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EXISTENCE AND CONCENTRATION OF POSITIVE GROUND STATES FOR SCHRÖDINGER-POISSON EQUATIONS WITH COMPETING POTENTIAL FUNCTIONS

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Abstract. This article concerns the Schrödinger-Poisson equation

$$-\varepsilon^2\Delta u + V(x)u + K(x)\phi u = P(x)|u|^{p-1}u + Q(x)|u|^{q-1}u, \quad x \in \mathbb{R}^3,$$
$$-\varepsilon^2\Delta\phi = K(x)u^2, \quad x \in \mathbb{R}^3,$$

where $3 < q < p < 5 = 2^* - 1$. We prove that for all $\varepsilon > 0$, the equation has a ground state solution. The methods used here are based on the Nehari manifold and the concentration-compactness principle. Furthermore, for $\varepsilon > 0$ small, these ground states concentrate at a global minimum point of the least energy function.

1. Introduction

We study the Schrödinger-Poisson equation

$$\begin{gathered} -\varepsilon^2\Delta u + V(x)u + \phi u = f(x,u), \quad x \in \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = u^2, \quad x \in \mathbb{R}^3. \end{gathered} \tag{1.1}$$

This equation has attracted much attention, and still is a stimulating field of research for mathematicians and for physicists. System (1.1) was first introduced in [2] and has been a study object of interest for nonlinear analysis. As a physical model, it describes a charged particle interacting with its own electrostatic field in quantum. And it can be a model to describe semiconductor theory, nonlinear optics and plasma physics. The presence of the nonlinear term $f(x,u)$ simulates the interaction between many particles and external nonlinear perturbations. In fact, it can be described by coupling the nonlinear Schrödinger and Maxwell equations and so it is also known as the Schrödinger-Maxwell system. We refer the readers to [2] and the references therein for the physical aspects of problem (1.1). Especially, the semi-classical state solutions describe the transition from Quantum Mechanics to Newtonian Mechanics from the point of view of physics.

In recent years, (1.1) with $V(x) \equiv 1$ and $\varepsilon = 1$ has been studied under variant assumptions on f. See for example [1, 3, 5, 7, 8, 9], and the references therein. In [12], the authors consider Schrödinger-Poisson equation with a non-constant

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potential and double parameters perturbation:

$$\begin{aligned} -\varepsilon^2\Delta u + V(x)u + \phi u = u^5 + f(u), \quad x \in \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = u^2, \quad u(x) > 0, \quad x \in \mathbb{R}^3. \end{aligned} \tag{1.2}$$

They use mountain pass to prove that (1.2) has a ground state solution which possesses the concentrating phenomenon, concentrating around global minimum of the potential V in the semi-classical limit. In [28], the authors studied the existence of positive ground state via the Nehari manifold methods. They multiply the nonlinearity by a potential $b(x)$, that is,

$$\begin{aligned} -\varepsilon^2\Delta u + V(x)u + \phi u = u^5 + b(x)f(u), \quad x \in \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = u^2, \quad u(x) > 0, \quad x \in \mathbb{R}^3. \end{aligned} \tag{1.3}$$

As for the concentration of ground state solutions, naturally, there is a competition between the linear potential $V(x)$ and the nonlinear potential $b(x)$, i.e., $V(x)$ wants to attract ground state solutions to its minimum points but $b(x)$ wants to attract ground state solutions to its maximum points. For instance in [4], a potential $K(x)$ before non-local term was added, i.e.,

$$\begin{gathered} -\Delta u + V(x)u + K(x)\phi u = a(x)|u|^{p-1}u, \quad x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, \quad x \in \mathbb{R}^3, \end{gathered}$$

under suitable assumptions, for $p \in (3, 5)$, the authors also obtain a positive ground state solution or positive solution. For other results for this system, see for example [10, 13, 15, 18, 21, 22, 23, 30, 31, 32, 33] and the references therein. We should mention that for the equation

$$-\varepsilon^2\Delta u + V(x)u = |u|^{p-2}u, \quad x \in \mathbb{R}^3, \tag{1.4}$$

if we let $\varepsilon^2 = \lambda^{-1}$, $v = \lambda^{-\frac{1}{p-2}}u$, then (1.4) can be written as

$$-\Delta v + \lambda V(x)v = |v|^{p-2}v, \quad x \in \mathbb{R}^3.$$

So in [33], the authors consider the system

$$\begin{gathered} -\Delta u + \lambda V(x)u + K(x)\phi u = a(x)|u|^{p-2}u, x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, x \in \mathbb{R}^3, \end{gathered}$$

where $2 < p < 6$, the potential V can be sign-changing. Under suitable conditions, they show some concentrations when $\lambda \to \infty$. See [22] for the generalized extensible beam equations.

Motivated by above works, and [29], we study the existence and concentration of ground states to (1.1) with competing potentials. More precisely, we are concerned with the Schrödinger-Poisson equation

$$\begin{aligned} -\varepsilon^2\Delta u + V(x)u + K(x)\phi u = P(x)|u|^{p-1}u + Q(x)|u|^{q-1}u, \quad x \in \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = K(x)u^2, \quad x \in \mathbb{R}^3, \end{aligned} \tag{1.5}$$

where $3 < q < p < 5 = 2^* - 1$, V, K and P are continuous and bounded positive functions. Q is continuous function and maybe change sign, even be negative. Using the Nehari manifold and the concentration-compactness, we shall prove that the above problem admits a ground state. Furthermore, we want to prove that these ground states concentrate at a point which locates on the middle ground of the competing potential functions $P(x)$ and $Q(x)$ as $\varepsilon \to 0^+$ via a concentration-compactness argument similar to [29].

The previous results of existence and concentration of Schrödinger-Poisson problems (see e.g. [12, Theorem 1.1], [27, Theorems 1.1, 1.2, 1.3], [28, Theorem 1.1]) can not be applied directly to (1.5) when $Q \neq 0$ is not a constant potential, especially, when Q is sign-changing or negative. To state our main results, we use the following assumptions:

(A1) $V(x) \in C^1(\mathbb{R}^3, \mathbb{R})$,
$$0 < V_0 := \inf_{x\in\mathbb{R}^3} V(x) \le V(x) \le V_\infty := \lim_{|x|\to\infty} V(x) < \infty$$
and $V(x) \not\equiv V_\infty$.

(A2) $K(x) \in C(\mathbb{R}^3, \mathbb{R})$,
$$0 < K_0 := \inf_{x\in\mathbb{R}^3} K(x) \le K(x) \le K_\infty := \lim_{|x|\to\infty} K(x) < \infty$$
and $K(x) \not\equiv K_\infty$.

(A3) $P(x) \in C^1(\mathbb{R}^3, \mathbb{R})$ and $P(x) \ge P_\infty := \lim_{|x|\to\infty} P(x) > 0$ and $P(x) \not\equiv P_\infty$.

(A4) $Q(x) \in C^1(\mathbb{R}^3, \mathbb{R})$ and $Q(x) \ge Q_\infty := \lim_{|x|\to\infty} Q(x)$, Q is allowed to change sign or be negative and $Q(x) \not\equiv Q_\infty$.

For each $s \in \mathbb{R}^3$, we consider the following problem with parameter $s \in \mathbb{R}^3$,
$$\begin{gathered} -\Delta u + V(s)u + \phi u = P(s)|u|^{p-1}u + Q(s)|u|^{q-1}u, \quad x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, \quad x \in \mathbb{R}^3. \end{gathered} \tag{1.6}$$
Denote the corresponding energy functional by I^s and the corresponding least energy by
$$C(s) = c(V(s), P(s), Q(s)) := \inf\{I^s(u) : u \text{ is a nontrivial solution of } (1.6)\}.$$
It is well known that $C(s)$ is well defined. Our main result is as follows.

