Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 87, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

OSCILLATORY BEHAVIOR FOR NONLINEAR HOMOGENEOUS NEUTRAL DIFFERENCE EQUATIONS OF SECOND ORDER WITH COEFFICIENT CHANGING SIGN

AJIT KUMAR BHUYAN, LAXMI NARAYAN PADHY, RADHANATH RATH

Abstract. In this article, we obtain sufficient conditions so that all solutions of the neutral difference equation

$$\Delta^2\big(y_n - p_n L(y_{n-s})\big) + q_n G(y_{n-k}) = 0,$$

and all unbounded solutions of the neutral difference equation

$$\Delta^2\big(y_n - p_n L(y_{n-s})\big) + q_n G(y_{n-k}) - u_n H(y_{\alpha(n)}) = 0$$

are oscillatory, where $\Delta y_n = y_{n+1} - y_n$, $\Delta^2 y_n = \Delta(\Delta y_n)$. Different types of super linear and sub linear conditions are imposed on G to prevent the solution approaching zero or $\pm\infty$.

1. Introduction

In this article, we obtain sufficient conditions so that all solutions of the neutral difference equation

$$\Delta^2\big(y_n - p_n L(y_{n-s})\big) + q_n G(y_{n-k}) = 0, \quad n \ge n_0, \tag{1.1}$$

and all unbounded solutions of the neutral difference equation

$$\Delta^2\big(y_n - p_n L(y_{n-s})\big) + q_n G(y_{n-k}) - u_n H(y_{\alpha(n)}) = 0, \quad n \ge n_0 \tag{1.2}$$

are oscillatory, where Δ is the forward difference operator $\Delta y_n = y_{n+1} - y_n$, $\Delta^2 y_n = \Delta(\Delta y_n)$, $\{q_n\}$ and $\{u_n\}$ are sequences of real numbers with $q_n > 0$, $u_n \ge 0$, and $G, H, L \in C(\mathbb{R}, \mathbb{R})$. We assume that $\alpha(n) < n - 1$ and it approaches ∞ as $n \to \infty$, and s, k are positive integers. Further, we assume that

$$\begin{gathered} G(-x) = -G(x), \quad H(-x) = -H(x), \quad L(-x) = -L(x), \quad \forall x \in \mathbb{R} \\ xG(x) > 0, \quad xH(x) > 0, \quad xL(x) > 0 \quad \forall x > 0. \end{gathered} \tag{1.3}$$

Some of the following assumptions are used later in this article.

- (A1) There exists $\delta > 0$ such that for each $x > 0$, $L(x) \le \delta x$;
- (A2) $q_n > 0$ and $\sum_{n=n_0}^{\infty} q_n = \infty$;
- (A3) $\sum_{n=n_1}^{\infty} q_n^* = \infty$, where $q^* = \min\{q_n, q_{n-s}\}$;
- (A4) $\liminf_{n\to\infty} q_n > 0$;
- (A5) G is non decreasing;

2010 *Mathematics Subject Classification.* 39A10, 39A12.
Key words and phrases. Oscillatory solution; nonoscillatory solution; asymptotic behavior; difference equation.

Submitted June 3, 2020. Published August 12, 2020.

- (A6) $\sum_{n=n_0}^{\infty} nu_n < \infty$;
- (A7) H is bounded.

For the sequence $\{p_n\}$ we state the following conditions:

$$0 \le p_n \le p, \tag{1.4}$$

$$0 \le p_n \le 1, \tag{1.5}$$

$$-p \le p_n < 0, \tag{1.6}$$

$$p_n \text{ changes sign and } -p \le p_n \le p, \tag{1.7}$$

$$1 \le p_n \le p, \tag{1.8}$$

$$-1 < -b \le p_n \le 0, \tag{1.9}$$

where p and b are positive constants.

As of now, many researchers all over the world are engaged to find necessary or sufficient conditions for oscillation or non oscillation for neutral difference equations, because of its important applications in different fields of science and technology. For the fundamentals and some recent results on the subject, one may go through the monograph [1, 5] and the research articles [2, 4, 12, 14] and the references cited there in. Sufficient conditions are found, in [3, 4, 7, 12, 13, 14, 15, 16], and more recently in [2, 3], so that every solutions of the non linear neutral difference equation

$$\Delta^2\big(y_n - p_n y_{n-s}\big) + q_n G(y_{n-k}) - u_n H(y_{n-r}) = f_n, \quad n \ge n_0, \tag{1.10}$$

(or of its particular case $u_n \equiv 0$, $f_n \equiv 0$) oscillates or tends to zero or to $\pm\infty$ at ∞. The asymptotic behavior of the solution is probably due to the presence of the forcing term f_n in (1.10).

The objective of this work is to find sufficient conditions so that all solutions of (1.2) are oscillatory under different cases of $p_n > 0$, $p_n < 0$ or p_n changing sign. For that, we had to prevent the bounded solutions of (1.2) from approaching zero by imposing a sub linear condition (4.4) or (4.1) on G as well as stop the unbounded solution of (1.2) from approaching $\pm\infty$ by imposing a super linear condition (3.5) or (3.2) on G. Then the results for (1.2) are applied to study the oscillatory behavior of the unbounded solutions of neutral difference equation

$$\Delta^2\big(y_n - p_n L(y_{n-s})\big) + v_n G(y_{n-k}) = 0, \quad n \ge n_0, \tag{1.11}$$

where v_n changes sign. Our results generalize and extend some results in [2, 11].

Let n_0 be a fixed nonnegative integer. Let $\rho = \min\big\{n_0 - s, n_0 - k, \inf_{n \ge n_0}\{\alpha(n)\}\big\}$. By a solution of (1.2) we mean a real sequence $\{y_n\}$ which is defined for all integers $n \ge \rho$ and satisfies (1.2) for $n \ge n_0$. Clearly if the initial condition

$$y_n = a_n \quad \text{for } \rho \le n \le n_0 + 1, \tag{1.12}$$

is given then equation (1.2) has a unique solution satisfying (1.12). A non trivial solution $\{y_n\}$ of (1.2) is said to be oscillatory if for every positive integer $n_0 > 0$, there exists $n \ge n_0$ such that $y_n y_{n+1} \le 0$, otherwise $\{y_n\}$ is said to be non-oscillatory.

