Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 85, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

STABILITY OF INITIAL-BOUNDARY VALUE PROBLEM FOR QUASILINEAR VISCOELASTIC EQUATIONS

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Abstract. We investigate the stability of the initial-boundary value problem for the quasilinear viscoelastic equation

$$\begin{gathered} |u_t|^\rho u_{tt} - \Delta u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0, \quad \text{in } \Omega \times (0,+\infty), \\ u = 0, \quad \text{in } \partial\Omega \times (0,+\infty), \\ u(\cdot,0) = u_0(x), \quad u_t(\cdot,0) = u_1(x), \quad \text{in } \Omega, \end{gathered}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$, ρ is a positive real number, and $g(t)$ is the relaxation function. We present a general polynomial decay result under some weak conditions on g, which generalizes and improves the existing related results. Moreover, under the condition $g'(t) \leq -\xi(t)g^p(t)$, we obtain uniform exponential and polynomial decay rates for $1 \leq p < 2$, while in the previous literature only the case $1 \leq p < 3/2$ was studied. Finally, under a general condition $g'(t) \leq -H(g(t))$, we establish a fine decay estimate, which is stronger than the previous results.

1. Introduction

In this article, we consider the stability of the initial-boundary value problem for quasilinear viscoelastic equations,

$$\begin{gathered} |u_t|^\rho u_{tt} - \Delta u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0, \quad \text{in } \Omega \times (0,+\infty), \\ u = 0, \quad \text{in } \partial\Omega \times (0,+\infty), \\ u(\cdot,0) = u_0(x), \quad u_t(\cdot,0) = u_1(x), \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\partial\Omega$, ρ is a positive real number, and $g(t)$ the relaxation function.

In [16], under the assumption that the bounded C^1-function $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies

$$1 - \int_0^{+\infty} g(t)ds > 0, \quad g'(t) \leq -\xi g^p(t), \quad 1 \leq p < \frac{3}{2}, \tag{1.2}$$

where $\xi > 0$ is a constant, Messaoudi and Tatar obtained decay rates in [16, Theorem 3.1].

2010 *Mathematics Subject Classification.* 35Q74, 35B35, 74H55, 74H40, 93D15.
Key words and phrases. Quasilinear viscoelastic equation; polynomial and exponential decay; relaxation function; uniform decay.

Submitted November 11, 2019. Published July 30, 2020.

More recently, Messaoudi and Al-Khulaifi [13] improved this result [16, Theorems 3.1] by using the assumption that the non-increasing differentiable function $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies

$$1-\int_0^{+\infty} g(t)ds > 0, \quad g'(t) \le -\xi(t)g^p(t), \quad 1 \le p < \frac{3}{2}, \tag{1.3}$$

here $\xi(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing differentiable function with $\xi(0) > 0$.

Messaoudi and Mustafa [14] also studied problem (1.1) and the corresponding decay results were obtained for the following condition on $g(t)$,

$$g'(t) \le -H(g(t)), \quad t \ge 0, \tag{1.4}$$

where H is a positive function and satisfies some conditions (see details in [14, hypotheses (A2) and (A3)]).

For more related information on the stability of problem (1.1) and some related equations or systems, we refer the reader to [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 18, 19, 20, 21, 22, 23, 24] and references therein.

In this article, we investigate the stability for problem (1.1) by using more general (weaker) assumptions on the relaxation functions $g(t)$. We establish ideal stability theorems with exact uniform polynomial decay rates t^{-1} for the solutions to this problem, under some basic conditions (see Theorem 3.2). Furthermore, in Theorems 3.4 and 3.6, our results hold for all $1 \le p < 2$, while in the previous literature only the case: $1 \le p < \frac{3}{2}$ was studied. Therefore, all of our results, with much weaker conditions on the relaxation function $g(t)$, are optimal so far.

In the next section, we prove some estimates (lemmas) which will be used in Section 3. Finally, we will state and prove our main results in Section 3.

2. Basic estimates

In this article we use the following assumptions:

(A1) $0 < \rho$, if $n = 1, 2$; and

$$0 < \rho \le \frac{2}{n-2}, \quad \text{if } n \ge 3;$$

(A2) $g(t) : [0, +\infty) \to [0, +\infty)$ is a non-increasing differentiable function with

$$\operatorname{meas}(\mathfrak{J}_0) = 0, \quad g(0) > 0, \quad g'(t) \le 0, \quad \mu_0 > 0,$$

where

$$\mathfrak{J}_0 := \{s \ge 0;\ g(s) > 0, g'(s) = 0\} = 0, \quad \mu_0 := 1 - \int_0^{+\infty} g(t)\,dt.$$

