\overline{\mathcal{V}_1\widehat{\varphi}}.\end{aligned}$$

Also we use

$$\begin{aligned}&\frac{2}{t}\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}+2\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})(\mathcal{V}_2\partial_t\Psi_k\widehat{\varphi})\overline{\mathcal{V}_1\widehat{\varphi}}+\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2(\overline{\mathcal{V}_1\partial_t\Psi_k\widehat{\varphi}})\\ &+2\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})\overline{\mathcal{V}_1\widehat{\varphi}}(\mathcal{V}_2\partial_t\widehat{\varphi})+\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\partial_t\widehat{\varphi}}+\mathcal{V}_K^*\Lambda'(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1 i\xi\widehat{\varphi}}\end{aligned}$$

$$- \mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1 i\Lambda\widehat{\varphi}} \approx 0.$$

Hence we obtain

$$\begin{aligned}\partial_t\big(\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}\big) &\approx i\Lambda\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}} - 2\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})(\mathcal{V}_2 i\Lambda\widehat{\varphi})\overline{\mathcal{V}_1\widehat{\varphi}} \\ &\approx i\Lambda\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}} - 2i\mathcal{V}_K^*\Lambda(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}.\end{aligned}$$

On the domain $|\xi| \le 8|\eta|$ we represent

$$\begin{aligned}& i\Lambda\mathcal{V}_K^*\chi(\frac{\xi}{4\eta})(\mathcal{V}_2\widehat{\varphi})^2\mathcal{V}_1\widehat{\varphi} \\ &= 2\frac{\Lambda}{\xi}\mathcal{V}_K^*\chi(\mathcal{V}_2 i\xi\widehat{\varphi})(\mathcal{V}_2\widehat{\varphi})\overline{\mathcal{V}_1\widehat{\varphi}} + \frac{\Lambda}{\xi}\mathcal{V}_K^*\chi(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1 i\xi\widehat{\varphi}} + \frac{\Lambda}{\xi}\mathcal{V}^*(\mathcal{A}_0\chi K)(\mathcal{V}_2\widehat{\varphi})^2\mathcal{V}_1\widehat{\varphi} \\ &\approx 2i\frac{\Lambda}{\xi}\mathcal{V}_K^*\chi\eta(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}.\end{aligned}$$

Thus we obtain $\partial_t(\mathcal{V}_K^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}) \approx i\mathcal{V}_K^* Z(\xi,\eta)(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}$, where

$$Z(\xi,\eta) = (\Lambda(\xi) - 2\Lambda(\eta))\Big(1 - \chi(\frac{\xi}{4\eta})\Big) + 2\Big(\frac{\Lambda(\xi)}{\xi} - \frac{\Lambda(\eta)}{\eta}\Big)\eta\chi(\frac{\xi}{4\eta}).$$

Next we choose $K_3(\xi,\eta) = \frac{((i\xi)^2-(i\eta)^2)\eta^2}{Z(\xi,\eta)}$ and $K_4(\xi,\eta) = \frac{(|\xi|(i\xi)-|\eta|(i\eta))\eta^2}{Z(\xi,\eta)}$, then we have $[(i\xi)^2, \mathcal{V}^*]\eta^2((\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}) \approx \partial_t(\mathcal{V}_{K_3}^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}})$ and $[|\xi|(i\xi), \mathcal{V}^*]\eta^2((\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}) \approx \partial_t(\mathcal{V}_{K_4}^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}})$. By Lemma 4.3 we have the estimate $\|\mathcal{V}_{K_j}^*\phi\|_{L^2} \le C\||\Lambda''|^{1/2}\eta\phi\|_{L^2}$ for $j = 3, 4$. Then equations (5.2) and (5.3) yield

$$\begin{aligned}\mathcal{L}\Big(\mathcal{I}_a u - \frac{it}{3}u^2\overline{w_{K_2}} - \frac{4i}{3}\mathcal{U}(t)\mathcal{F}^{-1}\mathcal{V}_{K_4}^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}\Big) &\approx 2\partial_x(|u|^2\mathcal{I}_a u) + \partial_x(u^2\overline{\mathcal{I}_a u}), \\ \mathcal{L}\Big(\partial_x^{-1}\mathcal{I}_b u - \frac{t}{4}u^2\overline{w_{K_1}} + \frac{7i}{6}\mathcal{U}(t)\mathcal{F}^{-1}\mathcal{V}_{K_3}^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}\Big) &\approx 2|u|^2\mathcal{I}_b u + u^2\overline{\mathcal{I}_b u}.\end{aligned}$$