2. Some lemmas

In this section, we present some lemmas to be applied in next section.

Lemma 2.1. [5, Theorem 7.6.1, page 184] *Let $\{r_n\}$ be a non negative sequence of real numbers, k a positive integer and*

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} r_i > \Big(\frac{k}{k+1}\Big)^{k+1}. \tag{2.1}$$

Then the following statements are true.

- (a) $\Delta x_n + r_n x_{n-k} \le 0$ *has no eventually positive solutions, which implies* $\Delta x_n + r_n x_{n-k} \ge 0$ *has no eventually negative solutions.*
- (b) $\Delta x_n - r_n x_{n+k} \ge 0$ *has no eventually positive solutions, which implies* $\Delta x_n - r_n x_{n+k} \le 0$ *has no eventually negative solutions.*

Lemma 2.2. *Suppose that* (A6) and (A7) *hold, and y_n is an eventually positive solution of* (1.2). *Then the sequence*

$$c_n = -\sum_{i=n}^{\infty}(i-n+1)u_i H(y_{\alpha(i)}) \tag{2.2}$$

satisfies

$$\lim_{n\to\infty} c_n = 0, \quad c_n \le 0, \quad \Delta c_n \ge 0, \tag{2.3}$$

for n large enough, and

$$\Delta^2 c_n = -u_n H(y_{\alpha(n)}). \tag{2.4}$$

Proof. Clearly, applying Δ^2 to (2.2), we obtain $\Delta^2 c_n = -u_n H(y_{\alpha(n)})$. By (A6) and (A7), $\sum_{i=n}^{\infty} i u_i H(y_{\alpha(i)}) < \infty$. Comparing this infinite series with (2.2), we show that $\{c_n\}$ converges absolutely to zero. The other statements follow easily. □

Note that if y_n is eventually negative, then $c_n \ge 0$ and $\Delta c_n \le 0$. Next, we prove an important lemma to be used later.

Lemma 2.3. *Let* (A1), (A6), (A7) *hold, y_n be an eventually positive solution of* (1.2), *and c_n be defined by* (2.2). *Then for the sequences*

$$z_n = y_n - p_n L(y_{n-s}), \tag{2.5}$$

$$w_n = z_n + c_n \tag{2.6}$$

we have the following statements:

- (a) *If* (A2) *and* (A5) *hold and p_n satisfy* (1.4), *then either $\Delta w_n < 0$ for large n which implies*

$$\lim_{n\to\infty} w_n = -\infty, \tag{2.7}$$

or $\Delta w_n > 0$ for large n which implies

$$\lim_{n\to\infty} w_n = 0, \tag{2.8}$$

$$w_n < 0, \quad \lim_{n\to\infty} \Delta w_n = 0. \tag{2.9}$$

- (b) *If in addition $p\delta \le 1$, then only* (2.8) *and* (2.9) *hold.*

Proof. Suppose that y_n is an eventually positive solution of (1.2). Then there exits an integer $n_1 \ge n_0$ such that $y_n > 0$, $y_{n-s} > 0$, y_{n-k} and $y_{\alpha(n)} > 0$ for $n \ge n_1$. Then setting c_n, z_n and w_n as in (2.2), (2.5), (2.6), and using (1.2), (2.5), (2.6), and Lemma 2.2, we obtain

$$\Delta^2 w_n = -q_n G(y_{n-k}) \le 0 \quad \text{for } n > n_1\,. \tag{2.10}$$

Then Δw_n is decreasing. Hence Δw_n is monotonic and of single sign for n large enough. It follows that either $\Delta w_n < 0$ or $\Delta w_n > 0$. If $\Delta w_n < 0$, then w_n is decreasing, and using that Δw_n is decreasing, we have

$$\lim_{n\to\infty} \Delta w_n = -\infty. \tag{2.11}$$

If $\Delta w_n > 0$, then w_n is increasing, and using that Δw_n is decreasing, we have

$$\lim_{n\to\infty} \Delta w_n = \zeta \quad \text{(a finite number)}. \tag{2.12}$$

Let us prove part (a). If (2.11) holds then clearly (2.7) follows. If (2.12) holds then, summing (2.10) from $n_2 > n_1$ to ∞ we obtain

$$\sum_{n=n_2}^{\infty} q_n G(y_{n-k}) < \infty, \tag{2.13}$$

which by using (A2) yields

$$\liminf_{n\to\infty} y_n = 0. \tag{2.14}$$

Then we find a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to 0$ as $k \to \infty$. Now using (1.4), (A1) and Lemma 2.2 we obtain

$$w_{n_k} < y_{n_k} + c_{n_k} \to 0 \quad \text{as } k \to \infty \tag{2.15}$$

and

$$w_{n_k+s} > -p\delta y_{n_k} + c_{n_k+s} \to 0 \quad \text{as } k \to \infty. \tag{2.16}$$

Since w_n is monotonic, it follows that $\lim_{n\to\infty} w_n = 0$, which is (2.8). Then (2.9) follows from (2.8). The proof of part (a) is complete.

To prove part (b) of the lemma, we show that (2.7) cannot happen; therefore (2.8) and (2.9) must occur. To obtain a contradiction, let us assume that $\lim_{n\to\infty} w_n = -\infty$. Note that from (2.6) and Lemma 2.2 we have

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} z_n; \tag{2.17}$$

thus $\lim_{n\to\infty} z_n = -\infty$. This implies that for large n, there exists $\eta > 0$, however large, such that for $n \geq n_3$ implies $z_n < -\eta$ which implies by (A1) that $y_n < -\eta + p\delta y_{n-s} < y_{n-s}$. Then y_n is bounded. Consequently z_n and w_n are bounded, which contradicts (2.7). As a result, (2.7) cannot hold and so, (2.8) holds, which implies (2.9). The proof is complete. □

Remark 2.4. If y_n is an eventually negative solution of (1.2), then using (1.3), we observe that $x_n = -y_n$ is a positive solution of (1.2). So that all the oscillation results for the positive solutions also apply to negative solutions.