In the sequel, $C, C_i > 0$, $i = 1, 2, \dots$ represent positive constants which are possibly different in different places. We denote

$$G(t) := \int_t^{+\infty} g(s)ds, \quad \text{for } t \ge 0;$$

$$M(\delta) := \int_0^{+\infty} \frac{g(s)}{K_\delta(s)}ds, \quad K_\delta(s) := \frac{-g'(s)}{g(s)} + \delta,$$

where $\delta \in (0, 1)$ is a constant. We define

$$I_1(t) := \int_\Omega \int_0^t G(t-s)|\nabla u(s)|^2\,ds\,dx,$$

$$I_2(t) := M(\delta)\Big(\delta\int_\Omega\int_0^t G(t-s)|\nabla u(s)|^2\,ds\,dx + E(t)\Big).$$

Lemma 2.1. *For $t\ge 0$,*

$$\frac{d}{dt}I_1(t) \le -\frac{1}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx + 2G(0)\int_\Omega|\nabla u(t)|^2dx, \quad (2.1)$$

and

$$\begin{aligned}\frac{d}{dt}I_2(t) \le& -\frac{1}{2}M(\delta)\int_\Omega\int_0^t K_\delta(t-s)g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\\ &+ 2\delta M(\delta)G(0)\int_\Omega|\nabla u(t)|^2dx.\end{aligned} \quad (2.2)$$

Moreover,

$$\delta M(\delta)\to 0, \quad \text{as } \delta\to 0. \quad (2.3)$$

Proof. Noting that

$$-(a\pm b)^2 \le -\frac{1}{2}a^2 + b^2,$$

we see by a direct calculation that, for $t\ge 0$,

$$\begin{aligned}\frac{d}{dt}I_1(t) &= -\int_\Omega\int_0^t g(t-s)|\nabla u(s)|^2\,ds\,dx + G(0)\int_\Omega|\nabla u(t)|^2dx\\ &= -\int_\Omega\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)-\nabla u(t)|^2\,ds\,dx\\ &\quad + G(0)\int_\Omega|\nabla u(t)|^2dx\\ &\le -\frac{1}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\\ &\quad + \int_\Omega\int_0^t g(t-s)|\nabla u(t)|^2\,ds\,dx + G(0)\int_\Omega|\nabla u(t)|^2dx\\ &\le -\frac{1}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx + 2G(0)\int_\Omega|\nabla u(t)|^2dx.\end{aligned}$$

This means that (2.1) holds.

From the definition of $K_\delta(s)$, (2.1) and (3.2), it follows that

$$\begin{aligned}\frac{d}{dt}I_2(t) &\le -\frac{1}{2}M(\delta)\int_\Omega\int_0^t (\delta g(t-s)+g'(t-s))\,|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\\ &\quad + 2\delta M(\delta)G(0)\int_\Omega|\nabla u(t)|^2dx\\ &\le -\frac{1}{2}M(\delta)\int_\Omega\int_0^t K_\delta(t-s)g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\\ &\quad + 2\delta M(\delta)G(0)\int_\Omega|\nabla u(t)|^2dx.\end{aligned}$$

According to [8, P. 1525, lines 8-10], we know that (2.3) is true. Thus, we completed the proof. □

We define

$$F_1(t) := \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u\,dx + \int_\Omega \nabla u\cdot\nabla u_t\,dx,$$

Lemma 2.2. *For $t \ge 0$,*

$$\begin{aligned} \frac{d}{dt}F_1(t) \le -\frac{\mu_0}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2\mu_0}\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\Big)^2 dx \\ + \int_\Omega |\nabla u_t|^2 dx + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2}dx. \end{aligned} \tag{2.4}$$

Proof. Clearly, we can rewrite the first equation in (1.1) as

$$|u_t|^\rho u_{tt} - \Delta u_{tt} - \Big(1-\int_0^t g(s)ds\Big)\Delta u - \int_0^t g(t-s)\left(\Delta u(t)-\Delta u(s)\right)ds = 0. \tag{2.5}$$

It follows from (2.5) that

$$\begin{aligned} \frac{d}{dt}F_1(t) &= \Big(1-\int_0^t g(s)ds\Big)\int_\Omega u\Delta u dx + \int_\Omega u(t)\int_0^t g(t-s)\left(\Delta u(t)-\Delta u(s)\right)\,ds\,dx \\ &\quad + \int_\Omega |\nabla u_t|^2 dx + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2}dx \\ &= -\Big(1-\int_0^t g(s)ds\Big)\int_\Omega |\nabla u|^2 dx \\ &\quad - \int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\,ds\,dx \\ &\quad + \int_\Omega |\nabla u_t|^2 dx + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2}dx \\ &\le -\mu_0\int_\Omega |\nabla u|^2 dx + \frac{\mu_0}{2}\int_\Omega |\nabla u|^2 dx \\ &\quad + \frac{1}{2\mu_0}\int_\Omega \left(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right)^2 dx \\ &\quad + \int_\Omega |\nabla u_t|^2 dx + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2}dx \\ &\le -\frac{\mu_0}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2\mu_0}\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\Big)^2 dx \\ &\quad + \int_\Omega |\nabla u_t|^2 dx + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2}dx. \end{aligned}$$

