Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 79, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

PERIOD FUNCTIONS AND CRITICAL PERIODS OF PIECEWISE LINEAR SYSTEM

SHAOQING WANG, JIAZHONG YANG

Communicated by Shaobo Gan

Abstract. In this article, we study a rich and complex phenomena of planar piecewise linear systems having two domains of which the separation curves are not straight. We show that for positive integers n and m and non-negative integers $n_1, n_2, \dots, n_m$, there exist two types of piecewise linear systems: one has a period annulus possessing exactly n critical periods; the other has m period annuli possessing exactly $n_1, n_2, \dots, n_m$ critical periods. Moreover, an algebraic curve can be chosen as the separation line in system of the first type.

1. Introduction

One of the important problems in qualitative theory of planar differential equations is studying centers of systems. Here an isolated singularity O is called a *center* if there is a neighborhood U of O where all trajectories in $U \setminus O$ are closed. By a *period annulus*, we mean the largest such neighborhood. A period annulus Ω of a center O can be parameterized by an analytic curve Γ which is transversal to the orbits in Ω and we denote by $T(\xi)$, called *period function*, the minimal positive period of the orbit passing through $\xi \in \Gamma$. In the literature, value $\xi_0 \in \Gamma$ is said to be a *critical period* if $T'(\xi_0) = 0$. It is not difficult to verify that the number of critical periods is independent of the choice of its parametrization. In particular, O is called an *isochronous center* if $T(\xi)$ is a constant function.

Related to the center and period function, many efforts are made to determine the uniform maximal number $H(n)$ of critical periods, analogue to Hilbert's sixteenth problem which asks for the uniform upper bound of the number of limit cycles (see [9] and reference therein), in all polynomial systems $\dot{x} = P_n(x, y)$, $\dot{y} = Q_n(x, y)$, where $P_n(x, y)$, $Q_n(x, y)$ are two polynomials of degree at most n. Meanwhile, many interesting results have been obtained. For example, the authors in [1] proved that for any given polynomial system, there are at most a finite number of critical periods in a period annulus contained in a compact region. And the lower bound of $H(n)$, found in [2] to be linear with respect to n, was improved later in [6] to be a quadratic function of n.

2010 *Mathematics Subject Classification.* 34C05, 34C25, 34C27.
Key words and phrases. Piecewise linear system; period annulus; period function; critical period.

Submitted June 22, 2020. Published July 27, 2020.

Attention is also paid to other aspects of center and period function, such as to isochronicity which can be seen in [3, 8, 10] and reference therein, to bifurcation of critical periods (see for instance [7]), as well as to application (see [5, 11]).

In general, more complex phenomena could appear in planar polynomial systems as their degrees increase, meanwhile a planar piecewise linear system with a complicated separation line can also present rich dynamic behaviors, see for instance [13] about number of limit cycles in a piecewise linear system of which the separation line is not a straight line (see its definition below). By the way, piecewise smooth perturbations of an isochronous center also draw some attention, readers are referred to [12].

Although there are many attractive results about center and critical periods in planar polynomial systems, to the best of our knowledge, only a few related conclusions have been drawn in planar piecewise systems, especially in planar piecewise linear systems. Thus in this paper, the authors focus on the latter. More precisely, this paper aims at the number of critical periods of the piecewise linear systems.

Here, for convenience, we give some notation of piecewise systems, which can be traced back to Filippov [4]. If the whole plane $\mathbb{R}^2$ is divided into two domains I_1 and I_2 by a continuous line γ called a *separation line* in this paper, and smooth systems X_1 and X_2 are set in I_1 and I_2 respectively, then a *piecewise smooth system* $X = (X_1, X_2)$ comes into being. In particular, $X = (X_1, X_2)$ is called a *piecewise linear system* provided that X_1, X_2 are both linear systems.

As for the points on the separation line γ, they can be classified into four types: *sliding points*;*escaping points*;*sewing points*, and *tangent points*. By a sliding point (resp, escaping point), we mean a point $p \in \gamma$ such that $X_1(p)$ and $X_2(p)$ point inward (resp, outward) γ. A point $p \in \gamma$ is said to be a sewing point if $X_1(p)$ and $X_2(p)$ are transversal to γ and point to the same direction. And a tangent point $p \in \gamma$ means either $X_1(p)$ or $X_2(p)$ is tangent to γ.

