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SOLUTIONS TO MEAN CURVATURE EQUATIONS IN WEIGHTED STANDARD STATIC SPACETIMES

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Abstract. In this article, we study the solutions for the mean curvature equation in a weighted standard static spacetime, $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$, having a warping function ρ whose weight function f does not depend on the parameter $t \in \mathbb{R}$. We establish a f-parabolicity criterion to study the rigidity of spacelike hypersurfaces immersed in $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ and, in particular, of entire Killing graphs constructed over the Riemannian base $\mathbb{P}^n$. Also we give applications to weighted standard static spacetimes of the type $\mathbb{G}^n \times_\rho \mathbb{R}_1$, where $\mathbb{G}^n$ is the Gaussian space.

1. Introduction

Standard static spacetimes are part of the so called stationary spacetimes. Let us recall that a *stationary spacetime* is a time-orientable Lorentzian manifold $(\overline{M}^{n+1}, \overline{g})$ where there exists an infinitesimal symmetry given by a timelike Killing vector field Y (see [27]). The existence of Y enables us to define around each point a coordinate system $(t, x_1, \dots, x_n)$ such that Y coincides with the coordinate vector field $\partial/\partial t$ on its domain of definition and such that the components of the metric tensor in these coordinates are independent of t. When we normalize Y we obtain an observers vector field $Z = Y/\sqrt{-\overline{g}(Y,Y)}$. These observers measure a metric tensor that does not change with time. Furthermore, if this timelike Killing vector field is also irrotational (that is, the distribution $Y^\perp$ of all smooth vector fields on $\overline{M}^{n+1}$ that are orthogonal to Y is involutive), then a local warped product structure appears and the spacetime is called *static* (for more details see, for instance, [1]). In fact, when this structure is global this spacetime is known as a *standard static spacetime*. More precisely, a standard static spacetime $(\overline{M}^{n+1}, \overline{g})$ endowed with a globally defined timelike Killing vector field Y is isometric to the warped product

$$(\mathbb{P}^n \times_\rho \mathbb{R}_1 \,, \pi^*_{\mathbb{P}^n}(\widetilde{g}) + (\rho \circ \pi_{\mathbb{P}^n})^2 \pi^*_{\mathbb{R}}(-dt^2)\,)$$

where $\pi_{\mathbb{P}^n}$ and $\pi_{\mathbb{R}}$ denote the canonical projections from $\mathbb{P}^n \times \mathbb{R}_1$ onto each factor, $\widetilde{g}$ is the Riemannian metric on the base $\mathbb{P}^n$, $\mathbb{R}_1$ is the manifold $\mathbb{R}$ endowed with the metric $-dt^2$ and $\rho = \sqrt{-\overline{g}(Y,Y)}$ is the warping function. In this context, it

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is known that any static spacetime is locally isometric to a standard static one (see [23, Proposition 12.38]). Conversely, Sánchez [29] and more recently Aledo, Romero and Rubio [1] obtained some sufficient conditions for a static spacetime to be standard. Other properties on the geometry of standard static spacetimes were studied by Sánchez [28, 29, 30]. The importance of standard static spacetimes also comes from the fact that they include some classical spacetimes, such as the $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{L}^{n+1}$, Einstein static universe as well as models that describe an universe where there is only a spherically symmetric non-rotating mass, as a star or a black hole, like exterior Schwarzschild spacetime and some regions of Reissner-Nordström spacetime (see, for example, [5, 17]).

On the other hand, the study of spacelike hypersurfaces immersed with constant mean curvature in a spacetime has attracted the interest of a considerable group of geometers as evidenced by the amount of works that it has generated in the last decades. This is due not only to its mathematical interest, but also to its relevance in General Relativity. For example, constant mean curvature spacelike hypersurfaces are particularly suitable for studying the propagation of gravitational radiation. See, for instance, [21, 31] for a summary of several reasons justifying this interest. From the mathematical point of view, the study of the geometry of constant mean curvature spacelike hypersurfaces is mostly due to the fact that they exhibit nice Calabi-Bernstein type properties. More precisely, this study had its beginnings when Bernstein [6] proved that the only entire minimal graphs in the 3-dimensional Euclidean space $\mathbb{R}^3$ are planes. In the Lorentzian setting, there is an analogue result to Bernstein's theorem, which states that the only entire maximal graphs in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$ are the spacelike planes. This result was firstly proved by Calabi [7], and extended to the general n-dimensional case by Cheng and Yau [9].

A natural extension to the Calabi-Bernstein problem is to determine a reasonable set of sufficient conditions which guarantee the uniqueness (or nonexistence) of complete spacelike hypersurfaces immersed into a certain ambient spacetime. When such a spacetime is a standard static spacetime $\mathbb{P}^n \times_\rho \mathbb{R}_1$, there is a remarkable family of spacelike hypersurfaces, namely, the spacelike slices $\mathbb{P}^n \times \{t_0\}$, with $t_0 \in \mathbb{R}$, which are totally geodesics constituting a foliation for the ambient spacetime. Therefore, it is natural to approach Calabi-Bernstein problems in a standard static spacetime. In this branch, the first author together with Lima Jr, de Lima and Medeiros [12] extended a technique due to Romero et al. [25] to establish sufficient conditions to guarantee the parabolicity of complete spacelike hypersurfaces in $\mathbb{P}^n \times_\rho \mathbb{R}_1$ whose Riemannian base $\mathbb{P}^n$ has parabolic universal Riemannian covering and, as applications, they obtain uniqueness results concerning these hypersurfaces. Afterwards, Pelegrín, Romero and Rubio [24] also studied complete spacelike hypersurfaces in spatially parabolic standard static spacetimes. In this context, they used a similar parabolicity criterion to give new uniqueness and nonexistence results for these spacelike hypersurfaces and to solve new Calabi-Bernstein-type problems.

At this point, we recall that a *weighted manifold* $\mathfrak{M}^{n+1}_f$ is a semi-Riemannian manifold $(\mathfrak{M}^{n+1}, \mathfrak{g})$ endowed with a weighted volume form $d\mu = e^{-f} d\mathfrak{M}$, where the *weight function* f is a real-valued smooth function on $\mathfrak{M}^{n+1}$ and $d\mathfrak{M}$ is the volume element induced by the metric $\mathfrak{g}$ (for details see, for instance, [4, 22]). Concerning the weighted product space $\mathbb{G}^n \times \mathbb{R}_1$, where $\mathbb{G}^n$ stands for the so-called *Gaussian space* which is nothing but that the Euclidian space $\mathbb{R}^n$ endowed with the Gaussian probability density $e^{-f(x)} = (2\pi)^{-(n+1)/2} e^{-|x|^2/2}$, $x \in \mathbb{R}^n$, An et al. [2] extended the classical Bernstein's theorem showing that the only weighted maximal graphs $\Sigma(z)$ of smooth functions $z(x) = t$ over $\mathbb{G}^n$, with $\sup_{\Sigma(z)} |Dz|_{\mathbb{G}} < 1$, are the affine hyperplanes $t =$ constant.

Motivated by the works described above, our purpose in this paper is to obtain uniqueness results related to the mean curvature equation for entire Killing graphs constructed over the Riemannian base $\mathbb{P}^n$ of a weighted standard static spacetime $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ having warping function ρ and whose weight function f does not depend on the parameter $t \in \mathbb{R}$ (see Section 5). For this, in Section 2 we recall some basic facts about spacelike hypersurfaces immersed in a weighted standard static spacetime. Afterwards, in Section 3 we establish a suitable f-parabolicity criterion and, under appropriate constraints on the Bakry-Émery Ricci tensor and on the f-mean curvature, in Section 4 we study the rigidity of spacelike hypersurfaces immersed in $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$. Finally, we point out that, in Section 5, applications of our main results to weighted standard static spacetimes of the type $\mathbb{G}^n \times_\rho \mathbb{R}_1$ are also given.

2. Weighted standard static spacetimes

Along this paper, we will consider an $(n+1)$-dimensional Lorentz manifold $\overline{M}^{n+1}$ with Lorentzian metric $g = g(\cdot,\cdot)$ and endowed with a Killing timelike vector field Y. Here timelike referred to a vector field means that $Y_p \in T_p\overline{M}$ is a timelike (and so nonzero) vector for each $p \in \overline{M}^{n+1}$. On the other hand, Killing mean that the $\mathcal{L}_Y g = 0$, where $\mathcal{L}_Y$ stands for the Lie derivative of g in the direction of Y.