This completes the proof. □

Now, we define

$$F_2(t) := \int_\Omega \Big(\Delta u_t - \frac{1}{\rho+1}|u_t|^\rho u_t\Big)\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx.$$

Lemma 2.3. *There is a constant $C_1 > 0$ such that, for $t \geq t_0$,*

$$\begin{aligned}
&\frac{d}{dt}F_2(t)\\
&\leq -\frac{G(0)}{2(\rho+1)}\int_\Omega |u_t(t)|^{\rho+2}dx - \frac{G(0)}{2}\int_\Omega |\nabla u_t(t)|^2 dx\\
&\quad + \frac{\mu_0 G(0)}{16}\int_\Omega |\nabla u(t)|^2 dx + C_1\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\Big)^2 dx\\
&\quad - C_1\int_\Omega\int_0^t g'(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx,
\end{aligned} \tag{2.6}$$

where t_0 a positive large number so that

$$\int_0^{t_0} g(s)ds = \frac{3G(0)}{4}.$$

Proof. By (2.5), we obtain

$$\begin{aligned}
&\frac{d}{dt}F_2(t)\\
&= -\frac{1}{\rho+1}\int_0^t g(s)ds\int_\Omega |u_t(t)|^{\rho+2}dx + \int_0^t g(s)ds\int_\Omega u_t(t)\Delta u_t(t)dx\\
&\quad + \int_\Omega u_t\int_0^t g'(t-s)(\Delta u(t)-\Delta u(s))\,ds\,dx\\
&\quad - \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx\\
&\quad - \Big(1-\int_0^t g(s)ds\Big)\int_\Omega \Delta u(t)\cdot\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx\\
&\quad - \int_\Omega\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \qquad (2.7)\\
&= -\frac{1}{\rho+1}\int_0^t g(s)ds\int_\Omega |u_t(t)|^{\rho+2}dx - \int_0^t g(s)ds\int_\Omega |\nabla u_t(t)|^2 dx\\
&\quad - \int_\Omega \nabla u_t\cdot\int_0^t g'(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&\quad - \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx\\
&\quad + \Big(1-\int_0^t g(s)ds\Big)\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx\\
&\quad + \int_\Omega\Big(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\Big)^2 dx.
\end{aligned}$$

Next, let us to estimate the third, fourth and fifth terms on the right of (2.7). First we estimate the fourth term. By Young's and Holder's inequality, for any $\zeta_1 > 0$, we have

$$-\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx$$

$$\le \frac{1}{\rho+1}\zeta_1\int_\Omega |u_t|^{2\rho+2}dx - \frac{g(0)}{4\zeta_1(\rho+1)}\int_\Omega\int_0^t g'(t-s)|u(t)-u(s)|^2\,ds\,dx.$$

By (A1), (A2) and the Sobolev embedding inequality, we obtain

$$\int_\Omega |u_t|^{2\rho+2}dx \le C_s(2E(0))^\rho \int_\Omega |\nabla u_t|^2 dx\,.$$

By Poincaré's inequality, we have

$$-\int_\Omega\int_0^t g'(t-s)|u(t)-u(s)|^2\,ds\,dx \le -C_p\int_\Omega\int_0^t g'(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx,$$

where C_p is the Poincaré's constant and C_s the Sobolev embedding constant. Therefore,

$$\begin{aligned} &-\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx \\ &\le \frac{C_s}{\rho+1}(2E(0))^\rho\zeta_1\int_\Omega|\nabla u_t|^2dx \\ &\quad -\frac{g(0)C_p}{4\zeta_1(\rho+1)}\int_\Omega\int_0^t g'(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx. \end{aligned} \tag{2.8}$$

Now, we estimate the third and fifth terms. It is not hard to see that, for any $\zeta_2, \zeta_3 > 0$,

$$\begin{aligned} &-\int_\Omega \nabla u_t\cdot\int_0^t g'(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\ &\le \zeta_2\int_\Omega|\nabla u_t|^2dx - \frac{g(0)}{4\zeta_2}\int_\Omega\int_0^t g'(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\Big(1-\int_0^t g(s)ds\Big)\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\,dx \\ &\le \zeta_3\int_\Omega|\nabla u(t)|^2dx + \frac{1}{4\zeta_3}\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\Big)^2dx. \end{aligned} \tag{2.10}$$

Thus, combining (2.8), (2.9), (2.10) with (2.7), we know that

$$\begin{aligned} &\frac{d}{dt}F_2(t) \\ &\le -\frac{1}{\rho+1}\int_0^t g(s)ds\int_\Omega |u_t(t)|^{\rho+2}dx \\ &\quad -\Big(\int_0^t g(s)ds - \zeta_2 - \frac{C_s}{\rho+1}(2E(0))^\rho\zeta_1\Big)\int_\Omega|\nabla u_t(t)|^2dx \\ &\quad -\Big(\frac{g(0)}{4\zeta_2}+\frac{g(0)C_p}{4\zeta_1(\rho+1)}\Big)\int_\Omega\int_0^t g'(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx \\ &\quad +\zeta_3\int_\Omega|\nabla u(t)|^2dx + \Big(1+\frac{1}{4\zeta_3}\Big)\int_\Omega\Big(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\Big)^2dx. \end{aligned}$$

Setting

$$\zeta_1 = \frac{(\rho+1)G(0)}{8C_s(2E(0))^\rho},\quad \zeta_2 = \frac{G(0)}{8},\quad \zeta_3 = \frac{\mu_0 G(0)}{16},$$

we obtain the estimate (2.6). This completes the proof. □

3. Main results and their proofs

We firstly state an existence and uniqueness result for problem (1.1), which can be proved by using similar arguments as in [4, 15] so we omit it here.