In a piecewise system, a *period annulus*, similar to that in an analytic system, is also a simply connected neighbourhood of a point O, called a Σ-*center*, in which all orbits are closed around O and intersect the separation line only at sewing points. Similarly, period function and critical period can also be defined in a piecewise system. With these notation, three main results of this paper can be stated as follows:

Theorem 1.1. *For any integer $n \geq 1$, there exists a piecewise linear system with only one period annulus which has exactly n critical periods. Moreover, an algebraic curve can be chosen as the separation line.*

For the case of more than one annuli, we have another two results:

Theorem 1.2. *For any integer $m \geq 2$, there is a piecewise linear system with exactly m period annuli which have no critical period, namely, the period functions are all monotonic.*

Theorem 1.3. *For any integer $m \geq 2$ and any m-tuple $(n_1, n_2, \dots, n_m)$ of non-negative integers, there exists a piecewise linear system with exactly m period annuli Ω_1, Ω_2, $\dots$, Ω_m, which have exactly n_1, n_2, $\dots$, n_m critical periods, respectively.*

This article is organized as follows. In section 2, we give the proof of Theorem 1.1 while the construction processes of the piecewise linear systems in Theorem 1.2 and 1.3 are given in section 3 and 4 respectively.

2. Proof of Theorem 1.1

Let f be a function defined on an interval $I \subset \mathbb{R}$. We denote the graph of f by $\mathrm{Gr}(f) = \{(x, f(x)) : x \in I\}$. In this section, f is an even polynomial on $\mathbb{R}$,

$$f(x) = c\sum_{i=0}^{n} \frac{a_i}{2i+1} x^{2i+2}, \tag{2.1}$$

where the coefficients $a_1, a_2, \dots, a_n$ are chosen such that

$$\sum_{i=0}^{n} a_i y^i = (y-1^2)(y-2^2)\dots(y-n^2), \quad \forall y \in \mathbb{R}, \tag{2.2}$$

and $c > 0$ is a constant number to be determined below. Then the whole plane is parted by $\mathrm{Gr}(f)$ into two domains: upper domain $I = \{(x,y) : y > f(x), x \in \mathbb{R}\}$ and lower domain $II = \{(x,y) : y < f(x), x \in \mathbb{R}\}$. See Figure 1 for its diagram.

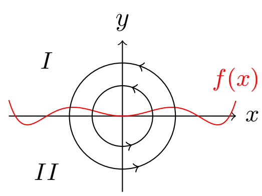

Figure 1. Diagram of the piecewise linear system in Section 2.

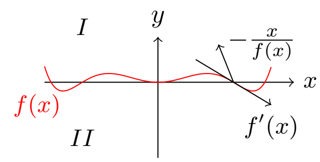

Figure 2.

In domains I and II, we set linear systems of center type

$$X_1 : \begin{cases} \dot{x} = -ay, \\ \dot{y} = ax, \end{cases}$$

and

$$X_2 : \begin{cases} \dot{x} = -by, \\ \dot{y} = bx, \end{cases}$$

where a and b are two distinct positive numbers. Thus a piecewise linear system $X = (X_1, X_2)$ of center-center type has been obtained. In the systems X_1 and X_2, the trajectories are all circular arcs centered at $(0,0)$, consequently, the orbits of X are also circles (see Figure 1).

Below we show that there is exactly a period annulus in X. As

$$f'(x) = c\sum_{i=0}^{n} \frac{2i+2}{2i+1} a_i x^{2i+1},$$

and

$$-\frac{x}{f(x)} = \begin{cases} -\frac{x}{c\sum_{i=0}^{n} \frac{a_i}{2i+1} x^{2i+2}}, & \text{if } f(x) \neq 0, \\ \infty, & \text{if } f(x) = 0, \end{cases}$$

represent the slopes of the graph of $f(x)$ and the orbit of X_1 (or X_2) at $(x, f(x))$ respectively (see Figure 2 for its diagram), then $c > 0$ can be chosen sufficiently small such that

$$|f'(x)| < |\frac{x}{f(x)}|, \quad |x| \le F_0,$$

where F_0 is the largest positive real root of $f'(x)$. Since when $x > F_0$, $f'(x) > 0 > -x/f(x)$, there is a period annulus in X containing the whole plane with period function $T(x)$.

The following is some information about $T(x)$ and its derivatives.