Theorem 1.1. (I) *Suppose that* (A1)–(A4) *are satisfied, and* $3 < q < p < 5 = 2^* - 1$. *Then for each* $\varepsilon > 0$, (1.5) *has a positive ground state solution* u_ε.

(II) *Let the assumptions in* (I) *be satisfied and let* $K(x) = 1$. *Then for* $\varepsilon > 0$ *small,*

(1) *the positive ground state solution* u_ε *obtained in* (I) *possesses at most one local (hence global) maximum point* x_ε *in* $\mathbb{R}^3$ *such that*
$$\lim_{\varepsilon\to 0^+} C(x_\varepsilon) = \inf_{s\in\mathbb{R}^3} C(s).$$

(2) $x_\varepsilon \to x_0$, $w_\varepsilon(x) := u_\varepsilon(\varepsilon x + x_\varepsilon)$ *converges in* $H^1(\mathbb{R}^3)$ *to a positive ground state solution of*
$$\begin{gathered} -\Delta u + V(x_0)u + \phi u = P(x_0)|u|^{p-1}u + Q(x_0)|u|^{q-1}u, \quad x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, \quad x \in \mathbb{R}^3. \end{gathered}$$

(3) *there exist* $C_1, C_2 > 0$ *such that*
$$u_\varepsilon(x) \le C_1 e^{-C_2|\frac{x-x_\varepsilon}{\varepsilon}|}.$$

Remark 1.2. We point out that the potential K appears both in the first equation and in the second equation of (1.5) which is used in (2.5). In Appendix, we will explain why we let $K(x) = 1$.

This article is organized as follows. In section 2, we verify the existence of ground states. In section 3, we are devoted to prove the properties of ground states including exponential decay and concentration.

2. Existence of ground states

Under our assumptions, for the existence of ground states, without loss of generality, we may assume that $\varepsilon = 1$. Then (1.5) becomes

$$\begin{gathered} -\Delta u + V(x)u + K(x)\phi u = P(x)|u|^{p-1}u + Q(x)|u|^{q-1}u, \quad x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, \quad x \in \mathbb{R}^3. \end{gathered} \tag{2.1}$$

Let $H^1(\mathbb{R}^3)$ denote the usual Sobolev space endowed with the standard scalar product and norm

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)dx, \quad \|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx.$$

The set $D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ is a Hilbert space endowed with the standard scalar product and norm

$$(u, v) = \int_{\mathbb{R}^3} \nabla u \nabla v dx, \quad \|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

For $u \in H^1(\mathbb{R}^3)$, we focus on the equation

$$-\Delta \phi = K(x)u^2.$$

It is well known that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta \phi_u = K(x)u^2.$$

Furthermore, we have

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|} dy. \tag{2.2}$$

Substituting this into (2.1), we can rewrite (2.1) as

$$-\Delta u + V(x)u + K(x)\phi_u(x)u = P(x)|u|^{p-1}u + Q(x)|u|^{q-1}u. \tag{2.3}$$

We formally formulate problem (2.3) in a variational way as

$$\begin{aligned} I(u) =& \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \\ & - \frac{1}{p+1} \int_{\mathbb{R}^3} P(x)|u|^{p+1}dx - \frac{1}{q+1} \int_{\mathbb{R}^3} Q(x)|u|^{q+1}dx, \quad u \in H^1(\mathbb{R}^3). \end{aligned}$$

For simplicity, define

$$N(u) := \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx, \quad \|u\|_s^s := \int_{\mathbb{R}^3} |u|^s dx.$$

Lemma 2.1. (1) *If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ and*

$$N(u) \le \liminf_{n\to\infty} N(u_n), N(u_n - u) = N(u_n) - N(u) + o_n(1). \tag{2.4}$$

(2) *$\phi_u \ge 0, \|\phi_u\|_{D^{1,2}} \le C\|u\|^2$, and $\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \le C\|u\|^4_{12/5} \le C\|u\|^4$.*
(3) *$\phi_{tu} = t^2\phi_u$, for all $t \in \mathbb{R}$.*

Proof. (1) It is easy to obtain the conclusions by the method in [7] with slight modification (see also [4]). The last splitting property can be obtained by [32, Lemma 2.1].

(2) Noting that K is positive, it is easy to check the conclusions.

(3) It follows form a direct computation, we omit it here. □

In view of Lemma 2.1, we can see that $I \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$. A direct computation, we have

$$\begin{aligned} \langle N'(u), \varphi \rangle &= \frac{1}{2} \int_{\mathbb{R}^3} K(x)\phi_u u\varphi dx + \frac{1}{2} \int_{\mathbb{R}^3} K(x)\phi_{\sqrt{u\varphi}} u^2 dx \\ &= \int_{\mathbb{R}^3} K(x)\phi_u u\varphi dx \quad \text{(by Fubini's theorem).} \end{aligned} \tag{2.5}$$

Thus, for all $\varphi \in H^1(\mathbb{R}^3)$, we have

$$\begin{aligned} \langle I'(u), \varphi \rangle = &\int_{\mathbb{R}^3} \nabla u \nabla \varphi dx + \int_{\mathbb{R}^3} V(x)u\varphi dx + \int_{\mathbb{R}^3} K(x)\phi_u u\varphi dx \\ &- \int_{\mathbb{R}^3} P(x)|u|^{p-1}u\varphi dx - \int_{\mathbb{R}^3} Q(x)|u|^{q-1}u\varphi dx. \end{aligned} \tag{2.6}$$

We define the Nehari manifold of I, as

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : \gamma(u) = 0\}, \tag{2.7}$$

where

$$\gamma(u) = \langle I'(u), u \rangle. \tag{2.8}$$

The next lemma shows that $\mathcal{N} \neq \emptyset$.

Lemma 2.2. *Suppose that $u \neq 0$ and $3 < q < p < 2^* - 1$. Then there is a unique $t = t(u) > 0$ such that $tu \in \mathcal{N}$ and $I(ru) < I(tu)$ if $r \neq t$.*

Proof. Set

$$f(t) := I(tu) = \frac{At^2}{2} + \frac{Bt^4}{4} - \frac{Ct^{p+1}}{p+1} - \frac{Dt^{q+1}}{q+1},$$

where

$$A = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx, \quad B = \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx,$$
$$C = \int_{\mathbb{R}^3} P(x)|u|^{p+1}dx, \quad D = \int_{\mathbb{R}^3} Q(x)|u|^{q+1}dx.$$

Then $f'(t) = At + Bt^3 - Ct^p - Dt^q$, and hence $f'(t) > 0$ for t small and $f'(t) < 0$ for t large. Hence there is $t = t(u) > 0$ such that $f'(t) = 0$. Thus

$$\gamma(tu) = tf'(t) = 0,$$

which implies $tu \in \mathcal{N}$. The uniqueness follows from the fact that the equation

$$A + Bt^2 - Ct^{p-1} - Dt^{q-1} = 0$$

has a unique positive solution. □

The next lemma is crucial for proving our results.

Lemma 2.3. *There exists $C > 0$ such that for any $u \in \mathcal{N}$,*

$$I(u) \geq C \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx.$$

Proof. Since V, K and P are positive, it follows from $\gamma(u) = 0$ that

$$\begin{aligned} I(u) =& I(u) - \frac{1}{q+1}\langle I'(u), u\rangle \\ =& \Big(\frac{1}{2} - \frac{1}{q+1}\Big)\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx + \Big(\frac{1}{4} - \frac{1}{q+1}\Big)\int_{\mathbb{R}^3} K(x)\phi_u u^2 dx \\ &+ \Big(\frac{1}{q+1} - \frac{1}{p+1}\Big)\int_{\mathbb{R}^3} P(x)|u|^{p+1}dx \\ \geq& C\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx. \end{aligned}$$

The proof is complete. □

According to Lemma 2.3, we can define

$$c^* := \inf_{u\in\mathcal{N}} I(u).$$

Then $c^* \geq 0$. Furthermore, the following lemma shows that $c^* > 0$.

