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SPATIAL DYNAMICS OF A NONLOCAL BISTABLE REACTION DIFFUSION EQUATION

BANG-SHENG HAN, MENG-XUE CHANG, YINGHUI YANG

ABSTRACT. This article concerns a nonlocal bistable reaction-diffusion equation with an integral term. By using Leray-Schauder degree theory, the shift functions and Harnack inequality, we prove the existence of a traveling wave solution connecting 0 to an unknown positive steady state when the support of the integral is not small. Furthermore, for a specific kernel function, the stability of positive equilibrium is studied and some numerical simulations are given to show that the unknown positive steady state may be a periodic steady state. Finally, we demonstrate the periodic steady state indeed exists, using a center manifold theorem.

1. INTRODUCTION

In this article, we consider the integro-differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ku^2(1 - \phi * u) - bu \quad \text{in } \mathbb{R} \times (0, \infty), \tag{1.1}$$

where

$$(\phi * u)(x) := \int_{\mathbb{R}} \phi(x - y)u(y, t)dy,$$

and $\phi(x)$ satisfies

$$\phi(x) \geq 0, \quad \phi(0) > 0, \quad \phi(x) = \phi(-x), \quad \int_{\mathbb{R}} \phi(x)dx = 1, \quad \int_{\mathbb{R}} x^2\phi(x)dx < \infty. \tag{1.2}$$

Such equations describe the nonlocal consumption of resource in the population dynamics, and in which $u(\geq 0)$ means the density of the population (see [4, 16]). The terms in (1.1) are interpreted as follows: the term ku^2 $(k > 0)$, represents the reproduction of the population, which is in direct proportion to the density square under the sexual case, and to the available resources; the integral term describes nonlocal consumption of the resources; the term $-bu$ corresponds to mortality of the population. Similar equations also arise in species evolution [4], ecology [15, 19], adaptive dynamics [16] (see also [9, 10, 17]), and Brownian motion [31].

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If the kernel ϕ tends to be a δ-function, equation (1.1) becomes the classical reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ku^2(1-u) - bu \quad \text{in } \mathbb{R}\times(0,\infty). \tag{1.3}$$

Moreover, if $k^2 - 4kb > 0$, equation (1.3) is bistable and the nonlinearity $f(u) = ku^2(1-u) - bu$ has three zeros:

$$u_+ = 0, \quad u_0 = \frac{k - \sqrt{k^2 - 4kb}}{2k}, \quad u_- = \frac{k + \sqrt{k^2 - 4kb}}{2k}.$$

In this case, we know (1.3) has a globally asymptotically stable traveling wave solution of the form $u(x,t) = u(x - ct)$ with the limits $u \to u_\pm$ as $x \to \pm\infty$, where the constant c denotes the wave speed and is unique up to translations in space (see e.g. [28] and there references therein).

Recently, much attention was devoted to introducing a nonlocal effect into the nonlinear reaction term (in fact, now researches about the nonlocality mainly focus on the monostable case, see [1, 2, 5, 6, 14, 13, 18, 23, 24, 27], the results for the bistable case are relatively seldom). When the support of the function ϕ is sufficiently small, Apreutesei et al [4] explored the property of the Fredholm operator

$$Lu = u'' + cu' + [2kw(1 - \phi * w) - b]u - kw^2(\phi * u),$$

where w satisfies

$$w'' + cw' + kw^2(1 - \phi * w) - bw = 0.$$

And they used it to prove that (1.1) admits traveling wave solutions connecting u_+ to u_-. Similar results can be established by using the method of Wang et al [29], where the quasi-monotonicity conditions are needed.

When the support of the integral is not small, by using a topological degree for the proper Fredholm operator, Demin and Volpert [11] proved that the equation admits monotone traveling wave solution connecting 0 to $\frac{1}{2} + \sqrt{\frac{1}{4} - \alpha}$ when the nonlinearities with the form of $u(\phi * u)(1-u) - \alpha u$. Alfaro et al [3] researched the case of the nonlinearities with the form of $u(u-\theta)(1-\phi*u)$. For more results about bistable reaction-diffusion equation can be referred to [8, 25, 26, 30]. It should be pointed out that the difficulties caused by different nonlinear term (the integral located at different place) are different. And the methods used to overcome these difficulties are also different.

More recently, the research about (1.1) has made some progresses. By using sub- and super-solutions for an appropriate monotone operator and cut-off approximation, Li et al [22] proved that equation (1.1) exists monotone traveling wave solution for the monostable case. However, there is no result for the bistable case when the support of ϕ is not small. The purpose of this paper is to find (at least partially) the traveling wave solution of (1.1) for the bistable case by developing the methods of Alfaro et al [3], Apreutesei et al [4] and Han et al [20]. In contrast with [3, 4, 20], the main difficulty in the study of (1.1) is to get a priori estimate of wave speed. To overcome such difficulty, we study an evolution equation and obtain some estimations. Furthermore, using the Leray-Schauder degree theory, we prove that equation (1.1) exists the traveling wave solution connecting 0 to an unknown positive steady state. In addition, in order to more clearly describe the behavior of the solution about (1.1), we give the stability analysis and numerical simulations. Now, we state our main results.

Theorem 1.1. *There exists a traveling wave solution* (c, u) *such that*

$$-cu' = u'' + ku^2(1 - \phi * u) - bu \quad in\ \mathbb{R}, \tag{1.4}$$

with the boundary conditions

$$\lim_{x\to+\infty} u(x) = 0, \quad \liminf_{x\to-\infty} u(x) > 0. \tag{1.5}$$

In above theorem, we prove that (1.1) admits traveling wave solutions connecting 0 to an unknown positive state, while we do not consider the relationship between u_0 and the value of this unknown positive state at the negative infinite. In fact, our main purpose is to study whether the unknown steady state can be periodic, so the results of Theorem 1.1 can be used in future work.

In addition, the numerical result about (1.1) is shown in [4] when the support of ϕ is not small. For completeness, we give the result when the support of the integral is small.

Remark 1.2 ([4, Theorem 4.1])**.** Assume that ϕ satisfies (1.2). Then there exists $\varepsilon_0 > 0$ such that (1.1) admits a solution $(u_\varepsilon, c_\varepsilon) \in C^{2+\alpha}(\mathbb{R}) \times \mathbb{R}$ satisfying (1.4) for any $|\varepsilon| < \varepsilon_0$, and with the following boundary conditions

$$\lim_{x\to+\infty} u_\varepsilon = 0, \quad \lim_{x\to-\infty} u_\varepsilon = u_-.$$

Next we study the stability of $u = u_-$ when the kernel ϕ takes some special forms.

Theorem 1.3.
(i) *If the kernel has the form* $\phi(x) = \frac{1}{2}e^{-|x|}$*, then* (4.3) *has Turing bifurcation around* $(u, v) = (\frac{k+\sqrt{k^2-4kb}}{2k}, \frac{k+\sqrt{k^2-4kb}}{2k})$ *at* $b = b_c$*, where* b_c *is defined in* (4.7)*.*
(ii) *If the kernel has the form* $\phi(x) = Ae^{-a|x|} - e^{-|x|}$*, then* (4.11) *has Turing bifurcation around* $(u, v, w) = (\frac{k+\sqrt{k^2-4kb}}{2k}, 3\frac{k+\sqrt{k^2-4kb}}{2k}, -2\frac{k+\sqrt{k^2-4kb}}{2k})$ *at* $b = b_c$*, where* b_c *is defined in* (4.15)*.*

In Theorem 1.3, we use the kernel with two specific forms, and use the linear stability analysis to take into account the stability of the state $u = u_-$. Moreover, when $u(x, 0)$ takes the form of (4.8), through numerical simulation, we show that the wave can connect 0 to a periodic steady state. Next, we show that (1.1) indeed admits periodic steady state. Previously, we gave some other assumptions on the kernel ϕ. After linearizing equation (1.1) around $u = u_-$, we can obtain the dispersion relation

$$d(\lambda, \sigma, k, b) := -\sigma^2 + b - ku_-^2\widehat{\phi}(\sigma) - \lambda. \tag{1.6}$$

For $\phi(x)$ satisfying (1.2), we also assume that there exists a unique $\sigma_c > 0$, $k_c > 0$ and $b_c > 0$ so that

(i) $d(0, \sigma_c, k_c, b_c) = 0$.
(ii) $\partial_\sigma d(0, \sigma_c, k_c, b_c) = 0$.
(iii) $\partial_{\sigma\sigma} d(0, \sigma_c, k_c, b_c) < 0$.

Then, the result about the existence of stationary periodic solutions of (1.1) can be stated as follows.

Theorem 1.4. *Assume that* $\phi(x)$ *satisfies* (1.2) *and* $d(\lambda, \sigma, k, b)$ *satisfies the three conditions* (i)-(iii) *above. Let* $k = k_c + \varepsilon^2$*,* $b = b_c + \frac{\varepsilon^2}{2}$ *and* $\sigma = \sigma_c + \delta$*. Then there* *exists an* $\varepsilon_0 > 0$ *such that equation* (1.1) *has a* $2\pi/\sigma$*-periodic solution with leading expansion of the form*

$$u_{\varepsilon,\delta}(x) = u_- + \sqrt{\Lambda}\cos((\sigma_c + \delta)x) + O(|Q|)$$

for all $\varepsilon \in (0, \varepsilon_0]$ *and* δ *satisfying*

$$\begin{aligned}&\Big(-\frac{k_c - 2b_c + \sqrt{k_c^2 - 4k_c b_c}}{4}\widehat{\phi}''(k_c) - 1\Big)\delta^2\\ &< \frac{(4b_c + \varepsilon^2)\widehat{\phi}(\sigma_c + \delta)}{\sqrt{k_c^2 - 4k_c b_c} + \sqrt{k_c^2 - 4k_c b_c - 4b_c\varepsilon^2 - \varepsilon^4}}\varepsilon^2 + \frac{\varepsilon^2}{2},\end{aligned}$$

where Λ *is defined in* (5.5) *and* Q *is defined in* (5.7).

This article is organized as follows. In Section 2 and 3, we prove Theorem 1.1. In Section 4, we research the stability of the state $u = u_-$, that is Theorem 1.3. In Section 5, we give the proof of Theorem 1.4. Finally, further discussions are made in Section 6.

2. Existence of traveling wave solutions

In this section, we construct a traveling wave solution of (1.1). Specifically, in subsection 2.1, we use the method of [3, 7] to give a priori estimates of solution u in a finite domain. In subsection 2.2, we construct a solution (c, u).

2.1. **A priori estimates of solution u in a finite domain.** For $a > 0$ and $0 \le \tau \le 1$, we seek a function $u = u_\tau^a \in C^2([-a, a], \mathbb{R})$ and a speed $c = c_\tau^a \in \mathbb{R}$ satisfying

$$T_\tau(a) : \begin{cases} -u'' - cu' = ku(u - u_0)(u_- - u) + \tau k u^2(u - \phi * \widehat{u}) \quad \text{in}(-a, a), \\ u(-a) = u_-, \quad u(0) = \varepsilon/2, \quad u(a) = 0, \end{cases} \tag{2.1}$$

where

$$\widehat{u} = \begin{cases} u_-, & \text{in } (-\infty, -a), \\ u, & \text{in } (-a, a), \\ u_+, & \text{in } (a, \infty), \end{cases}$$

and the number ε will be determined later. Firstly we introduce a homotopy from $T_0(a)$ (a local problem) to $T_\tau(1)$ (a nonlocal problem). Secondly, we obtain a solution of $T_1(a)$ by using a Leray-Schauder degree.