Lemma 2.1. *Let Y be a linear system having a center at $(0,0)$,*

$$Y : \begin{cases} \dot{x} = -ay, \\ \dot{y} = ax, \end{cases}$$

where $a > 0$. Then the time between $A(x,y)$ $(x > 0)$ and $B(0, \sqrt{x^2+y^2})$ which are in the same orbit of Y (see Figure 3) is $\frac{1}{a}(\frac{\pi}{2} - \arctan\frac{y}{x})$. Moreover, the time between $C(0, -\sqrt{x^2+y^2})$ and $A(x,y)$ $(x > 0)$ is $\frac{1}{a}(\frac{\pi}{2} + \arctan\frac{y}{x})$.

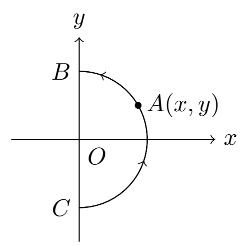

FIGURE 3. Diagram for Lemma 2.1

Proof. The linear system Y can be transformed, by direct computation, into the system in polar coordinates,

$$\dot{r} = 0,$$
$$\dot{\theta} = a,$$

where $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$.

Since $A(x,y)$ and $B(0, \sqrt{x^2+y^2})$ correspond to $(\sqrt{x^2+y^2}, \arctan(y/x))$ and $(\sqrt{x^2+y^2}, \frac{\pi}{2})$ respectively in polar coordinates, thus the time from $A(x,y)$ $(x > 0)$ to $B(0, \sqrt{x^2+y^2})$ is $\frac{1}{a}(\frac{\pi}{2} - \arctan\frac{y}{x})$.

Similarly, the second result of this lemma holds. $\square$

Lemma 2.2. *The sign of $T'(x)$ depends on $(f(x)/x)'$.*

Proof. Based on the trajectories of X_1 and X_2, the closed orbits of X have to intersect the graph of $f(x)$. By Lemma 2.1 and the even property of $f(x)$, we have that

$$T(x) = 2\Big(\frac{1}{a}\big(\frac{\pi}{2} - \arctan\frac{f(x)}{x}\big) + \frac{1}{b}\big(\frac{\pi}{2} + \arctan\frac{f(x)}{x}\big)\Big),$$

then by direct computation,

$$T'(x) = \big(\frac{2}{b} - \frac{2}{a}\big)\frac{(\frac{f(x)}{x})'}{1 + (\frac{f(x)}{x})^2}.$$

Thus the conclusion follows. $\square$

Now we prove Theorem 1.1. The expression (2.1) implies that

$$
\begin{aligned}
(\frac{f(x)}{x})' &= \Big(c\sum_{i=0}^{n}\frac{a_i}{2i+1}x^{2i+1}\Big)', \\
&= c\sum_{i=0}^{n} a_i x^{2i}, \\
&= c(x^2-1^2)(x^2-2^2)\dots(x^2-n^2),
\end{aligned} \tag{2.3}
$$

which shows that $\big(\frac{f(x)}{x}\big)'$ has n distinct positive zeros that are all simple: $1,\dots,n$.

We calculate the second derivative of $T(x)$ as follows:

$$
T''(x) = (\frac{2}{b}-\frac{2}{a})\Big(\frac{(\frac{f(x)}{x})''}{1+(\frac{f(x)}{x})^2} - \frac{2(\frac{f(x)}{x})(\frac{f(x)}{x})'^2}{(1+(\frac{f(x)}{x})^2)^2}\Big). \tag{2.4}
$$

Since $(\frac{f(x)}{x})' = 0$, while $(\frac{f(x)}{x})'' \neq 0$, at $x = 1,2,\dots,n$, combining Lemma 2.2, (2.3) and (2.4), we obtain that $1,\dots,n$ are exactly n zeros of $T'(x)$ which are simple. Namely, the annulus in X possesses exactly n critical periods. Consequently, Theorem 1.1 has been proved.

3. Proof of Theorem 1.2

In this section, we shall first prove Theorem 1.2 in the case of $m = 3$ since all essential ideas and methods applied in the general case can be highlighted in this special case. Here, we use another kind of piecewise linear system of focus-center type with a more complicated separation line, which is the red line given in Figure 4.