We observe that the distribution $\mathcal{D}$ of all smooth vector fields of $\overline{M}^{n+1}$ that are orthogonal to Y, defined at each point by

$$\overline{M}^{n+1} \ni p \mapsto \mathcal{D}(p) = \{v \in T_p\overline{M} : g(v, Y_p) = 0\},$$

is of constant rank and integrable. Given a Riemannian integral leaf $\mathbb{P}^n$ of that distribution $\mathcal{D}$, let $\Psi : \mathbb{I} \times \mathbb{P}^n \to \overline{M}^{n+1}$ be the flow generated by Y with initial values in $\mathbb{P}^n$, where $\mathbb{I}$ is a maximal interval of definition. Without loss of generality, in what follows we will consider $\mathbb{I} = \mathbb{R}$. In this setting, our space $\overline{M}^{n+1}$ can be regarded as the *standard static spacetime* $\mathbb{P}^n \times_\rho \mathbb{R}_1$ (cf. [23, Proposition 12.38]), that is, the Lorentzian product manifold $\mathbb{P}^n \times \mathbb{R}_1$ endowed with the warping metric

$$\langle \cdot,\cdot \rangle = \pi^*_{\mathbb{P}^n}(\langle \cdot,\cdot \rangle_{\mathbb{P}^n}) + (\rho \circ \pi_{\mathbb{P}^n})^2 \pi^*_{\mathbb{R}}(-dt^2), \tag{2.1}$$

where $\pi_{\mathbb{P}^n}$ and $\pi_{\mathbb{R}}$ denote the canonical projections from $\mathbb{P}^n \times \mathbb{R}_1$ onto each factor, $\langle \cdot,\cdot \rangle_{\mathbb{P}^n}$ is the induced Riemannian metric on the base $\mathbb{P}^n$, $\mathbb{R}_1$ is the manifold $\mathbb{R}$ endowed with the metric $-dt^2$ and

$$\rho = |Y| = \sqrt{-\langle Y, Y \rangle} > 0$$

is the warping function. We mean by $C^\infty(\mathbb{P}^n \times_\rho \mathbb{R}_1)$ the ring of real functions of class C^∞ on $\mathbb{P}^n \times_\rho \mathbb{R}_1$ and by $\mathfrak{X}(\mathbb{P}^n \times_\rho \mathbb{R}_1)$ the $C^\infty(\mathbb{P}^n \times_\rho \mathbb{R}_1)$-module of vector fields of class C^∞ on $\mathbb{P}^n \times_\rho \mathbb{R}_1$. The Levi-Civita connections of $\mathbb{P}^n \times_\rho \mathbb{R}_1$ and $\mathbb{P}^n$ will be denoted by $\overline{\nabla}$ and $\widetilde{\nabla}$, respectively.

Now, in the configuration described above, let $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ be a *weighted standard static spacetime*, namely, a standard static spacetime $\mathbb{P}^n \times_\rho \mathbb{R}_1$ endowed with a weighted volume form $d\overline{\sigma} = e^{-f} d\overline{v}$, where $f \in C^\infty(\mathbb{P}^n \times_\rho \mathbb{R}_1)$ is a real-valued function, called *weight function* (or *density function*), and $d\overline{v}$ is the volume element induced by the warping metric $\langle\cdot,\cdot\rangle$ defined in (2.1). For $\big(\mathbb{P}^n \times_\rho \mathbb{R}_1\big)_f$, we recall that the Bakry-Émery-Ricci tensor $\overline{\mathrm{Ric}}_f$ is defined by

$$\overline{\mathrm{Ric}}_f = \overline{\mathrm{Ric}} + \overline{\mathrm{Hess}} f, \tag{2.2}$$

where $\overline{\mathrm{Ric}}$ and $\overline{\mathrm{Hess}}$ stand for the Ricci tensor and the Hessian operator in $\mathbb{P}^n \times_\rho \mathbb{R}_1$, respectively.

Throughout this work, we will deal with complete spacelike hypersurfaces

$$\psi : \Sigma^n \hookrightarrow (\mathbb{P}^n \times_\rho \mathbb{R}_1)_f,$$

namely, isometric immersions from a (connected) n-dimensional Riemannian manifold Σ^n into weighted static spacetime $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$. In this setting, the Levi-Civita connection of Σ^n will be denoted by ∇. As $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ is time-oriented by the timelike vector field Y and $x : \Sigma^n \hookrightarrow (\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ is a spacelike hypersurface, then Σ^n is orientable (cf. [23, Proposition 5.26]) and one can choose a globally defined unit normal vector field N on Σ^n having the same time-orientation of $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ (cf. [23, Proposition 5.29]), that is,

$$\langle Y, N\rangle < 0. \tag{2.3}$$

Such N is said the *future-pointing Gauss map* of Σ^n. Let A denote the shape operator of Σ^n with respect to N. So that at each $p \in \Sigma^n$ A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$ given by

$$A_p v = -\overline{\nabla}_v N.$$

According to Gromov [16], the *weighted mean curvature* (or simply the *f-mean curvature*) H_f of Σ^n is given by

$$nH_f = nH - \langle \overline{\nabla} f, N\rangle, \tag{2.4}$$

where $H = -\frac{1}{n}\,\mathrm{tr}(A)$ denotes the standard mean curvature of Σ^n with respect to its orientation N. Moreover, we say that $\psi : \Sigma^n \hookrightarrow (\mathbb{P}^n \times_\rho \mathbb{R})_f$ is *f-maximal* when its f-mean curvature vanishes identically.

The *f-divergence* on Σ^n is $\mathrm{div}_f : C^\infty(\Sigma^n) \to C^\infty(\Sigma^n)$, defined by

$$\mathrm{div}_f(X) = \mathrm{div}\, X - \langle \nabla f, X\rangle,$$

where $\mathrm{div}(\cdot)$ denotes the standard divergence on Σ^n. We define the *f-Laplacian* (also called the *drift Laplacian*) of Σ^n by $\Delta_f : C^\infty(\Sigma^n) \to C^\infty(\Sigma^n)$, as

$$\Delta_f(u) = \mathrm{div}_f(\nabla u) = \Delta u - \langle \nabla f, \nabla u\rangle \tag{2.5}$$

where Δ is the standard Laplacian on Σ^n.

In what follows, associated with a spacelike hypersurface $\psi : \Sigma^n \hookrightarrow (\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$, we will consider two particular smooth functions, namely, the (vertical) *height function*

$$h = (\pi_{\mathbb{R}})\big|_{\Sigma^n} : \Sigma^n \to \mathbb{R} \tag{2.6}$$

and the *angle function* $\Theta : \Sigma^n \to \mathbb{R}$ defined as

$$\Theta(p) = \langle N(p), Y(p)\rangle, \tag{2.7}$$

where N is the future-pointing Gauss map of Σ^n and Y is the Killing vector field on $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$. From (2.3), we note that Θ will be always a negative function on Σ^n.

We have that
$$\nabla h = -\frac{1}{\rho^2} Y^\top, \tag{2.8}$$
where $(\cdot)^\top$ denote the projections of a smooth vector field in $\mathfrak{X}(\mathbb{P}^n \times_\rho \mathbb{R}_1)$ on $\mathfrak{X}(\Sigma^n)$. Moreover,
$$N^* = N + \frac{1}{\rho^2}\Theta Y, \tag{2.9}$$
where $(\cdot)^*$ denote the projections of a smooth vector field in $\mathfrak{X}(\mathbb{P}^n \times_\rho \mathbb{R}_1)$ on $\mathfrak{X}(\mathbb{P}^n)$. Hence, from (2.8) and (2.9) it is not difficult to verify that
$$|\nabla h|^2 = \frac{1}{\rho^2}|N^*|^2_{\mathbb{P}^n}. \tag{2.10}$$

3. An f-parabolicity criterion for spacelike hypersurfaces in $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$

Romero, Rubio and Salamanca [26] investigated the parabolicity of complete spacelike hypersurfaces in GRW spacetimes whose Riemannian fiber has a parabolic universal Riemannian covering. In this setting, they were able to guarantee the parabolicity of complete spacelike hypersurfaces, under suitable boundedness assumptions on the warping function and on the hyperbolic angle function of these hypersurfaces. Our aim in this section is just, following the ideas of [11], to obtain an extension of this parabolicity criterion to the context of standard static spacetimes.

A smooth function u on a weighted manifold Σ^n_f is said to be *f-superharmonic* if $\Delta_f u \le 0$. Taking this into account, the weighted manifold Σ^n_f is called *f-parabolic* if there is no nonconstant, nonnegative, f-superharmonic function on Σ^n.