Lemma 2.4. *There exists $r^* > 0$ such that $\|u\| > r^*$, for all $u \in \mathcal{N}$.*

Proof. For any $u \in \mathcal{N}$, we have

$$0 = \langle I'(u), u\rangle \geq \|u\|^2 - C_1\|u\|^4 - C_2\|u\|^{p+1} - C_3\|u\|^{q+1},$$

from which we obtain that $\|u\| > r^* > 0$, which completes the proof. □

Let $\{u_n\} \subset \mathcal{N}$ be a minimizing sequence of c^*, i.e. $I(u_n) \to c^*$ as $n \to \infty$. In the light of Lemma 2.3, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Extracting a subsequence if necessary, we have $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $u_n(x) \to u(x)$ a. e. in $\mathbb{R}^3$. Up to a subsequence, we have the following lemma.

Lemma 2.5. *$\int_{\mathbb{R}^3} |u_n|^{p+1}dx$ has a positive lower bound with respect to n, that is,*

$$\int_{\mathbb{R}^3} |u_n|^{p+1}dx \not\to 0.$$

Proof. Suppose to the contrary $\int_{\mathbb{R}^3} |u_n|^{p+1}dx \to 0$. Invoking the interpolation inequality, we obtain

$$\int_{\mathbb{R}^3} |u_n|^{q+1}dx \to 0.$$

So we have

$$\begin{aligned} c^* &= \lim_{n\to\infty} I(u_n) \\ &= \lim_{n\to\infty}\Big(I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle\Big) \\ &= \lim_{n\to\infty}\Big[-\frac{1}{2}\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2dx + \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\int_{\mathbb{R}^3} P(x)|u_n|^{p+1}dx \\ &\quad + \Big(\frac{1}{2} - \frac{1}{q+1}\Big)\int_{\mathbb{R}^3} Q(x)|u_n|^{q+1}dx\Big] \leq 0. \end{aligned}$$

This is a contradiction. □

Now, we can assume (extracting a subsequence, if necessary) that

$$\int_{\mathbb{R}^3} |u_n|^{p+1}dx + \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^2dx \xrightarrow{n\to\infty} \alpha \in (0,\infty).$$

We apply the concentration-compactness principle (see [20] or [14]) to

$$\rho_n := |u_n|^{p+1} + \phi_{u_n}|u_n|^2$$

to obtain $u_n \to u$ in $L^{p+1}(\mathbb{R}^3)$. By the concentration-compactness lemma, up to a subsequence, there are three possibilities:

1 (compactness). For any $\epsilon > 0$, there is a $R > 0$ and $\{x_n\} \subset \mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N\setminus B_R(x_n)} \rho_n dx < \epsilon.$$

2 (vanishing). For any $R > 0$, it holds

$$\lim_{n\to\infty}\Big(\sup_{x\in\mathbb{R}^N}\int_{B_R(x)} \rho_n|dx\Big) = 0.$$

3 (dichotomy). There exists a $\widetilde{\beta} \in (0,\alpha)$, such that for all $\epsilon > 0$, there is a $R > 0$, $\{x_n\} \subset \mathbb{R}^N$ and a sequence $R \le R_n \to \infty$ satisfied: for n large enough,

$$\Big|\int_{B_{R_n}(x_n)} \rho_n dx - \widetilde{\beta}\Big| < \epsilon, \quad \Big|\int_{\mathbb{R}^N\setminus B_{R_n}(x_n)} \rho_n dx - (\alpha - \widetilde{\beta})\Big| < \epsilon.$$

It is sufficient to show that vanishing and dichotomy do not occur.

Lemma 2.6. *The vanishing does not occur.*

Proof. The desired result follows from the vanishing Lemma in [14] (see also [24, Lemma 1.21]) and Lemma 2.5. □

Lemma 2.7. *The dichotomy does not occur.*

Proof. According to concentration-compactness principle,we can suppose that there exists a subsequence of $\{\rho_n\}$, still denote $\{\rho_n\}$, $\beta \in (0,1]$ and $\{x_n\} \subset \mathbb{R}^3$ such that for each $\epsilon > 0$, there exist $r_\epsilon > 0$, $r_\epsilon < r_n$, $r_n < r_{n+1} \xrightarrow{n\to\infty} \infty$ satisfying

$$\liminf_{n\to\infty}\int_{B_{r_n}(x_n)} \rho_n(x)dx \ge \alpha\beta - \epsilon, \tag{2.9}$$

$$\liminf_{n\to\infty}\int_{B^c_{2r_n}(x_n)} \rho_n(x)dx \ge (1-\beta)\alpha - \epsilon. \tag{2.10}$$

We only need to prove $\beta = 1$ to exclude dichotomy. Actually, $\beta = 1$ is can be done by using the next lemma. □

Let ϕ_n be a cut-off function such that $\phi_n \equiv 1$ in $B_{r_n}(x_n)$ and $\phi_n \equiv 0$ in $B^c_{2r_n}(x_n)$. Write

$$u_n = \phi_n u_n + (1-\phi_n)u_n := v_n + w_n.$$

Lemma 2.8. $\lim_{n\to\infty}\|w_n\| = 0$.

Proof. The proof is similar to that in [19] with slight modification. Since it has sign-changing potential Q, here we give the details for completeness. By direct calculations, we obtain

$$\Big|\int_{\mathbb{R}^3}(|\nabla u_n|^2-|\nabla v_n|^2-|\nabla w_n|^2)dx\Big| = 2\Big|\int_{\mathbb{R}^3}\nabla v_n\nabla w_n dx\Big| = o_n(1), \tag{2.11}$$

$$\Big|\int_{\mathbb{R}^3}V(x)(u_n^2-v_n^2-w_n^2)dx\Big| = o_n(1), \tag{2.12}$$

$$\Big|\int_{\mathbb{R}^3}P(x)\left(|u_n|^{p+1}-|v_n|^{p+1}-|w_n|^{p+1}\right)dx\Big| = o_n(1), \tag{2.13}$$

$$\Big|\int_{\mathbb{R}^3}Q(x)\left(|u_n|^{q+1}-|v_n|^{q+1}-|w_n|^{q+1}\right)dx\Big| = o_n(1). \tag{2.14}$$

We have the splitting property

$$N(u_n) = N(v_n)+N(w_n)+o_n(1). \tag{2.15}$$

In fact, by a direct computation, we obtain

$$N(u_n) = N(v_n)+N(w_n)+o_n(1)+\frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)|v_n(y)|^2|w_n(x)|^2}{4\pi|x-y|}\,dy\,dx.$$

Furthermore,

$$\begin{aligned}
&\int_{\{x:|x|\ge r_n\}}\int_{\{y:|y|\le 2r_n\}}\frac{|v_n(y)|^2|w_n(x)|^2}{4\pi|x-y|}\,dy\,dx\\
&=\int_{\{x:|x|\ge r_n\}}\int_{\{y:r_n\le|y|\le 2r_n\}}\frac{|v_n(y)|^2|w_n(x)|^2}{4\pi|x-y|}\,dy\,dx\\
&\quad+\int_{\{x:2r_n\ge|x|\ge r_n\}}\int_{\{y:|y|<r_n\}}\frac{|v_n(y)|^2|w_n(x)|^2}{4\pi|x-y|}\,dy\,dx\\
&\quad+\int_{\{x:|x|>2r_n\}}\int_{\{y:|y|<r_n\}}\frac{|v_n(y)|^2|w_n(x)|^2}{4\pi|x-y|}\,dy\,dx\\
&\le\int_{\{y:r_n\le|y|\le 2r_n\}}\phi_{u_n}u_n^2\,dx+\int_{\{x:r_n\le|x|\le 2r_n\}}\phi_{u_n}u_n^2dx+\frac{\int_{\mathbb{R}^3}u_n^2dx\int_{\mathbb{R}^3}u_n^2\,dx}{4\pi r_n}\\
&=o_n(1).
\end{aligned}$$