Lemma 2.5. *Let y_n be an eventually positive solution of* (1.2), *with w_n as in* (2.6). *Then the following statements hold.*

(a) *If* (2.7) *holds, then* (2.10) *implies*

$$\Delta w_{n+1} + q_n G(y_{n-k}) \leq 0, \tag{2.18}$$

which further implies

$$\Delta z_{n+1} + q_n G(y_{n-k}) \leq 0\,. \tag{2.19}$$

(b) *If* (2.8) *holds, then* (2.10) *implies*

$$\Delta w_n - q_n G(y_{n-k}) \geq 0. \tag{2.20}$$

Proof. If (2.7) holds then $\Delta w_n < 0$ and $\Delta w_n^2 < 0$. We write (2.10) as

$$\Delta w_{n+1} + q_n G(y_{n-k}) = \Delta w_n \le 0.$$

Thus, (2.18) holds. From (2.6), it follows that $\Delta w_{n+1} = \Delta z_{n+1} + \Delta c_{n+1}$. Therefore (2.18) implies $\Delta z_{n+1} + q_n G(y_{n-k}) = -\Delta c_{n+1} \le 0$ by Lemma 2.2. Hence (a) is proved. Let us prove (b). If (2.8) holds then (2.9) follows as a consequence, which implies $w_n < 0$ and $\Delta w_n > 0$. Using (2.9), we write (2.10), as

$$-\Delta w_n + q_n G(y_{n-k}) = -\Delta w_{n+1} \le 0,$$

which implies

$$\Delta w_n - q_n G(y_{n-k}) = \Delta w_{n+1} \ge 0.$$

This proves of (b), and completes the proof. □

Lemma 2.6. *Let* (A1), (A3), (A6), (A7) *hold. Assume that there exists* $\lambda > 0$ *such that for all* $x, y \in \mathbb{R}$ *with* $x + y > 0$, *we have*

$$G(x) + G(y) \ge \lambda G(x + y). \tag{2.21}$$

Further, we assume that

$$G(x)G(y) \ge G(xy) \quad \text{for all } x, y > 0. \tag{2.22}$$

Let y_n *be an eventually positive solution of* (1.2). *Define* c_n, z_n *and* w_n *as in* (2.2), (2.5) *and* (2.6) *respectively. If* p_n *satisfies* (1.6) *or* (1.7), *then* $\lim_{n\to\infty} w_n = 0$. *Consequently,* (2.9) *holds.*

Proof. Suppose y_n is an eventually positive or eventually negative solution of (1.2) and p_n satisfies (1.6). From (1.2), using (2.6), (2.5), (2.2) and Lemma 2.2, we obtain (2.10). This implies w_n and Δw_n are monotonic and single sign. Hence, it follows that (2.17) holds and let $\lim_{n\to\infty} z_n = \beta$. Clearly, $z_n > 0$ by (1.6). This implies, β in (2.17), cannot be in negative. If $\beta > 0$, then then there exists a positive scalar χ such that $z_n > \chi > 0$ for large n. Clearly, $\Delta w_n > 0$, otherwise, $\beta = -\infty$, a contradiction. Since Δw_n is decreasing, $\lim_{n\to\infty} \Delta w_n$ exists. If $x > y$ then using (1.3) and (2.21), we note that $0 < \lambda G(x-y) \le G(x) + G(-y) = G(x) - G(y)$. Thus, (A5) holds, i.e; G is non decreasing. Then using (A5), (A1) and (1.6) in (2.5), we have

$$z_n \le y_n + p\delta y_{n-s}. \tag{2.23}$$

From (2.10), by using (A3), (2.21), (2.22) and (2.23) it follows that

$$\begin{aligned} 0 &\ge \Delta^2 w_n + q_n G(y_{n-k}) + G(p\delta)[\Delta^2 w_{n-s} + q_n G(y_{n-s-k})] \\ &\ge \Delta^2 w_n + G(p\delta)\Delta^2 w_{n-s} + q_n^*\big(G(y_{n-k}) + G(p\delta)G(y_{n-s-k})\big) \\ &\ge \Delta^2 w_n + G(p\delta)\Delta^2 w_{n-s} + \lambda q_n^*\big(G(z_{n-k})\big) \\ &\ge \Delta^2 w_n + G(p\delta)\Delta^2 w_{n-s} + \lambda G(\chi) q_n^* \end{aligned} \tag{2.24}$$

for $n \ge n_2 > n_1$. Then taking summation in (2.24) from n_2 to $l - 1$ and then letting $l \to \infty$, we obtain a contradiction to (A3). Thus $\beta = \lim_{n\to\infty} w_n = 0$, which implies (2.9).

Suppose p_n satisfies (1.7). If $\beta > 0$ then proceeding as above, we obtain a similar contradiction. If $\beta < 0$ then using (1.7), we have $w_n \ge -p\delta y_{n-s} + c_n$. This implies $y_n \ge \frac{c_{n+s}}{p\delta} - \frac{w_{n+s}}{p\delta}$. Then taking limit inferior on both sides of this inequality, we obtain

$$\liminf_{n\to\infty} y_n \ge \liminf_{n\to\infty} \frac{c_{n+s}}{p\delta} + \liminf_{n\to\infty} \frac{-w_{n+s}}{p\delta} \ge -\beta/p\delta > 0.$$

In the above we used $\lim_{n\to\infty} c_n = 0$ and $\lim_{n\to\infty} w_n = \beta < 0$. For $-\beta/(3p\delta) = \epsilon > 0$, we find $n_3 \geq n_2$ such that $n > n_3$ implies $y_n > 2\epsilon$. As $c_n \to 0$, from (2.6) it follows that $p_n L(y_{n-s}) > y_n + c_n > \epsilon > 0$. This further implies $p_{n+s} > \frac{\epsilon}{L(y_n)} \geq \frac{\epsilon}{\delta y_n} > 0$, for $n \geq n_3$, which contradicts that p_n changes sign. Thus β cannot be in negative, hence $\lim_{n\to\infty} w_n = \beta = 0$. Consequently (2.9) holds. Similarly, if y_n be an eventually negative solution of (1.2) then proceeding with substitution $x_n = -y_n$ and taking note of Remark 2.4, it could be shown $\beta = \lim_{n\to\infty} w_n = 0$ and the proof is complete. □

Next we have the following remark, which would be helpful in proving results concerned with neutral equation (1.1).