On the other hand, given a weighted manifold Σ^n_f we define, for any compact subset $K \subset \Sigma^n$, the *f-capacity* of K as
$$\operatorname{cap}_f(K) = \inf\Big\{ \int_\Sigma |\nabla u|^2 e^{-f} d\Sigma : u \in \operatorname{Lip}_0(\Sigma) and u|_K \equiv 1\Big\},$$
where $\operatorname{Lip}_0(\Sigma)$ is the set of all compactly supported Lipschitz functions on Σ^n. The following statement relates the notion of f-capacity to the concept of f-parabolicity (see [15, Proposition 2.1]).

Lemma 3.1. *The weighted manifold Σ^n_f is f-parabolic if and only if $\operatorname{cap}_f(K) = 0$ for any compact set $K \subset \Sigma^n$.*

Let us recall that given two Riemannian manifolds (Σ', g') and (Σ, g), a diffeomorphism φ from Σ' onto Σ is called a *quasi-isometry* if there exists a constant $c \ge 1$ such that
$$c^{-1}|v|_{g'} \le |d\varphi(v)|_g \le c|v|_{g'}$$
for all $v \in T_p\Sigma', p \in \Sigma'$ (see [19] for more details). In this case, given a smooth function $f : \Sigma \to \mathbb{R}$, we can reason as in [14, Section 5] to verify that the $(f \circ \varphi)$-capacity of the compact subsets in Σ' changes under a quasi-isometry at most by a constant factor of the f-capacity of the compact subsets in Σ. The following statement corresponds to [11, Lemma 1].

Lemma 3.2. *Keeping the same notation above, we have:*

(a) *Given a quasi-isometry $\varphi : \Sigma' \to \Sigma$, Σ is f-parabolic if and only if Σ' is $(f \circ \varphi)$-parabolic;*

(b) *Let $\widetilde{\Sigma}$ be the universal Riemannian covering of Σ with canonical projection $\pi_\Sigma : \widetilde{\Sigma} \to \Sigma$. If $\widetilde{\Sigma}$ is $(f \circ \pi_\Sigma)$-parabolic, then Σ is f-parabolic.*

Recall that every connected manifold Σ has an universal covering, that is, there exist a simply connected manifold $\widetilde{\Sigma}$ (called a universal covering of Σ) and a smooth map $\widetilde{\pi} : \widetilde{\Sigma} \to \Sigma$ (called a covering map) such that each point $p \in \Sigma$ has a connected neighborhood U that is evenly covered by $\widetilde{\pi}$, that is, $\widetilde{\pi}$ maps each component of $\widetilde{\pi}^{-1}(U)$ diffeomorphically onto U (for more details, see [23, Appendix A]). Moreover, if Σ is a Riemannian manifold, then it is possible to give $\widetilde{\Sigma}$ a Riemannian structure such that the covering map $\widetilde{\pi} : \widetilde{\Sigma} \to \Sigma$ is a local isometry. In this case, $\widetilde{\Sigma}$ is said a universal Riemannian covering of Σ (see [13, pg. 152]).

From now on, we will denote by $\widetilde{\mathbb{P}}$ the universal Riemannian covering of base $\mathbb{P}^n$ with projection $\widetilde{\pi} : \widetilde{\mathbb{P}} \to \mathbb{P}^n$ and $\widetilde{f}$ will denote the composition $f \circ \widetilde{\pi}$. In this setting, a standard static spacetime $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ will be said *spattialy f-parabolic* if the universal Riemannian covering $\widetilde{\mathbb{P}}$ of its base $\mathbb{P}^n$ is $\widetilde{f}$-parabolic.

Proposition 3.3. *Let $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ be a weighted standard static spacetimes which is spatially $\widetilde{f}$-parabolic. If $\psi : \Sigma^n \hookrightarrow \overline{\mathbb{P}}^{n+1}$ is a spacelike hypersurface such that the function $\eta := \frac{\Theta}{\rho}$ is bounded on it, then Σ^n is f-parabolic.*

Proof. From Lemma 3.2 we have that

(i) f-parabolicity is invariant under a quasi-isometry;
(ii) if the universal Riemannian covering $\widetilde{\Sigma}$ of Σ^n is $(f \circ \pi_\Sigma)$-parabolic, then Σ^n is also f-parabolic.

Denoting $\pi = \pi_{\mathbb{P}} \circ \psi : \Sigma^n \to \mathbb{P}^n$, for any tangent vector $v \in T\Sigma$ we have

$$\langle v, v\rangle = \langle \pi_* v, \pi_* v\rangle_{\mathbb{P}} - \rho^2 \langle h_* v, h_* v\rangle_{\mathbb{R}} \leq c\langle \pi_* v, \pi_* v\rangle_{\mathbb{P}},$$

where $c = \sup_\Sigma \eta^2 \geq 1$. In particular, by previous inequality we see that $\pi_{*,p} : T_p\Sigma \to T_{\pi(p)}M$ is a isomorphism for every $p \in \Sigma^n$. Then, from inverse function theorem we obtain that π is a local diffeomorphism and applying [13, Lemma 7.3.3] (see also [20, Lemma 8.8.1]) we can conclude that π is a covering map and that $\mathbb{P}^n$ is complete.

On the other hand, using the Cauchy-Schwartz inequality we see that

$$\langle \nabla h, v\rangle^2 \leq \langle \nabla h, \nabla h\rangle \langle v, v\rangle$$

and, consequently, since $h_* v = dh(v) = \langle \nabla h, v\rangle$, we have

$$\begin{aligned}\langle v, v\rangle &= \langle \pi_* v, \pi_* v\rangle_{\mathbb{P}} - \rho^2 \langle h_* v, h_* v\rangle_{\mathbb{R}} \\ &= \langle \pi_* v, \pi_* v\rangle_{\mathbb{P}} - \rho^2 \langle \nabla h, v\rangle^2 \\ &\geq \langle \pi_* v, \pi_* v\rangle_{\mathbb{P}} - \rho^2 |\nabla h|^2 \langle v, v\rangle;\end{aligned}$$

that is,

$$\langle v, v\rangle (1 + \rho^2 |\nabla h|^2) \geq \langle \pi_* v, \pi_* v\rangle_{\mathbb{P}}.$$

By definition of the function η and from (2.10) we obtain

$$\langle v, v\rangle \geq \frac{1}{\eta^2} \langle \pi_* v, \pi_* v\rangle_{\mathbb{P}}$$

From our hypothesis we conclude that

$$c^{-1} \langle \pi_* v, \pi_* v\rangle_{\mathbb{P}} \leq \langle v, v\rangle \leq c\langle \pi_* v, \pi_* v\rangle_{\mathbb{P}}. \tag{3.1}$$

So, let $\widetilde{\Sigma}$ be the universal Riemannian covering of Σ^n with projection $\pi_\Sigma : \widetilde{\Sigma} \to \Sigma^n$. Then, the map $\pi_0 = \pi \circ \pi_\Sigma : \widetilde{\Sigma} \to \mathbb{P}^n$ is a covering map. Now, if $\widetilde{M}$ is the universal Riemannian covering of $\mathbb{P}^n$ with projection $\widetilde{\pi} : \widetilde{M} \to \mathbb{P}^n$, then there exists a diffeomorphism $\varphi : \widetilde{\Sigma} \to \widetilde{M}$ such that $\widetilde{\pi} \circ \varphi = \pi_0$. Moreover, φ is a quasi-isometry. Indeed, if $v \in T\widetilde{\Sigma}$, we have from (3.1) that

$$\begin{aligned}
\langle \varphi_* v, \varphi_* v \rangle_{\widetilde{M}} &= \langle \widetilde{\pi}_*(\varphi_* v), \widetilde{\pi}_*(\varphi_* v) \rangle_M \\
&= \langle (\pi_0)_* v, (\pi_0)_* v \rangle_M \\
&= \langle \pi_*((\pi_\Sigma)_* v), \pi_*((\pi_\Sigma)_* v) \rangle_M \\
&\le c \langle (\pi_\Sigma)_* v, (\pi_\Sigma)_* v \rangle_\Sigma \\
&= c \langle v, v \rangle_{\widetilde{\Sigma}}.
\end{aligned}$$

Analogously, we obtain $\langle \varphi_* v, \varphi_* v \rangle_{\widetilde{M}} \ge c^{-1} \langle v, v \rangle_{\widetilde{\Sigma}}$. Therefore, since the universal Riemannian covering of $\mathbb{P}^n$ is parabolic, it follows that the universal Riemannian covering of Σ^n is parabolic and, hence, Σ^n must be also parabolic. □

4. Rigidity results for spacelike hypersurfaces in $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$

It follows from [8] that in a weighted timelike geodesically complete spacetime $\overline{M}^{n+1}_f$ that contains a timelike line, with $\overline{\mathrm{Ric}}_f(X, X) \ge 0$ for all timelike vector field X and whose weight function f is bounded, the weight function f must be constant along timelike line of $\overline{M}^{n+1}_f$. Consequently, in any weighted standard static spacetime $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ having nonnegative Bakry-Émery-Ricci tensor for timelike vector fields and with bounded weight function f, we have that f does not depend on the parameter of the flow associated with the Killing vector field $\frac{\partial}{\partial t} \equiv Y$.