We denote $\Phi_a = \mathcal{I}_a u - \frac{it}{3}u^2\overline{w_{K_2}} - \frac{4i}{3}\mathcal{U}(t)\mathcal{F}^{-1}\mathcal{V}_{K_4}^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}$ and $\Phi_b = \partial_x^{-1}\mathcal{I}_b u - \frac{t}{4}u^2\overline{w_{K_1}} + \frac{7i}{6}\mathcal{U}(t)\mathcal{F}^{-1}\mathcal{V}_{K_3}^*(\mathcal{V}_2\widehat{\varphi})^2\overline{\mathcal{V}_1\widehat{\varphi}}$, then we obtain $\mathcal{L}\Phi_a \approx 2\partial_x|u|^2\Phi_a + \partial_x u^2\overline{\Phi_a}$ and $\mathcal{L}\Phi_b \approx 2|u|^2\partial_x\Phi_b + u^2\overline{\partial_x\Phi_b}$. Integration in time of these equation yields $\|\Phi_a\|_{L^2} + \|\Phi_b\|_{L^2} \le \varepsilon + C\varepsilon^3 t^\gamma$. Consequently, we obtain $\|\mathcal{I}_a u\|_{L^2} + \|\partial_x^{-1}\mathcal{I}_b u\|_{L^2} \le \varepsilon + C\varepsilon^3 t^\gamma$. This is the desired contradiction. The proof is complete. □

6. Proof of Theorem 1.1

By Lemmas 5.2, 5.5 and 5.6 we see that a priori estimate $\|u\|_{X_T} \le C\varepsilon$ holds for all $T > 0$. Therefore the existence of global solutions of the Cauchy problem (1.1) follows by a standard continuation argument.

Now we turn to the proof of the asymptotic formulas for the solutions u of the Cauchy problem (1.1). We need to compute the check the asymptotic behavior of the function $\widehat{\varphi}(t,\xi)$. As in the proof of Lemma 5.2 we obtain on the domain $|\xi| \ge t^{\gamma-\frac{1}{3}}$

$$\partial_t\widehat{\varphi} = |\widehat{\xi}|^6\langle\widehat{\xi}\rangle^{-6}\frac{i\xi}{t\Lambda''(\xi)}|\widehat{\varphi}|^2\widehat{\varphi} + O\Big(t^{-1}|\xi t^{1/3}|\langle\xi t^{1/3}\rangle^{-2}\|\widehat{\varphi}\|_Y^3\Big).$$

Hence for a new dependent variable $z(t,\xi) = \widehat{\varphi}(t,\xi)\Psi(t,\xi)$, where

$$\Psi(t,\xi) = \exp\Big(-\frac{i\xi}{|\Lambda''(\xi)|}\int_1^t |\widehat{\varphi}(\tau,\xi)|^2\frac{|\xi\tau^{1/3}|^6}{\tau\langle\xi\tau^{\frac{1}{3}}\rangle^6}d\tau\Big),$$

we find $\partial_t z(t,\xi) = O\big(t^{-1}|\xi t^{1/3}|\langle \xi t^{1/3}\rangle^{-2}\|\widehat{\varphi}\|_Y^3\big)$ on the domain $|\xi| \geq t^{\gamma-\frac{1}{3}}$. Integration in time yields

$$|z(t,\xi) - z(s,\xi)| \leq C\varepsilon^3|\xi| \int_t^s \frac{d\tau}{\tau^{2/3}(1+\xi^2\tau^{2/3})} \leq C\varepsilon^3 \int_{t^{3\gamma}}^{\infty} \frac{dz}{z^{\frac{4}{3}}} \leq C\varepsilon^3 t^{-\gamma}$$