For convenience, in the sequel, we often replace u with $\widehat{u}$. If

$$u(x_l) = \min_{x \in [-a,a]} u(x)$$

and $-u'' - cu' = ku(u - u_0)(u_- - u)$ on a neighborhood of x_l, we obtain $u \equiv u_l$ by the maximum principle. But $u \equiv u_l$ is obviously impossible. So any solution of $T_\tau(a)$ satisfies $u \ge 0$, and by the maximum principle we obtain

I cahanged $P_\tau(a)$ to $T_\tau(a)$. Please check it

$$u > 0, \quad -u'' - cu' = ku(u - u_0)(u_- - u) + \tau k u^2(u - \phi * u) \quad \text{in } (-a, a). \tag{2.2}$$

The following lemma gives a priori bounds for u.

Lemma 2.1. *There exist* $M(\phi) > u_-$ *and* $a_0 > 0$*, such that for every* $0 \le \tau \le 1$ *and* $a \ge a_0$*, any solution of* $T_\tau(a)$ *satisfies*

$$0 \le u(x) \le M, \quad \forall x \in [-a, a].$$

Proof. If $\tau = 0$, that is $T_0(a)$,

$$-u'' - cu' = ku(u-u_0)(u_- - u) \quad \text{in } (-a,a),$$
$$u(-a) = u_-, \quad u(0) = \frac{\varepsilon}{2}, \quad u(a) = 0,$$

we can obtain directly $0 \le u(x) \le u_- \le M$. Now, for $0 < \tau \le 1$, assume $\overline{M} := \max_{x\in[-a,a]} u(x) > u_-$ (otherwise, the conclusion is trued). Since $u(-a) = u_-$ and $u(a) = 0$, there exists a point $x_m \in (-a,a)$, such that $u(x_m) = \overline{M}$. By evaluating (2.2) at x_m, we obtain $(\phi * u)(x_m) \le \frac{kM-b}{kM} < 1$. In addition, from $u \ge 0$, we obtain

$$\begin{aligned} -u'' - cu' &= ku(u-u_0)(u_- - u) + \tau k u^2 (u - \phi * u) \\ &\le ku^2 \le k\overline{M}^2. \end{aligned} \tag{2.3}$$

Firstly, we consider the case of $c < 0$. Multiplying $e^{-|c|z}$ on (2.3) and integrating from x to x_m yields

$$\int_x^{x_m} (u'(z)e^{-|c|z})' dz \ge -\int_x^{x_m} k\overline{M}^2 e^{-|c|z} dz.$$

Since $u'(x_m) = 0$, separating $u'(x)$ and integrating from x to x_m, we obtain

$$\int_x^{x_m} u'(z) dz \le -\frac{k\overline{M}^2}{|c|} \int_x^{x_m} (e^{-|c|(x_m - z)} - 1) dz.$$

According to $u(x_m) = M$ and separating $u(x)$, we have

$$u(x) \ge \overline{M}[1 - k\overline{M}(x - x_m)^2 g(|c|(x_m - x))],$$

where $g(y) := \frac{e^{-y}+y-1}{y^2}$. It is clear that $g(y) \le 1/2$ for $y > 0$, which implies

$$u(x) \ge \overline{M}\big[1 - \frac{k\overline{M}}{2}(x - x_m)^2\big] \quad \forall x \in [-a, x_m]. \tag{2.4}$$

From $u(-a) = u_-$ it follows that

$$1 > u_- \ge \overline{M}\big[1 - \frac{k\overline{M}}{2}(a + x_m)^2\big]. \tag{2.5}$$

Now take $a_0 = 1/\sqrt{k\overline{M}}$ and let

$$x_0 := \frac{1}{\sqrt{k\overline{M}}}.$$

If $x_m \in (-a, -a + x_0)$, inequality (2.5) shows that $\overline{M} \le (1 - \frac{1}{2})^{-1} = 2$. If $x_m \in [-a + x_0, a)$, using (2.4) yields

$$1 \ge (\phi * u)(x_m) \ge \int_0^{x_0} \phi(z) u(x_m - z) dz \ge \overline{M} \int_0^{x_0} \phi(z)\Big(1 - \frac{k\overline{M}}{2} z^2\Big) dz.$$

From the definition of x_0, we obtain

$$1 \ge \frac{\overline{M}}{2} \int_0^{1/\sqrt{k\overline{M}}} \phi(z) dz \ge \frac{\overline{M}}{2} \int_0^{1/\sqrt{k\overline{M}}} (\phi(0) - \|\phi'\|_{L^\infty(-1,1)} z) dz,$$

which implies

$$\overline{M} \le \frac{(4k + \|\phi'\|)^2}{4k\phi^2(0)}.$$

Choose $M = (4k + \|\phi'\|)^2/(4k\phi^2(0))$ and then we can complete the proof of the case $c < 0$. The case $c > 0$ can be proved in a similar way by integrating on $[x_m, a]$ rather than on $[-a, x_m]$. Lastly if $c = 0$, by integrating twice the inequality $-u'' \le kM^2$ on $[x, x_m]$, we immediately achieve (2.4). The rest of the process is similar to the above. This completes the proof. □

Next, we show a priori estimates for c.

Lemma 2.2. (i) *For each $\varepsilon \in (0, \frac{u_-}{4})$, there exists $a_0(\varepsilon) > 0$ such that, for all $0 \le \tau \le 1$ and $a \ge a_0$, any solution of $T_\tau(a)$ satisfies $c \le 2\sqrt{kM} =: c_{\max}$, where M is defined in Lemma 2.1.*

(ii) *For any $a > 0$, there exists $\widehat{c}_{\min}(a) > 0$, such that, for all $0 \le \tau \le 1$, any solution (c, u) of $T_\tau(a)$ satisfies $c \ge -\widehat{c}_{\min}(a)$.*

(iii) *There exists $c_{\min} > 0$ and $a_0 > 0$, such that, for all $a \ge a_0$, any solution (c, u) of $T_1(a)$ satisfies $c \ge -c_{\min}$.*

Proof. (i) Note that $u(\ge 0)$ satisfies

$$-u'' - cu' = ku(u - u_0)(u_- - u) + \tau ku^2(u - \phi * u) \le ku^2 \le kMu. \tag{2.6}$$

We use the contrapositive method, by assuming that $c > 2\sqrt{kM}$. We define $\varphi_A(x) = Ae^{-\sqrt{kM}x}$ which satisfies

$$-c\varphi_A' - \varphi_A'' > kM\varphi_A. \tag{2.7}$$

Because of $u(x) \in L^\infty(-a, a)$, we have $u(x) < \varphi_A(x)$ when $A > 0$ is sufficiently large and $u(x) > \varphi_A(x)$ when $A < 0$. Then, we can define

$$A_0 = \inf\{A : \varphi_A(x) > u(x) \ for\ all\ x \in [-a, a]\}.$$

Obviously, there exists $x_0 \in [-a, a]$ such that $\varphi_{A_0}(x_0) = u(x_0)$ and $A_0 > 0$. Using (2.6), (2.7) and the maximum principle, we know that $x_0 \notin (-a, a)$. Because $A_0 > 0$, we have $x_0 = -a$. Combining this with $\varphi_{A_0}(-a) = u_-$, we have $A_0 = u_- e^{-\sqrt{kM}a}$. However, $u(0) \le \varphi_{A_0}(0) = u_- e^{-\sqrt{kM}a} < u(0) = \varepsilon/2$ when $a > \frac{1}{\sqrt{kM}}(\ln 2u_- - \ln \varepsilon)$, there must be $c \le 2\sqrt{kM}$. Choose $a_0 \ge \frac{1}{\sqrt{kM}}(\ln 2u_- - \ln \varepsilon)$ and then we completes the proof of (i).

(ii) Giving $a > 0$, the solution of $T_\tau(a)$ satisfies

$$-u'' - cu' + (M^2 + 1)u \ge 0,$$

and $u(-a) = u_-, u(a) = 0$. In view of $M^2 + 1 \ge 0$, by the comparison principle we know that $u \ge v$, where v satisfies

$$\begin{gathered} -v'' - cv' + (M^2 + 1)v = 0, \\ v(-a) = u_-,\ v(a) = 0. \end{gathered}$$

From precise calculation,

$$v(x) = \frac{u_-}{e^{-\lambda^+ a} - e^{(\lambda^+ - 2\lambda^-)a}} e^{\lambda^+ x} - \frac{e^{(\lambda^+ - \lambda^-)a}}{e^{-\lambda^+ a} - e^{(\lambda^+ - 2\lambda^-)a}} e^{\lambda^- x},$$

where $\lambda^\pm = \frac{-c \pm \sqrt{c^2 + 4(M^2+1)}}{2}$. We see that $v(0) \to u_-$ as $c \to -\infty$. Thus, for any $a > 0$, there exists $\widehat{c}_{\min}(a) > 0$ such that $c \le -\widehat{c}_{\min}(a)$ which implies $\varepsilon/2 < v(0) \le u(0)$ and leads u not to be the solution of $T_\tau(a)$. Thus, any solution (c, u) of $T_\tau(a)$ with $0 \le \tau \le 1$ requires $c \ge -\widehat{c}_{\min}(a)$. This proves (ii).

(iii) To obtain a priori lower bound for c of the equation

$$\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} + ku^2(1-\phi * u) - bu = 0, \quad x \in (-a,a),$$

we take into account the evolution equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} + ku^2(1-\phi * u) - bu. \tag{2.8}$$

So the solution we need is a stationary solution of (2.8). Take a solution $u(x,t)$ of equation (2.8) and let $v = u - u_0$, where $u_0 = \frac{k-\sqrt{k^2-4kb}}{2k} < 1/2$ and u_0 satisfies $u_0(1-u_0) = b/k$. Then v satisfies

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + c\frac{\partial v}{\partial x} + k(v+u_0)^2(1-(\phi * v) - u_0) - bu.$$

Suppose that $v < 0$ and $0 < u < 1$. Then

$$\begin{aligned} &k(v+u_0)^2(1-(\phi * v) - u_0) - bu \\ &= ku(v+u_0)(1-(\phi * v) - u_0) - ku_0(1-u_0)u \\ &< -k(\phi * v). \end{aligned} \tag{2.9}$$

Next, we analyze the equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + c\frac{\partial z}{\partial x} - k(\zeta * z), \tag{2.10}$$

where

$$(\zeta * z)(x) = \int_{\mathbb{R}} \zeta(x-y)z(y,t)dy,$$

and $\zeta(x)$ is a piecewise constant function. That is to say

$$\zeta(x) = \begin{cases} M = \sup_x \phi(x), & \text{if } x \in [-N,N], \\ 0, & \text{otherwise.} \end{cases}$$

where $[-N,N]$ is the support of the function $\zeta(x)$. Next we seek a solution of the equation

$$\chi'' + c_0\chi' - k(\zeta * \chi) = 0, \tag{2.11}$$

where c_0 may be different from c and $\chi(x)$ has an exponential form, such as $\chi(x) = -e^{\lambda x}$, then

$$\lambda^2 + c_0\lambda - \frac{kM}{\lambda}(e^{\lambda N} - e^{-\lambda N}) = 0.$$

For all M and N, this equation has a solution λ if c_0 is sufficiently small. Choosing this values of λ and c_0 and researching the corresponding solution $\zeta(x)$ of (2.10), then $z(x,t) = \chi(x-(c_0-c)t)$ satisfies (2.9), which has a constant outline and transfer to the right with the speed $c_0 - c$. On the other hand, the function $\widetilde{u}(x,t) = z(x,t) + w_0$ satisfies

$$\frac{\partial \widetilde{u}}{\partial t} = \frac{\partial^2 \widetilde{u}}{\partial x^2} + c\frac{\partial \widetilde{u}}{\partial x} - k(\zeta * (\widetilde{u} - w_0)). \tag{2.12}$$

Now let us compare the solution $u(x,t)$ of (2.8) with $\widetilde{u}(x,t)$ of (2.11), we know that $u(x,t) \to u_\pm$ as $x \to \pm\infty$ for all $t \geq 0$ and $\widetilde{u}(x,t)$ is strictly decreasing, converging to w_0 as $x \to -\infty$ and exponentially growing at $-\infty$.

