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GROUND STATE SOLUTIONS FOR QUASILINEAR SCHRÖDINGER EQUATIONS WITH PERIODIC POTENTIAL

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Abstract. This article concerns the quasilinear Schrödinger equation

$$-\Delta u - u\Delta(u^2) + V(x)u = K(x)|u|^{2\cdot 2^*-2}u + g(x,u), \quad x \in \mathbb{R}^N,$$
$$u \in H^1(\mathbb{R}^N), \quad u > 0,$$

where V and K are positive, continuous and periodic functions, $g(x,u)$ is periodic in x and has subcritical growth. We use the generalized Nehari manifold approach developed by Szulkin and Weth to study the ground state solution, i.e. the nontrivial solution with least possible energy.

1. Introduction and statement of main results

In this article, we study the Schrödinger equation

$$\begin{gathered} -\Delta u - u\Delta(u^2) + V(x)u = K(x)|u|^{2\cdot 2^*-2}u + g(x,u), \quad x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad u > 0, \end{gathered} \tag{1.1}$$

where $V, K : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^+$ are continuous functions. Note that $2 \cdot 2^* = \frac{4N}{N-2}$ corresponds to the critical exponent for problem (1.1).

Recent mathematical studies have focused on existence of solutions of (1.1) with $K(x) \equiv 0$ and $g(x,s) = |s|^{p-1}s$ with $4 \le p+1 < 2\cdot 2^*$, $N \ge 3$ for example in [7, 8, 12]. The quasilinear Schrödinger equations (1.1) are derived as models of several physical phenomena, see e.g. [7, 8] for an explanation. The existence of a positive ground state solution has been proved by Poppenberg, Schmitt and Wang [12] and Liu and Wang [7] by using the constrained minimization argument. Liu and Wang [8] established the existence of a positive solution of an equation of type (1.1) for every positive μ (in front of the nonlinear term) in an Orlicz space framework via the Mountain Pass Theorem. Colin and Jeanjean [2] gave a simple and short proof of the result of [8], which did not use Orlicz spaces, but rather developed in the usual $H^1(\mathbb{R}^N)$. In [6, 18], (1.1) with ε^2 in front of Δu and $u\Delta(u^2)$, has been studied, with g of subcritical and critical growth. It was shown that there exists a positive solution u_ε which concentrated at a local minimum of V as $\varepsilon \to 0$. There was also a result about existence of infinitely many solutions for (1.1) in [5] and

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existence of multi-bump solutions was shown for a quasilinear Schrödinger equation which is more general than (1.1) in [9].

For problems with critical nonlinearities, see [10, 11, 15, 14, 18, 20] and the references therein. Moameni [10, 11] considered (1.1) for $N = 2$ and $N \geq 3$ under different condition about g and V and obtained a nonnegative solution. Shi and Chen [15] obtained a positive solution by using the Mountain Pass Theorem in combination with the concentration-compactness principle. Silva and Vieira [13, 14] considered the quasilinear asymptotically periodic equation with subcritical or critical growth, used a version of the Mountain Pass Theorem without compactness condition to get a Cerami sequence associated with the minimax level to get a nontrivial critical point. Xue in [20] took into account the same asymptotically periodic equation as [14] and got a ground state solution. We take advantage of generalized Nehari manifold of [5, 16] to gain the ground state solution of (1.1) which is different from [5, 14] and a innovation point.

Setting $G(x,u) = \int_0^u g(x,s)ds$, we assume that V, K and g satisfy the following:

- (H1) V is continuous, 1-periodic in x_i, $1 \leq i \leq N$, and there exists a constant $a_0 > 0$ such that $V(x) \geq a_0$ for all $x \in \mathbb{R}^N$;
- (H2) K is continuous, 1-periodic in x_i, $1 \leq i \leq N$,
  - (i) $K(x) \geq K_{\min} > 0$ for all $x \in \mathbb{R}^N$,
  - (ii) $K(x) - K(x_0) = O(|x - x_0|^{N-2})$ as $x \to x_0$, $K(x_0) = \max_{\mathbb{R}^N} K(x)$;
- (H3) g is continuous, 1-periodic in x_i, $1 \leq i \leq N$, $|g(x,u)| \leq a(1 + |u|^{p-1})$ for some $a > 0$ and $4 < p < 2 \cdot 2^*$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = \infty$ if $N = 1$ or $N = 2$;
- (H4) $g(x,u) = o(u)$ uniformly in x as $u \to 0$;
- (H5) $u \mapsto g(x,u)/u^3$ is positive for $u \neq 0$, non-increasing on $(-\infty, 0)$ and non-decreasing on $(0, +\infty)$;
- (H6) $G(x,u)/u^4 \to \infty$ uniformly in x as $|u| \to \infty$, if $N \geq 10$;
- (H7) there exists an open bounded set $\Omega \in \mathbb{R}^N$, containing x_0 given by (H2), such that $G(x,u)/u^{2\cdot 2^* - 1} \to \infty$, as $|u| \to \infty$, uniformly in Ω, if $3 \leq N < 10$.