Theorem 3.1. *Let* (A1) *and* (A2) *hold. Then for any* $u_0 \in H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, *the problem* (1.1) *has a unique global solution on* $[0,\infty)$ *with the regularity*

$$u \in C^1\left(\mathbb{R}^+; H_0^1(\Omega)\right).$$

We introduce the energy functional

$$\begin{aligned} E(t) :=& \frac{1}{\rho+2}\int_\Omega |u_t|^{\rho+2}dx + \frac{1}{2}\int_\Omega |\nabla u_t|^2 dx + \frac{1}{2}\Big(1 - \int_0^t g(s)ds\Big)\int_\Omega |\nabla u|^2 dx \\ &+ \frac{1}{2}\int_\Omega\int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|^2\,ds\,dx. \end{aligned} \tag{3.1}$$

Then, for $t \geq 0$,

$$\begin{aligned} \frac{d}{dt}E(t) &= -\frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega\int_0^t g'(t-s)|\nabla u(t) - \nabla u(s)|^2\,ds\,dx \\ &\leq \frac{1}{2}\int_\Omega\int_0^t g'(t-s)|\nabla u(t) - \nabla u(s)|^2\,ds\,dx, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} E(t) \sim& \int_\Omega \left(|u_t|^{\rho+2} + |\nabla u_t|^2 + |\nabla u|^2\right)dx \\ &+ \int_\Omega\int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|^2\,ds\,dx. \end{aligned} \tag{3.3}$$

The following is our general uniform decay theorem for the solution energy of problem (1.1).

Theorem 3.2. *Let* (A1) *and* (A2) *hold. Then, for* $u_0, u_1 \in H_0^1(\Omega)$, *the solution energy* $E(t)$ *of the problem* (1.1) *satisfies*

$$\int_0^{+\infty} E(t) \leq CE(0), \quad t \geq 0,$$
$$E(t) \leq CE(0)(t+1)^{-1}, \quad t \geq 0,$$

where $C > 0$ *is a constant.*

Proof. The proof is mainly based on the construction of an auxiliary function $L(t)$ satisfying

$$L(t_0) \leq CE(0), \quad L(t) \geq 0, \quad t \geq 0,$$

and

$$\frac{d}{dt}L(t) \leq -\epsilon_0 E(t), \quad t \geq t_0. \tag{3.4}$$

Clearly, integrating (3.4) we obtain the desired estimate. Now, we apply the lemmas obtained in the previous section to construct this auxiliary function $L(t)$. We define

$$J(t) := NE(t) + F_1(t) + \frac{4}{G(0)}F_2(t).$$

By the definitions of $F_1(t)$ and $F_2(t)$ and a simple calculation, we see that, there is a constant $c_0 > 0$ such that, for $t \geq 0$,

$$|F_1(t)|, |F_1(t)| \leq c_0 E(t).$$

Taking $N > 8C_1/G(0)$ large enough, we obtain

$$c_1E(t) \le J(t) \le c_2E(t), \quad t \ge 0,$$

where $c_1, c_2 > 0$ are constants.

Thus, by (2.4), (2.6) and (3.2), for $t \ge t_0$, we have

$$\begin{aligned} \frac{d}{dt}J(t) \le & -\frac{\mu_0}{4}\int_\Omega |\nabla u(t)|^2dx - \frac{1}{\rho+1}\int_\Omega |u_t(t)|^{\rho+2}dx - \int_\Omega |\nabla u_t|^2dx \\ & + \Big(\frac{4C_1}{G(0)} + \frac{1}{2\mu_0}\Big)\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|ds\Big)^2 dx. \end{aligned} \tag{3.5}$$

Moreover,

$$\begin{aligned} &\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|ds\Big)^2 dx \\ &\le \int_\Omega \int_0^t \frac{g(s)}{K_\delta(s)}ds \int_0^t K_\delta(t-s)g(t-s)|\nabla u(t) - \nabla u(s)|^2\,ds\,dx \\ &\le M(\delta)\int_\Omega\int_0^t K_\delta(t-s)g(t-s)|\nabla u(t) - \nabla u(s)|^2\,ds\,dx. \end{aligned}$$