Thus we choose a constant h such that, for all $x \in \mathbb{R}$, $u(x,0) < \widetilde{u}(x-h,0)$. After that, we prove that $u(x,t) < \widetilde{u}(x-h,t)$ for all $x \in \mathbb{R}$ and $t \geq 0$. Assume it does not hold, then there exists $t_0 > 0$ such that

$$\begin{gathered} u(x,t_0) \geq \widetilde{u}(x-h,t_0) \quad \text{for all } x \in \mathbb{R}, \\ u(x_0,t_0) \geq \widetilde{u}(x_0-h,t_0) \quad \text{for some } x_0 \in \mathbb{R}. \end{gathered}$$

From $\widetilde{u}(x_0-h,t_0) < u_0$, it follows that $v(x_0,t_0) < 0$, which holds on some neighborhood $\delta(x)$ of $x = x_0$. Moreover, using $0 < u(x,t) < 1$ and $\phi * v < 0$ and (2.9), we know that

$$ku^2(1-\phi*u) - bu < -k\phi*v = -k\phi*(u-u_0) \leq -k\zeta*(\widetilde{u}-u_0), \quad x \in \delta(x_0).$$

which is a contradiction with (2.8) and (2.12). If there exists a stationary solution $u(x)$ such that (2.8), then put $u(x,0) = \chi(x)$ and obtain $u(x) = u(x,t) > \widetilde{u}(x,t)$. Thus, $c_0 - c < c$, where $c_0/2$ is chosen above.

To calculate lower bound of the speed c, similar to the above process, we construct $\chi(x) \to w_0$ as $x \to \infty$ and it exponentially decrease as $x \to \infty$. It spread to the left with (a certain speed) $c - c_0$. This completes the proof. □

Remark 2.3. Using above method, we cannot obtain a priori upper bound for c, because we can not get the specific nature of u as $x \to -\infty$. In addition, for $\tau > 0$ above method is true. If we discuss the case of $\tau = 0$ again, we can get a consistent with the lower bound of $c_{\min}$.

2.2. **Construction of a solution** (c,u)**.** We will construct a solution (c,u) of $T_1(a)$ by using Leray-Schauder degree argument in the following Proposition.

Proposition 2.4. *There exists $K > 0$ and $a_0 > 0$, such that, for all $a \geq a_0$, a solution (c,u) of $T_1(a)$, i.e.*

$$\begin{gathered} -u'' - cu' = ku^2(1-\phi*u) - bu \quad \text{in } (-a,a), \\ u(-a) = u_-, \quad u(0) = \frac{\varepsilon}{2}, \quad u(a) = 0, \\ u > 0, \quad \text{on } (-a,a). \end{gathered} \tag{2.13}$$

and

$$\|u\|_{C^2(-a,a)} \leq M, \quad -c_{\min} \leq c \leq c_{\max}.$$

Proof. Give $v \geq 0$ defined on $(-a,a)$ and satisfying the boundary conditions $v(-a) = u_-$ and $v(a) = 0$. We consider a family of linear problems

$$F_\tau(a): \begin{cases} -u'' - cu' = ku(u-u_0)(u_- - u) + \tau k v^2(v - \phi * v) & \text{in } (-a,a), \\ u(-a) = u_-, \quad u(a) = 0. \end{cases}$$

Define $L_\tau : \mathbb{R} \times C^{1,\alpha}(-a,a) \to \mathbb{R} \times C^{1,\alpha}(-a,a)$ as

$$L_\tau : (c,v) \mapsto \big(\frac{\varepsilon}{2} - v(0) + c, u_\tau^c := \text{ the solution of } F_\tau(a)\big).$$

where the norm is $\|(c,v)\|_X := \max(|c|, \|v\|_{C^{1,\alpha}})$. To find the nontrivial part of the kernel of $Id - L_1$, we construct (c,u) of $T_1(a)$. Then the Leray-Schauder topological can be applied here because L_τ is compact and continuity depends on the parameter $0 \leq \tau \leq 1$. Define the set

$$E := \big\{(c,v) : -c_{\min}(a) - 1 < c < c_{\max} + 1, \ v > 0, \ \|v\|_{C^{1,\alpha}} < M + 1\big\} \subset X,$$

where $c_{\min}(a)$ and $c_{\max}$ are defined in Lemma 2.2, and M is defined in Lemma 2.1. From Lemmas 2.1 and 2.2, it is easy to see that there exists $a_0 > 0$ such that, for any $a \geq a_0$, any $0 \leq \tau \leq 1$, the operator $Id - L_\tau$ can not vanish on the boundary ∂E. Therefore, by the homotopy invariance of the degree, we obtain

$$\deg(Id - L_1, E, 0) = \deg(Id - L_0, E, 0).$$

In addition, from the graph of $-u'' - cu' = ku(u - u_0)(u_- - u)$, it follows that u_0^c is decreasing with respect to c. Consequently, by using two additional homotopies (see [7, p.2834] or [3] for details), we have $\deg(Id - L_0, E, 0) = -1$, thus $\deg(Id - L_1, E, 0) = -1$ which implies that there is a solution $(c, u) \in E$ of $T_1(a)$. Finally, it follows from Lemma 2.2 that $c_{\max} \geq c \geq -c_{\min}$. This completes the proof. □

Remark 2.5 (Existence of solution on $\mathbb{R}$)**.** Equipped with the solution (c, u) of $T_1(a)$ in Proposition 2.4, now let $a \to \infty$. This enables to construct, by passing to a subsequence $a_n \to \infty$, a speed $-c_{\min} \leq c^* \leq c_{\max}$ and a function $U : \mathbb{R} \mapsto (0, M)$ in $C_b^2(\mathbb{R})$ such that

$$\begin{gathered} -U'' - cU' = kU^2(1 - \phi * U) - bU, \\ U(0) = \frac{\varepsilon}{2}. \end{gathered}$$

3. Behavior of the solution at $\pm\infty$

In Section 2, we prove that there exists a traveling wave solution of (1.1) on $\mathbb{R}$, in order to complete it, we also need to research the behavior of the solution at $\pm\infty$. Firstly, we rewrite equation (1.1) as

$$\begin{aligned} -cu' - u'' &= ku^2(1 - \phi * u) - bu \\ &= ku(u - u_0)(u_- - u) + ku^2(u - \phi * u). \end{aligned} \tag{3.1}$$

Secondly, we show that the solution in the box cannot attain $\varepsilon/2$ except that at $x = 0$.

Proposition 3.1. *For each $a \geq a_0$, the solution (c, u) of* (2.13) *satisfies*

$$u(x) = \frac{\varepsilon}{2} \text{ if and only if } x = 0.$$

Proof. It follows from Proposition 2.4, that there exists a pair of (c_τ, u_τ) satisfying

$$\begin{gathered} -u_\tau'' - c_\tau u_\tau' = ku_\tau(u_\tau - u_0)(u_- - u_\tau) + \tau k u_\tau^2(u_\tau - \phi * u_\tau) \text{ in } (-a, a), \\ u_\tau(-a) = u_-, \quad u_\tau(0) = \frac{\varepsilon}{2}, \quad u_\tau(a) = 0, \end{gathered}$$

and depending continuously upon $0 \leq \tau \leq 1$. For $\tau = 0$, the solution u_τ of

$$\begin{gathered} -u_\tau'' - c\tau u_\tau' = ku_\tau(u_\tau - u_0)(u_- - u_\tau) \text{ in } (-a, a), \\ u_\tau(-a) = u_-, \ u_\tau(0) = \frac{\varepsilon}{2}, \ u_\tau(a) = 0, \end{gathered}$$

satisfies $u_\tau(x) = \varepsilon/2$ if and only if $x = 0$. Thus we can define

$$\tau^* := \sup\big\{0 \leq \tau \leq 1, \forall \sigma \in [0, \tau], u_\sigma(x) = \frac{\varepsilon}{2} \text{ iff } x = 0\big\}.$$

If it does not hold, then there exists a $x^* \neq 0$ such that $u_{\tau^*}(x^*) = \varepsilon/2$. Therefore, we consider the following two cases:

Case 1: $x^* < 0$, and $u_{\tau^*} > \varepsilon/2$ on $(x^*, 0)$. It follows from the definition of τ^* that $u_{\tau^*} \geq \varepsilon/2$ on $(-a, 0)$, which implies that $u'_{\tau^*}(x^*) = 0$. In addition, for the interval $(x^*, 0)$, we consider the equation

$$-u''_{\tau^*} - c_{\tau^*}u'_{\tau^*} = ku_{\tau^*}(u_{\tau^*} - u_0)(u_- - u_{\tau^*}) + \tau^* ku^2_{\tau^*}(u_{\tau^*} - \phi * u_{\tau^*}) \leq 0$$

on $(x^*, 0)$. When u_{τ^*} is small, $(u_{\tau^*} - u_0)(u_- - u_{\tau^*}) < 0$, $u_{\tau^*} - \phi * u_\tau$ is bounded, and $u^2_{\tau^*}$ is a high-end terms. Furthermore, by the weak maximum principle, we know that u_{τ^*} can reach the extremum on x^* or 0, which leads to a contradiction.

Case 2: $x^* > 0$ and $u_{\tau^*} < \varepsilon/2$. By the definition of τ^*, one must have $u_{\tau^*} < \varepsilon/2$ on $(0, a)$ which implies that $u'_{\tau^*}(x^*) = 0$. In addition,

$$-u''_{\tau^*} - c_{\tau^*}u'_{\tau^*} = ku_{\tau^*}(u_{\tau^*} - u_0)(u_- - u_{\tau^*}) + \tau^* ku^2_{\tau^*}(u_{\tau^*} - \phi * u_{\tau^*}) \leq 0$$

on $(0, x^*)$, and $u(x^*) = \max_{x\in(0,x^*)} u(x)$. In view of the Hopf's Lemma, it has $u'_{\tau^*}(x^*) > 0$, which causes a contradiction. Thus u_{τ^*} attains $\varepsilon/2$ only at $x = 0$.