We note that if u_0 is a solution of (1.1), then so is the element $u_0(\cdot - k)$ under the action of $\mathbb{Z}^N$, set $\mathcal{O}(u_0) = \{u_0(\cdot - k) : k \in \mathbb{Z}^N\}$, $\mathcal{O}(u_0)$ is called the orbit of u_0 with respect to the action of $\mathbb{Z}^N$. Two solutions u_1 and u_2 of (1.1) are said to be geometrically distinct if $\mathcal{O}(u_1)$ and $\mathcal{O}(u_2)$ are disjoint. Now we state our main result.

Theorem 1.1. *Suppose that* (H1)–(H7) *hold, then problem* (1.1) *has a ground state solution.*

2. Preliminary results

In this section, we present the variational results which will be used in the proof of Theorem 1.1. We observe that (1.1) is formally the Euler-Lagrange equation associate with the energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1+2u^2)|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 - \frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N} K(x)|u|^{2\cdot 2^*} - \int_{\mathbb{R}^N} G(x,u).$$

From the variational point of view, the first difficulty associated with problem (1.1) is finding an appropriate function space where the functional J is well defined. To avoid such difficulty, we use the change of variable introduced by [8], that is, we consider $v = f^{-1}(u)$, where f is defined by

$$
\begin{gathered}
f'(t) = \frac{1}{(1+2f^2(t))^{1/2}} \quad \text{on } [0,+\infty), \\
f(t) = -f(-t) \quad \text{on } (-\infty, 0],
\end{gathered}
$$

having the following properties, which have been proved in [2, 3].

Lemma 2.1. *The function f satisfies the following properties:*

1. *f is uniquely defined, C^∞ and invertible;*
2. *$|f'(t)| \le 1$ for all $t \in \mathbb{R}$;*
3. *$|f(t)| \le |t|$ for all $t \in \mathbb{R}$;*
4. *$f(t)/t \to 1$ as $t \to 0$;*
5. *$f(t)/\sqrt{t} \to 2^{1/4}$ as $t \to +\infty$;*
6. *$f(t)/2 \le t f'(t) \le f(t)$ for all $t \ge 0$;*
7. *$|f(t)| \le 2^{1/4}|t|^{1/2}$ for all $t \in \mathbb{R}$;*
8. *$f^2(t)/2 \le f(t)f'(t)t \le f^2(t)$ for all $t \in \mathbb{R}$;*
9. *there exists a positive constant C such that*
$$
|f(t)| \ge \begin{cases} C|t|, & |t| \le 1, \\ C|t|^{1/2}, & |t| \ge 1; \end{cases}
$$
10. *$|f(t)f'(t)| \le 1/\sqrt{2}$ for all $t \in \mathbb{R}$.*

As a consequence of Lemma 2.1, the following has been proved in [4, 14].

Corollary 2.2. (i) *The function $f(t)f'(t)t^{-1}$ is strictly decreasing for all $t > 0$.*
(ii) *The function $f^p(t)f'(t)t^{-1}$ is strictly increasing for all $p \ge 3$ and $t > 0$.*
(iii) *The function $f^{2\cdot 2^*-1}(t)f'(t)t^{-1}$ is strictly increasing for all $t > 0$.*

In [4, 14] it is stated that the functions in Corollary 2.2 are respectively decreasing and increasing, but it is easy to see from the proofs there that they are strictly decreasing and strictly increasing.