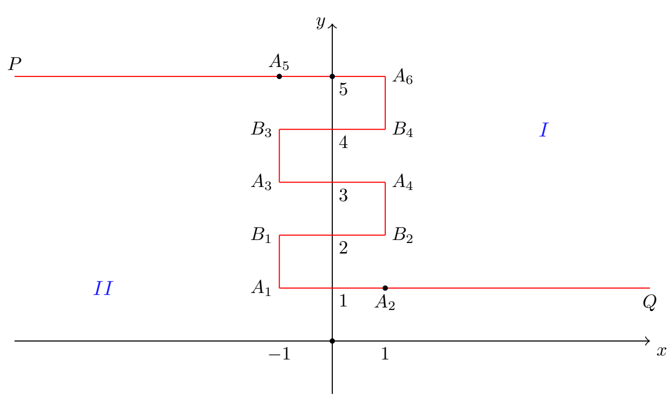

Figure 4. Separation line in the proof of Theorem 1.2

Specifically, in Figure 4, all lines are straight lines which are parallel to the two axes. And all points A_i, B_i $(i = 1,\dots,5)$ are at integer coordinates. Meanwhile $\overline{A_6A_5P}$ and $\overline{A_1A_2Q}$ are two rays. Now the polygonal line (the red line in Figure 4) $\gamma := \overline{PA_5A_6} \cup \overline{A_6B_4} \cup \overline{B_3B_4} \cup \overline{B_3A_3} \cup \overline{A_3A_4} \cup \overline{A_4B_2} \cup \overline{B_1B_2} \cup \overline{B_1A_1} \cup \overline{A_1A_2Q}$, i.e.

$$
\gamma := \overline{PA_5A_6B_4B_3A_3A_4B_2B_1A_1A_2Q}
$$

forming a separation line separates the whole plane into two parts: domain I and II, which can be seen in Figure 4.

In domain I, we set a linear system X_1 of center type:

$$X_1 : \begin{cases} \dot{x} = -y, \\ \dot{y} = x. \end{cases} \tag{3.1}$$

Meanwhile, a linear system X_2 of saddle type, of which the orbits are hyperbolas, is put in part II:

$$X_2 : \begin{cases} \dot{x} = y, \\ \dot{y} = x. \end{cases} \tag{3.2}$$

Then we obtain a piecewise linear system $X = (X_1, X_2)$, of which some trajectories are depicted in Figure 5 (blue trajectories are in the domain I, and black ones in domain II).

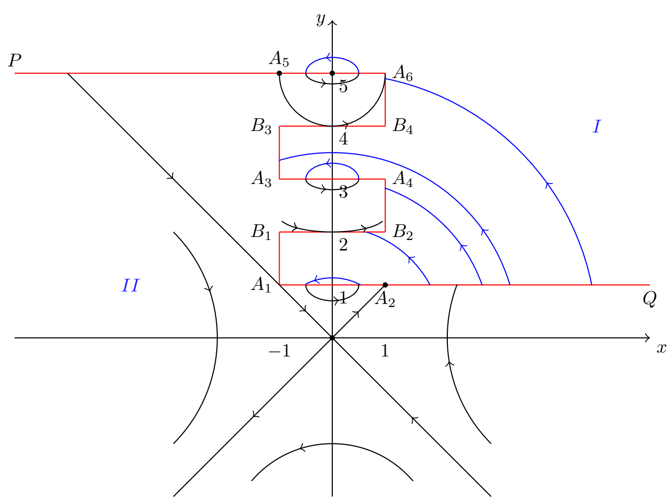

FIGURE 5. Some orbits of piecewise linear system X in Section 3

In more detail, the trajectories of $X = (X_1, X_2)$ in Figure 5 show that A_6B_4, B_3B_4, B_3A_3, A_4B_2, B_1B_2 and B_1A_1 all consist of sliding or escaping points (see the dashed lines in Figure 6), consequently, the closed orbits can only intersect three lines: PA_5A_6, A_3A_4 and A_1A_2Q. By symmetry of A_1A_2, A_3A_4 and A_5A_6 with respect to y-axis, there exist only three period annuli Ω_1, Ω_2 and Ω_3 containing A_1A_2, A_3A_4 and A_5A_6 respectively (for details see Figure 6). Denote by $T_i(x)$ the period function of period annulus Ω_i for each $i = 1, 2, 3$.

The remaining work is to show that $T_1'(x)$, $T_2'(x)$ and $T_3'(x)$ have no zero in interval $(0, 1)$. Without loss of generality, it is suffice to consider $T_1'(x)$. The first task is to give the expression of $T_1(x)$.

For convenience, let these three straight lines A_1A_2, A_3A_4 and A_5A_6 be graphs of three functions $f_1(x) \equiv 1$, $f_2(x) \equiv 3$ and $f_3(x) \equiv 5$, $x \in [-1, 1]$, respectively.