The rest is to prove $\tau^* = 1$. By contradiction, which assuming that $0 \leq \tau^* < 1$. It follows from the definition of τ^* that there exists a sequence (τ_n, x_n), such that $\tau_n \downarrow \tau^*$, $x_n \neq 0$, and $u_{\tau_n}(x_n) = \varepsilon/2$. Furthermore, take the sequence x_n converging to a limit x^* which implies that $u_{\tau^*}(x^*) = \varepsilon/2$. Thus $x^* = 0$ implies $x_n \to 0$. Then for some $-1 \leq c_n \leq 1$, we have

$$\frac{\varepsilon}{2} = u_{\tau_n}(x_n) = u_{\tau_n}(0) + u'_{\tau_n}(0)x_n + C_n\|u_{\tau_n}\|_{C^{1,\alpha}}|x_n|^{1+\alpha};$$

that is

$$|u'_{\tau_n}(0)| \leq |C_n|\|u_{\tau_n}\|_{C^{1,\alpha}}|x_n|^\alpha \leq C|x_n|^\alpha \to 0 \quad \text{as } n \to \infty,$$

By the continuity of (u'_τ) about τ, we know that $u'_\tau(0) = 0$. It has $u_\tau > \varepsilon/2$ on $(-a, 0)$, which is a contradiction. So $\tau^* = 1$ and this completes the proof. □

Remark 3.2. From Proposition 3.1, we know that $u(x) > \varepsilon/2$ on $(-a, 0)$ and $u(x) < \varepsilon/2$ on $(0, a)$. Thus let $a \to +\infty$, we can get

$$u(x) \begin{cases} \geq \varepsilon/2, & x \in (-\infty, 0], \\ \leq \varepsilon/2, & x \in [0, +\infty). \end{cases}$$

Lemma 3.3. *For all $K > 0$, and $\alpha < \beta$, there exists $\varepsilon = \varepsilon(K, \alpha, \beta) > 0$ such that if u is a solution of* (3.1) *with $c \in [\alpha, \beta]$, $0 < u \leq K$ and $\inf_{x\in\mathbb{R}} u(x) > 0$, then $\inf_{x\in\mathbb{R}} u(x) > \varepsilon$.*

Proof. Assume that there exists (c_n, u_n) satisfying (3.1). Let

$$\beta_n := \inf_{\mathbb{R}} u_n > 0, \quad \text{for all } n,$$

satisfying $\beta_n \to 0$ as $n \to +\infty$. Define $v_n = \frac{u_n}{\beta_n}$. Since $\inf_{\mathbb{R}} v_n = 1$ for all n, we assume that there exists a sequence $x_n \in \mathbb{R}$ such that $v_n(x_n) \leq 1 + \frac{1}{n}$. Let $w_n = \frac{1}{v_n(x+x_n)}$ satisfying

$$-\frac{c_n w'_n}{w^2_n} - \frac{w_n w''_n - 2w'_n}{w^2_n} = \frac{k\beta_n}{w^2_n}(1 - \phi * \widetilde{u}_n) - b\frac{\beta_n}{w_n},$$

where $\widetilde{u}_n = u(x + x_n)$. Note that $\sup_{x\in\mathbb{R}} u_n(x) \leq K$ and the coefficients above are uniformly bounded (with respect to n), therefore one can extract $w_{n_k}(x) \to w_\infty(x)$ (locally uniformly) and $c_n \to \widetilde{c} \in [\alpha, \beta]$. Moreover based on $\widetilde{u}_n(0) \leq \beta_n(1 + \frac{1}{n})$, it follows from the Harnack inequality that $\widetilde{u}_n(x) \to 0$ locally uniformly in x. In addition, $w_\infty(x)$ satisfies

$$-\frac{c_n w'_\infty}{w_\infty^2} - \frac{w_\infty w''_\infty - 2w'_\infty}{w_\infty^2} = -\frac{b}{w_\infty}.$$

From the strong maximum principle together with $w_\infty(0) = 1$ and $w_\infty(x) \leq 1$, it gets that $w_\infty(x) \equiv 1$, which is a contradiction with $b \neq 0$. □

Lemma 3.4. *Let u satisfy* (3.1) *with initial conditions $u(0) = \varepsilon/2$. Then there exists a sequence x_n, such that $|x_n| \to +\infty$ and $u(x_n) \to 0$ as $n \to +\infty$.*

Proof. If $\inf_{\mathbb{R}} u > 0$, it follows from Lemma 3.3 and $u \leq K$ that $\inf_{\mathbb{R}} u \geq \varepsilon$, which is a contradict with $u(0) = \varepsilon/2$. □

Proposition 3.5. *Let (c, u) be the solution of $T_1(\infty)$ which constructed in the end of Section* 2*, then*

$$\lim_{x\to+\infty} u(x) = 0,$$

and when $x > 0$, $u(x)$ is monotonically decreasing to zero.

Proof. Assume that the conclusion does not hold. Combining with Lemma 3.4, there exists $\{x_n\} > 0$, such that $u(x_n) = \max_{x\in\mathbb{R}} u(x)$, then

$$-cu'(x_n) - u''(x_n) \geq 0. \tag{3.2}$$

In addition, it follows from Remark 3.2 that $u(x) < \varepsilon/2$ when $x > 0$. As long as ε small enough, we can get $ku(1 - \phi * u) - b < 0$, so

$$ku^2(1 - \phi * u) - bu = u\{ku(1 - \phi * u) - b\} < 0, \quad \text{for } x > 0. \tag{3.3}$$

From (3.2) and (3.3), we obtain a contradiction. Thus $\lim_{x\to+\infty} u(x) = 0$ and $u(x)$ is monotonically decreasing to zero when $x > 0$. □

Proof of Theorem 1.1. From Remark 2.5, we know that there exists a pair of (c, u) satisfying (1.4). The remains to prove that u satisfies the condition (1.5). It follows from Proposition 3.5 that

$$\lim_{x\to+\infty} u(x) = 0.$$

Next, we prove the behavior of the solution $u(x)$ at negative infinity. From Remark 3.2, we know that $u(x) \geq \varepsilon/2$ for all $x \in (-\infty, 0]$. Then

$$\liminf_{x\to-\infty} u(x) \geq \frac{\varepsilon}{2} > 0.$$

So the condition (1.5) is established. This completes the proof. □

4. Linear stability analysis and numerical simulations

In Section 2 and 3, we prove that (1.1) admits a traveling wave solution connecting 0 to an unknown positive steady state when the support of ϕ is not small. In this section, we mainly study the behavior of the unknown state. Firstly, we show the stability of the positive steady state by using the linear stability analysis. Secondly, the special form of the solution for (1.1) is proved through using numerical simulations. For convenience, equation (1.1) can be written as

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + ku^2(1 - (\phi_\sigma * u)) - bu, \tag{4.1}$$

for $x \in \mathbb{R}$, $t > 0$, where $k > 0$, $d > 0$, $b > 0$. Here, we only consider two specific kernel functions (K1) and (K2).

(K1) $\phi(x) = \phi_\sigma(x) = \frac{1}{2}e^{-|x|}$, where $\sigma > 0$ is a constant.

Let

$$v(t,x) = (\phi_\sigma * u)(t,x). \tag{4.2}$$

Its second order derivative with respect to x is

$$v_{xx} = -\frac{1}{\sigma^2}(u - v).$$

Then (4.1) may be replaced by

$$\begin{aligned} u_t &= du_{xx} + ku^2(1-v) - bu, \\ 0 &= v_{xx} + \frac{1}{\sigma^2}(u - v), \end{aligned} \tag{4.3}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$. System (4.3) has three equilibria: $(0,0)$,

$$\Big(\frac{k-\sqrt{k^2-4kb}}{2k}, \frac{k-\sqrt{k^2-4kb}}{2k}\Big), \quad \Big(\frac{k+\sqrt{k^2-4kb}}{2k}, \frac{k+\sqrt{k^2-4kb}}{2k}\Big).$$

Setting $u = \frac{k+\sqrt{k^2-4kb}}{2k} + \widetilde{u}$ and $v = \frac{k+\sqrt{k^2-4kb}}{2k} + \widetilde{v}$, substituting them into (4.3) and linearizing gives

$$\begin{aligned} \widetilde{u}_t &= d\widetilde{u}_{xx} + b\widetilde{u} - \frac{1}{2}(k - 2b + \sqrt{k^2-4kb})\widetilde{v}, \\ 0 &= \widetilde{v}_{xx} + \frac{1}{\sigma^2}(\widetilde{u} - \widetilde{v}). \end{aligned} \tag{4.4}$$

Firstly we consider the stability of the point $(\frac{k+\sqrt{k^2-4kb}}{2k}, \frac{k+\sqrt{k^2-4kb}}{2k})$, which is equivalent to judging the sign of η about the characteristic equation

$$\begin{vmatrix} \eta - b & \frac{k-2b+\sqrt{k^2-4kb}}{2} \\ -\frac{1}{\sigma^2} & \eta + \frac{1}{\sigma^2} \end{vmatrix} = 0;$$

that is

$$\eta^2 + (\frac{1}{\sigma^2} - b)\eta + \frac{k - 4b + \sqrt{k^2-4kb}}{2\sigma^2} = 0. \tag{4.5}$$

From (4.5), we know that η is negative when σ is sufficiently small, and hence $(u,v) = (\frac{k+\sqrt{k^2-4kb}}{2k}, \frac{k+\sqrt{k^2-4kb}}{2k})$ is stable. However, η may greater than 0 as σ increasing, which implies the uniform steady state will loss the stability.

Next, we consider whether the equation (4.1) will occur Hopf bifurcation or Turing bifurcation around the equilibrium point $(u,v) = (\frac{k+\sqrt{k^2-4kb}}{2k}, \frac{k+\sqrt{k^2-4kb}}{2k})$ when this equilibrium point is unstable, that is to prove (i) of Theorem 1.3.

Proof of (i) of Theorem 1.3. Take a test function of the form

$$\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} = \sum_{k=1}^{\infty} \begin{pmatrix} C_k^1 \\ C_k^2 \end{pmatrix} e^{\lambda t + ilx}, \tag{4.6}$$

where l is real. Then substituting (4.6) into (4.5) yields

$$\begin{vmatrix} b - dl^2 - \lambda & -\frac{k-2b+\sqrt{k^2-4kb}}{2} \\ -\frac{1}{\sigma^2} & -\frac{1}{\sigma^2} - l^2 \end{vmatrix} = 0.$$

Thus

$$(\lambda - b + dl^2)(\frac{1}{\sigma^2} + l^2) + \frac{k - 2b + \sqrt{k^2 - 4kb}}{2\sigma^2} = 0,$$

which is equivalent to

$$\lambda = b - dl^2 - \frac{k - 2b + \sqrt{k^2 - 4kb}}{2 + 2\sigma^2 l^2}.$$

Note that this implies λ is real for all values of b and thus Hopf bifurcations from the uniform state $(u, v) = \big(\frac{k+\sqrt{k^2-4kb}}{2k}, \frac{k+\sqrt{k^2-4kb}}{2k}\big)$ of the equation (4.3) are impossible. Moreover, as b increases it is possible for λ to pass through 0, indicating a loss of stability of the uniform steady state. For a fixed value of the wave number l, this occurs when

$$\Big(\frac{2}{\sigma^2} + l^2\Big)^2 b^2 - \Big(\frac{4dl^2 + k}{\sigma^4} + \frac{6dl^4 + l^2 k}{\sigma^2} + 2dl^6\Big)b + \frac{d^2 l^4 + kdl^2}{\sigma^4} + d^2 l^8 + \frac{2d^2 l^6 + dkl^4}{\sigma^2} = 0.$$

For convenience, let

$$A = (\frac{2}{\sigma^2} + l^2)^2, \quad B = \frac{4dl^2 + k}{\sigma^4} + \frac{6dl^4 + l^2 k}{\sigma^2} + 2dl^6,$$
$$C = \frac{d^2 l^4 + kdl^2}{\sigma^4} + d^2 l^8 + \frac{2d^2 l^6 + dkl^4}{\sigma^2}.$$

Then

$$b = b_c := \frac{B \pm \sqrt{B^2 - 4C}}{2A}, \tag{4.7}$$

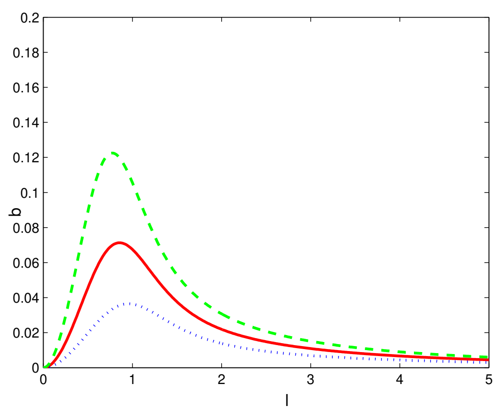

FIGURE 1. The phase shows the relation between b and l at the different value of σ with the parameter values $d = 1,\ k = 5$. The green, red, blue curve respectively represents $\sigma = 1.2,\ 1,\ 0.8$, (For interpretation of the reference to color in this figure legend, the reader is referred to the web version of this article.)