So, after the change of variables from J, we obtain the functional

$$
\begin{aligned}
I(v) = &\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(v) \\
&- \frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N} K(x)|f(v)|^{2\cdot 2^*} - \int_{\mathbb{R}^N} G(x, f(v)),
\end{aligned} \tag{2.1}
$$

which is well defined in $H^1(\mathbb{R}^N)$ and belongs to C^1 under the hypotheses (H1)–(H4). Moreover, the critical points of I are the weak solutions of the problem

$$
-\Delta v + V(x)f(v)f'(v) = K(x)|f(v)|^{2\cdot 2^*-2} f(v)f'(v) + g(x, f(v))f'(v), \tag{2.2}
$$

for $v \in H^1(\mathbb{R}^N)$; that is

$$
\begin{aligned}
\langle I'(v), w\rangle = &\int_{\mathbb{R}^N} \nabla v \nabla w + \int_{\mathbb{R}^N} V(x) f(v) f'(v) w \\
&- \int_{\mathbb{R}^N} K(x)|f(v)|^{2\cdot 2^*-2} f(v)f'(v)w - \int_{\mathbb{R}^N} g(x, f(v))f'(v)w,
\end{aligned} \tag{2.3}
$$

for all $v, w \in H^1(\mathbb{R}^N)$. It has been shown in [2] that if $v \in H^1(\mathbb{R}^N)$ is a critical point of the functional I, then $u = f(v) \in H^1(\mathbb{R}^N)$ and u is a solution of (1.1).

We also observe that for obtaining a nonnegative solution for (2.1), we set $g(x,s) = 0$ for all $x \in \mathbb{R}^N$, $s < 0$. Indeed, let v be a critical point of I. Taking $w = -v^-$, where $v^- = \max\{-v, 0\}$, we obtain

$$\int_{\mathbb{R}^N} (|\nabla v^-|^2 + V(x)f(v)f'(v)(-v^-))dx = 0.$$

Since $f(v)(-v^-) \geq 0$, we have

$$\int_{\mathbb{R}^N} |\nabla v^-|^2 dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \frac{V(x)f(v)(-v^-)}{\sqrt{1+2f^2(v)}} dx = 0.$$

Hence we conclude that $v^- = 0$ almost everywhere in $\mathbb{R}^N$ and, therefore, $v = v^+ \geq 0$. As $u = f(v)$, we conclude that u is a nonnegative solution for the Problem (1.1).

Here, we consider the space $H^1(\mathbb{R}^N)$ endowed with one of the following norms:

$$\|u\| = \Big(\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2\Big)^{1/2}.$$

Let

$$M = \{v \in H^1(\mathbb{R}^N) \setminus \{0\} : \langle I'(v), v\rangle = 0\}.$$

Recall that M is called the Nehari manifold. We do not know whether M is of class C^1 under our assumptions and therefore we cannot use minimax theory directly on M. To overcome this difficulty, we employ an argument developed in [16, 17].

3. Proof of Theorem 1.1

For the rest of this article, we assume that (H1)–((H7) hold. Firstly, (H3) and (H4) imply that for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|g(x,u)| \leq \varepsilon|u| + C_\varepsilon|u|^{p-1}, \quad \text{for all } u \in \mathbb{R}. \tag{3.1}$$

And using (H4) and (H5), one can easily check that

$$G(x,u) \geq 0 \quad \text{and} \quad g(x,u)u \geq 4G(x,u) > 0 \quad \text{if } u \neq 0. \tag{3.2}$$

For $t > 0$, let

$$\begin{aligned} h(t) = I(tu) &= \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(tu) \\ &\quad - \frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N} K(x)|f(tu)|^{2\cdot 2^*} - \int_{\mathbb{R}^N} G(x, f(tu)). \end{aligned}$$

Lemma 3.1. *For each $u \in H^1(\mathbb{R}^N)\setminus\{0\}$, there exists a unique $t_u = t(u) > 0$ such that $m(u) := t_u u \in M$, $I(m(u)) = \max I(\mathbb{R}^+ u)$.*

Proof. By (3.1) and Lemma 2.1 (7), for ε sufficiently small we have

$$\begin{aligned} h(t) &\geq \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(tu) - \frac{|K|_\infty}{2\cdot 2^*}\int_{\mathbb{R}^N} |f(tu)|^{2\cdot 2^*} \\ &\quad - \frac{\varepsilon}{2}\int_{\mathbb{R}^N} f^2(tu) - \frac{C_\varepsilon}{p}\int_{\mathbb{R}^N} |f(tu)|^p \\ &\geq \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - C_1 t^{2^*}\int_{\mathbb{R}^N} |u|^{2^*} - C_2 t^{p/2}\int_{\mathbb{R}^N} |u|^{p/2}, \end{aligned}$$