Therefore, (2.15) holds. So putting together (2.11)-(2.15), we obtain

$$|I(tu_n)-I(tv_n)-I(tw_n)|\le t^2o_n(1)+t^4o_n(1)+t^{p+1}o_n(1)+t^{q+1}o_n(1). \tag{2.16}$$

Let $t(v_n)$ and $t(w_n)$ be the positive values which maximize $f(t):=I(tv_n)$ and $I(tw_n)$. Firstly, we discuss the case $t(v_n)\le t(w_n)$ (the other case will be treated later). In this case,

$$I(tw_n)\ge 0,\quad \text{for } t\le t(v_n)\le t(w_n). \tag{2.17}$$

Our next aim is to find suitable bounds for the sequence $\{t(v_n)\}$. We claim that there exist $0<\underline{t}<1<\bar{t}$ independent of n such that $t(v_n)\in(\underline{t},\bar{t})$.

In fact, we already know that

$$\int_{\mathbb{R}^3}P(x)|u_n|^{p+1}dx \xrightarrow{n\to\infty} A\in(0,\infty). \tag{2.18}$$

Since Q is allowed to change sign or be negative, we only have

$$\int_{\mathbb{R}^3} Q(x)|u_n|^{q+1}dx \xrightarrow{n\to\infty} B \in (-\infty,\infty). \tag{2.19}$$

Case 1. $B > 0$. Take

$$\bar{t} = \big[\frac{(p+1)M}{A}\big]^{\frac{1}{p-3}},$$

where A is from (2.18) and M is large enough such that $\bar{t} > 1$ and moreover

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2)dx + \frac{1}{2}\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2 dx \le M. \tag{2.20}$$

Thus, for n large enough, we have

$$\begin{aligned} I(\bar{t}u_n) \le& \frac{\bar{t}^4}{2}\Big(M - \frac{2\bar{t}^{p-3}}{p+1}\int_{\mathbb{R}^3} P(x)|u_n|^{p+1}dx\Big) \\ \le& -M\frac{\bar{t}^4}{2} + o_n(1) < 0. \end{aligned} \tag{2.21}$$

Case 2. $B \le 0$. Note that

$$a_n = \int_{\mathbb{R}^3}\Big(|\nabla u_n|^2 + V(x)u_n^2\Big)dx, \quad b_n = \int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2 dx,$$
$$c_n = -\int_{\mathbb{R}^3} Q(x)|u_n|^{q+1}dx$$

are bounded. We can choose $M_1 > 0$ independent of n, such that

$$a_n + \frac{1}{2}b_n + \frac{2}{q+1}c_n \le M_1.$$

Take

$$\bar{t} = \Big[\frac{(p+1)M_1}{A}\Big]^{\frac{1}{p-q}}.$$

And let M_1 be large enough such that $\bar{t} > 1$. For n large enough, we have

$$\begin{aligned} I(\bar{t}u_n) \le& \frac{\bar{t}^{q+1}}{2}\Big(M_1 - \frac{2\bar{t}^{p-q}}{p+1}\int_{\mathbb{R}^3} P(x)|u_n|^{p+1}dx\Big) \\ \le& -M\frac{\bar{t}^{q+1}}{2} + o_n(1) < 0. \end{aligned} \tag{2.22}$$

For the case $B = 0$, it is easy to obtain a similar result.

Thus, by (2.16), for all $\epsilon > 0$, for n large enough, we obtain

$$I(\bar{t}u_n) \ge I(\bar{t}v_n) + I(\bar{t}w_n) - \epsilon. \tag{2.23}$$

Taking into account (2.21), choosing a smaller $\epsilon > 0$ if necessary, it holds that

$$I(\bar{t}v_n) + I(\bar{t}w_n) < 0.$$

It follows that $I(\bar{t}v_n) < 0$, or $I(\bar{t}w_n) < 0$. By Lemma 2.2 and $t(v_n) \le t(w_n)$, it holds that $t(v_n) \le \bar{t}$.

For the lower positive bound, we also need to discuss two cases.

Case 1. $B > 0$. Take

$$\underline{t} = \Big(\frac{c^*}{M}\Big)^{1/2},$$

where M comes from as in (2.20) and large enough. Note that $\underline{t} < 1$, for any $t < \underline{t}$,

$$I(tu_n) \le \frac{t^2}{2}\Big(\int_{\mathbb{R}^3}(|\nabla u_n|^2 + V(x)u_n^2)dx + \frac{1}{2}\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2dx\Big) \le \frac{c^*}{2}.$$

Case 2. $B \le 0$. Note that there exists a $L > 0$, such that

$$-L \le \int_{\mathbb{R}^3} Q(x)|u_n|^{q+1}dx.$$

We take

$$\underline{t} = \min\big\{\big(\frac{c^*}{2M}\big)^{1/2}, \big(\frac{c^*}{2L}\big)^{\frac{1}{q+1}}\big\}.$$

So that it holds

$$\begin{aligned} I(tu_n) \le& \frac{t^2}{2}\Big(\int_{\mathbb{R}^3}(|\nabla u_n|^2 + V(x)u_n^2)dx + \frac{1}{2}\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2dx\Big) \\ &- \frac{t^{q+1}}{2}\int_{\mathbb{R}^3} Q(x)|u_n|^{q+1}dx \\ \le& \frac{c^*}{2}. \end{aligned}$$

Similarly to (2.23), jointly with (2.17),

$$I(t(v_n)u_n) \ge I(t(v_n)v_n) + I(t(v_n)w_n) \ge c^* - \epsilon.$$

So, by choosing a small $\epsilon > 0$, for n large enough, $I(t(v_n)u_n) > c^*/2$. Thus we obtain the lower bound of $t(v_n)$.

For all $t \in (0, t(v_n))$, noting that $t(v_n) \le t(w_n)$, combining with (2.16), we have

$$\begin{aligned} I(tw_n) \le& I(t(v_n)w_n) \\ =& I(t(v_n)u_n) - I(t(v_n)v_n) + o_n(1) \\ \le& I(u_n) - c^* + o_n(1) = o_n(1), \end{aligned}$$

where $\{u_n\} \subset \mathcal{N}$ and $\lim_{n\to\infty} I(u_n) = c^*$ are used in the last inequality.

Moreover, it is well-known that there exists a $D > 0$ independent of n such that

$$\frac{1}{p+1}\int_{\mathbb{R}^3} P(x)|w_n|^{p+1}dx \le \frac{D}{2},\ \textit{and}\ \frac{1}{q+1}\int_{\mathbb{R}^3} Q(x)|w_n|^{q+1}dx \le \frac{D}{2}.$$

Observing that $0 < t < \underline{t} < 1$ and $q < p$, one has

$$\begin{aligned} o_n(1) \ge& I(tw_n) \\ \ge& \frac{t^2}{2}l_n - t^{q+1}\frac{D}{2} - t^{p+1}\frac{D}{2} \\ \ge& \frac{t^2}{2}l_n - Dt^{q+1}, \end{aligned}$$

where

$$l_n = \int_{\mathbb{R}^3}(|\nabla w_n|^2 + V(x)w_n^2)dx$$

are bounded. Let $t = \big(\frac{l_n}{4D}\big)^{\frac{1}{q-1}}$. Taking D large enough if necessary such that $t \in (0, \underline{t})$, we obtain

$$\frac{t^2}{2}l_n - Et^{q+1} = \frac{t^2}{4}l_n.$$

Therefore,

$$o_n(1) \geq I(tw_n) \geq Cl_n^{2q-1}.$$

In the case $t(v_n) > t(w_n)$, we can argue analogously to conclude that

$$\lim_{n\to\infty} \|v_n\| = 0.$$

This contradicts (2.9). Evidence now allows us to conclude, that $\lim_{n\to\infty} \|w_n\| = 0$. □

Lemma 2.9. *c^* is achieved.*

Proof. The proof is divided into two cases.