For discussing the front, we mainly consider the case $b = \frac{B-\sqrt{B^2-4C}}{2A}$. Next, we consider the relation between b and l, we can easily know there exists a l_c, such that

b has a maximum $b_{\max} = b(l_c)$. To see the relation of l and b more clearly, in there, we take the parameter d, k, σ with specific value, and obtain the diagram about b and l (see Figure 1). Thus as b increases through $b_{\max}$, the uniform steady state $(u, v) = (\frac{k+\sqrt{k^2-4kb}}{2k}, \frac{k+\sqrt{k^2-4kb}}{2k})$ loses stability and it is anticipated that a new, non-uniform steady state will appear having a spatial structure similar to $\exp(ilx)$. This prove (i) of Theorem 1.3. □

Similar to the process above, we obtain the relation between σ and l (we omit the process). Following we research the influence of σ for the solution of the equation (4.1) by using the numerical simulation.

Before our numerical simulation, the initial value problem needs to be developed first. We define the initial condition of $u(x, t)$ as

$$u(x,0) = \begin{cases} u_-, & \text{for } x \le L_0, \\ 0, & \text{for } x > L_0. \end{cases} \tag{4.8}$$

From (4.2), we know that $v(x, 0)$ is determined by

$$v(x,0) = \int_{\mathbb{R}} \frac{1}{2\sigma} e^{-\frac{|x-y|}{\sigma}} u(y,0)dy; \tag{4.9}$$

then

$$v(x,0) = \begin{cases} u_- - \frac{u_-}{2} e^{x-\frac{L_0}{\sigma}}, & \text{for } x \le L_0, \\ u_- e^{-\frac{x-L_0}{\sigma}}, & \text{for } x > L_0. \end{cases} \tag{4.10}$$

The zero-flux boundary conditions were applied here. Along with (4.8)-(4.10), system (4.4) can be simulated through the pdepe package in Matlab (see Figure 2).

From Figure 2 we see that equation (4.1) admits monotone traveling wave solution connecting 0 to $\frac{k+\sqrt{k^2-4kb}}{2k}$ when σ is small. As σ increasing, the solution will occur a 'hump', and the travelling wave loses its monotone. Moreover, as σ being much larger, the 'hump' is being much steeper. If the value of σ is continue to increase, then the stability of the state $u = \frac{k+\sqrt{k^2-4kb}}{2k}$ will lose. Furthermore, a periodic steady state will replace the state $u = \frac{k+\sqrt{k^2-4kb}}{2k}$.

(K2) $\phi(x) = \phi_\sigma(x) = \frac{A}{\sigma} e^{-\frac{a}{\sigma}|x|} - \frac{1}{\sigma} e^{-\frac{|x|}{\sigma}}$, where $A = \frac{3a}{2} > 0$, $a \in (\frac{2}{3}, \sqrt{\frac{2}{3}})$.

Define

$$v(t,x) = \Big(\frac{A}{\sigma} e^{-\frac{a}{\sigma}|x|} * u\Big)(t,x), w(t,x) = \Big(\frac{1}{\sigma} e^{-\frac{|x|}{\sigma}} * u\Big)(t,x).$$

and let $\phi_{\sigma v} = \frac{A}{\sigma} e^{-\frac{a}{\sigma}|x|}$, $\phi_{\sigma w} = \frac{1}{\sigma} e^{-\frac{|x|}{\sigma}}$. Then

$$v_{xx} = -\frac{1}{\sigma^2}(3a^2 u - a^2 v), \quad w_{xx} = -\frac{1}{\sigma^2}(-2u - w).$$

So equation (4.1) reduces to the system

$$\begin{aligned} u_t &= du_{xx} + f_1(u,v,w), \\ 0 &= v_{xx} + f_2(u,v,w), \\ 0 &= w_{xx} + f_3(u,v,w), \end{aligned} \tag{4.11}$$

where $f_1(u, v, w) = ku^2(1 - v - w) - bu$, $f_2(u, v, w) = \frac{1}{\sigma^2}(3a^2 u - a^2 v)$, $f_3(u, v, w) = \frac{1}{\sigma^2}(-2u - w)$. Obviously, systems (4.11) has three equilibria

$$(u_1^*, v_1^*, w_1^*) = (0,0,0), \quad (u_2^*, v_2^*, w_2^*) = (u_0, 3u_0, -2u_0),$$
$$(u_3^*, v_3^*, w_3^*) = (u_-, 3u_-, -2u_-).$$

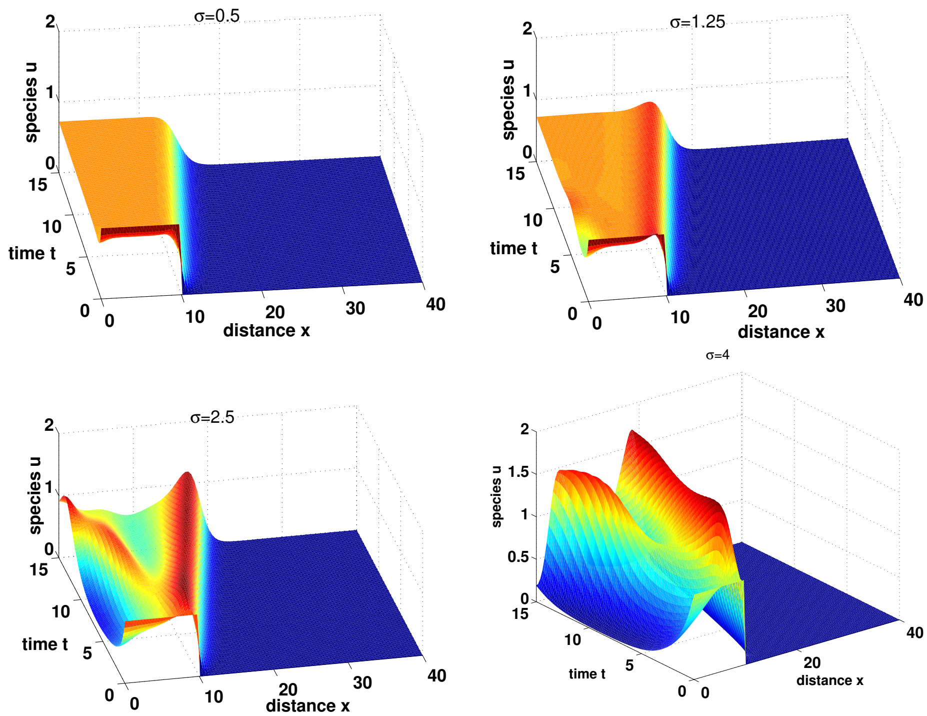

FIGURE 2. Numerical simulations of the time evolution and space evolution for the bistable nonlocal equation (4.1) with kernel $\phi_\sigma(x) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}$. The computational domain is $x \in [0, 40]$, $t \in [0, 15]$. The parameter values: $d = 1$, $k = 5$, $b = 1$, σ are followed by 0.5, 1.25, 2.5, 4.

We will mainly analyze system (4.11) to get the dynamical behavior of system (4.1). Now, linearizing system (4.11) around (u_3^*, v_3^*, w_3^*) we can get

$$\begin{aligned} u_t &= du_{xx} + a_{11}u + a_{12}v + a_{13}w, \\ 0 &= v_{xx} + a_{21}u + a_{22}v + a_{23}w, \\ 0 &= w_{xx} + a_{31}u + a_{32}v + a_{33}w. \end{aligned} \tag{4.12}$$

where

$$\begin{gathered} a_{11} = \frac{\partial f_1}{\partial u}\Big|_{(u_3^*, v_3^*, w_3^*)} = b, \quad a_{12} = \frac{\partial f_1}{\partial v}\Big|_{(u_3^*, v_3^*, w_3^*)} = -u_- + b, \\ a_{13} = \frac{\partial f_1}{\partial w}\Big|_{(u_3^*, v_3^*, w_3^*)} = -u_- + b, \quad a_{21} = \frac{\partial f_2}{\partial u}\Big|_{(u_3^*, v_3^*, w_3^*)} = \frac{3a^2}{\sigma^2}, \\ a_{22} = \frac{\partial f_2}{\partial v}\Big|_{(u_3^*, v_3^*, w_3^*)} = -\frac{a^2}{\sigma^2}, \quad a_{23} = \frac{\partial f_2}{\partial w}\Big|_{(u_3^*, v_3^*, w_3^*)} = 0, \\ a_{31} = \frac{\partial f_3}{\partial u}\Big|_{(u_3^*, v_3^*, w_3^*)} = -\frac{2}{\sigma^2}, \quad a_{32} = \frac{\partial f_2}{\partial v}\Big|_{(u_3^*, v_3^*, w_3^*)} = 0, \\ a_{33} = \frac{\partial f_3}{\partial w}\Big|_{(u_3^*, v_3^*, w_3^*)} = -\frac{1}{\sigma^2}. \end{gathered}$$

Similarly, we first consider the stability of the equilibrium point (u_3^*, v_3^*, w_3^*), which is equivalent to judge the sign of η about the characteristic equation

$$\begin{vmatrix} \eta - b & \frac{1}{2}(k-2b+\sqrt{k^2-4kb}) & \frac{1}{2}(k-2b+\sqrt{k^2-4kb}) \\ -\frac{3a^2}{\sigma^2} & \eta + \frac{a^2}{\sigma^2} & 0 \\ \frac{2}{\sigma^2} & 0 & \eta + \frac{1}{\sigma^2} \end{vmatrix} = 0.$$

That is

$$(\eta - b)\big(\eta + \frac{a^2}{\sigma^2}\big)\big(\eta + \frac{1}{\sigma^2}\big) + \frac{1}{2\sigma^2}(k - 2b + \sqrt{k^2-4kb})\big((3a^2-2)\eta + \frac{a^2}{\sigma^2}\big). \quad (4.13)$$

From (4.13) we know that η is negative when σ is sufficiently small, and hence $(u,v,w) = \big(\frac{k+\sqrt{k^2-4kb}}{2k},\ 3\frac{k+\sqrt{k^2-4kb}}{2k},\ -2\frac{k+\sqrt{k^2-4kb}}{2k}\big)$ is stable. However, the uniform steady state will lose stability as σ increasing. Next, we consider whether the equation (4.1) will has Hopf bifurcation or Turing bifurcation around the equilibrium point $(u,v,w) = \big(\frac{k+\sqrt{k^2-4kb}}{2k},\ 3\frac{k+\sqrt{k^2-4kb}}{2k},\ -2\frac{k+\sqrt{k^2-4kb}}{2k}\big)$ when this equilibrium point is unstable.