where the constants C_1, C_2 are independent of t. Since $u \neq 0$ and $p > 4$, it is easy to see that $h(t) > 0$ whenever $t > 0$ is small enough. On the other hand, using Lemma 2.1 (3) and (6), we have

$$
\begin{aligned}
&h(t)\\
&\le \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{t^2}{2}\int_{\mathbb{R}^N}V(x)u^2 - \frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N}K(x)|f(tu)|^{2\cdot 2^*} - \int_{\mathbb{R}^N}G(x,f(tu))\\
&\le \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{t^2}{2}\int_{\mathbb{R}^N}V(x)u^2 - \frac{t^{2^*}}{2\cdot 2^*}\int_{\mathbb{R}^N}K(x)\frac{|f(tu)|^{2\cdot 2^*}}{|tu|^{2^*}}\cdot |u|^{2^*},
\end{aligned}
$$

then we can easily show that $h(t) \to -\infty$ as $t \to \infty$. Therefore, $\max_{t>0} h(t)$ achieved at some $t_u = t(u) > 0$ so that $h'(t_u) = 0$ and then $t_u u \in M$.

The condition $h'(t) = 0$ is equivalent to

$$
\begin{aligned}
\int_{\mathbb{R}^N}|\nabla u|^2 = \int_{u\neq 0}\Big[&\frac{K(x)|f(tu)|^{2\cdot 2^*-2}f(tu)f'(tu)}{tu}\\
&+\frac{g(x,f(tu))f'(tu)}{tu} - \frac{V(x)f(tu)f'(tu)}{tu}\Big]u^2.
\end{aligned}
$$

Let

$$
\xi(s) := \frac{K(x)|f(s)|^{2\cdot 2^*-2}f(s)f'(s)}{s} + \frac{g(x,f(s))f'(s)}{s} - \frac{V(x)f(s)f'(s)}{s}.
$$

By (H5) and Corollary 2.2 (ii), the function

$$
\frac{g(x,f(s))f'(s)}{s} = \frac{g(x,f(s))}{f^3(s)}\cdot\frac{f^3(s)f'(s)}{s}
$$

is strictly increasing for $s > 0$. Hence also $s \mapsto \xi(s)$ is strictly increasing according to Corollary 2.2 (i) and (iii). So there exists a unique $t_u > 0$ such that $h'(t_u) = 0$. $\square$

Lemma 3.2. (1) *There exists $\rho > 0$ such that $c = \inf_M I \ge \inf_{S_\rho} I > 0$, where $S_\rho = \{u \in H^1(\mathbb{R}^N) : \|u\| = \rho\}$.*
(2) *$\|u\|^2 \ge 2c$ for all $u \in M$.*

Proof. (1) According to [5] that $\int_{\mathbb{R}^N}|\nabla u|^2 + \int_{\mathbb{R}^N}V(x)f^2(u) \ge C\|u\|^2$ whenever $\|u\| \le \rho$. By (3.1) and Lemma 2.1 (3) and (7) we have

$$
\begin{aligned}
\int_{\mathbb{R}^N}G(x,f(u)) &\le \frac{\varepsilon}{2}\int_{\mathbb{R}^N}f^2(u) + \frac{C_\varepsilon}{p}\int_{\mathbb{R}^N}|f(u)|^p\\
&\le \frac{\varepsilon}{2}\int_{\mathbb{R}^N}|u|^2 + \frac{CC_\varepsilon}{p}\int_{\mathbb{R}^N}|u|^{p/2}\\
&\le C\varepsilon\|u\|^2 + CC_\varepsilon\|u\|^{p/2},
\end{aligned}
$$

and

$$
\int_{\mathbb{R}^N}K(x)|f(u)|^{2\cdot 2^*} \le 2^{2^*/2}|K|_\infty\int_{\mathbb{R}^N}|u|^{2^*} \le C\|u\|^{2^*},
$$

therefore, for sufficiently small ε,

$$
I(u) \ge C\|u\|^2 - C\|u\|^{p/2} - C\|u\|^{2^*},
$$

and then $\inf_{S_\rho} I > 0$ is obtained when ρ is small enough. The inequality $\inf_M I \ge \inf_{S_\rho} I$ is a consequence of Lemma 3.1 since for every $u \in M$ there is $s > 0$ such that $su \in S_\rho$ and $I(t_u u) \ge I(su)$.