Case (1): $\{x_n\}$ is bounded. In this case, by Lemmas 2.6 and 2.7, $u_n \to u \not\equiv 0$ in $L^s(\mathbb{R}^3)$ for all $s \in (2,6)$. Combining with Lemma 2.2, there is a unique $t > 0$ such that $\gamma(tu) = 0$ and hence

$$c^* \leq I(tu) \leq \liminf_{n\to\infty} I(tu_n) \leq \liminf_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(u_n) = c^*.$$

Case (2): $\{x_n\}$ is unbounded. Set $z_n = u_n(\cdot + x_n)$, and we have $z_n \rightharpoonup z$ in $H^1(\mathbb{R}^3)$. It is easy to show $z_n \to z$ in $L^2(\mathbb{R}^3)$ by following the same method in [19, step 4]. By interpolation inequalities, we obtain $z_n \to z \not\equiv 0$ in $L^s(\mathbb{R}^3)$ for all $s \in [2,6)$. Using Lebesgue dominated convergence theorem, one has

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{R}^3} V(x)u_n^2 dx &= \lim_{n\to\infty} \int_{\mathbb{R}^3} V(x+x_n)z_n^2 dx = V_\infty \int_{\mathbb{R}^3} z^2 dx \\
&\geq \int_{\mathbb{R}^3} V(x)z^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^3} V(x)z_n^2 dx.
\end{aligned}$$

Similarly,

$$\begin{gathered}
\lim_{n\to\infty} \int_{\mathbb{R}^3} P(x)|u_n|^{p+1} dx \leq \lim_{n\to\infty} \int_{\mathbb{R}^3} P(x)|z_n|^{p+1} dx, \\
\lim_{n\to\infty} \int_{\mathbb{R}^3} Q(x)|u_n|^{q+1} dx \leq \lim_{n\to\infty} \int_{\mathbb{R}^3} Q(x)|z_n|^{q+1} dx.
\end{gathered}$$

Especially, for $N(u_n)$, Lebesgue dominated convergence theorem can be used, so it also holds

$$\lim_{n\to\infty} N(u_n) \geq \lim_{n\to\infty} N(z_n).$$

Thus, similarly to case (1),

$$c^* \leq I(t(z)z) \leq \liminf_{n\to\infty} I(t(z)z_n) \leq \liminf_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(u_n) = c^*.$$

The proof is complete. □

Similar to [16, Lemma 2.5] (see also [19, Theorem 2.2], we have the following result.

Lemma 2.10. *Suppose that $u \in \mathcal{N}$ and $I(u) = c^*$, then u is a weak solution of* (2.1).

In view of Lemma 2.9 and Lemma 2.10, we can define

$$c^{**} = \inf\{I(u) : u \text{ is a nontrivial solution of } (2.1)\}.$$

The next lemma shows that the functional I satisfies the mountain pass geometry.

Lemma 2.11. *The functional I satisfies*
(1) *there exist α, $\rho > 0$ such that $I(u) \geq \alpha$ for all $\|u\| = \rho$.*
(2) *there exists $e \in H^1(\mathbb{R}^3) \setminus \overline{B_\rho(0)}$, such that $I(e) < 0$.*

Proof. Since K is positive and P, Q are bounded, by the Sobolev embedding and result (2) in Lemma 2.1, we have

$$I(u) \geq \frac{1}{2}\|u\|^2 - C_1\|u\|^{p+1} - C_2\|u\|^{q+1}.$$

Hence we can choose some α, $\rho > 0$ such that $I(u) \geq \alpha$ for all $\|u\| = \rho$. For $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, we have

$$f(t) := I(tu) = \frac{At^2}{2} + \frac{Bt^4}{4} - \frac{Ct^{p+1}}{p+1} - \frac{Dt^{q+1}}{q+1} < 0,$$

for $t > 0$ large enough, where A, B, C, D are similar to Lemma 2.2. Choose $e = t_0 u$ for some suitable t_0. □

As a consequence of the Mountain Pass lemma without $(PS)_c$ condition, we define the constant

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)) > 0,$$

where

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Also we define

$$c^{***} = \inf_{v\in H^1(\mathbb{R}^3)\{0\}} \max_{t_0} I(tv).$$

Using the mountain pass value c and c^{***} as the connections, we can prove that the minimizer u is a ground state.

Lemma 2.12. $c = c^* = c^{**} = c^{***}$.

Proof. The original research should be attributed to Rabinowitz (see [17, Proposition 3.11]). For the convenience of readers, we sketch the proof.

For any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, by Lemma 2.2, the ray $R_t = \{tu : \ t \geq 0\}$ intersects the Nehari manifold $\mathcal{N}$ once and only once at $t(u)u$, where $t(u)$ is given in Lemma 2.2. This implies that $c^* = c^{***}$. Next, we show $c^* = c^{**}$. Obviously, $c^* \leq c^{**}$. On the other hand, since u is a nontrivial solution of (2.1), it holds that $I(u) \leq max_{t\geq 0} I(tu)$. So $c^{**} \leq c^{***}$. Now we check $c = c^*$. Since for all $\gamma \in \Gamma, \gamma(0) = 0, I(\gamma(1)) < 0$, it follows that γ crosses $\mathcal{N}$. We obtain $c \geq c^*$. On the other hand, for fixed $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, choosing suitable $\alpha = \alpha(u)$ large enough, let $g_u(t) = t\alpha u$. We have $g_u \in \Gamma$. So

$$c^{***} = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \max_{t\in[0,1]} I(g_u(t)) \geq c,$$

which completes the proof. □

Lemma 2.13. *The ground state u can be nonnegative. Furthermore, $u > 0$.*

Proof. Since

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx = \int_{\mathbb{R}^3} |\nabla |u||^2 dx,$$

we may assume that u obtained in Lemma 2.10 is nonnegative. Furthermore, a Moser iteration argument implies that $u \in L^s(\mathbb{R}^3)$ for $2 \leq s \leq \infty$ ([12, Proposition 3.3]). It follows from standard arguments in [6] that $u \in C^{1,\beta}_{\rm loc}(\mathbb{R}^3)$ for some $\beta > 0$.