Motivated by this fact, we will consider standard static spacetimes $\mathbb{P}^n \times_\rho \mathbb{R}_1$ endowed with a weight function f not depending on the parameter $t \in \mathbb{R}$, that is, $\langle \overline{\nabla} f, Y \rangle = 0$. For sake of simplicity, we will denote such an ambient space by $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$.

In this section, we will apply the Proposition 3.3 in order to obtain rigidity results for spacelike hypersurfaces in $\mathbb{P}^n \times_\rho \mathbb{R}_1$. For this, we will need of the following key proposition, which provides an explicit formula for the drift Laplacian of the angle function Θ defined in (2.7).

Proposition 4.1. *Let $\psi : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be an immersed spacelike hypersurface and let $\Theta \in C^\infty(\Sigma^n)$ be the angle function defined in* (2.7). *Then*

$$\Delta_f \Theta = n Y^\top(H_f) + \Big(\widetilde{\mathrm{Ric}}_f(N^*, N^*) - \frac{1}{\rho} \widetilde{\mathrm{Hess}} \rho(N^*, N^*) + \Theta^2 \frac{\widetilde{\Delta}_f(\rho)}{\rho^3} + |A|^2 \Big) \Theta.$$

Proof. Firstly, since Y is a Killing vector field, for any $X \in \mathfrak{X}(\Sigma^n)$, we have

$$\langle \nabla \Theta, X \rangle = X(\Theta) = X(\langle N, Y \rangle) = \langle \overline{\nabla}_X N, Y \rangle + \langle N, \overline{\nabla}_X Y \rangle = \langle -A(Y^\top) - \overline{\nabla}_N Y, X \rangle,$$

which assures that

$$\nabla \Theta = -A(Y^\top) - (\overline{\nabla}_N Y)^\top. \tag{4.1}$$

On the other hand, from (2.4) we note that

$$\begin{aligned}
n Y^\top(H) &= Y^\top(n H_f + \langle \overline{\nabla} f, N \rangle) \\
&= n Y^\top(H_f) + Y^\top(\langle \overline{\nabla} f, N \rangle) \\
&= n Y^\top(H_f) + \langle Y, \overline{\mathrm{Hess}} f(N) \rangle + \Theta \overline{\mathrm{Hess}} f(N, N) - \langle A(Y^\top), \overline{\nabla} f \rangle,
\end{aligned} \tag{4.2}$$

where we used the decomposition $Y = Y^\top - \Theta\, N$.

Moreover, since f is supposed to be invariant along the flow determinate by Y, from (4.1) we obtain that

$$\begin{aligned}\langle \nabla\Theta, \overline{\nabla} f\rangle &= -\langle A(Y^\top) + (\overline{\nabla}_N Y)^\top, \overline{\nabla} f\rangle \\ &= -\langle A(Y^\top), \overline{\nabla} f\rangle - \langle \overline{\nabla}_N Y, \overline{\nabla} f\rangle \\ &= -\langle A(Y^\top), \overline{\nabla} f\rangle + \langle Y, \overline{\nabla}_N \overline{\nabla} f\rangle \\ &= -\langle A(Y^\top), \overline{\nabla} f\rangle + \langle Y, \overline{\mathrm{Hess}} f(N)\rangle.\end{aligned} \tag{4.3}$$

Substituting (4.3) in (4.2) we obtain

$$nY^\top(H) = nY^\top(H_f) + \Theta\overline{\mathrm{Hess}} f(N,N) + \langle \nabla\Theta, \overline{\nabla} f\rangle. \tag{4.4}$$

From [3, Proposition 2.12] we have

$$\Delta\Theta = nY^\top(H) + \Theta(\mathrm{Ric}(N,N) + |A|^2), \tag{4.5}$$

Thus, from (2.2), (2.5), (4.5) and (4.4) we obtain

$$\Delta_f\Theta = nY^\top(H_f) + (\overline{\mathrm{Ric}}_f(N,N) + |A|^2)\Theta. \tag{4.6}$$

Now, if we consider the decomposition $N = N^* + N^\perp$ of N, where $(\cdot)^\perp$ denote the projection of a vector field in $\mathfrak{X}(\mathbb{P}^n \times_\rho \mathbb{R}_1)$ on $\mathfrak{X}(\mathbb{R}_1)$, we have

$$\begin{aligned}\overline{\mathrm{Hess}} f(N,N) &= \langle \overline{\nabla}_N \overline{\nabla} f, N\rangle \\ &= \langle \overline{\nabla}_N \widetilde{\nabla} f, N^* + N^\perp\rangle \\ &= \widetilde{\mathrm{Hess}} f(N^*,N^*) + \frac{1}{\rho}\langle \widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle |N^\perp|^2 \\ &= \widetilde{\mathrm{Hess}} f(N^*,N^*) - \frac{1}{\rho^3}\langle \widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle \Theta^2.\end{aligned} \tag{4.7}$$

From [23, Corollary 7.43] we obtain

$$\overline{\mathrm{Ric}}(N,N) = \widetilde{\mathrm{Ric}}(N^*,N^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\rho(N^*,N^*) + \Theta^2\frac{\widetilde{\Delta}(\rho)}{\rho^3}. \tag{4.8}$$

Hence, from (2.2), (4.7) and (4.8), we have

$$\overline{\mathrm{Ric}}_f(N,N) = \widetilde{\mathrm{Ric}}_f(N^*,N^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\rho(N^*,N^*) + \Theta^2\frac{\widetilde{\Delta}_f(\rho)}{\rho^3} \tag{4.9}$$

Therefore, from (4.9) and (4.6) we obtain the desired result. □

Now, we are in position to present our first rigidity theorem.

Theorem 4.2. *Let $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a weighted standard static spacetimes which is spatially $\widetilde{f}$-parabolic. Suppose that $\widetilde{\mathrm{Ric}}_f \geq 0$, the warping function ρ is convex and $\langle \widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle \leq 0$. Let $\psi : \Sigma^n \hookrightarrow \overline{\mathbb{P}}^{n+1}$ be an immersed spacelike hypersurface with constant f-mean curvature H_f such that its angle function Θ is bounded and $\inf_\Sigma \rho > 0$. Then, Σ^n is totally geodesic and ρ is a positive constant. In addition, if $\widetilde{\mathrm{Ric}}_f$ is positive at some point $p_0 \in \Sigma^n$, then Σ^n is contained in a slice $\mathbb{P}^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.*

Proof. Since H_f is constant, from Proposition 4.1, we have the formula

$$\Delta_f\Theta = \Big(\widetilde{\mathrm{Ric}}_f(N^*,N^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\rho(N^*,N^*) + \Theta^2\frac{\widetilde{\Delta}_f(\rho)}{\rho^3} + |A|^2\Big)\Theta. \tag{4.10}$$

Let us observe that at points where N^* is different from zero we have

$$\frac{1}{\rho}\widetilde{\mathrm{Hess}}\rho(N^*,N^*) = \frac{|N^*|^2}{\rho}\widetilde{\mathrm{Hess}}\rho(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}) = \frac{\Theta^2-\rho^2}{\rho^3}\widetilde{\mathrm{Hess}}\rho(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|})\,.$$

Taking a local orthonormal frame $\{E_1 = \frac{N^*}{|N^*|}, E_2, \dots, E_n\}$ tangent to $\mathbb{P}^n$, we also have

$$\frac{\Theta^2}{\rho^3}\widetilde{\Delta}(\rho) = \frac{\Theta^2}{\rho^3}\widetilde{\mathrm{Hess}}\rho(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}) + \frac{\Theta^2}{\rho^3}\sum_{i=2}^n\widetilde{\mathrm{Hess}}\rho(E_i,E_i).$$

Then

$$-\frac{1}{\rho}\widetilde{\mathrm{Hess}}\rho(N^*,N^*) + \frac{\Theta^2}{\rho^3}\widetilde{\Delta}(\rho) = \frac{1}{\rho}\widetilde{\mathrm{Hess}}\rho(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}) + \frac{\Theta^2}{\rho^3}\sum_{i=2}^n\widetilde{\mathrm{Hess}}\rho(E_i,E_i)$$

and, from (2.5), we obtain

$$\begin{aligned} &-\frac{1}{\rho}\widetilde{\mathrm{Hess}}\rho(N^*,N^*) + \frac{\Theta^2}{\rho^3}\widetilde{\Delta}_f(\rho) \\ &= \frac{1}{\rho}\widetilde{\mathrm{Hess}}\rho(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}) + \frac{\Theta^2}{\rho^3}\sum_{i=2}^n\widetilde{\mathrm{Hess}}\rho(E_i,E_i) - \frac{\Theta^2}{\rho^3}\langle\widetilde{\nabla}f,\widetilde{\nabla}\rho\rangle \geq 0, \end{aligned} \tag{4.11}$$

where in the last step we use the convexity of ρ and the hypothesis $\langle\widetilde{\nabla}f,\widetilde{\nabla}\rho\rangle \leq 0$.