Remark 2.7. Lemmas 2.3, 2.5 and 2.6 hold for $u_n \equiv 0$. In that case $c_n = 0$ and $w_n = z_n$.

The following Lemmas follow from Lemmas 2.3, 2.5, and 2.6 as a consequence of the above remark.

Lemma 2.8. *Assume* (A1) *holds. Let y_n be an eventually positive solution of* (1.1), *and z_n be defined as in* (2.5). *Then*

$$\Delta^2 z_n = -q_n G(y_{n-k}) \leq 0, \tag{2.25}$$

and the following statements hold.

(a) *If* (A2), (A5) *hold and p_n satisfies* (1.4), *then either $\Delta w_n < 0$ for large n which implies*

$$\lim_{n\to\infty} z_n = -\infty, \tag{2.26}$$

or $\Delta w_n > 0$ for large n which implies

$$\lim_{n\to\infty} z_n = 0, \tag{2.27}$$

$$z_n < 0, \quad \Delta z_n > 0, \quad \lim_{n\to\infty} \Delta z_n = 0. \tag{2.28}$$

(b) *If in addition $\delta \leq 1$ and if p_n satisfy* (1.5), *then only* (2.27) *and* (2.28) *hold.*

Lemma 2.9. *If y_n is any eventually positive solution of* (1.1), *with z_n as in* (2.5), *then the following statements hold.*

(a) *If* (2.26) *holds then,* (2.25) *implies* (2.19), *i.e;*

$$\Delta z_{n+1} + q_n G(y_{n-k}) \leq 0.$$

(b) *If* (2.27) *holds, then* (2.25) *implies*

$$\Delta z_n - q_n G(y_{n-k}) \geq 0. \tag{2.29}$$

Lemma 2.10. *Let* (A1), (A3), (2.21), *and* (2.22) *hold, let y_n be an eventually positive or eventually negative solution of* (1.1), *and let z_n be as in* (2.5). *If p_n satisfies* (1.6) *or* (1.7) *then $\lim_{n\to\infty} z_n = 0$. Consequently, $z_n < 0$, $\Delta z_n > 0$ for $y_n > 0$ and $z_n > 0$, $\Delta z_n < 0$ for $y_n < 0$.*

3. Main results part I

In this section, we find sufficient conditions, so that, all unbounded solutions of (1.2) oscillate.

Remark 3.1 ([6, Remark 4.8]). Assumption (A4) and the condition

$$\sum_{j=1}^{\infty} q_{n_j} = \infty, \text{ where } q_{n_j} \text{ is any subsequence of } q_n \tag{3.1}$$

are equivalent.

Theorem 3.2. *Let* (A1), (A4)–(A7) *hold, and* $s > k+1$, (1.4) *be satisfied. If*

$$\Big|\int_a^{\infty} \frac{du}{G(u)}\Big| < \infty, \quad \forall a \in \mathbb{R}, \tag{3.2}$$

then every unbounded solution of (1.2) *oscillates.*

Proof. To obtain a contradiction, let y_n be an eventually positive solution of (1.2). Setting z_n, w_n and c_n as in (2.5), (2.6) and (2.2) respectively, we obtain (2.10). Note that (A4) implies (A2). Hence, by Lemma 2.3(a), we observe that either (2.7) or (2.8) holds.

First we consider the case when (2.7) holds. Using Lemma 2.5(a), we show that (2.10) implies (2.19). From (2.7), (2.17) and Lemma 2.2, it follows that $\lim_{n\to\infty} z_n = -\infty$, which implies $\Delta z_n < 0$ and $z_n < 0$ for large n. If $p_n = 0$ then $z_n = y_n < 0$, a contradiction. Hence $p_n > 0$. From (2.5), we find $y_{n-k} \geq -z_{n+s-k}/(p\delta)$. Using this in (2.19), we obtain

$$\Delta z_{n+1} + q_n G\Big(\frac{-z_{n+s-k}}{p\delta}\Big) \leq 0. \tag{3.3}$$

Note that $-z_n/(p\delta) = v_n$ implies $\Delta z_n = -p\delta \Delta v_n$. Then, substituting this expression in the above, we obtain

$$p\delta \Delta v_{n+1} - q_n G(v_{n+s-k}) \geq 0.$$

Note that $v_n > 0$, $\lim_{n\to\infty} v_n = \infty$ and v_n is increasing. Dividing both sides by $G(v_{n+s-k})$, we obtain

$$p\delta \frac{\Delta v_{n+1}}{G(v_{n+s-k})} \geq q_n. \tag{3.4}$$

Then writing $\Delta v_{n+1} = \int_{v_{n+1}}^{v_{n+2}} dx$, where $v_{n+1} \leq x \leq v_{n+2}$, and using $s-k \geq 2$, we obtain

$$q_n \leq p\delta \int_{v_{n+1}}^{v_{n+2}} \frac{dx}{G(x)}.$$

Summing n_2 to $l-1$, and then taking limit $l \to \infty$, we obtain

$$\sum_{n=n_2}^{\infty} q_n \leq p\delta \int_{v_{n_2+1}}^{\infty} \frac{dx}{G(x)} < \infty\,,$$

by (3.2), which contradicts (A2).

Now we consider the case when (2.8) holds. Consequently, we obtain (2.9). Then taking summation in (2.10) from n_2 to ∞ we find (2.13). As y_n is unbounded, we can find a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ which approaches ∞ as $j \to \infty$. Then there exists $\eta > 0$ such that $y_{n_j} > \eta$ for large j. Then $\sum_{j=n_3}^{\infty} q_{n_j} G(y_{n_j}) > G(\eta) \sum_{j=n_3}^{\infty} q_{n_j} \to$

$+\infty$ by (A4). This contradicts (2.13) which follows from (2.8). The proof for the case $y_n < 0$, and unbounded is similar. Thus, the proof is complete. □

Theorem 3.3. *Let* (A1), (A4)–(A7), *and* (3.2), $s > k+1$, (1.8) *be satisfied. Then every unbounded solution of* (1.2) *oscillates.*

The proof of the above theorem is similar to that of theorem 3.2; we omit it.