A closed orbit in the period annulus Ω_1 passing through $(x, f_1(x))$ $(0 < x < 1)$ consists of two trajectories L_1 and L_2 (see Figure 7): L_1 is located in the domain I,

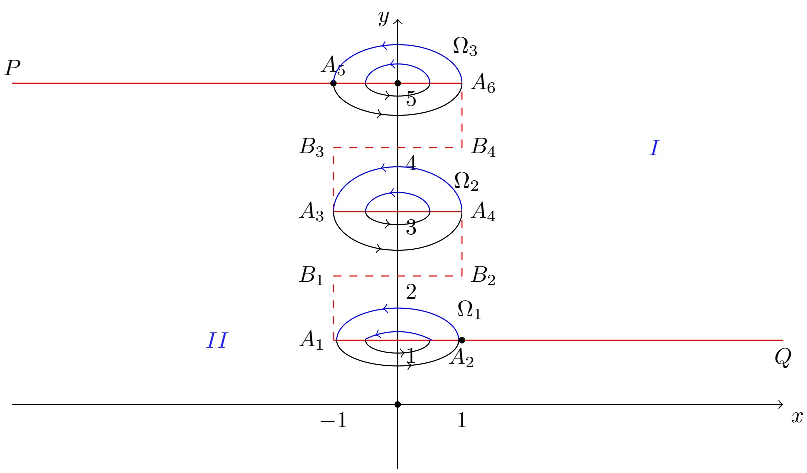

FIGURE 6. Three period annuli in Section 3

from $(x, f_1(x))$ to $(-x, f_1(x))$ and conversely, L_2 is from $(-x, f_1(x))$ to $(x, f_1(x))$, in domain II.

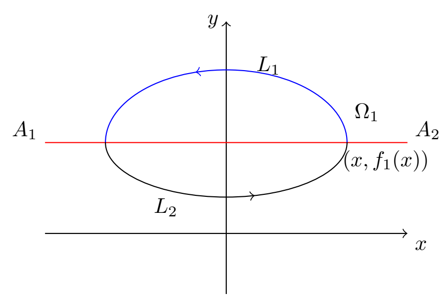

FIGURE 7. A closed orbit in Ω_1

By Lemma 2.2, the time in trajectory L_1 is

$$t_1(x) = 2\Big(\frac{\pi}{2} - \arctan\frac{f_1(x)}{x}\Big), \quad x \in (0,1). \tag{3.3}$$

As for the time $t_2(x)$ in L_2, we have the following lemma.

Lemma 3.1.

$$t_2(x) = \ln\frac{f_1(x)+x}{f_1(x)-x}, \quad x \in (0,1). \tag{3.4}$$

Proof. By the orthogonal transformation

$$\begin{aligned} u &= \frac{y-x}{\sqrt{2}},\\ v &= \frac{x+y}{\sqrt{2}}, \end{aligned} \tag{3.5}$$

system X_2 is changed into the following standard linear system of saddle type

$$\begin{aligned}\dot{u} &= -u,\\ \dot{v} &= v,\end{aligned} \tag{3.6}$$

and time $t_2(x)$ can be obtained as follows:

$$\int_{\frac{f_1(x)+x}{\sqrt{2}}}^{\frac{f_1(x)-x}{\sqrt{2}}} dt = -\int_{\frac{f_1(x)+x}{\sqrt{2}}}^{\frac{f_1(x)-x}{\sqrt{2}}} \frac{du}{u} = \ln\frac{f_1(x)+x}{f_1(x)-x},$$

Thus Lemma 3.1 holds. □

From expressions (3.3) and (3.4), we obtain

$$T_1(x) = t_1(x) + t_2(x) = 2\big(\frac{\pi}{2} - \arctan\frac{f_1(x)}{x}\big) + \ln\frac{f_1(x)+x}{f_1(x)-x}, \tag{3.7}$$

thus $T_1'(x)$ can be obtained by direct computations,

$$\begin{aligned}T_1'(x) &= -2\frac{(\frac{f_1(x)}{x})'}{1+(\frac{f_1(x)}{x})^2} + \frac{f_1(x)-x}{f_1(x)+x}\,\frac{(\frac{f_1(x)}{x})'(\frac{f_1(x)}{x}-1) - (\frac{f_1(x)}{x})'(\frac{f_1(x)}{x}+1)}{(\frac{f_1(x)}{x}-1)^2}\\ &= -\Big(\frac{2}{(\frac{f_1(x)}{x})^2+1} + \frac{2}{(\frac{f_1(x)}{x})^2-1}\Big)\Big(\frac{f_1(x)}{x}\Big)',\end{aligned}$$

From simplicity above conclusion can be summarized in the following statement.