(2) For $u \in M$, by Lemma 2.1 (3),

$$
\begin{aligned}
c &\le I(u) \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(u) \\
&\quad - \frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N} K(x)|f(u)|^{2\cdot 2^*} - \int_{\mathbb{R}^N} G(x,u) \\
&\le \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(u) \le \frac{1}{2}\|u\|^2.
\end{aligned}
$$

□

Lemma 3.3. *If $\mathcal{V}$ is a compact subset of $H^1(\mathbb{R}^N)\setminus\{0\}$, then m maps $\mathcal{V}$ into bounded set in $H^1(\mathbb{R}^N)$.*

Proof. We may assume without loss of generality that $\mathcal{V} \subset S$. Arguing by contradiction, suppose there exist $u_n \in \mathcal{V}$ and $v_n = m(u_n) = t_{u_n} u_n$ such that $\|v_n\| \to \infty$ as $n \to \infty$. Passing to a subsequence, there is $u \in H^1(\mathbb{R}^N)$ with $\|u\| = 1$ such that $u_n \to u \in S$. Since $|v_n(x)| \to \infty$ if $u(x) \neq 0$, then by (H6), Lemma 2.1 (5) and Fatou's lemma that

$$
\int_{\mathbb{R}^N} \frac{G(x, f(v_n))}{t_{u_n}^2} = \int_{\mathbb{R}^N} \frac{G(x, f(v_n)) u_n^2}{v_n^2} = \int_{\mathbb{R}^N} \frac{G(x, f(v_n))}{f^4(v_n)} \cdot \frac{f^4(v_n)}{v_n^2} \cdot u_n^2 \to \infty.
$$

By Lemma 2.1 (3),

$$
0 \le \frac{I(v_n)}{t_{u_n}^2} \le \frac{1}{2} - \int_{\mathbb{R}^N} \frac{G(x, f(v_n))}{t_{u_n}^2} \to -\infty,
$$

a contradiction. □

Recall that S is the unit sphere in $H^1(\mathbb{R}^N)$ and define the mapping $m : S \to M$ by setting

$$
m(w) := t_w w,
$$

where t_w is as in Lemma 3.1. Note that $\|m(w)\| = t_w$. Lemma 3.4 below is taken from [17, Proposition 8 and Corollary 10]. That the hypotheses in [17] are satisfied is a consequence of Lemmas 3.1, 3.2 and 3.3. Indeed, if $h(t) = I(tw)$ and $w \in S$, then $h'(t) > 0$ for $0 < t < t_w$ and $h'(t) < 0$ for $t > t_w$ by Lemma 3.1, $t_w \ge \delta > 0$ by Lemma 3.2 and $t_w \le R$ for $w \in \mathcal{V} \subset S$ by Lemma 3.3.

Lemma 3.4. *The mapping m is continuous. Moreover, the mapping m is a homeomorphism between S and M, and the inverse of m is $m^{-1}(u) = u/\|u\|$.*

We shall consider the functional $\Psi : S \to \mathbb{R}$ given by

$$
\Psi(w) = I(m(w)).
$$

Lemma 3.5. (1) *$\Psi \in C^1(S, \mathbb{R})$ and*

$$
\langle \Psi'(w), z\rangle = \|m(w)\| \langle I'(m(w)), z\rangle
$$

for all $z \in T_w(S) = \{v \in H^1(\mathbb{R}^N), \langle v, w\rangle = 0\}$.

(2) *If (w_n) is a Palais-Smale sequence for Ψ, then $(m(w_n))$ is a Palais-Smale sequence for I. If $(u_n) \subset M$ is a bounded Palais-Smale sequence for I , then $(m^{-1}(u_n))$ is a Palais-Smale sequence for Ψ.*

(3) *w is a critical point of Ψ if and only if $m(w)$ is a nontrivial critical point of I. Moreover, the corresponding values of Ψ and I coincide and* $\inf_S \Psi = \inf_M I$.

Proof. The argument is similar to that [16, Proposition 2.9 and Corollary 2.10] except that we do not claim I is coercive on M. But we obtain the following claim.

Claim: Each Palais-Smale sequence for I is bounded. First of all we observe that if a sequence $(u_n) \subset H^1(\mathbb{R}^N)$ satisfies

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x) f^2(u_n) \le A$$

for some constant $A > 0$, then it is bounded in $H^1(\mathbb{R}^N)$.