Proof (ii) of Theorem 1.3. Similar to [17], we define

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \sum_{k=1}^{\infty} \begin{pmatrix} C_k^1 \\ C_k^2 \\ C_k^3 \end{pmatrix} e^{\lambda t + ilx}, \quad (4.14)$$

where λ is the growth rate of perturbations in time t, l is the wave speed. So, substituting equation (4.14) into equation (4.12), we can obtain

$$\det A = \begin{vmatrix} a_{11} - dl^2 - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - l^2 & a_{23} \\ a_{31} & a_{32} & a_{33} - l^2 \end{vmatrix} = 0.$$

Then

$$\begin{vmatrix} b - dl^2 - \lambda & -\frac{1}{2}(k-2b+\sqrt{k^2-4kb}) & -\frac{1}{2}(k-2b+\sqrt{k^2-4kb}) \\ \frac{3a^2}{\sigma^2} & -\frac{a^2}{\sigma^2} - l^2 & 0 \\ -\frac{2}{\sigma^2} & 0 & -\frac{1}{\sigma^2} - l^2 \end{vmatrix} = 0.$$

which is equivalent to

$$\begin{aligned} &(b - dl^2 - \lambda)\big(\frac{a^2}{\sigma^2} + l^2\big)\big(\frac{1}{\sigma^2} + l^2\big) - \frac{3a^2}{2\sigma^2}(k - 2b + \sqrt{k^2-4kb})\big(\frac{1}{\sigma^2} + l^2\big) \\ &+ \frac{1}{2}\big(k - 2b + \sqrt{k^2-4kb}\big)\big(\frac{a^2}{\sigma^2} + l^2\big)\frac{2}{\sigma^2} = 0. \end{aligned}$$

Note that this implies λ being real for all values b and thus Hopf bifurcation from the uniform state $(u_-, 3u_-, -2u_-)$ of system (4.11) are impossible. Moreover, as b increases it is possible loss of stability of the uniform steady state. For a fixed k this occurs when

$$\begin{aligned} &(b - dl^2)\big(\frac{a^2}{\sigma^2} + l^2\big)\big(\frac{1}{\sigma^2} + l^2\big) + \frac{1}{2\sigma^2}\big(k - 2b + \sqrt{k^2-4kb}\big) \\ &\times \big(-\frac{a^2}{\sigma^2} + (2 - 3a^2)l^2\big) = 0, \end{aligned}$$

which is equivalent to

$$Bb^2 + Cb + D = 0,$$

where

$$B = \Big(\frac{2a^2}{\sigma^4} + \frac{(4a^2-1)l^2}{\sigma^2} + l^4\Big)^2,$$

$$\begin{aligned} C = 2\Big(\frac{2a^2}{\sigma^4} + \frac{(4a^2-1)l^2}{\sigma^2} + l^4\Big)\Big(-\frac{2dl^2a^2+a^2k}{2\sigma^4} \\ + \frac{-2dl^2(a^2l^2+l^2)+kl^2(2-3a^2)}{2\sigma^2} - dl^6\Big) + \frac{k}{\sigma^4}\Big(-\frac{a^2}{\sigma^2}+(2-3a^2)l^2\Big)^2, \end{aligned}$$

$$\begin{aligned} D = \Big(-\frac{2dl^2a^2+a^2k}{2\sigma^4} + \frac{-2dl^2(a^2l^2+l^2)+kl^2(2-3a^2)}{2\sigma^2} - dl^6\Big)^2 \\ - \frac{k^2}{4\sigma^4}(-\frac{a^2}{\sigma^2}+(2-3a^2)l^2)^2. \end{aligned}$$

Then

$$b = b_c := \frac{-C-\sqrt{C^2-4BD}}{2B}, \quad \text{or} \quad b = b_c := \frac{-C+\sqrt{C^2-4BD}}{2B}. \tag{4.15}$$

Next, we consider the relation between b and l. For convenience, from the expression of equation (4.15) we only consider $b = \frac{-C+\sqrt{C^2-4BD}}{2B}$. We can easily to know that b has a minimum, that is, there is a l_c, such that $b_{\min} = b(l_c)$. In order to see the relation of l and b more clearly, we take the parameter $d,\ k,\ \sigma$ with specific value, and obtain the figure about b and l (see Figure 3).

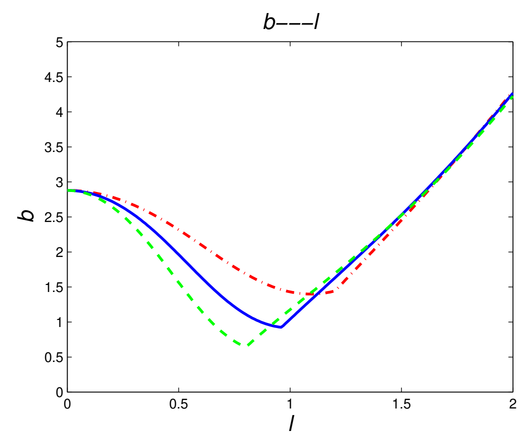

FIGURE 3. The phase show the relation between b and l at the different value of σ, the parameter value is $d = 1,\ k = 5$ and the green, blue and red curve respectively represents $\sigma = 1.2,\ 1,\ 0.8$.

Thus as b increased beyond $b_{\min}$, the uniform steady state

$$(u,v,w) = \Big(\frac{k+\sqrt{k^2-4kb}}{2k},\ 3\frac{k+\sqrt{k^2-4kb}}{2k},\ -2\frac{k+\sqrt{k^2-4kb}}{2k}\Big)$$

looses stability and it is anticipated that a new, non-uniform steady state will appear having a spatial structure similar to $\exp(ilx)$. This prove (ii) of Theorem 1.3. $\square$

Similar to the above process, we can also get the relation between σ and l (we omit the process). Following we study the influence of σ for the solution of the equation (4.1) by using the numerical simulation.

As for (K1), we first develop the initial value problem. Set

$$u(x,0) = \begin{cases} u_-, & \text{for } x \le L_0, \\ 0, & \text{for } x > L_0. \end{cases} \tag{4.16}$$

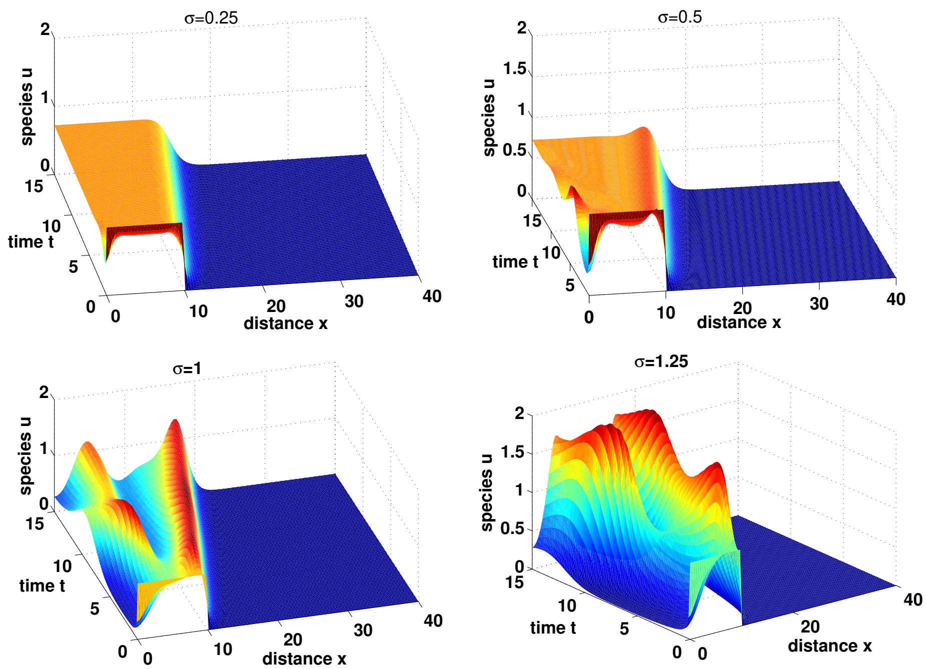

FIGURE 4. Numerical simulations of the time evolution and space evolution for the bistable nonlocal equation (4.1) with kernel $\phi_\sigma(x) = \frac{A}{\sigma}e^{-\frac{a}{\sigma}|x|} - \frac{1}{\sigma}e^{-\frac{|x|}{\sigma}}$. The computational domain is $x \in [0,40]$, $t \in [0,15]$. The parameter values: $d=1$, $k=5$, $b=1$, σ is followed by 0.25, 0.5, 1, 1.25.

From the definition of $v(x,t)$, we have

$$v(x,0) = \int_{\mathbb{R}} \frac{A}{\sigma} e^{-\frac{a|x-y|}{\sigma}} u(y,0)dy.$$

Then

$$v(x,0) = \begin{cases} 3u_- - \frac{3u_-}{2} e^{\frac{a(x-L_0)}{\sigma}}, & \text{for } x \le L_0, \\ \frac{3u_-}{2} e^{-\frac{a(x-L_0)}{\sigma}}, & \text{for } x > L_0. \end{cases} \tag{4.17}$$

Similarly, we know that

$$w(x,0) = \int_{\mathbb{R}} \frac{1}{\sigma} e^{-\frac{|x-y|}{\sigma}} u(y,0)dy\,.$$

Then

$$w(x,0)=\begin{cases}2u_- - u_- e^{\frac{x-L_0}{\sigma}}, & \text{for } x\le L_0,\\ u_- e^{-\frac{x-L_0}{\sigma}}, & \text{for } x> L_0.\end{cases} \tag{4.18}$$

With (4.16)–(4.18) and the zero-flux boundary conditions, simulating results for (4.11) are also performed by pdepe in Matlab; see Figure 4.

From the previous analysis, we know that the state $u=u_-$ of the equation (4.1) may be unstable. So what steady state will occur around $u=u_-$? From the Figure 4, we can see that equation (4.1) will have a periodic steady state around $u=u_-$; that is to say, (4.1) admit a traveling wave solution connecting 0 to a periodic steady state.

5. Periodic stationary solutions

In Section 4, we showed that the wave can connect 0 to a period steady state through numerical simulations. In this section, we prove that equation (1.1) indeed admits stationary periodic solutions for $\sigma=\sigma_c,\ k=k_c$ and $b=b_c$.