for any $s > t > 0$ in the domain $|\xi|t^{1/3} \geq t^{\gamma}$. Then there exists a unique final state $z_+ \in L^\infty$ such that $\|z(t) - z_+\|_{L^\infty(|\xi|\geq t^{\gamma-\frac{1}{3}})} \leq C\varepsilon^3 t^{-\gamma}$ for all $t > 0$. We write $\int_1^t |\widehat{\varphi}(\tau,\xi)|^2 \frac{d\tau}{\tau} = \int_1^t |z(\tau,\xi)|^2 \frac{d\tau}{\tau} = |z_+|^2 \log t + \Phi(t)$. For the remainder term $\Phi(t)$, we have $\Phi(s) - \Phi(t) = \int_t^s \big(|z(\tau)|^2 - |z(s)|^2\big)\frac{d\tau}{\tau} + \big(|z(s)|^2 - |z_+|^2\big)\log\frac{s}{t}$. Then we obtain $\|\Phi(s) - \Phi(t)\|_{L^\infty(|\xi|\geq t^{\gamma-\frac{1}{3}})} \leq C\varepsilon^3 t^{-\gamma}$ for any $s > t > 0$. Hence there exists a unique real-valued function $\Phi_+ \in L^\infty$ such that $\|\Phi(t) - \Phi_+\|_{L^\infty(|\xi|\geq t^{\gamma-\frac{1}{3}})} \leq C\varepsilon^3 t^{-\gamma}$ for all $t > 0$. Therefore,

$$\Big\|\Psi(t,\xi) - \exp\Big(\frac{i\xi}{|\Lambda''(\xi)|}(|z_+|^2\log t + \Phi_+)\Big)\Big\|_{L^\infty(|\xi|\geq t^{\gamma-\frac{1}{3}})} \leq Ct^{-\gamma}$$

for all $t > 0$. Thus for large time, $\|\widehat{\varphi}(t,\xi) - z_+\Psi(t,\xi)\|_{L^\infty(|\xi|\geq t^{\gamma-\frac{1}{3}})} \leq Ct^{-\gamma}$ and

$$\Big\|z_+\Psi(t,\xi) - W_+\exp\Big(\frac{i\xi}{|\Lambda''(\xi)|}|z_+|^2\log t\Big)\Big\|_{L^\infty(|\xi|\geq t^{\gamma-\frac{1}{3}})} \leq Ct^{-\gamma}$$

with $W_+ = z_+\exp(\frac{i\xi}{|\Lambda''(\xi)|}\Phi_+)$. Thus, $\widehat{\varphi}(t,\xi) = W_+\exp\big(\frac{3i\xi}{|\Lambda''(\xi)|}|W_+|^2\log t\big) + O(t^{-\delta})$ on the domain $|\xi| \geq t^{\gamma-\frac{1}{3}}$ with some $\delta > 0$. Using the factorization formulas we have $u = \mathcal{D}_t\mathcal{B}M\mathcal{V}\widehat{\varphi}$. Then by Lemma 3.9 we have

$$\begin{aligned}
u(t,x) &= \mathcal{D}_t\mathcal{B}MA_0\widehat{\varphi}(t) + O\Big(t^{-1/3}\langle t^{1/3}\mu(\frac{x}{t})\rangle^{-3/4}\Big)\\
&= \frac{M}{\sqrt{it\Lambda''(\mu(\frac{x}{t}))}} W_+(\mu(\frac{x}{t}))\exp\Big(\frac{i\mu(\frac{x}{t})}{|\Lambda''(\mu(\frac{x}{t}))|}|W_+(\mu(\frac{x}{t}))|^2\log t\Big)\\
&\quad + O\Big(t^{-1/3}\langle t^{1/3}\mu(\frac{x}{t})\rangle^{-3/4}\Big) + O(t^{-\frac{1}{3}-\delta})
\end{aligned}$$

on the domain $|x| \geq t^{\frac{1}{3}+2\gamma}$.