To show this claim, we just need to show that $\int_{\mathbb{R}^N} u_n^2$ is bounded. In fact, by Lemma 2.1 (9) and (H1), we observe that

$$\int_{|u_n(x)|\le 1} u_n^2 \le \frac{1}{C^2} \int_{|u_n(x)|\le 1} f^2(u_n) \le \frac{1}{C^2 a_0} \int_{\mathbb{R}^N} V(x) f^2(u_n) \le \frac{A}{C^2 a_0}$$

and

$$\int_{|u_n(x)|>1} u_n^2 \le \int_{|u_n(x)|>1} |u_n|^{2^*} \le C\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2\Big)^{2^*/2} \le C A^{2^*/2}.$$

Therefore,

$$\int_{\mathbb{R}^N} u_n^2 = \int_{|u_n(x)|\le 1} u_n^2 + \int_{|u_n(x)|>1} u_n^2 \le C.$$

To complete the proof, we only need to show that $\int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x) f^2(u_n)$ is bounded.

Let $(u_n) \subset H^1(\mathbb{R}^N)$ be a Palais-Smale sequence for I at level $c \in \mathbb{R}$, i.e.

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0.$$

Then for n large enough, by Lemma 2.1 (6) and (8), and (3.2), we have

$$\begin{aligned}
c + o(1) &\ge I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(u_n) - \frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N} K(x)|f(u_n)|^{2\cdot 2^*} \\
&\quad - \int_{\mathbb{R}^N} G(x, f(u_n)) - \frac{1}{2}\Big[\int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x) f(u_n) f'(u_n) u_n \\
&\quad - \int_{\mathbb{R}^N} K(x)|f(u_n)|^{2\cdot 2^*-2} f(u_n) f'(u_n) u_n - \int_{\mathbb{R}^N} g(x, f(u_n)) f'(u_n) u_n\Big] \\
&\ge \Big(\frac{1}{4}\int_{\mathbb{R}^N} g(x, f(u_n)) f(u_n) - \int_{\mathbb{R}^N} G(x, f(u_n))\Big) \\
&\quad + \big(\frac{1}{4} - \frac{1}{2\cdot 2^*}\big)\int_{\mathbb{R}^N} K(x)|f(u_n)|^{2\cdot 2^*} \\
&\ge \frac{1}{2N}\int_{\mathbb{R}^N} K(x)|f(u_n)|^{2\cdot 2^*} \\
&\ge \frac{K_{\min}}{2N}\int_{\mathbb{R}^N} |f(u_n)|^{2\cdot 2^*},
\end{aligned}$$

which implies $\int_{\mathbb{R}^N} |f(u_n)|^{2\cdot 2^*} \le C$. By (H3) and (H4) that for each $\varepsilon > 0$ there exists C_ε such that $g(x,u) \le \varepsilon|u| + C_\varepsilon |u|^{22^*-1}$[1], then by Lemma 2.1 (6) and (8),

$$
\begin{aligned}
&\frac{1}{2}\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x) f^2(u_n)\Big)\\
&\le \int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x) f(u_n) f'(u_n) u_n\\
&= \langle I'(u_n),\, u_n\rangle + \int_{\mathbb{R}^N} K(x)|f(u_n)|^{2\cdot 2^*-2} f(u_n) f'(u_n) u_n + \int_{\mathbb{R}^N} g(x, f(u_n)) f'(u_n) u_n\\
&\le \int_{\mathbb{R}^N} K(x) |f(u_n)|^{2\cdot 2^*} + \int_{\mathbb{R}^N} g(x, f(u_n)) f(u_n) + o_n(1)\\
&\le |K|_\infty \int_{\mathbb{R}^N} |f(u_n)|^{2\cdot 2^*} + \varepsilon \int_{\mathbb{R}^N} f^2(u_n) + C_\varepsilon \int_{\mathbb{R}^N} |f(u_n)|^{2\cdot 2^*} + o_n(1).
\end{aligned}
$$

Let $\varepsilon \in (0, a_0/4)$. By the above inequality and the boundedness of $\int_{\mathbb{R}^N} |f(u_n)|^{2\cdot 2^*}$ we have

$$
\begin{aligned}
&\frac{1}{4}\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x) f^2(u_n)\Big)\\
&\le \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + \big(\frac{1}{2} - \frac{\varepsilon}{a_0}\big) \int_{\mathbb{R}^N} V(x) f^2(u_n)\\
&\le |K|_\infty \int_{\mathbb{R}^N} |f(u_n)|^{2\cdot 2^*} + C_\varepsilon \int_{\mathbb{R}^N} |f(u_n)|^{2\cdot 2^*} \le C,
\end{aligned}
$$

then (u_n) must be bounded. Then by Lemmas 3.1 and 3.2, one can follow the same line of the proof of [16, Corollary 2.10] to complete the present proof. $\square$