Using our constraint on $\widetilde{\mathrm{Ric}}_f$ and equation (4.11), it follows that Θ is a bounded f-superharmonic function on Σ^n. From Proposition 3.3, Σ^n is f-parabolic and, thus, Θ is constant on it. So, returning to (4.10), we obtain $|A|^2 = 0$, that is, Σ^n is totally geodesic. Now we claim that ρ is a positive constant. Indeed, for any $X \in T\Sigma$, we can write

$$X = X^* - \frac{\langle X,Y\rangle}{\rho^2}Y,$$

where X^* denotes the orthogonal projection of X onto $T\mathbb{P}$. Since Σ^n is totally geodesic, from [23, Proposition 7.35], we have

$$\begin{aligned} X(\Theta) &= \langle N,\overline{\nabla}_XY\rangle \\ &= \langle N,\overline{\nabla}_{X^*}Y\rangle - \frac{\langle X,Y\rangle}{\rho^2}\langle N,\overline{\nabla}_YY\rangle \\ &= \frac{1}{\rho}\langle X,\overline{\nabla}\rho\rangle\langle N,Y\rangle - \frac{1}{\rho}\langle X,Y\rangle\langle N,\overline{\nabla}\rho\rangle. \end{aligned}$$

Thus, from the above equation, we conclude that

$$\nabla\Theta = \frac{1}{\rho}(\Theta\overline{\nabla}\rho - \langle N,\overline{\nabla}\rho\rangle Y).$$

Since Θ is constant, taking into account that $\overline{\nabla}\rho$ and Y are linearly independent, it follows that ρ is a positive constant.

Furthermore, we have, again from (4.10), that $\widetilde{\mathrm{Ric}}_f(N^*, N^*)(p_0) = 0$. So, if $\widetilde{\mathrm{Ric}}_f$ is positive at some point $p_0 \in \Sigma^n$, then $N^*(p_0) = 0$. Consequently, using (2.10) it is not difficult to see that

$$|\nabla h|^2 = \frac{1}{\rho^2}|N^*|^2_{\mathbb{P}} = \frac{1}{\rho^2}\big(\frac{\Theta^2}{\rho^2} - 1\big) = 0,$$

which means that Σ^n is contained in a slice $\mathbb{P}^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$. □

In the next result, we treat the case where $\widetilde{\mathrm{Ric}}_f$ is not necessarily nonnegative.

Theorem 4.3. *Let $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a weighted standard static spacetimes which is spatially $\tilde{f}$-parabolic. Suppose that $\widetilde{\mathrm{Ric}}_f \geq -\kappa$, for some constant $\kappa > 0$, and that ρ is a convex warping function such that $\langle \widetilde{\nabla} f, \widetilde{\nabla} \rho\rangle \leq 0$. Let $\psi : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be an immersed spacelike hypersurface with constant f-mean curvature, bounded angle function Θ and such that $\inf_\Sigma \rho > 0$. If the height function h satisfies*

$$|\nabla h|^2 \leq \frac{\alpha}{\kappa\rho^2}|A|^2, \tag{4.12}$$

for some constant $\alpha \in (0, 1)$, then Σ^n is contained in a slice $\mathbb{P}^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.

Proof. Noting that H_f is constant, $\Theta < 0$ on Σ^n and taking into account our constraint on $\widetilde{\mathrm{Ric}}_f$, from (2.10) and (4.11) jointly with Proposition 4.1, we obtain

$$\Delta_f\Theta \leq (-\kappa\rho^2|\nabla h|^2 + |A|^2)\Theta. \tag{4.13}$$

Using the hypothesis (4.12), from (4.13) we obtain

$$\Delta_f(\Theta) \leq (1 - \alpha)|A|^2\Theta. \tag{4.14}$$

Hence, from (4.14) follows that Θ is a bounded f-superharmonic function on Σ^n. Since Proposition 3.3 guarantees that Σ^n is f-parabolic, Θ must be constant on Σ^n. So, returning to (4.14), we see that Σ^n is totally geodesic. Therefore, hypothesis (4.12) assures that h is constant on Σ^n, that is, there exists $t_0 \in \mathbb{R}$ such that $\Sigma^n \subset \mathbb{P}^n \times \{t_0\}$. □

Next we study specific weight functions that will be defined in terms of the warping function ρ. The following proposition give us an expression for the Laplacian of the height function h in terms of the weighted mean curvature $H_{\log\rho^2}$.

Proposition 4.4. *Let $\psi : \Sigma^n \hookrightarrow \mathbb{P}^n \times_\rho \mathbb{R}_1$ be an immersed spacelike hypersurface and let $h \in C^\infty(\Sigma^n)$ be the height function. Then*

$$\Delta h = -n\rho^{-2}\Theta\, H_{\log\rho^2}, \tag{4.15}$$

where Θ is the angle function and $H_{\log\rho^2}$ is the $\log\rho^2$-mean curvature of Σ^n.

Proof. Let $\{E_1, \dots, E_n\}$ be an orthonormal frame defined in a neighborhood of some point of Σ^n. From (2.8) we note that

$$\begin{aligned}
\rho^{-2}\operatorname{div}(\nabla h) &= \rho^{-2}\operatorname{div}(-\rho^{-2}\,Y^\top)\\
&= -\rho^{-2}\langle\nabla\rho^{-2}, Y^\top\rangle - \rho^{-4}\operatorname{div}(Y^\top)\\
&= \langle\nabla\rho^{-2}, \nabla h\rangle - \rho^{-4}\operatorname{div}(Y + \Theta N)\\
&= \langle\nabla\rho^{-2}, \nabla h\rangle - \rho^{-4}\sum_{i=1}^n\ langle\nabla_{E_i}(Y + \Theta N), E_i\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle \nabla\rho^{-2}, \nabla h\rangle - \rho^{-4}\sum_{i=1}^{n}\langle \overline{\nabla}_{E_i}(Y+\Theta N), E_i\rangle \\
&= \langle \nabla\rho^{-2}, \nabla h\rangle - \rho^{-4}\sum_{i=1}^{n}\underbrace{\langle \overline{\nabla}_{E_i}Y, E_i\rangle}_{0} - \rho^{-4}\sum_{i=1}^{n}\langle \overline{\nabla}_{E_i}(\Theta N), E_i\rangle \\
&= \langle \nabla\rho^{-2}, \nabla h\rangle - \rho^{-4}\sum_{i=1}^{n}\langle E_i(\Theta)\underbrace{\langle N, E_i\rangle}_{0} + \Theta\overline{\nabla}_{E_i}N, E_i\rangle \\
&= \langle \nabla\rho^{-2}, \nabla h\rangle + \rho^{-4}\Theta\,\mathrm{tr}(A) \\
&= \langle \nabla\rho^{-2}, \nabla h\rangle - n\rho^{-4}H\Theta.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Delta h = \mathrm{div}(\nabla h) &= \rho^2\langle \nabla\rho^{-2}, \nabla h\rangle - n\rho^{-2}H\Theta \\
&= \langle \nabla\log\rho^{-2}, -\rho^2\, Y^{\top}\rangle - n\rho^{-2}H\Theta \\
&= -\rho^{-2}\langle \overline{\nabla}\log\rho^{-2}, Y^{\top}\rangle - n\rho^{-2}H\Theta \\
&= -\rho^{-2}\langle \overline{\nabla}\log\rho^{-2}, Y+\Theta N\rangle - n\rho^{-2}H\Theta \\
&= -\rho^{-2}\underbrace{\langle \overline{\nabla}\log\rho^{-2}, Y\rangle}_{0} - \rho^{-2}\langle \overline{\nabla}\log\rho^{-2}, N\rangle\Theta - n\rho^{-2}H\Theta \\
&= -\rho^{-2}\Theta\{nH + \langle \overline{\nabla}(\log\rho^{-2}), N\rangle\} \\
&= -n\rho^{-2}\Theta\, H_{\log\rho^2},
\end{aligned}
$$

where in the last equality we used (2.4). □

In the next theorem, the weighted mean curvature $H_{\log\rho^2}$ of the spacelike hypersurface is not supposed to be constant. Indeed, we just assume a certain control on the sign of $H_{\log\rho^2}$. We recall that a *slab* of a standart static spacetime $\mathbb{P}^n\times_\rho\mathbb{R}_1$ is a region of the type

$$
\mathbb{P}^n\times_\rho[t_1,t_2] = \{(t,q)\in\mathbb{P}^n\times_\rho\mathbb{R}_1 : t_1\le t\le t_2\}.
$$