Firstly, linearizing (1.1) around $u=u_-$, that is to say, let $u=u_-+\widetilde{v}$, we have

$$\widetilde{v}_t=\widetilde{v}_{xx}+b\widetilde{v}-ku_-^2\phi*\widetilde{v}-ku_0u_-\widetilde{v}\phi*\widetilde{v}-k\widetilde{v}^2\phi*\widetilde{v}.$$

Up to a rescaling, let $\widetilde{v}(t,x)=v(t,\sigma x)$, we can obtain the new equation

$$v_t=\sigma^2v_{xx}+bv-ku_-^2\phi_\sigma*v-ku_0u_-v\phi_\sigma*v-kv^2\phi_\sigma*v,$$

where $\phi_\sigma(x)=\frac{1}{\sigma}\phi(\frac{x}{\sigma})$ and v is 2π-periodic in x. We define

$$\mathfrak{B}(\sigma,k,b)v:=\sigma^2v_{xx}+bv-ku_-^2\phi_\sigma*v,\quad \mathfrak{Q}(v,k,\sigma):=-ku_0u_-v\phi_\sigma*v-kv^2\phi_\sigma*v.$$

We obtain

$$v_t=\mathfrak{B}(\sigma,k,b)v+\mathfrak{Q}(v,k,\sigma). \tag{5.1}$$

If we define

$$\mathfrak{Y}:=L^2_{per}[0,2\pi]=\{u\in L^2_{loc}(\mathbb{R})|u(x+2\pi)=u(x),\ x\in\mathbb{R}\},$$
$$\Upsilon:=D(A)=H^2_{per}[0,2\pi]=\{u\in H^2_{loc}(\mathbb{R})|u(x+2\pi)=u(x),\ x\in\mathbb{R}\}$$

then we know that $\mathfrak{Q}:\Upsilon\to\Upsilon$ is smooth. Equation (5.1) can be written as

$$v_t=\mathfrak{B}_cv+\mathfrak{C}(\varepsilon,\delta)v+\mathfrak{Q}(v,k_c+\varepsilon^2,\sigma_c+\delta),$$

where $\mathfrak{B}_c=\mathfrak{B}(\sigma_c,k_c,b_c)$ and $\mathfrak{C}(\varepsilon,\delta)=\mathfrak{B}(\sigma_c+\delta,k_c+\varepsilon^2,b_c+\frac{\varepsilon^2}{2})-\mathfrak{B}(\sigma_c,k_c,b_c)$. Since $\mathfrak{B}_c$ is continuous and Υ is dense and compactly embedded into $\mathfrak{Y}$, that the resolvent of $\mathfrak{B}_c$ is compact and its spectrum $\sigma(\mathfrak{B}_c)$ only have eigenvalues λ. From (1.6), we know that

$$\sigma(\mathfrak{B}_c)=\{\lambda_n\in C|\lambda_n=-\sigma^2n^2+b-ku_-^2\widehat{\phi}(n\sigma),\ n\in Z\}. \tag{5.2}$$

Consequently,

$$\sigma(\mathfrak{B}_c)\cap i\mathbb{R}=\{0\},$$

and $\lambda=0$ which geometric multiplicity is two, and the corresponding eigenvectors are $\mathbf{e}(x):=e^{ix}$ and $\overline{\mathbf{e}}(x):=e^{-ix}$. In addition, the algebraic multiplicity is also two by computation. We define $\mathfrak{Y}_\mathfrak{c}:=\{\mathbf{e},\overline{\mathbf{e}}\}$ and the spectral projection $H_c:\mathfrak{Y}\to\mathfrak{Y}_\mathfrak{c}$ as

$$H_cu=\langle u,\mathbf{e}\rangle\mathbf{e}+\langle u,\overline{\mathbf{e}}\rangle\overline{\mathbf{e}},$$

where $\langle u,v\rangle=\frac{1}{2\pi}\int_0^{2\pi}u(x)\overline{v}(x)dx$. It follows from (5.2) that

$$\|(i\nu-\mathfrak{B}_c)^{-1}\|_{(id-H_c)\mathfrak{X}}\le\frac{C}{1+|\nu|},\quad \nu\in\mathbb{R},$$

where $C>0$ is a positive constant. So, by using the center manifold theorem (see [12, 14, 21]), we know that there exist $U\subset\mathfrak{Y}_{\mathfrak{c}},\ V\subset(id-H_c)\Upsilon,\ W\subset\mathbb{R}^2$, for any $m<\infty$, a C^m-map $\Phi:U\times W\to V$ having the following properties.

(i) $v(t)=\mathbf{B}(t)\mathbf{e}+\overline{\mathbf{B}}(t)\overline{\mathbf{e}}+\Phi(\mathbf{B}(t),\overline{\mathbf{B}}(t),\varepsilon,\delta)$, $t\in\mathbb{R}$.
(ii) $\|\Phi(\mathbf{B},\overline{\mathbf{B}},\varepsilon,\delta)\|_\Upsilon=O(|\varepsilon|^2|\mathbf{B}|+|\delta||\mathbf{B}|+|\mathbf{B}|^2)$.
(iii) $\frac{d\mathbf{B}}{dt}=g(\mathbf{B},\overline{\mathbf{B}},\varepsilon,\delta)=\mathbf{B}h(|\mathbf{B}|^2,\varepsilon,\delta)$, where h is a C^{m-1} and is a real-valued.

From the above three properties, we know that if we want to obtain the periodic steady solution of (5.1), only need to obtain the form of $\mathbf{B}$. Furthermore, if we want to obtain the form of $\mathbf{B}$, only need to obtain the form of h. So we next to find the form of h.

Lemma 5.1. *The Taylor expansion of the map h is*

$$\begin{aligned}h(|\mathbf{B}|^2,\varepsilon,\delta)&=\Big(-\frac{k_c-2b_c+\sqrt{k_c^2-4k_cb_c}}{4}\widehat{\phi}''(k_c)-1\Big)\delta^2+\frac{\varepsilon^2}{2}\\&\quad+\frac{(4b_c+\varepsilon^2)\widehat{\phi}(\sigma_c+\delta)}{\sqrt{k_c^2-4k_cb_c}+\sqrt{k_c^2-4k_cb_c-4b_c\varepsilon^2-\varepsilon^4}}\varepsilon^2\\&\quad+\varpi|\mathbf{B}|^2+O(|\delta|^3+|\varepsilon||\delta|+|\mathbf{B}|^4),\end{aligned}\tag{5.3}$$

where

$$\begin{aligned}\varpi:=&\frac{-4k_c^2+4k_c\sqrt{k_c^2-4k_cb_c}}{-(k_c-4b_c)-\sqrt{k_c^2-4k_cb_c}}+\frac{11k_c^2-124k_cb_c+11k_c\sqrt{k_c^2-4k_cb_c}}{-(k_c-4b_c)-\sqrt{k_c^2-4k_cb_c}}\widehat{\phi}(\sigma)\\&+\frac{2k_cb_c\widehat{\phi}(2\sigma)+2k_c(k_c-2b_c+\sqrt{k_c^2-4k_cb_c})(\widehat{\phi}^2(\sigma)+\widehat{\phi}(\sigma)\widehat{\phi}(2\sigma))}{-\sigma^2+b_c-\frac{k_c-2b_c+\sqrt{k_c^2-4k_cb_c}}{2}\widehat{\phi}(2\sigma)}\\&+\frac{k_c(k_c-2b_c+\sqrt{k_c^2-4k_cb_c})+6k_cb_c\widehat{\phi}(\sigma)}{-\sigma^2+b_c-\frac{k_c-2b_c+\sqrt{k_c^2-4k_cb_c}}{2}\widehat{\phi}(2\sigma)}\\&+\frac{8k_c(k_c-2b_c+\sqrt{k_c^2-4k_cb_c})}{-(k_c-4b_c)-\sqrt{k_c^2-4k_cb_c}}\widehat{\phi}^2(\sigma)<0.\end{aligned}$$

Proof. substituting $v(t)=\mathbf{B}(t)\mathbf{e}+\overline{\mathbf{B}}(t)\overline{\mathbf{e}}+\Phi(\mathbf{B}(t),\overline{\mathbf{B}}(t),\varepsilon,\delta)$ in (5.1) and comparing the coefficient about $\mathbf{Be}$, one obtains

$$\begin{aligned}\frac{d\mathbf{B}}{dt}=&\Big(-(\sigma_c+\delta)^2+b_c+\frac{\varepsilon^2}{2}-\frac{k_c-2b_c+\sqrt{k_c^2-4k_cb_c}-4b_c\varepsilon^2-\varepsilon^4}{2}\widehat{\phi}(\sigma_c+\delta)\Big)\mathbf{B}\\&+O_{(\varepsilon,\delta)}(\mathbf{B}|\mathbf{B}|^2).\end{aligned}$$

From the assumption of $d(\sigma,k,b)$, we know that

$$\begin{gathered}-\sigma_c^2+b_c-\frac{k_c-2b_c+\sqrt{k_c^2-4k_cb_c}}{2}\widehat{\phi}(\sigma_c)=0,\\-2\sigma_c-\frac{k_c-2b_c+\sqrt{k_c^2-4k_cb_c}}{2}\widehat{\phi}'(\sigma_c)=0.\end{gathered}$$

Therefore,

$$\begin{aligned}\zeta(\varepsilon,\delta) :=& -(\sigma_c+\delta)^2 + b_c + \frac{\varepsilon^2}{2} - \frac{k_c - 2b_c + \sqrt{k_c^2 - 4k_cb_c - 4b_c\varepsilon^2 - \varepsilon^4}}{2}\widehat{\phi}(\sigma_c+\delta)\\ =&\Big(-\frac{k_c - 2b_c + \sqrt{k_c^2 - 4k_cb_c}}{4}\widehat{\phi}''(k_c) - 1\Big)\delta^2 + \frac{\varepsilon^2}{2}\\ &+ \frac{(4b_c+\varepsilon^2)\widehat{\phi}(\sigma_c+\delta)}{\sqrt{k_c^2-4k_cb_c} + \sqrt{k_c^2 - 4k_cb_c - 4b_c\varepsilon^2 - \varepsilon^4}}\varepsilon^2 + O(|\varepsilon|^2|\delta| + |\delta|^3),\end{aligned}$$

as $(\varepsilon,\delta)\to(0,0)$.