Hence, by (3.5), for $t \ge t_0$, we see that

$$\begin{aligned} &\frac{d}{dt}J(t) \\ &\le -\frac{\mu_0}{4}\int_\Omega |\nabla u(t)|^2dx - \frac{1}{\rho+1}\int_\Omega |u_t(t)|^{\rho+2}dx - \int_\Omega |\nabla u_t|^2dx \\ &\quad + \Big(\frac{4C_1}{G(0)} + \frac{1}{2\mu_0}\Big)M(\delta)\int_\Omega\int_0^t K_\delta(t-s)g(t-s)|\nabla u(t) - \nabla u(s)|^2\,ds\,dx. \end{aligned} \tag{3.6}$$

Now we define

$$L(t) := J(t) + \frac{\mu_0}{32G(0)}I_1(t) + 2\Big(\frac{4C_1}{G(0)} + \frac{1}{2\mu_0}\Big)I_2(t).$$

Then, by (2.1), (2.2) and (3.6), for $t \ge t_0$, we obtain

$$\begin{aligned} \frac{d}{dt}L(t) \le & -\Big(\frac{3\mu_0}{16} - 4G(0)\Big(\frac{4C_1}{G(0)} + \frac{1}{2\mu_0}\Big)\delta M(\delta)\Big)\int_\Omega |\nabla u(t)|^2dx \\ & - \frac{1}{\rho+1}\int_\Omega |u_t(t)|^{\rho+2}dx - \int_\Omega |\nabla u_t|^2dx \\ & - \frac{\mu_0}{64G(0)}\int_\Omega\int_0^t g(t-s)|\nabla u(t) - \nabla u(s)|^2\,ds\,dx. \end{aligned} \tag{3.7}$$

Convergence (2.3) shows that there exists $\delta_0 > 0$ such that, for any $0 < \delta < \delta_0$,

$$\delta M(\delta) \le \frac{\mu_0}{64G(0)\big(\frac{4C_1}{G(0)} + \frac{1}{2\mu_0}\big)}.$$

Thus, by (3.7) and (3.3), we deduce that, for $0 < \delta < \delta_0$, there exists a constant $\epsilon_0 > 0$ such that, for $t \ge t_0$,

$$\frac{d}{dt}L(t) \le -\epsilon_0 E(t). \tag{3.8}$$

Since $L(t) \geq 0$ for $t \geq 0$, and $L(t_0) \leq CE(0)$, it follows by integrating (3.8) over $[t_0, \tau)$ that for any $\tau > t_0$,

$$\int_{t_0}^{\tau} E(t)dt \leq CE(0).$$

So,

$$\int_{0}^{+\infty} E(t)dt \leq CE(0). \tag{3.9}$$

Noting that $E'(t) \leq 0$, by(3.9), we obtain

$$E(t) \leq CE(0)(t+1)^{-1}, \quad t \geq 0.$$

This completes the proof. □

Remark 3.3. (1) As showed in Theorem 3.2, the polynomial decay rates can be obtained without the control conditions on $g'(t)$ used previously.

There are many functions $g(t)$ satisfying the assumptions (A2) without satisfying the previous restriction that $g(t)$ controls $g'(t)$ as in (1.2), (1.3) and (1.4). For example, if

$$g(t) = \big(\sqrt{2} + \sin t\big)e^{-t}, \quad t \geq 0,$$

then

$$\begin{aligned} g'(t) &= -\big(\sqrt{2} - \cos t + \sin t\big)e^{-t} \\ &= -\sqrt{2}\big(1 - \cos(t + \frac{\pi}{4})\big)e^{-t}, \quad t \geq 0. \end{aligned}$$

Clearly,

$$g'(t) \leq 0, \quad \text{for } t \geq 0;$$
$$g'(t) = 0, \quad \text{for } t = 2k\pi - \frac{\pi}{4}, \quad k = 1, 2, \dots .$$

Hence, $g(t)$ satisfies (A2), while $g(t)$ does not satisfy (1.2), (1.3) or (1.4). That is, $g'(t)$ is not controlled by $g(t)$.

Functions $g(t)$ as above have not been studied in the literature. However, we can treat the problem (1.1) with these general relaxation functions, and according to Theorem 3.2 here, we know the energy $E(t)$ of problem (1.1) decays at least at the rate $(t+1)^{-1}$.

(2) The decay rates given in Theorem 3.2 are optimal in a sense according to [13, Example 3.1, Remark 3.2] and [8, Remark 3.3(ii)].

When the derivative $g'(s)$ is controlled by the relaxation function $g(t)$, we can prove the following results.