Theorem 3.4. *Let* (A1), (A4)–(A7) *hold. Suppose* p_n *satisfies* (1.8), $s > k+1$, *and*

$$\liminf_{|x|\to\infty} \frac{G(x)}{x} > \gamma > 0. \tag{3.5}$$

Suppose that

$$\liminf_{n\to\infty} \sum_{i=n-s+k+1}^{n-1} q_i > \frac{p\delta}{\gamma}\Big(\frac{s-k-1}{s-k}\Big)^{s-k} \tag{3.6}$$

Then every unbounded solution of (1.2) *oscillates.*

Proof. To obtain a contradiction, let y_n be an eventually positive solution of (1.2). Proceeding as in the proof of theorem 3.2, we show that if (2.7) holds then

$$\Delta z_{n+1} + q_n G(-\frac{z_{n+s-k}}{p\delta}) \le 0.$$

Applying (3.5) to the above inequality, we obtain

$$\Delta z_{n+1} - \gamma q_n(\frac{z_{n+s-k}}{p\delta}) \le 0.$$

Note that $z_n < 0$ for large n. Substituting $(z_{n+1}/(p\delta)) = v_n$ and $\Delta z_{n+1} = p\delta\Delta v_n$, in the above we obtain

$$\Delta v_n - \frac{\gamma}{p\delta} q_n v_{n+s-k-1} \le 0.$$

Since $s-k-1 > 0$ this is an advanced difference inequality with a negative solution v_n, which contradicts Lemma 2.1(b).

Next consider the case that (2.8) holds. Proceeding as in the proof of theorem 3.2 we obtain a contradiction. The proof for the case $y_n < 0$, and unbounded is similar. Thus, the proof is complete. □

Remark 3.5. Condition (3.6) implies (A2). If (3.6) holds and (A2) fails, we have $\sum_{n=n_1}^{\infty} q_n < \infty$ which implies

$$\frac{p\delta}{\gamma}\Big(\frac{s-k-1}{s-k}\Big)^{s-k} < \liminf_{n\to\infty} \sum_{i=n-s+k}^{n-1} q_i \le \limsup_{n\to\infty}\Big(\sum_{i=n_1}^{n-1} q_i - \sum_{i=n_1}^{n-s+k-1} q_i\Big) = 0,$$

a contradiction.

Theorem 3.6. *Suppose* (A1), (A4)–(A7) *hold, and* (1.5) *and* $\delta \le 1$ *are satisfied. Then every unbounded solution of* (1.2) *oscillates.*

Proof. Let y_n be an unbounded and eventually positive solution of (1.2). Setting c_n, z_n and w_n as in (2.2), (2.5) and (2.6) respectively, we obtain (2.10). By Lemma 2.3(b), we have $\lim_{n\to\infty} w_n = 0$. Using this, unboundedness of y_n and (A4), and proceeding as in the last part of the proof of theorem 3.2 we obtain a contradiction. A similar contradiction could be obtained if y_n be an eventually negative and unbounded solution of (1.2). This completes the proof. □

Theorem 3.7. *Suppose* (A1), (A3), (A4), (A6), (A7) *hold,* (1.6) *or* (1.7)*, and* (2.21) *and* (2.22) *be satisfied. Then every unbounded solution of* (1.2) *oscillates.*

Proof. On the contrary suppose y_n be an eventually positive and unbounded solution of (1.2). Setting z_n and w_n as in (2.5) and (2.6), we obtain (2.10). Application of Lemma 2.6 yields $\beta = \lim_{n\to\infty} w_n = 0$, $w_n < 0$ and $\Delta w_n > 0$. Then using this, unboundedness of y_n, (A4) and proceeding as in the last part of the proof of theorem 3.2 we obtain a contradiction. A similar contradiction could be obtain if y_n be an eventually negative and unbounded solution of (1.2). This completes the proof. □

Note that the condition

$$\liminf_{n\to\infty} |x_n| > 0 \quad \text{implies} \quad \liminf_{n\to\infty} |G(x_n)| > 0. \tag{3.7}$$

is equivalent to

$$\liminf_{u\to\pm\infty} G(u) \neq 0 \tag{3.8}$$

and note that (A5) implies (3.8). Consequently, we quote a particular case of [13, theorem 2.5, p.236] for $f_n \equiv 0$ as our next result.

Theorem 3.8. *Suppose* (A2), (A6), (A7) *hold, and* (1.9) *and* (3.8) *are satisfied. If* $L(x) = x$*, then every non-oscillatory solution of* (1.2) *is bounded. Or equivalently every unbounded solution of* (1.2) *oscillates.*

4. Main results part II

In this section, we find sufficient conditions so that all solutions of (1.1) oscillate under condition (A2), which is less restrictive than (A4).

Theorem 4.1. *Suppose* (A1), (A2), (A5) *hold, and* (1.4) *and* $s < k$ *are satisfied. If*

$$\Big|\int_0^{\pm c} \frac{du}{G(u)}\Big| < \infty, \text{ for any finite positive } c \in \mathbb{R}, \tag{4.1}$$