Next we consider the self-similar region $|x| \leq t^{\frac{1}{3}+2\gamma}$. We represent

$$\begin{aligned}
u(t) &= \mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}\\
&= \sqrt{\frac{1}{2\pi}}\int_{|\xi|\geq 2t^{-\frac{1}{3}+\gamma}} e^{ix\xi - it\Lambda(\xi)}\widehat{\varphi}(t,\xi)d\xi\\
&\quad + \sqrt{\frac{1}{2\pi}}\int_{|\xi|\leq 2t^{-\frac{1}{3}+\gamma}} e^{ix\xi}\Big(e^{-it\Lambda(\xi)} - e^{-\frac{it}{3}|\xi|^3}\Big)\widehat{\varphi}(t,\xi)d\xi\\
&\quad + \sqrt{\frac{1}{2\pi}}\int_{|\xi|\leq 2t^{-\frac{1}{3}+\gamma}} e^{ix\xi - \frac{it}{3}|\xi|^3}\widehat{\varphi}(t,\xi)d\xi = I_1 + I_2 + I_3.
\end{aligned}$$

We show that the first term I_1 is a remainder in the region $|x| \leq t^{\frac{1}{3}+2\gamma}$. Integrating by parts we have

$$\begin{aligned}
I_1 &= C\frac{e^{ix\xi - it\Lambda(\xi)}\widehat{\varphi}(t,\xi)}{i(x - t\Lambda'(\xi))}\Big|_{|\xi|=2t^{-\frac{1}{3}+\gamma}} - C\int_{|\xi|\geq 2t^{-\frac{1}{3}+\gamma}} e^{ix\xi - it\Lambda(\xi)}\widehat{\varphi}(t,\xi)\frac{t\Lambda''(\xi)d\xi}{(x - t\Lambda'(\xi))^2}\\
&\quad - C\int_{|\xi|\geq 2t^{-\frac{1}{3}+\gamma}} e^{ix\xi - it\Lambda(\xi)}\frac{\widehat{\varphi}_\xi(t,\xi)d\xi}{i(x - t\Lambda'(\xi))}.
\end{aligned}$$

Hence

$$\begin{aligned}
|I_1| &\le Ct^{-\frac{1}{3}-2\gamma}\|\widehat{\varphi}\|_{L^\infty} + C\|\widehat{\varphi}\|_{L^\infty}\int_{|\xi|\ge 2t^{-\frac{1}{3}+\gamma}} \frac{t\Lambda''(\xi)}{(x-t\Lambda'(\xi))^2}d\xi \\
&\quad + Ct^{-1}\|\widehat{\varphi}_\xi\|_{L^2}\Big(\int_{|\xi|\ge 2t^{-\frac{1}{3}+\gamma}} \frac{d\xi}{\xi^4}\Big)^{1/2} \\
&\le Ct^{-\frac{1}{3}-2\gamma}\|\widehat{\varphi}\|_{L^\infty} + Ct^{-\frac{1}{2}-\frac{3}{2}\gamma}\|\widehat{\varphi}_\xi\|_{L^2} \\
&\le C\varepsilon t^{-\frac{1}{3}-\frac{3}{2}\gamma}.
\end{aligned}$$

The second term I_2 is also a remainder since $|I_2| \le Ct\|\widehat{\varphi}\|_{L^\infty}\int_{|\xi|\le 2t^{-\frac{1}{3}+\gamma}} |\xi|^4 d\xi \le C\varepsilon t^{-\frac{2}{3}+5\gamma} \le C\varepsilon t^{-\frac{1}{3}-\gamma}$, if $\gamma \le \frac{1}{18}$. Changing $\xi t^{1/3} = \widehat{\xi}$, $y = xt^{-1/3}$, we obtain for the last summand

$$\begin{aligned}
I_3 &= t^{-1/3}\sqrt{\frac{1}{2\pi}}\int_{|\widehat{\xi}|\le 2t^\gamma} e^{iy\widehat{\xi}}e^{-\frac{i}{3}|\widehat{\xi}|^3}\widehat{\varphi}(t,\widehat{\xi}t^{-1/3})d\widehat{\xi} \\
&= t^{-1/3}\sqrt{\frac{1}{2\pi}}\int_{|\widehat{\xi}|\le 2t^\gamma} e^{i\widetilde{S}(y,\widehat{\xi})}\widehat{\varphi}(t,\widehat{\xi}t^{-1/3})d\widehat{\xi} = t^{-1/3}F(t,y),
\end{aligned}$$

where $\widetilde{S}(y,\widehat{\xi}) = y\widehat{\xi} - \frac{1}{3}|\widehat{\xi}|^3$, $\Phi(t,\widehat{\xi}) = \widehat{\varphi}(t,\widehat{\xi}t^{-1/3})$, and

$$F(t,y) = \sqrt{\frac{1}{2\pi}}\int_{|\widehat{\xi}|\le 2t^\gamma} e^{i\widetilde{S}(y,\widehat{\xi})}\Phi(t,\widehat{\xi})d\widehat{\xi}.$$