Theorem 4.5. *Let $\mathbb{P}^n_{\log\rho^2}\times_\rho\mathbb{R}_1$ be a weighted standard static spacetimes which is spatially $\log\widetilde{\rho}^2$-parabolic. Let $\psi:\Sigma^n\hookrightarrow\mathbb{P}^n_{\log\rho^2}\times_\rho\mathbb{R}_1$ be an immersed spacelike hypersurface such that η is bounded. Suppose that the $\log\rho^2$-mean curvature $H_{\log\rho^2}$ and the function $\langle\nabla\rho,\nabla h\rangle$ have opposite signs. If Σ^n lies in a slab, then Σ^n is contained in a slice $\mathbb{P}^n\times\{t_0\}$, for some $t_0\in\mathbb{R}$.*

Proof. By (2.5) and from Proposition 4.4, we have

$$
\begin{aligned}
\Delta_{\log\rho^2}h &= -n\rho^{-2}\Theta\, H_{\log\rho^2} - \langle\nabla\log\rho^2,\nabla h\rangle \\
&= -n\rho^{-2}\Theta\, H_{\log\rho^2} - \frac{2}{\rho}\langle\nabla\rho,\nabla h\rangle.
\end{aligned}
$$

Taking into account that $H_{\log\rho^2}$ and $\langle\nabla\rho,\nabla h\rangle$ have opposite signs, we conclude that $\Delta_{\log\rho^2}h$ does not change sing. Therefore, since Proposition 3.3 guarantees the $\log\rho^2$-parabolicity of Σ^n, h must be constant and, consequently, Σ^n is contained in a slice $\mathbb{P}^n\times\{t_0\}$, for some $t_0\in\mathbb{R}$. □

We recall that a spacelike hypersurface Σ^n is said *f-maximal* if its f-mean curvature vanishes identically on it. In this setting, from Theorem 4.5 we also have the following result.

Corollary 4.6. *Let $\mathbb{P}^n_{\log\rho^2}\times_\rho\mathbb{R}_1$ be a weighted standard static spacetimes which is spatially $\log\widetilde{\rho}^2$-parabolic. Let $\psi:\Sigma^n\hookrightarrow\overline{\mathbb{P}}^{n+1}$ be a $\log\rho^2$-maximal spacelike hypersurface, contained in a slab, such that η is bounded. If the function $\langle\nabla\rho,\nabla h\rangle$ does not change sign, then Σ^n is contained in a slice $\mathbb{P}^n\times\{t_0\}$, for some $t_0\in\mathbb{R}$.*

Proceeding as above, we obtain the following rigidity result.

Theorem 4.7. *Let $\mathbb{P}^n_{\log\rho^{-2}}\times_\rho\mathbb{R}_1$ be a weighted standard static spacetimes which is spatially $\log\widetilde{\rho}^{-2}$-parabolic. Let $\psi:\Sigma^n\hookrightarrow\mathbb{P}^n_{\log\rho^{-2}}\times_\rho\mathbb{R}_1$ be a maximal spacelike hypersurface such that η is bounded and $\inf_\Sigma\rho>0$. If $\mathrm{Ric}_{\log\rho^{-2}}\geq\kappa$, for some constant $\kappa>0$, then Σ^n is contained in a slice $\mathbb{P}^n\times\{t_0\}$, for some $t_0\in\mathbb{R}$.*

Proof. Firstly, observe that, reasoning as in the proof of Proposition 4.4, we obtain

$$\begin{aligned}\Delta h=\mathrm{div}(\nabla h)&=\rho^2\langle\nabla\rho^{-2},\nabla h\rangle-n\rho^{-2}H\Theta\\&=\langle\nabla\log\rho^{-2},\nabla h\rangle-n\rho^{-2}H\Theta.\end{aligned}$$

Therefore, using (2.5), we obtain

$$\Delta_{\log\rho^{-2}}h=-n\rho^{-2}H\Theta. \tag{4.16}$$

Now, from Bochner's formula (see [32, page 378]) we have

$$\frac{1}{2}\Delta_{\log\rho^{-2}}|\nabla h|^2=|\operatorname{Hess}h|^2+\mathrm{Ric}_{\log\rho^{-2}}(\nabla h,\nabla h)+\langle\nabla\Delta_{\log\rho^{-2}}h,\nabla h\rangle. \tag{4.17}$$

Consequently, taking into account our restriction on $\mathrm{Ric}_{\log\rho^{-2}}$ and the assumption that Σ^n is maximal, from (4.16) and (4.17), we obtain

$$\frac{1}{2}\Delta_{\log\rho^{-2}}|\nabla h|^2\geq\mathrm{Ric}_{\log\rho^{-2}}(\nabla h,\nabla h)\geq\kappa|\nabla h|^2\geq0. \tag{4.18}$$

On the other hand, Proposition 3.3 guarantees that Σ^n is $\log\rho^{-2}$-parabolic. Since, from (2.10), $\inf_\Sigma\rho>0$ implies in the boundedness of $|\nabla h|$ and, consequently, in the boundedness of $|\nabla h|^2$, we conclude from $\log\rho^{-2}$- parabolicity of Σ^nthat $|\nabla h|^2$ is constant, and then $\Delta_{\log\rho^2}|\nabla h|^2=0$. Returning to (4.18), we obtain that $|\nabla h|=0$ and Σ^n is contained in a slice. □

5. Entire Killing graphs and the mean curvature equation in $\mathbb{P}^n_f\times_\rho\mathbb{R}_1$

According to [10], we define the *entire Killing graph* $\Sigma(z)$ associated with a smooth function $z\in C^\infty(\mathbb{P})$ as been the hypersurface given by

$$\Sigma(z)=\{\Psi(x,z(x)):x\in\mathbb{P}^n\}\subset\mathbb{P}^n\times_\rho\mathbb{R}_1.$$

The metric induced on $\mathbb{P}^n$ from the Lorentzian metric (2.1) via $\Sigma(z)$ is given by

$$\langle,\rangle_z=\langle,\rangle_{\mathbb{P}}-\rho^2dz^2.$$

Moreover, $\Sigma(z)$ is spacelike if, and only if, $\rho^2|Dz|^2_{\mathbb{P}}<1$, where Dz denotes the gradient of a function z with respect to the metric $\langle,\rangle_{\mathbb{P}}$ of $\mathbb{P}^n$. Indeed, if $\Sigma(z)$ is spacelike, then

$$0<\langle Dz,Dz\rangle_z=\langle Dz,Dz\rangle_{\mathbb{P}}-\rho^2\langle Dz,Dz\rangle^2_{\mathbb{P}}$$

and, hence, we conclude that $\rho^2|Dz|^2_{\mathbb{P}} < 1$. Conversely, if $\rho^2|Dz|^2_{\mathbb{P}} < 1$ and X is a vector field tangent to $\Sigma(z)$, we obtain, from Cauchy-Schwarz inequality,

$$\langle X, X\rangle_z = \langle X^*, X^*\rangle_{\mathbb{P}} - \rho^2\langle Dz, X^*\rangle^2_{\mathbb{P}} \geq \langle X^*, X^*\rangle_{\mathbb{P}}(1 - \rho^2|Dz|^2_{\mathbb{P}}),$$

where X^* is the orthogonal projection of X onto $T\mathbb{P}^n$. Thus, $\langle X, X\rangle_z \geq 0$ and $\langle X, X\rangle_z = 0$ if, and only if, $X = 0$.