Consequently, by Schauder estimate (see [11]), $u \in C^{2,\beta}_{\rm loc}(\mathbb{R}^3)$. Furthermore, u satisfies the equation

$$-\Delta u + c(x)u = P(x)u^p + Q(x)^+ u^q \geq 0,$$

where $c(x) := V(x) + K(x)\phi_u(x) + Q^-(x)u^q(x) > 0$ and $Q^\pm := \max\{\pm Q, 0\}$. Applying the strong maximum principle (see [11]), we have $u > 0$. □

3. Concentration of positive ground state

In this section, we are devoted to the concentration behaviour of the ground state solutions u_ε of (1.5) as $\varepsilon \to 0^+$. From now on, we assume $K(x) = 1$. Introducing the re-scaled transformation $x \mapsto \varepsilon x$ (i.e., $v_\varepsilon(x) := u(\varepsilon x)$), we can rewrite (1.5) as

$$\begin{gathered} -\Delta v_\varepsilon + V_\varepsilon(x)v_\varepsilon + \phi v_\varepsilon = P_\varepsilon(x)|v_\varepsilon|^{p-1}v_\varepsilon + Q_\varepsilon(x)|v_\varepsilon|^{q-1}v_\varepsilon, \quad x \in \mathbb{R}^3, \\ -\Delta\phi = v_\varepsilon^2, \quad x \in \mathbb{R}^3, \end{gathered} \tag{3.1}$$

where $V_\varepsilon(x) = V(\varepsilon x)$, and P_ε, Q_ε defined in a similar way. According to section 2, v_ε is positive ground state of (3.1). Let I_ε be the energy functional associated with (3.1) and $\mathcal{N}_\varepsilon$ be the corresponding Nehari manifold and set least energy $c_\varepsilon = \inf_{v\in\mathcal{N}_\varepsilon} I_\varepsilon(v)$. We need the following constant coefficients problem

$$-\Delta u + \mu u + \phi_u u = \xi|u|^{p-1}u + \tau|u|^{q-1}u, \tag{3.2}$$

where $\mu > 0$, $\xi > 0$, τ can positive or negative. In the same way, $I_{\mu\lambda\xi\tau}$, $\mathcal{N}_{\mu\xi\tau}$ and $c_{\mu\xi\tau}$ correspond to the energy functional, Nehari manifold, least energy associated with (3.2), respectively.

Similar to [17, Lemma 3.17] (see also [29, Lemma 2.2] or [26, Lemma 4.1]), we have the following result.

Lemma 3.1. *Suppose $\mu_1 \geq \mu_2$, $\xi_2 \geq \xi_1$ and $\tau_2 \geq \tau_1$. Then $c_{\mu_1\xi_1\tau_1} \geq c_{\mu_2\xi_2\tau_2}$. Furthermore, if one of inequalities is strict, then $c_{\mu_1\xi_1\tau_1} > c_{\mu_2\xi_2\tau_2}$.*

Since assume the potential functions V, P and Q are C^1, according to [29, (i) of Lemma 2.3, and Lemma 2.4], we have the following lemma.

Lemma 3.2. *The ground energy function $C(s)$ is locally Lipschitz continuous in $s \in \mathbb{R}^3$. If V, K, P and Q are constant functions, then the least energy depends continuously on them.*

Lemma 3.3. *There exists $C > 0$ independent with ε such that $c_\varepsilon \geq C$. Furthermore,*

$$\limsup_{\varepsilon\to 0^+} c_\varepsilon \leq \inf_{s\in\mathbb{R}^3} C(s). \tag{3.3}$$

Proof. By Lemma 3.1, $c_\varepsilon \geq c(\inf V, \|P\|_{L^\infty}, \|Q\|_{L^\infty}) > 0$, we only need to prove (3.3). The original idea is from the proof of [25, Lemma 2.2]. In view of our assumptions on V, P and Q, it holds that $\inf_{s\in\mathbb{R}^3} C(s)$ can be achieved by some s_0. Let u_0 be a ground state of

$$-\Delta u + V(s_0)u + \phi_u u = P(s_0)|u|^{p-1}u + Q(s_0)|u|^{q-1}u. \tag{3.4}$$

Denote the energy functional I^{s_0}. Take a sequence $\{y_k\}$ such that $C(y_k) \to C(s_0) = \inf_{s\in\mathbb{R}^3} C(s)$. For any $R > 0$, take a cut-off function φ_R with $\varphi_R = 1$ in $B_R(0)$ and $\varphi_R = 0$ in B^c_{R+1}. Set $v_R = \varphi_R u_0$ and $w(x) = v_R(x - \frac{y_k}{\varepsilon})$. Then there exists a unique $\theta > 0$ such that $\theta w \in \mathcal{N}_\varepsilon$ and $\theta \to 1$ as $R \to \infty$, $k \to \infty$ and $\varepsilon \to 0^+$. Since

$$c_\varepsilon = \inf_{v\in\mathcal{N}_\varepsilon} I_\varepsilon(v) \leq I_\varepsilon(\theta w)$$

$$
\begin{aligned}
=&\theta^2\Big\{I^{s_0}(w)+\frac{1}{2}\int_{\mathbb{R}^3}[V(\varepsilon x)-V(s_0)]w^2dx\\
&+\frac{1-\theta^{p-1}}{p+1}\int_{\mathbb{R}^3}P_\varepsilon(x)|w|^{p+1}dx+\frac{1-\theta^{q-1}}{q+1}\int_{\mathbb{R}^3}Q_\varepsilon(x)|w|^{q+1}dx\\
&+\frac{1}{p+1}\int_{\mathbb{R}^3}[P(s_0)-P_\varepsilon(x)]|w|^{p+1}dx+\frac{1}{q+1}\int_{\mathbb{R}^3}[Q(s_0)-Q_\varepsilon(x)]|w|^{q+1}dx\Big\}\\
=&\theta^2\Big\{I^{s_0}(v_R)+\frac{1}{2}\int_{\mathbb{R}^3}[V(\varepsilon x+y_k)-V(s_0)]v_R^2dx\\
&+\frac{1-\theta^{p-1}}{p+1}\int_{\mathbb{R}^3}P(\varepsilon x+y_k)|v_R|^{p+1}dx+\frac{1-\theta^{q-1}}{q+1}\int_{\mathbb{R}^3}Q(\varepsilon x+y_k)|v_R|^{q+1}dx\\
&+\frac{1}{p+1}\int_{\mathbb{R}^3}[P(s_0)-P(\varepsilon x+y_k)]|v_R|^{p+1}dx\\
&+\frac{1}{q+1}\int_{\mathbb{R}^3}[Q(s_0)-Q(\varepsilon x+y_k)]|v_R|^{q+1}dx\Big\}.
\end{aligned}
$$

Letting $R\to\infty$ and $k\to\infty$ in the above inequality, we have

$$
\limsup_{\varepsilon\to 0^+}c_\varepsilon\le\inf_{s\in\mathbb{R}^3}C(s).
$$

□

Lemma 3.4. *$\{v_\varepsilon\}$ is bounded in $H^1(\mathbb{R}^3)$.*

Proof. Since v_ε is minimizer of I_ε on $\mathcal{N}_\varepsilon$,

$$
\begin{aligned}
c_\varepsilon=&I_\varepsilon(v_\varepsilon)-\frac{1}{q+1}\langle I'_\varepsilon(v_\varepsilon),v_\varepsilon\rangle\\
=&\Big(\frac{1}{2}-\frac{1}{q+1}\Big)\int_{\mathbb{R}^3}(|\nabla v_\varepsilon|^2+V_\varepsilon(x)v_\varepsilon^2)dx\\
&+\Big(\frac{1}{4}-\frac{1}{q+1}\Big)\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx\\
&+\Big(\frac{1}{q+1}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^3}P_\varepsilon(x)|v_\varepsilon|^{p+1}dx.
\end{aligned}
$$

In view of Lemma 3.3, we obtain the desired conclusion. □

Lemma 3.5. *There exists $\varepsilon^*>0$ such that, for all $\varepsilon\in(0,\varepsilon^*)$, there exists $y_\varepsilon\in\mathbb{R}^3$ and R, $C>0$ such that*

$$
\int_{B_R(y_\varepsilon)}v_\varepsilon^2dx\ge C.
$$

Proof. We assume, for the sake of contradiction, that there is a sequence $\varepsilon_k\to 0$ as $k\to\infty$, such that for all $R>0$

$$
\lim_{k\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}v_{\varepsilon_k}^2dx=0.
$$