Theorem 3.4. *Let* (A1) *and* (A2) *hold, and*

$$g'(t) \leq -\xi(t)g^p(t), \quad t \geq 0, \tag{3.10}$$

where $\xi(t) : \mathbb{R}^+ \to \mathbb{R}^+$ *is a non-increasing differentiable function with* $\xi(0) > 0$ *and* $1 \leq p < 2$ *is a constant. Then there are constants* $C, \eta > 0$ *such that for* $t \geq 0$,

$$E(t) \leq \begin{cases} CE(0)e^{-\eta\int_0^t \xi(s)ds}, & p = 1, \\ CE(0)\Big(\frac{1}{1+\int_0^t \xi(s)ds}\Big)^{\frac{1}{p-1}} & 1 < p < 2. \end{cases} \tag{3.11}$$

Proof. A key idea in the proof is to construct a Lyapunov function satisfying $R(t) \sim E(t)$ and

$$\frac{d}{dt}R(t) \le -\epsilon_2 \xi(t) R^p(t).$$

To find this function, we will use the results of Theorem 3.2 and $J(t)$ defined above. Clearly,

$$\begin{aligned}&\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\Big)^2 dx\\ &\le G(0)\int_\Omega\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx.\end{aligned}$$

Thus, by (3.5) and (3.3), for $t \ge t_0$, we have

$$\frac{d}{dt}J(t) \le -\epsilon_1 E(t) + C_2\int_\Omega\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx, \tag{3.12}$$

where $\epsilon_1 > 0$ is a constant.

On the other hand, by Theorem 3.2, we know that

$$\int_0^{+\infty} E(t)dt \le CE(0), \quad \text{and} \quad E(t) \le CE(0)(t+1)^{-1}.$$

Since

$$\begin{aligned}&\int_\Omega\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\\ &\le \Big(\int_0^t\int_\Omega |\nabla u(t)-\nabla u(s)|^2 dx ds\Big)^{1-\frac{1}{p}}\Big(\int_\Omega\int_0^t g^p(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\Big)^{1/p}\\ &\le C\Big(\int_0^t (E(t)+E(s))ds\Big)^{1-\frac{1}{p}}\Big(\int_\Omega\int_0^t g^p(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\Big)^{1/p}\\ &\le CE^{1-\frac{1}{p}}(0)\Big(\int_\Omega\int_0^t g^p(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\Big)^{1/p},\end{aligned}$$

by (3.12) it follows that for $t \ge t_0$,

$$\frac{d}{dt}J(t) \le -\epsilon_1 E(t) + C_3E^{1-\frac{1}{p}}(0)\Big(\int_\Omega\int_0^t g^p(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\Big)^{1/p}. \tag{3.13}$$

Multiplying (3.13) by $\xi(t)E^{p-1}(t)$, for $t \ge t_0$, we obtain

$$\begin{aligned}&\xi(t)E^{p-1}(t)\frac{d}{dt}J(t)\\ &\le -\epsilon_1\xi(t)E^p(t)\\ &\quad + C_3E^{1-\frac{1}{p}}(0)\xi(t)E^{p-1}(t)\Big(\int_\Omega\int_0^t g^p(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\Big)^{1/p}\\ &\le -\frac{\epsilon_1}{2}\xi(t)E^p(t) + C_4\xi(t)\int_\Omega\int_0^t g^p(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx.\end{aligned} \tag{3.14}$$

Since $\xi(t), E(t)$ are non-increasing functions, from (3.10) it follows that for $t \ge 0$,

$$\begin{aligned}\frac{d}{dt}\big(\xi(t)E^{p-1}(t)J(t)\big) &= \xi(t)E^{p-1}(t)\frac{d}{dt}J(t) + J(t)\frac{d}{dt}\big(\xi(t)E^{p-1}(t)\big)\\ &\le \xi(t)E^{p-1}(t)\frac{d}{dt}J(t),\end{aligned}$$

and

$$\begin{aligned}
&\xi(t)\int_\Omega\int_0^t g^p(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\\
&\le \int_\Omega\int_0^t \xi(t-s)g^p(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\\
&\le -\int_\Omega\int_0^t g'(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\,dx\\
&\le -2\frac{d}{dt}E(t).
\end{aligned}$$

Hence, by (3.14), for $t\ge t_0$, we have

$$\frac{d}{dt}\big(\xi(t)E^{p-1}(t)J(t)+2C_4E(t)\big)\le -\frac{\epsilon_1}{2}\xi(t)E^p(t). \tag{3.15}$$

Now, we define

$$R(t):=\xi(t)E^{p-1}(t)J(t)+2C_4E(t).$$

Then, $R(t)\sim E(t)$. By (3.15), for $t\ge t_0$, we obtain

$$\frac{d}{dt}R(t)\le -\epsilon_2\xi(t)R^p(t),$$

where $\epsilon_2>0$ is a constant. This completes the proof. □

Remark 3.5. (1) Theorem 3.4 extends the results in [13, 14, 16], where $g'(t)$ was assumed to satisfy (3.10) with $p\in[1,3/2)$, since Theorem 3.4 holds for all $p\in[1,2)$. Moreover, the decay rates obtained in [13] are

$$\begin{gathered}
E(t)\le Ke^{-\lambda\int_{t_0}^t\xi(s)ds},\quad p=1,\\
E(t)\le K\Big(\frac{1}{1+\int_{t_0}^t\xi^{2p-1}(s)ds}\Big)^{\frac{1}{2p-2}},\quad 1<p<\frac{3}{2}.
\end{gathered}$$