Then every bounded solution of (1.1) *oscillates.*

Proof. On the contrary let y_n be a bounded eventually positive solution of (1.1). Setting z_n as in (2.5) we obtain (2.25). Then z_n is bounded and by Lemma 2.8(a), we find that (2.26) cannot hold because boundedness of z_n, as a result, (2.27) holds. Then (2.28) follows as a consequence which implies that $z_n < 0$ and increasing. If $p_n = 0$, then $z_n = y_n < 0$, is a contradiction. Hence $p_n > 0$. From (A1) and (1.4) it follows that $y_{n-k} \geq \frac{z_{n+s-k}}{-p\delta}$. Hence, (2.25) with Lemma 2.9 (b) yields

$$\Delta z_n - q_n G(z_{n+s-k}/(-p\delta)) \geq 0.$$

Substituting $v_n = z_n/(-p\delta)$, which implies $-p\delta\Delta v_n = \Delta z_n$, we find that

$$p\delta\Delta v_n + q_n G(v_{n+s-k}) \leq 0, \tag{4.2}$$

which together with $s < k$ and v_n is positive and decreasing, implies

$$p\delta\Delta v_n + q_n G(v_n) \leq 0,$$

Then dividing both sides of the above by $G(v_n)$ we obtain

$$\frac{p\delta\Delta v_n}{G(v_n)} + q_n \leq 0.$$

Then using $\Delta v_n = \int_{v_n}^{v_{n+1}} dx$ and taking $v_{n+1} \leq x \leq v_n$ we have

$$p\delta \int_{v_n}^{v_{n+1}} \frac{dx}{G(v_n)} + q_n \leq 0,$$

which implies, because of the nondecreasing character of G that

$$p\delta \int_{v_n}^{v_{n+1}} \frac{dx}{G(x)} + q_n \leq 0.$$

Summing from $n = n_1$ to $l-1$ we obtain

$$p\delta \int_{v_{n_1}}^{v_l} \frac{dx}{G(x)} + \sum_{n_1}^{l-1} q_n \leq 0.$$

As $l \to \infty$, $v_l \to 0$, and so in the limiting case, we obtain

$$\sum_{n_1}^{\infty} q_n \leq p\delta \int_0^{v_{n_1}} \frac{dx}{G(x)} < \infty$$

by (4.1), which contradicts (A2). The proof for the case y_n being eventually negative is similar and this completes the proof. □

Theorem 4.2. *Suppose* (A1), (A2), (A5) *hold, and* (1.5), (4.1) , $\delta \leq 1$ *and* $s < k$ *are satisfied. Then every solution of* (1.1) *oscillates.*

Proof. On the contrary, let y_n be an eventually positive solution of (1.1). Setting z_n as in (2.5), we obtain (2.25). Then by Lemma 2.8(b), we find that (2.27) holds. Then (2.28), follows as a consequence and $z_n < 0$. If $p_n = 0$ then $z_n = y_n < 0$ which is a contradiction. Hence from (2.25), and Lemma 2.9(b) we obtain

$$\Delta z_n - q_n G(y_{n-k}) \geq 0.$$

Using $\delta \leq 1$ and (1.5), we find $y_{n-k} \geq \frac{z_{n+s-k}}{-\delta} \geq -z_{n+s-k}$. Therefore,

$$\Delta z_n - q_n G(-z_{n+s-k}) \geq 0.$$

Substituting $-z_n = v_n$, which implies $\Delta z_n = -\Delta v_n$, in the above, we obtain

$$\Delta v_n + q_n G(v_{n+s-k}) \leq 0. \tag{4.3}$$

Then further using $s < k$ and $v_n > 0$ and decreasing, we obtain

$$\Delta v_n + q_n G(v_n) \leq 0.$$

Dividing both sides of the above inequality, by $G(v_n)$, we obtain

$$\frac{\Delta v_n}{G(v_n)} + q_n \leq 0.$$

Taking $v_{n+1} \leq v \leq v_n$ and using $\Delta v_n = \int_{v_n}^{v_{n+1}} dv$, we proceed as in the proof of theorem 4.1 to obtain

$$\sum_{n=n_1}^{\infty} q_n \leq \int_0^{v_{n_1}} \frac{dv}{G(v)} < \infty$$

by (4.1), which contradicts (A2). The proof for the case when y_n is eventually negative is similar. □

Theorem 4.3. *Suppose* (A1), (A2), (A5) *hold, and* (1.5)*,* $\delta \le 1$ *and* $s < k$ *are satisfied. If*

$$\liminf_{|x|\to 0} \frac{G(x)}{x} > \gamma > 0. \tag{4.4}$$

and

$$\liminf_{n\to\infty} \sum_{i=n-k+s}^{n-1} q_i > \frac{1}{\gamma}\Big(\frac{k-s}{k-s+1}\Big)^{k-s+1}, \tag{4.5}$$

then every solution of (1.1) *oscillates.*

Proof. As $s < k$ and $0 < \delta \le 1$ proceeding as in the proof of the theorem 4.2, we obtain the first order delay difference inequality (4.3), which by (4.4), yields

$$\Delta v_n + \gamma q_n v_{n+s-k} \le 0$$

which has a positive solution. This, contradicts Lemma 2.1(a). The proof for the case when y_n is eventually negative is similar. This completes the proof. □

Theorem 4.4. *Suppose* (A1), (A3) *hold, and* (1.6)*,* (2.21) *and* (2.22) *are satisfied. Then every solution of* (1.1) *oscillates.*

Proof. On the contrary, suppose y_n be an eventually positive solution of (1.1). Setting z_n as in (2.5), we obtain (2.25). Then applying Lemma 2.10, we obtain $\beta = \lim_{n\to\infty} z_n = 0$, which implies $z_n < 0$, a contradiction because $z_n \ge 0$ by (1.6). The proof for the case when y_n is eventually negative, is similar and thus, the proof is complete. □

Theorem 4.5. *Suppose* (A1), (A3) *hold, and* (1.7)*,* (2.21) *and* (2.22) *are satisfied. Then every solution of* (1.1) *oscillates.*

Proof. On the contrary, assume y_n be an eventually positive solution of (1.1). Setting z_n as in (2.5), we obtain (2.25). Then application of Lemma 2.10 yields $\beta = \lim_{n\to\infty} z_n = 0$. Consequently $\Delta z_n > 0$ and $z_n < 0$. Again, this would lead to $p_n > \frac{y_n}{L(y_{n-s})} > 0$ for large n, which is a contradiction, because p_n changes sign. For the proof of the case, when y_n is eventually negative, we may proceed with $x_n = -y_n$ and complete the proof. □