By the vanishing lemma, we have $v_{\varepsilon_k}\to 0$ in $L^s(\mathbb{R}^3)$ for $s\in(2,6)$. Since

$$
\begin{aligned}
c_{\varepsilon_k}=&I_{\varepsilon_k}(v_{\varepsilon_k})\\
=&I_{\varepsilon_k}(v_{\varepsilon_k})-\frac{1}{2}\langle I'_{\varepsilon_k}(v_{\varepsilon_k}),v_{\varepsilon_k}\rangle\\
=&-\frac{1}{2}\int_{\mathbb{R}^3}\phi_{v_{\varepsilon_k}}v_{\varepsilon_k}^2dx+\big(\frac{1}{2}-\frac{1}{p+1}\big)\int_{\mathbb{R}^3}P_{\varepsilon_k}(x)|v_{\varepsilon_k}|^{p+1}dx
\end{aligned}
$$

$$+\Big(\frac{1}{2}-\frac{1}{q+1}\Big)\int_{\mathbb{R}^3} Q_{\varepsilon_k}(x)|v_{\varepsilon_k}|^{q+1}dx,$$

combining this with the result (1) of Lemma 3.3, it follows that

$$0<\liminf_{k\to 0} c_{\varepsilon_k}=-\frac{1}{2}\liminf_{k\to 0}\int_{\mathbb{R}^3}\phi_{v_{\varepsilon_k}}v_{\varepsilon_k}^2dx\le 0.$$

This is a contradiction. □

For simplicity, we denote

$$w_\varepsilon(x):=v_\varepsilon(x+y_\varepsilon)=u_\varepsilon(\varepsilon x+\varepsilon y_\varepsilon), \tag{3.5}$$

so w_ε is a positive ground state solution to

$$-\Delta w_\varepsilon+V(\varepsilon x+\varepsilon y_\varepsilon)w_\varepsilon+\phi_{w_\varepsilon}w_\varepsilon=P(\varepsilon x+\varepsilon y_\varepsilon)w_\varepsilon^p+Q(\varepsilon x+\varepsilon y_\varepsilon)w_\varepsilon^q.$$

By Lemmas 3.4 and 3.5, we obtain $w_\varepsilon\rightharpoonup w_0$ in $H^1(\mathbb{R}^3)$, with $w_0\ge 0$, $w_0\not\equiv 0$. Furthermore, we will give next lemma which is used in Lemma 3.7.

Lemma 3.6. *$w_\varepsilon\to w_0$ in $L^{p+1}(\mathbb{R}^3)$.*

Proof. We define

$$\begin{aligned}\mu_\varepsilon(\mathbb{R}^3)=&\Big(\frac{1}{2}-\frac{1}{q+1}\Big)\int_{\mathbb{R}^3}(|\nabla v_\varepsilon|^2+V(\varepsilon x)v_\varepsilon^2)dx\\&+\Big(\frac{1}{4}-\frac{2}{q+1}\Big)\int_{\mathbb{R}^3}\phi_{v_\varepsilon}v_\varepsilon^2dx\\&+\Big(\frac{1}{q+1}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^3}P(\varepsilon x)|v_\varepsilon|^{p+1}dx.\end{aligned}$$

By following the same methods in [29, Lemma 3.3], we can exclude vanishing and dichotomy. By compactness conditions, for any $\eta>0$, there exists $\rho>0$ such that

$$\int_{B_\rho^c(0)}(|\nabla w_\varepsilon|^2+w_\varepsilon^2)dx<\eta.$$

By this and Sobolev embedding theorem, we complete the proof. □

Lemma 3.7. *$\{\varepsilon y_\varepsilon\}$ is bounded.*

Proof. Suppose to the contrary that, if necessary going to a subsequence, $\varepsilon y_\varepsilon\to\infty$. Denote

$$c^\infty=c(V_\infty,P_\infty,Q_\infty).$$

Since our assumptions potentials on V, P and Q, we have $c^\infty>\inf_{s\in\mathbb{R}^3}C(s)$. Using Lemma 3.2, we can choose $\epsilon>0$ small such that

$$C^\epsilon:=C(V_\infty-\epsilon,P_\infty+\epsilon,Q_\infty+\epsilon)>\inf_{s\in\mathbb{R}^3}C(s). \tag{3.6}$$

Let N^ϵ be the corresponding Nehari manifold for the C^ϵ. Thus there is a $\theta>0$ such that $\theta w_0\in N^\epsilon$. Based on [24, Lemma A.1], in view of Lemma 3.6, we can use Lebesgue dominated convergence theorem and Fatou's Lemma to obtain

$$\begin{aligned}C^\epsilon\le&\frac{\theta^2}{2}\int_{\mathbb{R}^3}|\nabla w_0|^2+(V_\infty-\epsilon)w_0^2dx+\frac{\theta^4}{4}\int_{\mathbb{R}^3}\phi_{w_0}w_0^2dx\\&-\frac{\theta^{p+1}}{p+1}\int_{\mathbb{R}^3}(P_\infty+\epsilon)|w_0|^{p+1}dx-\frac{\theta^{q+1}}{q+1}\int_{\mathbb{R}^3}(Q_\infty+\epsilon)|w_0|^{q+1}dx\end{aligned}$$

$$
\begin{aligned}
&\le \liminf_{\varepsilon\to 0^+}\Big\{\frac{\theta^2}{2}\int_{\mathbb{R}^3}[|\nabla w_\varepsilon|^2+V(\varepsilon x+\varepsilon y_\varepsilon)w_\varepsilon^2]dx\\
&\quad+\frac{\theta^4}{4}\int_{\mathbb{R}^3}\phi_{w_\varepsilon}w_\varepsilon^2dx-\frac{\theta^{p+1}}{p+1}\int_{\mathbb{R}^3}P(\varepsilon x+\varepsilon y_\varepsilon)|w_\varepsilon|^{p+1}dx\\
&\quad-\frac{\theta^{q+1}}{q+1}\int_{\mathbb{R}^3}Q(\varepsilon x+\varepsilon y_\varepsilon)|w_\varepsilon|^{q+1}dx\Big\}\\
&=\liminf_{\varepsilon\to 0^+}\Big(\frac{\theta^2}{2}(I_{\varepsilon,1}+I_{\varepsilon,2})+\frac{\theta^4}{4}I_{\varepsilon,3}-\frac{\theta^{p+1}}{p+1}I_{\varepsilon,4}-\frac{\theta^{q+1}}{q+1}I_{\varepsilon,5}\Big)\\
&:=\liminf_{\varepsilon\to 0^+}g_\varepsilon(\theta).
\end{aligned}
$$

Clearly, $g_\varepsilon(\theta)<g_\varepsilon(1)$ for $\theta\in(0.1)$. Therefore, combining this with Lemma 3.4, it holds

$$
C^\epsilon\le\liminf_{\varepsilon_k\to 0^+}c_{\varepsilon_k}\le\inf_{s\in\mathbb{R}^3}C(s).
$$

It contradicts (3.6). □

Without loss of generality, we assume that $\varepsilon y_\varepsilon\to x_0$ as $\varepsilon\to 0^+$. By a Moser iteration argument, we see that $w_\varepsilon\to w_0$ in $L^s(\mathbb{R}^3)$ for $2\le s\le\infty$, (see also [12, Proposition 3.3]. It follows that $w_\varepsilon\to w_0$ in $C^{1,\beta}_{\rm loc}(\mathbb{R}^3)$ for some $\beta>0$. Consequently, by Schauder estimate (see [11]), $w_\varepsilon\to w_0$ in $C^{2,\beta}_{\rm loc}(\mathbb{R}^3)$, and hence

$$
-\Delta w_0+V(x_0)w_0+\phi_{w_0}w_0=P(x_0)w_0^p+Q(x_0)w_0^q,\quad x\in\mathbb{R}^3.
$$

Denote by E the corresponding energy functional.