In addition, if

$$\int_0^{+\infty}\Big(\frac{1}{t\xi^{2p-1}(t)+1}\Big)dt<+\infty,\quad 1<p<\frac{3}{2}, \tag{3.16}$$

reference [13] shows the improved estimate

$$E(t)\le K\Big(\frac{1}{1+\int_{t_0}^t\xi^p(s)ds}\Big)^{\frac{1}{p-1}},\quad 1<p<\frac{3}{2}.$$

Since $\xi(t)$ is nonnegative and non-increasing, it is clear that $\xi^p(s)\lesssim\xi(s)$, and then

$$\Big(\frac{1}{1+\int_0^t\xi(s)ds}\Big)^{\frac{1}{p-1}}\lesssim\Big(\frac{1}{1+\int_{t_0}^t\xi^p(s)ds}\Big)^{\frac{1}{p-1}}.$$

Therefore, the decay rates given in Theorem 3.4 is stronger than the previous conclusion in the [13, Theorem 3.1] for all $p\in[1,2)$. On the other hand, we obtain the stronger estimate without the other restrictions on $\xi(t)$ (as (3.16) in [13, Theorem 3.1]). As can be seen, Theorem 3.4 here give stronger conclusions essentially under weaker conditions on $g(t)$.

(2) The decay rates given in Theorem 3.4 are optimal in according to [13, Example 3.1, Remark 3.2] and [8, Remark 3.3(ii)].

Theorem 3.6. *Let the assumptions of Theorem 3.2 hold, and*

$$g'(t) \le -H(g(t)), \quad t \ge 0, \tag{3.17}$$

where $H \in C^1(\mathbb{R}^+)$ is a positive function with $H(0) = 0$, and it is also a linear or strictly increasing and strictly convex C^2 function on $(0, r]$, for some $r < 1$. Then there are constants $k_1, k_2, k_3, \varepsilon_0 > 0$ such that

$$E(t) \le k_3 G^{-1}(k_1 t + k_2), \quad t \ge 0, \tag{3.18}$$

where

$$G(t) = \int_t^1 \frac{1}{sH'(\varepsilon_0 s)} ds.$$

Proof. By Theorem 3.2, we obtain

$$\int_0^{+\infty} E(t)dt \le CE(0) \quad \text{and} \quad E(t) \le CE(0)(t+1)^{-1}.$$

So,

$$\int_\Omega \int_0^t |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx \le CE(0) < +\infty. \tag{3.19}$$

According to (3.17) and (3.19), we can and do take $t_1 > t_0$ large enough such that for any $t \ge t_1$,

$$\int_\Omega \int_{t_1}^t |\nabla u(t) - \nabla u(s)|^2 \, ds \, dx < \min\{r, H(r)\}, \tag{3.20}$$

$$-\int_\Omega \int_0^{t-t_1} g'(t-s)|\nabla u(t) - \nabla u(s)|^2 \, ds \, dx < \min\{r, H(r)\}, \tag{3.21}$$

$$\int_\Omega \int_0^{t-t_1} g(t-s)|\nabla u(t) - \nabla u(s)|^2 \, ds \, dx < \min\{r, H(r)\}, \tag{3.22}$$

$$\max\{g(t), -g'(t)\} < \min\{r, H(r)\}. \tag{3.23}$$

Using (3.17), (3.20)-(3.23) and Jensen's inequality, for $t \ge t_1$, we obtain

$$\begin{aligned} &-\int_\Omega \int_0^{t-t_1} g'(t-s)|\nabla u(t) - \nabla u(s)|^2 \, ds \, dx \\ &\ge \int_\Omega \int_0^{t-t_1} H(g(t-s))|\nabla u(t) - \nabla u(s)|^2 \, ds \, dx \\ &\ge H\Big(\int_\Omega \int_0^{t-t_1} g(t-s)|\nabla u(t) - \nabla u(s)|^2 \, ds \, dx\Big). \end{aligned} \tag{3.24}$$

Then for $t \ge t_1$,

$$\begin{aligned} &\int_\Omega \int_0^{t-t_1} g(t-s)|\nabla u(t) - \nabla u(s)|^2 \, ds \, dx \\ &\le H^{-1}\Big(-\int_\Omega \int_0^{t-t_1} g'(t-s)|\nabla u(t) - \nabla u(s)|^2 \, ds \, dx\Big). \end{aligned} \tag{3.25}$$

Moreover, by [14, P. 1860, equation (3.24)], for $t \ge t_1$, we obtain

$$\frac{d}{dt} W_1(t) \le -\epsilon_3 E(t) + C_5 \int_\Omega \int_0^{t-t_1} g(t-s)|\nabla u(t) - \nabla u(s)|^2 \, ds \, dx, \tag{3.26}$$

where $W_1(t) \sim E(t)$ and $\epsilon_3 > 0$ is a constant.