Let us show that $F(t,y)$ is a Cauchy sequence on the domain $|y| \le t^\gamma$. We can write

$$\begin{aligned}
&\int_{|\widehat{\xi}|\le 2t_1^\gamma} e^{i\widetilde{S}(y,\widehat{\xi})}\Phi(t_1,\widehat{\xi})d\widehat{\xi} - \int_{|\widehat{\xi}|\le 2t_1^\gamma} e^{i\widetilde{S}(y,\widehat{\xi})}\Phi(t_2,\widehat{\xi})d\widehat{\xi} \\
&= \int_{t_1}^{t_2}\int_{|\widehat{\xi}|\le 2t_1^\gamma} e^{i\widetilde{S}(y,\widehat{\xi})}\partial_t\Phi(t,\widehat{\xi})d\widehat{\xi}dt.
\end{aligned}$$

Using the identity $\partial_t\Phi(t,\widehat{\xi}) = \partial_t\big(\widehat{\varphi}(t,\widehat{\xi}t^{-1/3})\big) = \frac{1}{3t}(3t\partial_t\widehat{\varphi} - \xi\partial_\xi\widehat{\varphi}) = \frac{1}{3}\widehat{\xi}t^{-4/3}(\xi^{-1}\widehat{\mathcal{P}}_b - b\xi^{-1}\widehat{\mathcal{I}}_b)\widehat{\varphi}$, and applying Lemmas 5.5, 5.6, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\|\partial_t\Phi(t,\widehat{\xi})\|_{L^1_{\widehat{\xi}}(|\widehat{\xi}|\le 2t^\gamma)} &\le Ct^{\frac{3}{2}\gamma-\frac{4}{3}}\|(\xi^{-1}\widehat{\mathcal{P}}_b - 2b\xi^{-1}\widehat{\mathcal{I}}_b)\widehat{\varphi}\|_{L^2_{\widehat{\xi}}(|\widehat{\xi}|\le 2t^\gamma)} \\
&\le Ct^{\frac{3}{2}\gamma-\frac{7}{6}}\|(\xi^{-1}\widehat{\mathcal{P}}_b - 2b\xi^{-1}\widehat{\mathcal{I}}_b)\widehat{\varphi}\|_{L^2_\xi} \\
&\le Ct^{\frac{3}{2}\gamma-\frac{7}{6}}(\|\partial_x^{-1}\mathcal{P}_b u\|_{L^2} + \|\partial_x^{-1}\mathcal{I}_b u\|_{L^2}) \le C\varepsilon t^{\frac{5}{2}\gamma-\frac{7}{6}}.
\end{aligned}$$

Integrating by parts as above we find

$$\begin{aligned}
&\int_{2t_1^\gamma\le|\widehat{\xi}|\le 2t_2^\gamma} e^{i\widetilde{S}(y,\widehat{\xi})}\Phi(t_2,\widehat{\xi})d\widehat{\xi} \\
&= \int_{2t_1^\gamma\le|\widehat{\xi}|\le 2t_2^\gamma} e^{i(y\widehat{\xi}-\frac{1}{3}|\widehat{\xi}|^3)}\widehat{\varphi}(t_2,\widehat{\xi}t_2^{-1/3})d\widehat{\xi} \\
&= e^{i\widetilde{S}(y,\widehat{\xi})}\frac{\widehat{\varphi}(t_2,\widehat{\xi}t_2^{-1/3})}{i(y-|\widehat{\xi}|\widehat{\xi})}\Big|_{|\widehat{\xi}|=2t_1^\gamma}^{|\widehat{\xi}|=2t_2^\gamma} - \int_{2t_1^\gamma\le|\widehat{\xi}|\le 2t_2^\gamma} e^{i\widetilde{S}(y,\widehat{\xi})}\widehat{\varphi}(t_2,\widehat{\xi}t_2^{-1/3})\partial_{\widehat{\xi}}\frac{1}{i(y-|\widehat{\xi}|\widehat{\xi})}d\widehat{\xi}
\end{aligned}$$