The function $g : \mathbb{P}^n \times \mathbb{R}_1 \to \mathbb{R}$ given by $g(x, t) = z(x) - t$ is such that $\Sigma(z) = \Psi(g^{-1}(0))$. Thus, for each vector field X tangent to $\mathbb{P}^n \times_\rho \mathbb{R}_1$, we have

$$X(g) = X^*(g) - \frac{1}{\rho^2}\langle X, \partial_t\rangle\partial_t(g) = \langle \frac{1}{\rho^2}\partial_t + Dz, X\rangle.$$

Hence,

$$\overline{\nabla} g = \frac{1}{\rho^2}\partial_t + Dz$$

is a normal vector field on $g^{-1}(0)$ and, consequently,

$$N_0 = \Psi_*(\overline{\nabla} g) = \frac{1}{\rho^2}Y + \Psi_*(Dz)$$

is a normal timelike vector field on $\Sigma(z)$. Since

$$|N_0| = \frac{(1 - \rho^2|Dz|^2_{\mathbb{P}})^{1/2}}{\rho},$$

it follows that

$$N = \frac{N_0}{|N_0|} = \frac{1}{\rho(1 - \rho^2|Dz|^2_{\mathbb{P}})^{1/2}}(Y + \rho^2\Psi_*(Dz)) \tag{5.1}$$

defines the future-pointing Gauss map of $\Sigma(z)$ such that its angle function is

$$\Theta = \langle N, Y\rangle = -\frac{\rho}{(1 - \rho^2|Dz|^2_{\mathbb{P}})^{1/2}} < 0. \tag{5.2}$$

Moreover, for each vector field X tangent to $\mathbb{P}^n$, the shape operator A of $\Sigma(z)$ with respect to N is given by

$$\begin{aligned} AX = &-\frac{\rho}{(1 - \rho^2|Dz|^2_{\mathbb{P}})^{1/2}}D_X Dz - \frac{\rho^3\langle D_X Dz, Dz\rangle}{(1 - \rho^2|Dz|^2_{\mathbb{P}})^{3/2}}Dz - \frac{\rho^2\langle D\rho, X\rangle|Dz|^2_{\mathbb{P}}}{(1 - \rho^2|Dz|^2_{\mathbb{P}})^{3/2}}Dz \\ &- \frac{\langle D\rho, X\rangle}{(1 - \rho^2|Dz|^2_{\mathbb{P}})^{1/2}}Dz - \frac{\langle Dz, X\rangle}{(1 - \rho^2|Dz|^2_{\mathbb{P}})^{1/2}}D\rho, \end{aligned} \tag{5.3}$$

where D denotes the Levi-Civita connections in $\mathbb{P}^n$.

So, it follows from (5.3) that the mean curvature H_z of a spacelike entire Killing graph $\Sigma(z)$ is given by

$$nH(z) = \operatorname{Div}(\frac{\rho Dz}{(1 + \rho^2|Dz|^2_{\mathbb{P}})^{1/2}}) + \frac{\langle Dz, D\rho\rangle}{(1 + \rho^2|Dz|^2_{\mathbb{P}})^{1/2}},$$

where Div stands for the divergence operator on $\mathbb{P}^n$ with respect to the metric $\langle,\rangle_{\mathbb{P}}$. A direct computation shows that the f-mean curvature is given by

$$n(H_z)_f = \operatorname{Div}_f(\frac{\rho Dz}{(1 - \rho^2|Dz|^2_{\mathbb{P}})^{1/2}}) + \frac{\langle Dz, D\rho\rangle}{(1 - \rho^2|Dz|^2_{\mathbb{P}})^{1/2}}.$$

From the previous discussion, an entire Killing graph $\Sigma(z)$ is spacelike with constant f-mean curvature C if, and only if, the function $z \in C^\infty(\mathbb{P})$ satisfies the following elliptic partial differential equation of f-divergence form

$$\begin{gathered}\operatorname{Div}_f\Big(\frac{\rho Dz}{(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}}\Big)+\frac{\langle Dz, D\rho\rangle}{(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}}=C, \quad \text{in } \mathbb{P}^n \\ \rho^2|Dz|^2_{\mathbb{P}}<1.\end{gathered} \tag{5.4}$$

In what follows, we will use the theorems obtained in the previous section, on entire Killing graph context, to obtain uniqueness results for equations of the type (5.4). We start by applying the Theorem 4.2 to get the following result.

Theorem 5.1. *Let $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a weighted standard static spacetimes which is spatially $\widetilde{f}$-parabolic with convex warping function ρ, $\langle\widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle \le 0$ and $\widetilde{\operatorname{Ric}}_f \ge 0$. If the entire Killing graph $\Sigma(z)$ associated with $z \in C^\infty(\mathbb{P})$ is such that $\rho|_{\Sigma(z)}$ is bounded and $\widetilde{\operatorname{Ric}}_f$ is positive at some point $p_0 \in \Sigma(z)$, then the only solutions of the problem*

$$\begin{gathered}\operatorname{Div}_f\Big(\frac{\rho Dz}{(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}}\Big)+\frac{\langle Dz, D\rho\rangle}{(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}}=C, \quad z\in C^\infty(\mathbb{P}) \\ \sup_{\Sigma(z)}(\rho^2|Dz|^2_{\mathbb{P}})<1,\end{gathered}$$

are constants.

Proof. Since we are supposing that $\sup\rho^2|Dz|^2_{\mathbb{P}} < 1$, from (5.2), the boundness of $\rho|_{\Sigma(z)}$ is equivalent to the boundness of Θ. Furthermore, we observe that the condition $\sup\rho^2|Dz|^2_{\mathbb{P}} < 1$ also implies the boundness of η. Indeed, using (5.2) again, we have that

$$\eta=\frac{1}{(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}}.$$

Hence, we can disregard the hypothesis $\inf_{\Sigma(z)}\rho > 0$ in Theorem 4.2 to obtain the present result. □

An important example of weighted Riemannian manifold is the so-called *Gaussian space* $\mathbb{G}^n$, which corresponds to the Euclidean space $\mathbb{R}^n$ endowed with the Gaussian probability measure

$$e^{-f}dx^2=(2\pi)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2}}dx^2.$$

Concerned with the weighted product space $\mathbb{G}^n \times \mathbb{R}_1$, An et al extended the classical Bernstein's theorem [6] showing that the only weighted minimal graphs $\Sigma^n(z)$ of functions $z(x_2, \cdots, x_{n+1}) = x_1$ over $\mathbb{G}^n$, with $\sup_{\Sigma(z)}|Dz|_{\mathbb{G}} < 1$, are the hyperplanes $x_1 = \text{constant}$ (see [2, Theorem 4]).

Taking into account this previous discussion, from Theorem 5.1 we obtain an extension of Theorem 4 of [2].

Corollary 5.2. *Consider the weighted standard static spacetime $\mathbb{G}^n \times_\rho \mathbb{R}_1$, where $\mathbb{G}^n$ is the Gaussian space and the warping function ρ is convex with $\langle\widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle \le 0$. If the entire Killing graph $\Sigma(z)$ associated with $z \in C^\infty(\mathbb{G})$ is such that $\rho|_{\Sigma(z)}$ is bounded, the only solutions of the problem*

$$\operatorname{Div}_f\Big(\frac{\rho Dz}{(1-\rho^2|Dz|^2_{\mathbb{G}})^{1/2}}\Big)+\frac{\langle Dz, D\rho\rangle}{(1-\rho^2|Dz|^2_{\mathbb{G}})^{1/2}}=C, \quad z\in C^\infty(\mathbb{G})$$

$$\sup_{\Sigma(z)}(\rho^2|Dz|^2_{\mathbb{G}}) < 1,$$

are constants.

Proof. We note that, since $\mathrm{Vol}_f(\mathbb{G}^n) = 1$, [18, Remark 3] guarantees that $\mathbb{G}^n$ is f-parabolic. Moreover, with a straightforward computation, we obtain that $\widetilde{\mathrm{Ric}}_f = 1$. Therefore, since $\mathbb{G}^n$ is also simply connected, the result follows from Theorem 5.1. □

The next result is an application of Theorem 4.3.

Theorem 5.3. *Let $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a weighted standard static spacetime which is spatially $\widetilde{f}$-parabolic with convex warping function ρ, $\langle \widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle \le 0$ and $\widetilde{\mathrm{Ric}}_f \ge -\kappa$, for some constant $\kappa > 0$. If the entire Killing graph $\Sigma(z)$ associated with z is such that $\rho|_{\Sigma(z)}$ is bounded and $\alpha \in (0,1)$ is a constant, the only solutions of the problem*

$$\begin{gathered}\mathrm{Div}_f\Big(\frac{\rho Dz}{(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}}\Big) + \frac{\langle Dz, D\rho\rangle}{(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}} = C, \quad z \in C^\infty(\mathbb{P}) \\ \sup_{\Sigma(z)}(\rho^2|Dz|^2_{\mathbb{P}}) < \frac{\alpha|A|^2}{\alpha|A|^2+\kappa},\end{gathered} \tag{5.5}$$

are constants.