Lemma 3.2. *For $i = 1, 2, 3$ and $x \in (0, 1)$, $T_i'(x)$ has the same sign as $-\big(\frac{f_i(x)}{x}\big)'$.*

Since $f_1(x) \equiv 1$, $f_2(x) \equiv 3$ and $f_3(x) \equiv 5$, $x \in (-1, 1)$, then Theorem 1.2 in the case of $m = 3$, as a straightforward corollary of Lemma 3.2, has been proved.

For the general case, the same method as above can be used to construct a separation line, as well as a piecewise linear system which has exactly m period annuli possessing no critical period.

4. Proof of Theorem 1.3

Without loss of generality, we assume that $m = 3$. For fixed triple (n_1, n_2, n_3) of non-negative integers, below we established a piecewise linear system having 3 period annuli Ω_1, Ω_2 and Ω_3 with exactly n_1, n_2 and n_3 critical periods respectively.

The method of the construction of piecewise linear system here is similar to that in the proof of Theorem 1.2, but the separation line given below is more complicated.

Here the separation line is the red curve in Figure 8. More exactly, in this figure, two straight lines D_1D_2, D_3D_4 and two rays C_5P, C_2Q are parallel to x-axis, while four lines C_1D_1, D_2C_4, C_3D_3 and D_4C_6 are straight and parallel to y-axis. The other three curves C_1C_2, C_3C_4, C_5C_6 are respectively graphs of even functions $f_1(x)$, $f_2(x)$ and $f_3(x)$ defined on different intervals, and k_1, k_2 and k_3 are three positive integers. Here

$$f_i(x) = \begin{cases} k_i + xg_i(x), & \text{if } n_i \neq 0,\\ k_i, & \text{if } n_i = 0,\end{cases} \quad i = 1, 2, 3, \tag{4.1}$$

where

$$g_i(x) = k_i\int_0^x \frac{1}{t^2}\big(1 - \frac{\cos(k_i^8t^4)}{1+\frac{t^4}{2}}\big)dt, \quad x \in \Big[-\frac{\sqrt[4]{n_i\pi}}{k_i^2}, \frac{\sqrt[4]{n_i\pi}}{k_i^2}\Big], \tag{4.2}$$

and k_1, k_2 and k_3 are sufficiently large such that the following two properties hold:

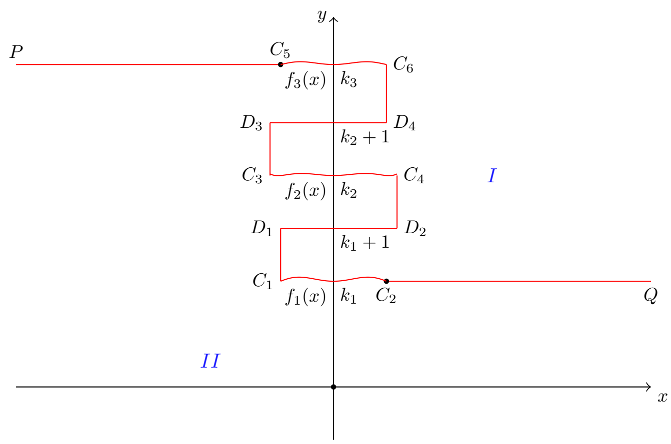

Figure 8. Separation line in the proof of Theorem 1.3

(i) $n_i\pi/k_i^4 \ll 1$;
(ii) $1/k_i^2 \ll 1/k_i$ for $i = 1, 2, 3$.

Then $|xg_i(x)| < 1$, when $x \in [-\sqrt[4]{n_i\pi}/k_i^2, \sqrt[4]{n_i\pi}/k_i^2]$. And the non-self-intersecting red curve $\gamma := \overline{PC_5} \cup \widetilde{C_5C_6} \cup \overline{C_6D_4} \cup \overline{D_4D_3} \cup \overline{D_3C_3} \cup \widetilde{C_3C_4} \cup \overline{C_4D_2} \cup \overline{D_2D_1} \cup \overline{D_1C_1} \cup \widetilde{C_1C_2} \cup \overline{C_2Q}$, i.e.

$$\gamma := \overline{PC_5C_6D_4D_3C_3C_4D_2D_1C_1C_2Q}$$

as a separation line cuts the whole plane into two domains I and II (see Figure 8).