Lemma 3.8. *$C(x_0)=\inf_{s\in\mathbb{R}^3}C(s)$. Furthermore, $w_\varepsilon\to w_0$ in $H^1(\mathbb{R}^3)$.*

Proof. It is similar to that in [29, Lemma 3.5]. For readers convenience, we sketch the proof. Using Fatou's lemma and (3.3), we obtain

$$
\begin{aligned}
\inf_{s\in\mathbb{R}^3}C(s)&\le C(x_0)\le E(w_0)-\frac{1}{q+1}\langle E'(w_0),w_0\rangle\\
&\le\liminf_{\varepsilon\to 0^+}\Big\{\Big(\frac12-\frac{1}{q+1}\Big)\int_{\mathbb{R}^3}[|\nabla w_\varepsilon|^2+V(\varepsilon x+\varepsilon y_\varepsilon)w_\varepsilon^2]dx\\
&\quad+\Big(\frac14-\frac{2}{q+1}\Big)\int_{\mathbb{R}^3}\phi_{w_\varepsilon}w_\varepsilon^2dx\\
&\quad+\Big(\frac{1}{q+1}-\frac{1}{p+1}\Big)\int_{\mathbb{R}^3}P(\varepsilon x+\varepsilon y_\varepsilon)w_\varepsilon^{p+1}dx\Big\}\\
&=\liminf_{\varepsilon\to 0^+}c_\varepsilon\le\inf_{s\in\mathbb{R}^3}C(s).
\end{aligned}
$$

It follows from the above inequalities and Lemma 3.6 that

$$
\int_{\mathbb{R}^3}(|\nabla w_\varepsilon|^2+V(\varepsilon x+\varepsilon y_\varepsilon)w_\varepsilon^2)dx\to\int_{\mathbb{R}^3}(|\nabla w_0|^2+V(x_0)w_0^2)dx \text{ as } \varepsilon\to 0^+.
$$

Which yields $w_\varepsilon\to w_0$ in $H^1(\mathbb{R}^3)$. □

As in [12, Lemma 3.8 and 3.9], the following lemmas hold.

Lemma 3.9. *For all $\varepsilon\in(0,\varepsilon^*)$, w_ε possesses at most one maximum point p_ε. Moreover $\lim_{|x|\to\infty}w_\varepsilon(x)=0$ uniformly on $\varepsilon\in(0,\varepsilon^*)$.*

Lemma 3.10. *There exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$u_\varepsilon(x) \le C_1 e^{-C_2|\frac{x-x_\varepsilon}{\varepsilon}|} \quad \text{for all } x \in \mathbb{R}^3.$$

Proof. The proof is similar that in [12, Lemma 3.11], but in our case potential Q is allowed to change sign or negative, we check it step by step. Denote

$$f_\varepsilon(x, w_\varepsilon) := P(\varepsilon x + \varepsilon y_\varepsilon) w_\varepsilon^p + Q(\varepsilon x + \varepsilon y_\varepsilon) w_\varepsilon^q.$$

Obviously,

$$\lim_{w_\varepsilon \to 0} = \frac{f_\varepsilon(x, w_\varepsilon)}{w_\varepsilon} = 0. \tag{3.7}$$

Thus by Lemma 3.9, there is $R_1 > 0$, independent of $\varepsilon \in (0, \varepsilon^*)$, such that

$$f_\varepsilon(x, w_\varepsilon) \le \frac{1}{4} V_0 w_\varepsilon.$$

Fix $\psi(x) = C_1 e^{-C_2|x|}$ with $C_2^2 = \frac{V_0}{2}$ and $C_1 e^{-C_2 R_1} \ge w_\varepsilon(x)$ for all $|x| = R_1$. It is obtained that

$$\Delta\psi \le C_2^2 \psi \le \frac{V_0}{2}\psi. \tag{3.8}$$

Hence

$$-\Delta w_\varepsilon + \frac{3}{4} V_0 w_\varepsilon \le \frac{1}{4} V_0 w_\varepsilon. \tag{3.9}$$

Define $\psi_\varepsilon = \psi - w_\varepsilon$, using (3.8)) and (3.9), we obtain

$$\begin{gathered} -\Delta\psi_\varepsilon + \frac{V_0}{2}\psi_\varepsilon \ge 0, \quad \text{if } |x| \ge R_1, \\ \psi_\varepsilon \ge 0, \quad \text{if } |x| = R_1, \\ \lim_{|x| \to \infty} \psi_\varepsilon(x) = 0. \end{gathered}$$

The maximum principle implies that $\psi_\varepsilon \ge 0$ in $|x| \ge R_1$ and we conclude that

$$w_\varepsilon(x) \le C_1 e^{-C_2|x|}, \quad \forall |x| \ge R_1, \text{ and all } \varepsilon \in (0, \varepsilon^*).$$

By Lemma 3.9, we have w_ε has a unique maximum point p_ε. Then v_ε has a unique maximum point $p_\varepsilon + y_\varepsilon$ and u_ε has a unique maximum point $x_\varepsilon = \varepsilon(p_\varepsilon + y_\varepsilon)$. Thus, we have

$$u_\varepsilon(x) = v_\varepsilon(\frac{x}{\varepsilon}) = w_\varepsilon(\varepsilon^{-1}x - y_\varepsilon) = w_\varepsilon(\varepsilon^{-1}x - \varepsilon^{-1}x_\varepsilon + p_\varepsilon) \le C_1 e^{-C_2|\frac{x-x_\varepsilon}{\varepsilon}|}.$$

□

Proof of Theorem 1.1. The existence result (I) of Theorem 1.1 follows directly from Section 2. The concentration results (1) and (2) of Theorem 1.1 follow from Lemma 3.8, the result (3) of Theorem 1.1 follows from Lemma 3.10. □

Remark 3.11. In our paper, we assume that our potentials V, P and Q are C^1 functions to ensure that $C(s)$ is continuous. It follows that $\inf_{s\in\mathbb{R}^3} C(s)$ can be achieved. Under our assumptions, the result shows that $C(s)$ is Lipschitz continuous (see Lemma 3.2), so we gesture V, P, Q can be weaker than C^1 functions.

4. Appendix

As mentioned in Remark 1.2, introducing the re-scaled transformation $x \mapsto \varepsilon x$, the system

$$\begin{gathered} -\varepsilon^2 \Delta u + V(x)u + K(x)\phi u = P(x)|u|^{p-1}u + Q(x)|u|^{q-1}u, \quad x \in \mathbb{R}^3, \\ -\varepsilon^2 \Delta \phi = K(x)u^2, \quad x \in \mathbb{R}^3, \end{gathered}$$

can be written as

$$\begin{gathered} -\Delta u + V(\varepsilon x)u + K(\varepsilon x)\phi u = P(\varepsilon x)|u|^{p-1}u + Q(\varepsilon x)|u|^{q-1}u, \quad x \in \mathbb{R}^3, \\ -\Delta \phi = K(\varepsilon x)u^2, \quad x \in \mathbb{R}^3. \end{gathered}$$

This system is different from (3.1) since in the second equation, ϕ is dependent on ε. In fact, we can write the corresponding energy functional as

$$\begin{aligned} I_\varepsilon(u) =& \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2)dx + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(\varepsilon x) \frac{K(\varepsilon y)u^2(y)}{|x-y|} u^2(x)\, dy\, dx \\ & - \frac{1}{p+1} \int_{\mathbb{R}^3} P(\varepsilon x)|u|^{p+1}dx - \frac{1}{q+1} \int_{\mathbb{R}^3} Q(\varepsilon x)|u|^{q+1}dx, u \in H^1(\mathbb{R}^3). \end{aligned}$$

For concentration, it is complicated since it appears terms $K(\varepsilon x)$ and $K(\varepsilon y)$.

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