$$-\int_{2t_1^\gamma\le|\widehat{\xi}|\le 2t_2^\gamma} e^{i\widetilde{S}(y,\widehat{\xi})}\frac{t_2^{-1/3}\widehat{\varphi}_\xi(t_2,\widehat{\xi}t_2^{-1/3})}{i(y-|\widehat{\xi}|\widehat{\xi})}d\widehat{\xi}$$

on the domain $|y|\le t_1^\gamma$ for $t_2>t_1$. Hence

$$\begin{aligned}
&|\int_{2t_1^\gamma\le|\widehat{\xi}|\le 2t_2^\gamma} e^{i\widetilde{S}(y,\widehat{\xi})}\Phi(t_2,\widehat{\xi})d\widehat{\xi}|\\
&\le Ct_1^{-2\gamma}\|\widehat{\varphi}\|_{L^\infty}+C\|\widehat{\varphi}\|_{L^\infty}\int_{2t_1^\gamma\le|\widehat{\xi}|\le 2t_2^\gamma}\frac{\widehat{\xi}d\widehat{\xi}}{(x-|\widehat{\xi}|\widehat{\xi})^2}\\
&\quad + Ct_2^{-1/3}\Big(\int_{2t_1^\gamma\le|\widehat{\xi}|\le 2t_2^\gamma}|\widehat{\varphi}(t_2,\widehat{\xi}t_2^{-1/3})|^2d\widehat{\xi}\Big)^{1/2}\Big(\int_{2t_1^\gamma\le|\widehat{\xi}|\le 2t_2^\gamma}\frac{d\widehat{\xi}}{\widehat{\xi}^4}\Big)^{1/2}\\
&\le Ct_1^{-2\gamma}\|\widehat{\varphi}\|_{L^\infty}+Ct_1^{-\frac{1}{6}-\frac{3}{2}\gamma}\|\widehat{\varphi}_\xi\|_{L^2}\le C\varepsilon t_1^{-\frac{3}{2}\gamma}.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
|F(t_1,y)-F(t_2,y)| &\le \Big|\int_{|\widehat{\xi}|\le 2t_1^\gamma} e^{i\widetilde{S}(y,\widehat{\xi})}\Phi(t_1,\widehat{\xi})d\widehat{\xi}-\int_{|\widehat{\xi}|\le 2t_1^\gamma} e^{i\widetilde{S}(y,\widehat{\xi})}\Phi(t_2,\widehat{\xi})d\widehat{\xi}\Big|\\
&\quad +|\int_{2t_1^\gamma\le|\widehat{\xi}|\le 2t_2^\gamma} e^{i\widetilde{S}(y,\widehat{\xi})}\Phi(t_2,\widehat{\xi})d\widehat{\xi}|\\
&\le C\varepsilon t_1^{-\frac{3}{2}\gamma}+\int_{t_1}^{t_2}\|\partial_t\Phi(t,\widehat{\xi})\|_{L^1_{\widehat{\xi}}(|\widehat{\xi}|\le 2t^\gamma)}dt\\
&\le C\varepsilon t_1^{-\frac{3}{2}\gamma}+C\varepsilon\int_{t_1}^{t_2}t^{\frac{5}{2}\gamma-\frac{7}{6}}dt\le C\varepsilon t_1^{-\delta}
\end{aligned}$$

for any $t_2\ge t_1$ with some small $\delta>0$. Hence there exists a limit $f_m(y)=\lim_{t\to\infty}F(t,y)$ such that $|F(t,y)-f_m(y)|\le C\varepsilon t^{-\delta}$ for $|y|\le t^\gamma$. Thus in the self-similar region $|x|\le t^{\frac{1}{3}+\gamma}$ we have the asymptotic formula $u(t,x)=t^{-\frac{1}{3}}f_m(xt^{-1/3})+O(t^{-\frac{1}{3}-\delta})$. Taking the limit $t\to\infty$ in equation (1.1) we can see that $t^{-1/3}f_m(xt^{-1/3})$ is the self-similar solution of equation (1.2). This completes the proof of the asymptotic part of Theorem 1.1.

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