Similar to the piecewise linear system in section 3, we also set linear systems

$$X_1 : \begin{cases} \dot{x} = -y, \\ \dot{y} = x, \end{cases}$$

and

$$X_2 : \begin{cases} \dot{x} = y, \\ \dot{y} = x, \end{cases}$$

in domains I and II respectively. Up to now, a piecewise linear system $X = (X_1, X_2)$ of center-saddle type has been constructed. We shall show in two steps that there exist three period annuli Ω_1, Ω_2, Ω_3 in X with n_1, n_2 and n_3 critical periods respectively.

• Step1: We show that there are three period annuli in piecewise system X. Similar to Figure 5 in Section 3, by the direction of trajectories of X_1 and X_2, possible closed orbits of X must pass through the graph of $f_1(x)$, $f_2(x)$ or $f_3(x)$. By Taylor expansion, we have

$$f_i(x) = k_i + \frac{2k_i}{3}x^4 + \circ(x^5), \quad |x| \ll 1,$$

thus

$$f_i'(x) = \frac{8k_i}{3}x^3 + \circ(x^4), \quad |x| \ll 1,$$

and

$$\frac{x}{f_i(x)} = \frac{x}{k_i} + \circ(x^4), \quad |x| \ll 1.$$

Thus if $|x| < \frac{\sqrt[4]{n_i\pi}}{k_i^2}$, $|f_i'(x)| < \frac{x}{f(x)}$. Meanwhile, analogue to the proof of Theorem 1.1 in section 2, $|f_i'(x)|$ (resp, $|\frac{x}{f_i(x)}|$) represents absolute value of the slope of the graph of $f_i(x)$ (resp, orbit of X_1 or X_2) at $(x, f_i(x))$ (see Figure 9 for the example of $f_1(x)$). Thus for $i = 1, 2, 3$, there must be a period annulus Ω_i containing the graph of even function $f_i(x)$ ($x \in [-\frac{\sqrt[4]{n_i\pi}}{k_i^2}, \frac{\sqrt[4]{n_i\pi}}{k_i^2}]$) with Σ-center $(0, k_i)$ (see Figure 10, the blue and black trajectories are in domains I and II respectively).

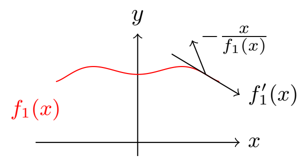

FIGURE 9.

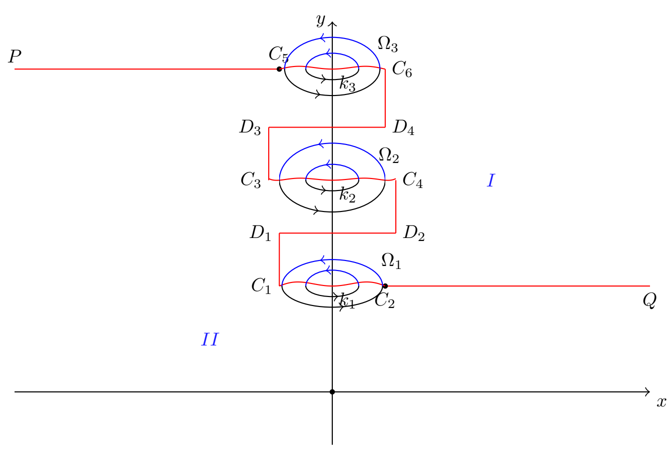

FIGURE 10. The three annuli in the proof of Theorem 1.3

- Step2. We will show that period annulus Ω_i has exactly n_i critical periods, $i = 1, 2, 3$.

We denote by $T_i(x)$ ($i = 1, 2, 3$) the period function of period annulus Ω_i. By the same method as that in the proof of Lemma 3.2, similar to formula (3.7), we have that

$$T_1(x) = 2(\frac{\pi}{2} - \arctan\frac{f_1(x)}{x}) + \ln\frac{f_1(x)+x}{f_1(x)-x}. \tag{4.3}$$

Thus

$$T_1'(x) = -\Big(\frac{2}{(\frac{f_1(x)}{x})^2+1} + \frac{2}{(\frac{f_1(x)}{x})^2-1}\Big)\Big(\frac{f_1(x)}{x}\Big)', \tag{4.4}$$

and

$$\begin{aligned} T_1''(x) =& -\Big(\frac{2}{(\frac{f_1(x)}{x})^2+1} + \frac{2}{(\frac{f_1(x)}{x})^2-1}\Big)\Big(\frac{f_1(x)}{x}\Big)'' \\ &+ \Big(\frac{4}{((\frac{f_1(x)}{x})^2+1)^2} + \frac{4}{((\frac{f_1(x)}{x})^2-1)^2}\Big)\Big(\frac{f_1(x)}{x}\Big)\Big((\frac{f_1(x)}{x})'\Big)^2. \end{aligned} \tag{4.5}$$