By (3.25) and (3.26), for $t \ge t_1$, we have

$$\begin{aligned} &\frac{d}{dt}W_1(t) \\ &\le -\epsilon_3 E(t) + C_5 H^{-1}\Big(-\int_\Omega \int_0^{t-t_1} g'(t-s)|\nabla u(t) - \nabla u(s)|^2\, ds\, dx\Big). \end{aligned} \tag{3.27}$$

Now, we define

$$W_2(t) := H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) W_1(t) + ME(t),$$

where $0 < \varepsilon_0 < r$, $M > 0$ are constants, which will be specific later.

Clearly, $W_2(t) \sim E(t)$ because of the assumption on H. Therefore, for $t \ge t_1$,

$$\begin{aligned} &\frac{d}{dt}W_2(t) \\ &= H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) \frac{d}{dt}W_1(t) + \varepsilon_0 \frac{E'(t)}{E(0)} H''\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) W_1(t) + ME'(t) \\ &\le C_5 H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) H^{-1}\Big(-\int_\Omega \int_0^{t-t_1} g'(t-s)|\nabla u(t) - \nabla u(s)|^2\, ds\, dx\Big) \\ &\quad - \epsilon_3 E(t) H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) + ME'(t), \end{aligned} \tag{3.28}$$

where we have used $E'(t) \le 0$, $H'' \ge 0$, and (3.27).

Next, we estimate the first term on the right of (3.28). Let $H^\star$ be the convex conjugate of H in the sense of Young (see [2, P. 61-64] and [14, P. 1863]). Then

$$H^\star(s) = s(H')^{-1}(s) - H[(H')^{-1}(s)], \quad s \in (0, H'(r)), \tag{3.29}$$

and it satisfies

$$ab \le H^\star(a) + H(b), \quad \text{for } a \in (0, H'(r)],\ b \in (0, r]. \tag{3.30}$$

Setting

$$a = H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big), \quad b = H^{-1}\Big(-\int_\Omega \int_0^{t-t_1} g'(t-s)|\nabla u(t) - \nabla u(s)|^2\, ds\, dx\Big),$$

and using (3.29), (3.30) and (3.21), we obtain

$$\begin{aligned} &H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) H^{-1}\Big(-\int_\Omega \int_0^{t-t_1} g'(t-s)|\nabla u(t) - \nabla u(s)|^2\, ds\, dx\Big) \\ &\le H^\star\Big(H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big)\Big) - \int_\Omega \int_0^{t-t_1} g'(t-s)|\nabla u(t) - \nabla u(s)|^2\, ds\, dx \\ &\le \varepsilon_0 \frac{E(t)}{E(0)} H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) - 2E'(t). \end{aligned} \tag{3.31}$$

From (3.28) and (3.31), it follows that for $t \ge t_1$,

$$\frac{d}{dt}W_2(t) \le -\left(\epsilon_3 E(0) - C_5 \varepsilon_0\right) \frac{E(t)}{E(0)} H'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) + (M - 2C_5)E'(t). \tag{3.32}$$

Therefore, if we take $M > 0$ large enough and $\varepsilon_0 > 0$ small sufficiently, then we obtain, for $t \ge t_1$,

$$\frac{d}{dt}W_2(t) \le -\epsilon_4 \widetilde{H}\Big(\frac{E(t)}{E(0)}\Big), \tag{3.33}$$

where $\epsilon_4 > 0$ is a constant and $\widetilde{H}(t) = tH'(\varepsilon_0 t)$. We define

$$W(t) := \gamma \frac{W_2(t)}{E(0)},$$

where $\gamma > 0$ small enough such that

$$W(t) < \frac{E(t)}{E(0)}.$$

Clearly, $W(t) \sim E(t) \sim W_2(t)$, and $\widetilde{H}(t), \widetilde{H}'(t) \geq 0$. So, by (3.33), we know that there exists $\epsilon_5 > 0$ such that for $t \geq t_1$

$$\frac{d}{dt} W(t) \leq -\epsilon_5 \widetilde{H}\left(W(t)\right). \tag{3.34}$$

This gives the estimate (3.18). Thus the proof is complete. □

Remark 3.7. In [14, Theorem 3.1], if the relaxation function $g(t)$ satisfies (3.17), then the decay rate is

$$E(t) \leq k_3 H_1^{-1}(k_1 t + k_2), \quad t \geq 0.$$

Detailed information about H_1 can be found in [14, Theorem 3.1]. In addition, if

$$\int_0^1 H_1(t)dt < +\infty, \tag{3.35}$$

then the improved estimate (3.18) iss obtained.

As showed in Theorem 3.6, the improved estimate (3.18) is directly obtained without the extra assumption condition (3.35) (except (3.17)). Therefore, Theorem 3.6 improves [14, Theorem 3.1] essentially, with weaker conditions on the relaxation function. Moreover, Theorem 3.6 gives stronger conclusions.

Acknowledgments. The work was supported by the NSF of China (11771091, 11971306, 11831011), by the China Postdoctoral Science Foundation (2018M632094), and by the Shanghai Key Laboratory for Contemporary Applied Mathematics (08DZ2271900).

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