Theorem 4.6. *Suppose* (A2), (A5) *hold,* $L(x) = x$*, and* p_n *satisfies* (1.9)*. Then every solution of* (1.1) *oscillates.*

Proof. On the contrary assume y_n be an eventually positive solution of (1.1). Setting z_n as in (2.5), we obtain (2.25). Note that (A5) implies (3.8). Then applying Theorem 3.8 for $u_n = 0$, we show that y_n is bounded, which implies z_n is bounded. As z_n is monotonic, $\lim_{n\to\infty} z_n = \beta \in \mathbb{R}$. Summing (2.25) from n_1 to ∞, we obtain (2.13), which implies $\liminf_{n\to\infty} y_n = 0$. By [9, Lemma 2.1], we have $\lim_{n\to\infty} z_n = 0$. As a consequence (2.28) holds, which implies $z_n < 0$. However, by (1.9) we have $z_n > 0$, a contradiction. The proof for the case $y_n < 0$ is similar. Thus proof is complete. □

Next, we give some examples to illustrate the results.

Example 4.7. Consider the neutral difference equation

$$\Delta^2\big(y_n - p y_{n-4}\big) + 18\Big(\frac{1}{2^{2n}} + 1 - \frac{p}{16}\Big)y_{n-1} - \Big(\frac{72}{2^{2n}} + \frac{9}{2^n}\Big)H(y_{n-3}) = 0 \tag{4.6}$$

where $|p| < 16$. Suppose $p = \pm 2$ or $p = \pm 1/2$. Here, $s = 4$, $k = 1$, $q_n = 18\big(\frac{1}{2^{2n}} + 1 - \frac{p}{16}\big)$, $u_n = \big(\frac{72}{2^{2n}} + \frac{9}{2^n}\big)$ and $H(u) = u/(1 + |u|)$. Clearly, the neutral difference equation (4.6) satisfies all the conditions of Theorems 3.4, 3.6, 3.7 and 3.8. As a result, it has an unbounded solution $y_n = 2^n(-1)^n$, which is oscillatory.

Example 4.8. Consider the neutral difference equation

$$\Delta^2\big(y_n - by_{n-2}\big) + \Big(9(1 - b/4)(2^n + 128) + 2^{-2n}\Big)G(y_{n-7}) - u_nH(y_{n-7}) = 0 \quad (4.7)$$

where $|b| < 4$ is suitably selected constant. Suppose $b = \pm 1/2$. Here, $s = 2, k = 7$, $u_n = \frac{2+2^{n-7}}{2^{2n}(1+2^{n-7})}$, $q_n = \big(9(1 - b/4)(2^n + 128) + 2^{-2n}\big)$, $G(u) = u/(1 + |u|)$ and $H(u) = u/(2 + |u|)$. Clearly, the neutral difference equation (4.7) satisfies all the conditions of Theorems 3.6 and 3.8. Consequently, it has an unbounded solution $y_n = 2^n(-1)^n$, which is oscillatory.

Example 4.9. Consider the neutral difference equation

$$\Delta^2\big(y_n - pL(y_{n-3})\big) + \Big[\frac{4(1 + a + p)}{(1 + a)(1 + \gamma)}\Big]G(y_{n-5}) = 0 \quad (4.8)$$

where $p > 0$ is any scalar. Here $L(x) = x/(a + |x|)$ and $G(u) = u(\gamma + |u|)$ where a and γ are positive constants. This neutral equation satisfies all the conditions of Theorems 4.3. As such, it has a solution $y_n = (-1)^n$, which is oscillatory.

Example 4.10. Consider the neutral difference equation

$$\Delta^2\big(y_n + p(-1)^nL(y_{n-5})\big) + 4y_{n-1}^{1/3} = 0 \quad (4.9)$$

where $p > 0$ is any scalar. Here p_n changes sign and satisfies (1.7). Further, $L(x) = x/(a + |x|)$. This neutral equation satisfies all the conditions of Theorem 4.5. Hence, it has a solution $y_n = (-1)^{3n}$, which is oscillatory.

It seems, no result in the literature, could be applied to the neutral equations (4.8)–(4.9) given in the examples above, because of the non linear term inside Δ^2,

5. Application to neutral difference equations with oscillating coefficients

In this section, we find sufficient conditions so that every unbounded solution of the second order neutral difference equation (1.11) oscillates, where v_n is allowed to change sign. Let $v_n^+ = \max\{v_n, 0\}$ and $v_n^- = \max\{-v_n, 0\}$. Then $v_n = v_n^+ - v_n^-$ and the equation (1.11) can be written as

$$\Delta^2\big[y_n - p_nL(y_{n-s})\big] + v_n^+G(y_{n-k}) - v_n^-G(y_{n-k}) = 0. \quad (5.1)$$

Now we proceed as in the previous section by setting $q_n = v_n^+$, $u_n = v_n^-$ and $H(x) = G(x)$. Assumptions (A4), (A3) and (A6) become

$$\sum_{n=n_0}^{\infty} v_n^+ = \infty. \quad (5.2)$$

$$\liminf_{n\to\infty} v_n^+ > 0. \quad (5.3)$$

$$\sum_{n=n_0}^{\infty} V_n^+ = \infty \quad \text{where } V_n^+ = \min\{v_n^+, v_{n-s}^+\}. \quad (5.4)$$

$$\sum_{n=n_0}^{\infty} n v_n^- < \infty. \tag{5.5}$$

respectively, which are feasible conditions. Therefore, the study of (1.11) reduces to the study of (5.1), which could be achieved, by following the study of (1.2) for different results in section 3. The following results for (1.11) (with v_n changing sign) follow from Theorems 3.6, 3.7 and 3.8, by replacing q_n by v_n^+, u_n by v_n^- and H by G.