From (4.4), the sign of $T_1'(x)$ corresponding to Ω_1 depends on $-\big(\frac{f_1(x)}{x}\big)'$. The expression of $f_1(x) = k_1 + xg_1(x)$ gives

$$\big(\frac{f_1(x)}{x}\big)' = \big(\frac{k_1}{x}\big)' + g_1'(x) = -\frac{k_1 - x^2 g_1'(x)}{x^2}.$$

With (4.2), it follows that

$$\big(\frac{f_1(x)}{x}\big)' = -\frac{k_1 \cos(k_1^8 x^4)}{x^2(1+\frac{x^4}{2})},$$

which has only n_1 simple zeros in $(0, \sqrt[4]{n_1\pi}/k_1^2)$:

$$\frac{\sqrt[4]{\pi/2}}{k_1^2}, \frac{\sqrt[4]{3pi/2}}{k_1^2}, \dots, \frac{\sqrt[4]{(n_1-\frac{1}{2})\pi}}{k_1^2}.$$

Consequently, by (4.4) and (4.5), $T_1'(x)$ has only n_1 zeros which are all simple in $(0, \sqrt[4]{n_1\pi}/k_1^2)$. Namely, the period annulus Ω_1 has exactly n_1 critical periods. Similarly, Ω_2 (resp, Ω_3) has exactly n_2 (resp, n_3) critical periods.

Combining step 1 and step 2, we have proved Theorem 1.3 in the case of $m = 3$. For a general m, the same method can be applied. Thus Theorem 1.3 has been proved.

Acknowledgements. This work was supported by the National Science Foundation of China (Grants NSFC 11671016 and 11471027). The authors would like to express our gratitude to the referees for their helpful suggestions.

References

[1] C. Chicone, F. Dumortier; *Finiteness for critical periods of planar analytic vector fields*, Nonlinear Anal., **20(4)** (1993), 315-335.

[2] A. Cima, A. Gasull, P. R. da Silva; *On the number of crital periods for planar polynomial systems*, Nonlinear Anal., **69** (2008), 1889-1903.

[3] C. Du, Y. Liu, H. Mi; *A class of ninth degree system with four isochronous centers*, Comp. Math. Appl., **56** (2008), 2609-2620.

[4] A. F. Filippov; *Differential equations with discontinuous right-hand sides*, Nauka, Moscow, (1985), transl, :Kluwer, Dordrecht, (1988).

[5] E. Freire, A. Gasull, A. Guillamon; *First derivative of the period function with applications*, J. Diff. Equ., **204** (2004), 139-162.

[6] A. Gasull, C. Liu, J. Yang; *On the number of critical periods for the planar polinomial systems of arbitrary degree*, J. Diff. Equ., **249** (2010), 684-692.

[7] A. Gasull, Y. Zhao; *Bifurcation of critical periods from the rigid quadratic isochronous vector field*, Bull. Sci. math., **132** (2008), 292-312.

[8] X. Jarque, J. Villadelprat; *Nonexistence of isochronous centers in planar polynomial Hamiltonian system of degree four*, J. Diff. Equ., **180** (2002), 334-373.

[9] C. Li; *Abelian integrals and limit cycles*, Qual. Theory Dyn. Syst., **11** (2012), 111-128.

[10] J. Llibre, V. G. Romanovski; *Isochronicity and linearizability of planar polynomial Hamiltonian systems*, J. Diff. Equ., **259** (2015), 1649-1662.

[11] F. Rothe; *The periods of the Volterra-Lotka system*, J. Reine Angew. Math., **355** (1985), 129-138.

[12] H. Song, L. Peng, Y. Cui; *Limit cycles in piecewise smooth perturbations of a quartic isochronous center*, Electron. J. Differential Equations, 2019, no. 107 (2019), 1-23.

[13] C. Zou, J. Yang; *Piecewise linear system with a center-saddle type singularity*, J. Math. Anal. Appl., **459** (2018), 453–463.

Shaoqing Wang
School of Mathematical Sciences, Peking University, Beijing 100871, China
Email address: wangshaoqing@pku.edu.cn

Jiazhong Yang
School of Mathematical Sciences, Peking University, Beijing 100871, China
Email address: jyang@math.pku.edu.cn