To obtain ϖ in (5.3), let $(\varepsilon,\delta)=(0,0)$. Note that

$$v(t) = \mathbf{B}(t)\mathbf{e} + \overline{\mathbf{B}}(t)\overline{\mathbf{e}} + \mathbf{B^2}(t)\mathbf{e_{2,0}} + \mathbf{B}\overline{\mathbf{B}}\mathbf{e_{1,1}} + \overline{\mathbf{B}}^{\mathbf{2}}\mathbf{e_{0,2}} + O(|\mathbf{B}|^3).$$

and ϖ is actually the coefficient of the term $\mathbf{B}|\mathbf{B}\cdot\overline{\mathbf{B}}|$. From (5.1), it follows that this term appears in $-k_cu_0v^2$, $-2k_cu_-v\phi * v$ and $-k_cv^2\phi * v$, thus

$$\begin{aligned}\varpi =&\langle -2k_cu_-(\mathbf{e}\cdot\phi * \mathbf{e_{1,1}} + \overline{\mathbf{e}}\cdot\phi * \mathbf{e_{2,0}} + \mathbf{e_{2,0}}\cdot\phi * \overline{\mathbf{e}} + \mathbf{e_{1,1}}\cdot\phi * \mathbf{e}),\mathbf{e}\rangle\\ &- \langle k(\mathbf{e}\cdot\mathbf{e}\cdot\phi * \overline{\mathbf{e}} + 2\overline{\mathbf{e}}\cdot\mathbf{e}\cdot\phi * \mathbf{e}) + k_cu_0(2\mathbf{e}\cdot\mathbf{e_{1,1}} + 2\overline{\mathbf{e}}\cdot\mathbf{e_{2,0}}),\mathbf{e}\rangle.\end{aligned} \tag{5.4}$$

Next, we need to compute $\mathbf{e_{1,1}}$ and $\mathbf{e_{2,0}}$. Straightforward computations show that

$$\begin{aligned}\mathbf{e_{1,1}}(x) &= \frac{2k_cu_0 + 4k_cu_-\widehat{\phi}(\sigma_c)}{b_c - k_cu_-^2} + Span(\mathbf{e},\overline{\mathbf{e}}),\\ \mathbf{e_{2,0}}(x) &= \frac{k_cu_0 + 2k_cu_-\widehat{\phi}(\sigma_c)}{-\sigma_c^2 + b_c - k_cu_-^2\widehat{\phi}(2\sigma_c)}e^{i2x} + Span(\mathbf{e},\overline{\mathbf{e}}).\end{aligned}$$

Using $\mathbf{e_{1,1}}$ and $\mathbf{e_{2,0}}$ in (5.4), we obtain the coefficient ϖ. This completes the proof. ☐

Proof of Theorem 1.4. For convenience, we define

$$\begin{aligned}\Lambda :=&\Big(-\frac{k_c - 2b_c + \sqrt{k_c^2 - 4k_cb_c}}{4}\widehat{\phi}''(k_c) - 1\Big)\frac{\delta^2}{\varpi} + \frac{\varepsilon^2}{2\varpi}\\ &+ \frac{(4b_c+\varepsilon^2)\widehat{\phi}(\sigma_c+\delta)}{\sqrt{k_c^2-4k_cb_c} + \sqrt{k_c^2 - 4k_cb_c - 4b_c\varepsilon^2 - \varepsilon^4}}\frac{\varepsilon^2}{\varpi} > 0.\end{aligned} \tag{5.5}$$

We aim at finding a nontrivial stationary solution $\mathbf{B_0}\in\mathbb{C}$ satisfying

$$0 = h(|\mathbf{B}|^2,\varepsilon,\delta). \tag{5.6}$$

Up to a rescaling $\mathbf{B_0} = \sqrt{\Lambda}\widetilde{\mathbf{B}}_\mathbf{0}$, equation (5.6) can be rewritten as

$$\Lambda\cdot(-\varpi + \varpi|\widetilde{\mathbf{B}}_\mathbf{0}|^2 + O(\sqrt{\Lambda})) = 0, \quad \text{as } \Lambda\to 0.$$

By using the implicit function theorem, we have

$$|\widetilde{\mathbf{B}}_\mathbf{0}| = 1 + O(\sqrt{\Lambda}), \quad \text{as } \Lambda\to 0.$$

So equation (5.1) admits periodic solutions of the form

$$v_{\varepsilon,\delta}(x) = \sqrt{\Lambda}\cos((\sigma_c+\delta)x) + O(|Q|),$$

where

$$
\begin{aligned}
Q =&\Big(-\frac{k_c - 2b_c + \sqrt{k_c^2 - 4k_c b_c}}{4}\widehat{\phi}''(k_c) - 1\Big)\delta^2 + \frac{\varepsilon^2}{2} \\
&+ \frac{(4b_c + \varepsilon^2)\widehat{\phi}(\sigma_c + \delta)}{\sqrt{k_c^2 - 4k_c b_c} + \sqrt{k_c^2 - 4k_c b_c - 4b_c\varepsilon^2 - \varepsilon^4}}\varepsilon^2,
\end{aligned}
\tag{5.7}
$$

for some $\varepsilon \in (0, \varepsilon_0]$ and δ satisfying

$$
\begin{aligned}
&\Big(-\frac{k_c - 2b_c + \sqrt{k_c^2 - 4k_c b_c}}{4}\widehat{\phi}''(k_c) - 1\Big)\delta^2 \\
&< \frac{(4b_c + \varepsilon^2)\widehat{\phi}(\sigma_c + \delta)}{\sqrt{k_c^2 - 4k_c b_c} + \sqrt{k_c^2 - 4k_c b_c - 4b_c\varepsilon^2 - \varepsilon^4}}\varepsilon^2 + \frac{\varepsilon^2}{2}.
\end{aligned}
$$

So we have the existence of periodic solutions of (1.1) that can be written as

$$
u_{\varepsilon,\delta}(x) = u_- + \sqrt{\Lambda}\cos((\sigma_c + \delta)x) + O(|Q|).
$$

This completes the proof. □

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References

[1] S. Ai; *Traveling wave fronts for generalized Fisher equations with spatio-temporal delays*, J. Differential Equations, **232** (2007), no. 1, 104–133.
[2] M. Alfaro, J. Coville; *Rapid travelling waves in the nonlocal Fisher equation connect two unstable states*, Appl. Math. Lett., **25** (2012), no. 12, 2095–2099.
[3] M. Alfaro, J. Coville, G. Raoul; *Bistable travelling waves for nonlocal reaction diffusion equations*, Discrete Contin. Dyn. Syst., **34** (2014), no. 5, 1775–1791.
[4] N. Apreutesei, A. Ducrot, V. Volpert; *Travelling waves for integro-differential equations in population dynamics*, Discrete Contin. Dyn. Syst. Ser. B, **11** (2009), no. 3, 541–561.
[5] P. Ashwin, M. Bartuccelli, T. Bridges, S. Gourley; *Travelling fronts for the KPP equation with spatio-temporal delay*, Z. Angew. Math. Phys., **53** (2002), no. 1, 103–122.
[6] X. Bao and W.-T. Li; *Propagation phenomena for partially degenerate nonlocal dispersal models in time and space periodic habitats*, Nonlinear Anal. Real World Appl., **51** (2020), 102975, 26 pp.
[7] H. Berestycki, G. Nadin, B. Perthame and L. Ryzhik; *The non-local Fisher-KPP equation: Travelling waves and steady states*, Nonlinearity, **22** (2009), no. 12, 2813–2844.
[8] H. Berestycki, N. Rodroffguez; *A non-local bistable reaction-diffusion equation with a gap*, Discrete Contin. Dyn. Syst., **37** (2017), no. 2, 685–723.
[9] N. Britton; *Spatial structures and periodic traveling waves in an integro-differential reaction-diffusion population model*, SIAM J. Appl. Math., **50** (1990), no. 6, 1663–1688.
[10] G.-S. Chen, S.-L. Wu; *Pulsating type entire solutions originating from three fronts for a bistable reaction-advection-diffusion equation in periodic media*, Nonlinear Anal. Real World Appl., **50** (2019), 498–518.
[11] I. Demin, V. Volpert; *Existence of waves for a nonlocal reaction-diffusion equation*, Math. Model. Nat. Phenom., **5** (2010), no. 5, 80–101.
[12] J. P. Eckmann, C. E. Wayne; *Propagating fronts and center manifold theorem*, Comm. Math. Phys., **136** (1991), no. 2, 285–307.
[13] J. Fang, X.- Q. Zhao; *Monotone wave fronts of the nonlocal Fisher-KPP equation*, Nonlinearity, **24** (2011), no. 11, 3043–3054.
[14] G. Faye, M. Holzer; *Modulated traveling fronts for a nonlocal Fisher-KPP equation: a dynamical systems approach*, J. Differential Equations, **258** (2015), no. 7, 2257–2289.

[15] J. E. Furter, M. Grinfeld; *Local vs nonlocal interactions in population dynamics*, J. Math. Biol., **27** (1989), no. 1, 65–80.
[16] S. Genieys, V. Volpert, P. Auger; *Pattern and waves for a model in population dynamics with nonlocal consumption of resources*, Math. Model. Nat. Phenom., **1** (2006), no. 1, 65–82.
[17] S. Gourley; *Travelling front solutions of a nonlocal Fisher equation*, J. Math. Biol., **41** (2000), no. 3, 272–284.
[18] F. Hamel, L. Ryzhik; *On the nonlocal Fisher-KPP equation: steady states, spreading speed and global bounds*, Nonlinearity, **27** (2014), no. 11, 2735–2753.
[19] B.-S. Han, Y. Yang, W.-J. Bo, H. Tang; *Global dynamics for a Lotka-Volterra competition diffusion system with nonlocal effects*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **30** (2020), no 5, 2050066.
[20] B.-S. Han, Z.-C. Wang and Z. Du; *Traveling waves for nonlocal Lotka-Volterra competition systems*, Discrete Contin. Dyn. Syst. Ser. B, **25** (2020), no 5, 1959–1983.
[21] M. Haragus, G. Iooss; *Local Bifurcations, Center Manifolds, and Normal Forms in Infinite Dimensional Dynamical Systems*, Universitext, Springer-Verlag London Ltd., London, 2011.
[22] J. Li, E. Latos, L. Chen; *Wavefronts for a nonlinear nonlocal bistable reaction-diffusion equation in population dynamics*, J. Differential Equations, **263** (2017), no. 10, 6427–6455.
[23] G. Nadin, B. Perthame and M. Tang; *Can a traveling wave connect two unstable states? The case of the nonlocal Fisher equation*, C. R. Math. Acad. Sci. Paris, **349** (2011), no. 9-10, 553–557.
[24] B. Sounvoravong, S. J. Guo, Y. Z. Bai; Bifurcation and stability of a diffusive SIRS epidemic model with time delay, Electron. J. Differential Equations, **2019** (2019), no. 45, 1–16.
[25] Y. Tian, X.-Q. Zhao; *Bistable traveling waves for a competitive-cooperative system with nonlocal delays*, J. Differential Equations, **264** (2018), no. 8, 5263–5299.
[26] S. Trofimchuk, V. Volpert; *Traveling Waves for a Bistable Reaction-Diffusion Equation with Delay*, SIAM J. Math. Anal., **50** (2018), no. 1, 1175–1199.
[27] C. Ou, J. Wu; *Traveling wavefronts in a delayed food-limited population model*, SIAM J. Math. Anal., **39** (2007), no. 1, 103–125.
[28] A. Volpert, V. Volpert, V. Volpert; *Travelling Wave Solutions of Parabolic Systems*, Translated from the Russian manuscript by James F. Heyda. Translations of Mathematical Monographs, 140. American Mathematical Society, Providence, RI, 1994.
[29] Z.- C. Wang, W.- T. Li, S. Ruan; *Travelling wave fronts in reaction-diffusion systems with spatio-temporal delays*, J. Differential Equations, **222** (2006), no. 1, 185–232.
[30] S.-L. Wu, C.-H. Hsu; *Entire solutions with annihilating fronts to a nonlocal dispersal equation with bistable nonlinearity and spatio-temporal delay*, J. Dynam. Differential Equations, **29** (2017), no. 2, 409–430.
[31] C. Zhang, B. Han; *Stability analysis of stochastic delayed complex networks with multi-weights based on Razumikhin technique and graph update theory*, Physica A, **538** (2020), 122827.

Bang-Sheng Han (corresponding author)
School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan, 611756, China
Email address: hanbangsheng@swjtu.edu.cn

Meng-Xue Chang
School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan, 611756, China
Email address: mengxue_chang@163.com

Yinghui Yang
School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan, 611756, China
Email address: yangyh8605@swjtu.edu.cn