Proof of Theorem 1.1. It follows from Lemma 3.2 (1) that $c = \inf_M I > 0$. By Ekeland's variational principle [19], there exists a Palais-Smale sequence $(w_n) \subset S$ for Ψ such that $\Psi(w_n) \to c$. Set $u_n = m(w_n)$, then from Lemma 3.5 (2), $(u_n) \subset M$ is a Palais-Smale sequence for I and $I(u_n) \to c$. According to the Claim in the proof of Lemma 3.5, (u_n) is bounded. Clearly, (u_n) is either
(i) Vanishing: For each $r > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^2 = 0,$$

or (ii) Non-vanishing: There exists $r, \delta > 0$ and a sequence $(y_n) \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 \ge \delta.$$

In (ii) we may assume $y_n \in \mathbb{Z}^N$ by taking a larger r if necessary. Suppose (ii) holds and let $\widetilde{u_n}(x) := u_n(x + y_n)$. Since I is invariant and ∇I is equivariant with respect to the $\mathbb{Z}^N$-action, $\widetilde{u_n} \rightharpoonup u$ after passing to a subsequence, $I'(u) = 0$ and since $\lim_{n\to\infty} \int_{B_r(0)} |u|^2 \ge \delta$, $u \ne 0$. So u is a nontrivial critical point of I. Therefore $u \in M$ and $I(u) \ge c$. Furthermore, from Lemma 2.1 (6) and (8), (3.2) and Fatou's lemma, we have

$$
\begin{aligned}
&c + o(1)\\
&= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(u_n) - \frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N} K(x)|f(u_n)|^{2\cdot 2^*} \\
&\quad - \int_{\mathbb{R}^N} G(x,f(u_n)) - \Big[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f(u_n)f'(u_n)u_n \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^N} K(x)|f(u_n)|^{2\cdot 2^*-2}f(u_n)f'(u_n)u_n - \frac{1}{2}\int_{\mathbb{R}^N} g(x,f(u_n))f'(u_n)u_n\Big] \\
&= \frac{1}{2}\int_{\mathbb{R}^N} V(x)[f^2(u_n) - f(u_n)f'(u_n)u_n] \\
&\quad + \Big(\frac{1}{2}\int_{\mathbb{R}^N} K(x)|f(u_n)|^{2\cdot 2^*-2}f(u_n)f'(u_n)u_n - \frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N} K(x)|f(u_n)|^{2\cdot 2^*}\Big) \\
&\quad + \Big(\frac{1}{2}\int_{\mathbb{R}^N} g(x,f(u_n))f'(u_n)u_n - \int_{\mathbb{R}^N} G(x,f(u_n))\Big) \\
&\ge \frac{1}{2}\int_{\mathbb{R}^N} V(x)[f^2(u) - f(u)f'(u)u] \\
&\quad + \Big(\frac{1}{2}\int_{\mathbb{R}^N} K(x)|f(u)|^{2\cdot 2^*-2}f(u)f'(u)u - \frac{1}{2\cdot 2^*}\int_{\mathbb{R}^N} K(x)|f(u)|^{2\cdot 2^*}\Big) \\
&\quad + \Big(\frac{1}{2}\int_{\mathbb{R}^N} g(x,f(u))f'(u)u - \int_{\mathbb{R}^N} G(x,f(u))\Big) + o(1) \\
&= I(u) - \frac{1}{2}\langle I'(u),u\rangle + o(1) \\
&= I(u) + o(1),
\end{aligned}
$$

which implies $I(u) \le c$. Hence $I(u) = c$ and thus u is a ground state solution of problem (1.1). Hence to complete the proof of Theorem 1.1, it remains to show that vanishing cannot occur. This will be done in the following three lemmas. $\square$

Before stating the next result, we recall that the best constant for Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is

$$
S = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{(\int_{\mathbb{R}^N} |u|^{2^*})^{2/2^*}}. \tag{3.3}
$$

Lemma 3.6 ([14])**.** *Suppose* (H1)–(H4) *are satisfied. Then*

1. $\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)[f^2(u_n) - f(u_n)f'(u_n)u_n] = 0$,
2. $\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)[u_n^2 - f(u_n)f'(u_n)u_n] = 0$,
3. $\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)[2^{2^*/2}|u_n|^{2^*} - |f(u_n)|^{2\cdot 2^*}] = 0$,
4. $\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)[\frac{1}{2}|f(u_n)|^{2^*-2}f(u_n)f'(u_n)u_n - \frac{1}{2}2^{\frac{2^*-2}{2}}|u_n|^{2^*}] = 0$.