Theorem 5.1. *Suppose that* (A1) *holds with* $\delta \le 1$, (A5) *holds,* p_n *satisfies* (1.5), G *is bounded, and* (5.3) *and* (5.5) *are satisfied. Then every unbounded solution of* (1.11) *(with* v_n *changing sign) oscillates.*

Theorem 5.2. *Suppose* p_n *satisfies* (1.6) *or* (1.7), G *is bounded,* (A1) *holds, and* (2.21), (2.22), (5.3), (5.4), *and* (5.5) *are satisfied. Then every unbounded solution of* (1.11) *(with* v_n *changing sign) oscillates.*

Theorem 5.3. *Suppose* p_n *satisfy* (1.9), G *is bounded,* (3.8), (5.2) *and* (5.5) *are satisfied. If* $L(x) = x$, *then every unbounded solution of* (1.11) *oscillates.*

6. Final comments

Before we close this article, we would like to give our concluding remarks, which may be helpful for further research. In this paper, some oscillatory results are obtained for the neutral difference equation (1.2) and (1.1) by imposing different super linear conditions like (3.5) or (3.2), and sublinear conditions like (4.4) or (4.1) on G. Note that the super linear condition (3.5) and the sub linear condition (4.4) on G include their corresponding linear case $G(x) = x$. Authors while studying the oscillatory and asymptotic behavior of (1.2) or (1.1), very often find difficulty in tackling, the case of $p_n \ge 1$, i.e; when (1.8) or (1.4) are satisfied. That is why, the results [10, Theorems 2.6 and 2.7] appear to be wrong, as the neutral equation

$$\Delta^2(y_n - 4y_{n-1}) + 4^{(n+1)/3} y_{n-2}^{1/3} = 0$$

satisfies all the conditions of the theorems, but, it admits a non oscillatory solution $y_n = 2^n$, which tends to ∞, as $n \to \infty$, contradicting the theorems. With the super linear G with (3.2) or (3.5), we proved in Theorems 3.2 and 3.4 that (A4) is sufficient for all unbounded solutions of (3.2) to be oscillatory which is more restrictive than (A2). Hence, one may extend this study to improve the results (Theorems 3.2 and 3.4)by attempting to answer the following problem.

Problem 6.1. *Suppose that* $L(x) = x$, *or* (A1) *holds, and* $1 \le p_n \le p$. *Assuming* (A2), (A5) *and* (3.2) *can we prove that every unbounded solution of* (1.1) *oscillates?*

Acknowledgment. The authors are obliged and thankful to the referee and the editor for their various suggestions to improve the presentation of this article.

References

[1] R. P. Agarwal; *Difference Equations and Inequalities*, Marcel Dekker, NewYork, 2000.

[2] Chittaranjan Behera, Radhanath Rath, Prayag Prasad Mishra; *Oscillation for second order neutral difference equations with variable delays,* International J. of Mathematical, Engineering and Management Sciences, **5**, (2020), no. 4, 663–681.

[3] Chittaranjan Behera, Radhanath Rath; *Oscillation and asymptotic behavior of second-order neutral delay difference equations with variable delays.* Int. J. of Diff. equ. **15** (2020), no. 1, 107–126.

[4] Chittaranjan Behera, Radhanath Rath, Prayag Prasad Mishra; *Oscillation and asymptotic behavior of a higher-order neutral delay difference equation with variable delays under* Δ^m (to appear in Math. Slovaca, **70** (Dec 2020), no. 6).

[5] I. Gyori, G. Ladas; *Oscillation Theory of Delay-Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[6] B. Karpuz, R. N. Rath, S. K. Rath; *On Oscillation and asymptotic behaviour of a higher order functional difference equation of neutral type,* Int. J. Difference Equ. **4**(2009), no. 1, 69–96.

[7] B. Karpuz, O. Ocalan, M. K. Yildiz; *Oscillation of a class of difference equations of second order,* Math. Comput. Modelling, **49** (2009), 912-917.

[8] W. G. Kelley, A. C. Peterson; *Difference Equations:An Introduction With Applications*, Academic Press, NewYork,1991.

[9] N. Parhi, A. K. Tripathy; *Oscillation of forced nonlinear neutral delay difference equations of first order*, Czech. Math. J. **53** (2003), 83-101.

[10] N. Parhi, A. K. Tripathy; *Oscillation of a class of non-linear neutral difference equations of higher order*, J. Math. Anal. Appl. **284**(2003), 756-774.

[11] N. Parhi, A. K. Tripathy; *Oscillation of a class of neutral difference equations of first order*, Journal of Difference Equations and Applications **9**(2003) no. 10, 933-946.

[12] R. N. Rath, B. L. S. Barik, S. K. Rath; *Oscillation of higher order neutral Functional Difference Equations with positive and negative coefficients,* Mathematica Slovaca, **60** (2010), no. 3, 361–384.

[13] R. N. Rath, C. R. Behera, A. K. Bhuyan; *Oscillatory and asymptotic behaviour of higher order neutral difference equations,* Acta Mathematica Academiae Paedagogicae Nyíregyháziensis, **31**(2015), no. 2, 233-248 .

[14] Radhanath Rath, Chittaranjan Behera; *Oscillatory and asymptotic behaviour of a first order neutral equation of discrete type with variable several delay under* Δ *sign.* International J. of Math. and Mathematical Sciences, Vol 2018, Article ID 4586176, https//doi.org10.1155/20184586176

[15] A. K. Tripathy, S. Panigrahi; *Oscillation in non linear neutral difference equations with positive and negative coefficients*,Int. J. of Difference equations **5**(2010), no. 2, 251–265.

[16] M. K. Yildiz; *Oscillation of a class of nonlinear difference equations of second order with oscillating coefficients,* Konuralp J. Math., **3**(2015), no. 2, 211-218.

Ajit Kumar Bhuyan
Dept. Of Mathematics, Sai international School, Bhubaneswar, Odisha, India
Email address: ajitbhuyan13@gmail.com

Laxmi Narayan Padhy
Dept. of Math and Computer Science, Konark Institute of Science and Technology, Bhubaneswar, Odisha, India
Email address: padhyln@gmail.com

Radhanath Rath (corresponding author)
VSSUT Burla, 768018. Retired Principalhallikote Autonomous College, Berhampur, 760001. Center Point Apartment, Flat A-203, Shailashree Vihar ph-7, 751024, Bhubaneswar, Odisha, India
Email address: radhanathmath@yahoo.co.in