Proof. From equation (5.11) we have

$$|N^*|^2_{\mathbb{P}} = \frac{\rho^2|Dz|^2_{\mathbb{P}}}{1-\rho^2|Dz|^2_{\mathbb{P}}}. \tag{5.6}$$

Then (2.10) and (5.6) give us the relation

$$|\nabla h|^2 = \frac{|Dz|^2_{\mathbb{P}}}{1-\rho^2|Dz|^2_{\mathbb{P}}}. \tag{5.7}$$

Now, using (5.7) we conclude that the hypothesis the hypothesis (4.12) is equivalent to

$$\rho^2|Dz|^2_{\mathbb{P}} \le \frac{\alpha|A|^2}{\alpha|A|^2+\kappa}.$$

Furthermore, since $\kappa > 0$, we have that that $\frac{\alpha|A|^2}{\alpha|A|^2+\kappa} \le 1$. Hence, the result follows from Theorem 4.3. □

Reasoning as in the Corollary 5.2, we have the following result.

Corollary 5.4. *Consider the weighted standard static spacetime $\mathbb{G}^n \times_\rho \mathbb{R}_1$, where $\mathbb{G}^n$ is the Gaussian space and the warping function ρ is convex with $\langle \widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle \le 0$. If the entire Killing graph $\Sigma(z)$ associated with $z \in C^\infty(\mathbb{G})$ is such that $\rho|_{\Sigma(z)}$ is bounded, then, for any constants $k > 0$ and $\alpha \in (0,1)$, the only solutions of the problem*

$$\begin{gathered}\mathrm{Div}_f(\frac{\rho Dz}{(1-\rho^2|Dz|^2_{\mathbb{G}})^{1/2}}) + \frac{\langle Dz, D\rho\rangle}{(1-\rho^2|Dz|^2_{\mathbb{G}})^{1/2}} = C, \quad z \in C^\infty(\mathbb{G}) \\ \sup_{\Sigma(z)}(\rho^2|Dz|^2_{\mathbb{G}}) < \frac{\alpha|A|^2}{\alpha|A|^2+\kappa},\end{gathered} \tag{5.8}$$

are constants.

From Theorem 4.5, we obtain the following result.

Theorem 5.5. *Let $\mathbb{P}^n_{\log\rho^2}\times_\rho\mathbb{R}_1$ be a weighted standard static spacetimes which is spatially $\log\widetilde{\rho}^2$-parabolic. If the entire Killing graph associated with z is such that $\langle\nabla\rho,\Psi_*(Dz)\rangle$ does not change sign, then the only bounded solutions of the problem*

$$\begin{gathered}\operatorname{Div}_{\log\rho^2}\Big(\frac{\rho Dz}{(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}}\Big)+\frac{\langle Dz,D\rho\rangle}{(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}}=C,\qquad z\in C^\infty(\mathbb{P})\\ \sup_{\Sigma(z)}(\rho^2|Dz|^2_{\mathbb{P}})<1.\end{gathered}\tag{5.9}$$

are constants.

Proof. Firstly, observe that

$$\begin{aligned}\langle\nabla\rho,\nabla N\rangle&=\nabla h(\rho)=-\frac{1}{\rho^2}Y^\top(\rho)\\&=-\frac{1}{\rho^2}Y^\top\big((-\langle Y,Y\rangle)^{1/2}\big)\\&=-\frac{1}{\rho^2}\big(\frac{1}{2}(-\langle Y,Y\rangle)^{1/2}Y^\top\langle Y,Y\rangle\big)\\&=-\frac{1}{2\rho^3}Y^\top\langle Y,Y\rangle\big)\\&=-\frac{1}{\rho^3}\langle\overline{\nabla}_{Y^\top}Y,Y\rangle\\&=-\frac{1}{\rho^3}\langle\overline{\nabla}_{Y+\Theta N}Y,Y\rangle\\&=-\frac{1}{\rho^3}\big(\underbrace{\langle\overline{\nabla}_YY,Y\rangle}_{0}+\langle\overline{\nabla}_{\Theta N}Y,Y\rangle\big)=-\frac{1}{\rho^3}\langle\overline{\nabla}_{\Theta N}Y,Y\rangle\\&=-\frac{\Theta}{\rho^3}\langle\overline{\nabla}_NY,Y\rangle\\&=-\frac{\Theta}{2\rho^3}N\langle Y,Y\rangle=-\frac{\Theta}{2\rho^3}N(\rho^2)\\&=-\frac{\Theta}{2\rho^3}-2\rho N^*(\rho)\\&=\frac{\Theta}{\rho^2}\langle\overline{\nabla}\rho,N^*\rangle.\end{aligned}\tag{5.10}$$

On the other hand, from (5.1), we have

$$N^*=N-N^\perp=\frac{\rho\Psi_*(Dz)}{(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}}.\tag{5.11}$$

Hence, from (5.10) and (5.11) we obtain

$$\langle\nabla\rho,\nabla N\rangle=\frac{\Theta}{\rho}\langle\overline{\nabla}\rho,\frac{\rho\Psi_*(Dz)}{(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}}\rangle=\frac{\Theta}{\rho(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}}\langle\overline{\nabla}\rho,\Psi_*(Dz)\rangle.$$

Therefore, $\langle\nabla\rho,\nabla N\rangle$ do not change of sign if and only if $\langle\overline{\nabla}\rho,\Psi_*(Dz)\rangle$ do not change of sign and the result follows from Corollary 4.6. $\square$

Taking

$$\rho = \left(e^{\frac{|x|^2}{2}+\log(2\pi)^{\frac{n}{2}}}\right)^{1/2} \tag{5.12}$$

in Theorem 5.5, we obtain the following consequence.

Corollary 5.6. *Consider the weighted standard static spacetime $\mathbb{G}^n \times_\rho \mathbb{R}_1$, where $\mathbb{G}^n$ is the Gaussian space and ρ is defined in (5.12). If the entire Killing graph associate to z is such that $\langle \nabla\rho, \Psi_*(Dz)\rangle$ does not change sign, then the only bounded solutions of the problem*

$$\begin{gathered}\operatorname{Div}_f\Big(\frac{\rho Dz}{(1-\rho^2|Dz|^2_{\mathbb{G}})^{1/2}}\Big) + \frac{\langle Dz, D\rho\rangle}{(1-\rho^2|Dz|^2_{\mathbb{G}})^{1/2}} = C, \quad z \in C^\infty(\mathbb{G})\\ \sup_{\Sigma(z)}(\rho^2|Dz|^2_{\mathbb{G}}) < 1,\end{gathered}$$

are constants.

Applying the Theorem 4.7 we obtain the following result.

Theorem 5.7. *Let $\mathbb{P}^n_{\log\rho^{-2}} \times_\rho \mathbb{R}_1$ be a weighted standard static spacetimes which is spatially $\log\widetilde{\rho}^{-2}$-parabolic. If the entire Killing graph associate to z is such that $|Dz|^2_{\mathbb{P}}$ is bounded and $\operatorname{Ric}_{\log\rho^{-2}} \geq \kappa$, for some constant $\kappa > 0$, then the only bounded solutions of the problem*

$$\begin{gathered}\operatorname{Div}\Big(\frac{\rho Dz}{(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}}\Big) + \frac{\langle Dz, D\rho\rangle}{(1-\rho^2|Dz|^2_{\mathbb{P}})^{1/2}} = 0, \quad z \in C^\infty(\mathbb{P})\\ \sup_{\Sigma(z)}(\rho^2|Dz|^2_{\mathbb{P}}) < 1,\end{gathered} \tag{5.13}$$

are constants.

Proof. We observe that if $z \in C^\infty(\mathbb{P})$ is solution of problem (5.13), then the entire Killing graph $\Sigma(z)$ is spacelike and maximal. Moreover using (5.7), we note that the boundness of $|\nabla h|^2$ follows from the boundness of $|Dz|^2_{\mathbb{P}}$. Then, the result follows from Theorem 4.7. □

Finally, considering

$$\rho = (e^{\frac{|x|^2}{2}+\log(2\pi)^{\frac{n}{2}}})^{-1/2} \tag{5.14}$$

in Theorem 5.7, we have the following result.

Corollary 5.8. *Consider the weighted standard static spacetime $\mathbb{G}^n \times_\rho \mathbb{R}_1$, where $\mathbb{G}^n$ is the Gaussian space and ρ is defined in (5.14). If the entire Killing graph associate to z is such that $|Dz|^2_{\mathbb{P}}$ is bounded and $\operatorname{Ric}_{\log\rho^{-2}} \geq \kappa$, for some constant $\kappa > 0$, then the only bounded solutions of the problem*

$$\begin{gathered}\operatorname{Div}\Big(\frac{\rho Dz}{(1-\rho^2|Dz|^2_{\mathbb{G}})^{1/2}}\Big) + \frac{\langle Dz, D\rho\rangle}{(1-\rho^2|Dz|^2_{\mathbb{G}})^{1/2}} = 0, \quad z \in C^\infty(\mathbb{G})\\ \sup_{\Sigma(z)}(\rho^2|Dz|^2_{\mathbb{G}}) < 1,\end{gathered}$$

are constants.

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