Lemma 3.7. *Suppose* (H1)–(H4) *are satisfied. If* $c \in (0, \frac{1}{2N}|K|_\infty^{(2-N)/2}S^{N/2})$*, then* (u_n) *cannot be vanishing.*

Proof. Suppose by contradiction that (u_n) is vanishing, then it follows from P.L. Lions' lemma (see [19, Lemma 1.21]) that $u_n \to 0$ in $L^s(\mathbb{R}^N)$ whenever $2 < s < 2^*$. Thus by (3.1), as in [13], we can deduce that

$$
\int_{\mathbb{R}^N} g(x,f(u_n))f'(u_n)u_n \to 0, \int_{\mathbb{R}^N} G(x,f(u_n)) \to 0. \tag{3.4}
$$

Since (u_n) is a $(Palais - Smale)_c$ sequence for the functional I, it follows that

$$
\begin{aligned}
c + o(1) &= I(u_n) - \langle I'(u_n), u_n\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^N} V(x)[f^2(u_n) - f(u_n)f'(u_n)u_n] \\
&\quad + \int_{\mathbb{R}^N} K(x)\big[\frac{1}{2}|f(u_n)|^{2^*-2}f(u_n)f'(u_n)u_n - \frac{1}{2\cdot 2^*}|f(u_n)|^{2\cdot 2^*}\big].
\end{aligned} \tag{3.5}
$$

From Lemma 3.6 (1), (3) and (4), we have

$$
c + o(1) = \int_{\mathbb{R}^N} K(x)|u_n|^{2^*}\Big[\frac{1}{2}2^{\frac{2^*-2}{2}} - \frac{2^{\frac{2^*-2}{2}}}{2^*}\Big] = \frac{2^{\frac{2^*-2}{2}}}{N}\int_{\mathbb{R}^N} K(x)|u_n|^{2^*}.
$$

Hence

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)|u_n|^{2^*} = \frac{Nc}{2^{\frac{2^*-2}{2}}} > 0. \tag{3.6}
$$

Consequently, using Lemma 3.6 (4), we have

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)|f(u_n)|^{2^*-2}f(u_n)f'(u_n)u_n = Nc. \tag{3.7}
$$

On the other hand, taking the first limit in (3.4), Lemma 3.6 (2) and the fact $\langle I'(u_n), u_n\rangle \to 0$, so that

$$
\lim_{n\to\infty}\Big[\int_{\mathbb{R}^N} K(x)|f(u_n)|^{2^*-2}f(u_n)f'(u_n)u_n - \|u_n\|^2\Big] = 0.
$$

Therefore, from (3.7), it follows that

$$
\lim_{n\to\infty}\|u_n\|^2 = Nc. \tag{3.8}
$$

By the definition of S in (3.3),

$$
\frac{1}{|K|_\infty}\int_{\mathbb{R}^N} K(x)|u_n|^{2^*} \le \int_{\mathbb{R}^N}|u_n|^{2^*} \le \Big(\frac{1}{S}\int_{\mathbb{R}^N}|\nabla u_n|^2\Big)^{2^*/2} \le \Big(\frac{\|u_n\|^2}{S}\Big)^{2^*/2}.
$$

Passing to the limit in the above inequality, in view of (3.6) and (3.8), one can obtain

$$
\frac{1}{|K|_\infty}\frac{Nc}{2^{\frac{2^*-2}{2}}} \le \Big(\frac{Nc}{S}\Big)^{2^*/2},
$$

that is

$$
c \ge \frac{1}{2N}|K|_\infty^{(2-N)/2}S^{N/2},
$$

which is contradicts to the assumption that $c < \frac{1}{2N}|K|_\infty^{(2-N)/2}S^{N/2}$, then the proof is complete. ☐

Lemma 3.8 ([14]). *Suppose that* (H1)–(H7) *are satisfied. Then there exists* $u \in H^1(\mathbb{R}^N)\setminus\{0\}$ *such that*

$$
\max_{t\ge 0} I(tu) < \frac{1}{2N}|K|_\infty^{(2-N)/2}S^{N/2}.
$$

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