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LOCAL EXISTENCE AND BLOW-UP CRITERION FOR THE TWO AND THREE DIMENSIONAL IDEAL MAGNETIC BÉNARD PROBLEM

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Abstract. In this article, we consider the ideal magnetic Bénard problem in both two and three dimensions and prove the existence and uniqueness of strong local-in-time solutions, in H^s for $s > \frac{n}{2} + 1$, $n = 2, 3$. In addition, a necessary condition is derived for singularity development with respect to the BMO-norm of the vorticity and electrical current, generalizing the Beale-Kato-Majda condition for ideal hydrodynamics.

1. Introduction

The magnetic Bénard problem with full viscosity is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p_* = (\mathbf{b}\cdot\nabla)\mathbf{b} + \theta e_n, \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta - \kappa\Delta\theta = \mathbf{u}\cdot e_n, \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.2}$$

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{b} - \mu\Delta\mathbf{b} = (\mathbf{b}\cdot\nabla)\mathbf{u}, \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.3}$$

$$\nabla\cdot\mathbf{u} = 0 = \nabla\cdot\mathbf{b}, \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.4}$$

with initial conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \theta(x,0) = \theta_0(x), \quad \mathbf{b}(x,0) = \mathbf{b}_0(x) \quad \text{in } \mathbb{R}^n,$$

where $n = 2, 3$. Here $\mathbf{u} : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ is the velocity field, $\theta : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is the temperature, $\mathbf{b} : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ is the magnetic field, p_* is the total pressure field where $p_* = p + \frac{1}{2}|\mathbf{b}|^2$, $p : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is the pressure. e_n denotes the unit vector along the n^{th} direction. The term θe_n represents buoyancy force on fluid motion and $\mathbf{u}\cdot e_n$ signifies the Rayleigh-Bénard convection in a heated inviscid fluid. $\nu \geq 0$, $\mu \geq 0$ and $\kappa \geq 0$ denote the coefficients of kinematic viscosity, magnetic diffusion and thermal diffusion respectively.

The global-in-time regularity in two-dimensions of the above problem when ν, μ and $\kappa > 0$ is known for a long time [14]. Because of the parabolic couplings, it is indeed possible to rewrite the above system in the abstract framework of the Navier-Stokes equations and then use the standard solvability techniques (see Temam [30]).

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In three-dimensions, one can at-most expect local-in-time solvability result with arbitrary initial data and global-in-time result for sufficiently small initial data, much like the Navier-Stokes equations. In [9], the authors obtained the global well-posedness of two-dimensional magnetic Bénard problem without thermal diffusivity and with vertical or horizontal magnetic diffusion. Moreover, the authors prove global regularity and some conditional regularity of strong solutions with mixed partial viscosity. This work provides an extension of an earlier result [33] on the global regularity with full dissipation and magnetic diffusion. It is worthwhile to note that there are very few literatures available where the case $\nu = \kappa = \mu = 0$ has been discussed in two and three dimensions for the magnetic Bénard problem.

However, for the ideal magneto hydrodynamic (MHD) equations, i.e. when $\theta \equiv 0$ and $\nu = \mu = 0$, in (1.1)-(1.3), the local-in-time existence of strong solutions have been proved by Schmidt [28] and Secchi [29], when the initial data is in H^m for integer $m > 1 + n/2$. Schmidt [28] obtained the well-posedness and regularity of maximal solutions and continuous dependence on forcing terms and initial data (using a regularization procedure). Caflisch, Klapper and Steele [7] derived a criteria for energy conservation and helicity conservation for weak solutions of ideal MHD equations. The authors in [7] extended the Beale-Kato-Majda [3] criterion to the three-dimensional ideal MHD equations by showing that for sufficiently regular initial data the condition

$$\int_0^T (\|\nabla \times \mathbf{u}(\tau)\|_{\mathrm{L}^\infty} + \|\nabla \times \mathbf{b}(\tau)\|_{\mathrm{L}^\infty})\, d\tau < \infty,$$

ensures that the solution can be continued beyond time T, where $\nabla \times \mathbf{u}$ is the fluid vorticity, $\nabla \times \mathbf{b}$ is the electrical current.

On the other hand, for the ideal Boussinesq system, i.e. when $\mathbf{b} \equiv \mathbf{0}$, $\nu = \kappa = 0$, and the Rayleigh-Bénard convection term $\mathbf{u} \cdot e_n$ is absent in (1.1)-(1.3), only local-in-time existence results are available even in two-dimensions. It was proved in [8] that if the initial data $(\mathbf{u_0}, \theta_0) \in H^3_\sigma(\mathbb{R}^2) \times H^3(\mathbb{R}^2)$, then local-in-time classical solutions exist and is unique. Moreover, Beale-Kato-Majda type criterion for blow-up of smooth solutions is established in [8]. More precisely, they proved that the smooth solution exists on $[0, T]$ if and only if $\nabla\theta \in L^1(0, T; L^\infty(\mathbb{R}^2))$. For the three-dimensional Boussinesq system, a very few results on local-in-time existence and blow-up criterion are available (e.g. see [15, 16, 26, 31]). However, in the very particular case of the axisymmetric initial data, global-in-time well-posedness has been proven in three-dimensions by Abidi et al [1]. In recent work [23], authors proved local-in-time existence and uniqueness of strong solutions in H^s for real $s > n/2 + 1$ for the ideal Boussinesq equations in $\mathbb{R}^n$, $n = 2, 3$ and established Beale-Kato-Majda type blow-up criterion with respect to the BMO-norm of the vorticity.

In this work, we consider the ideal magnetic Bénard problem (i.e. when $\nu = \kappa = \mu = 0$) in both two and three dimensions and prove local-in-time existence and uniqueness of the strong solutions when the initial data $(\mathbf{u_0}, \theta_0, \mathbf{b}_0) \in H^s_\sigma(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \times H^s_\sigma(\mathbb{R}^n)$, where $s > n/2 + 1$. We prove when $s > n/2 + 1$, BMO-norms of the vorticity, electrical current and that of the gradient of the temperature (i.e. $\nabla\times\mathbf{u}, \nabla\times\mathbf{b}, \nabla\theta \in L^1(0, T; BMO)$) control the breakdown of smooth solutions of the above systems. However, we later show that under suitable additional assumption on θ_0, one can completely relax the condition on gradient of the temperature and the conditions $\nabla \times \mathbf{u}, \nabla \times \mathbf{b} \in L^1(0, T; BMO)$ are sufficient to ensure that the smooth solution persists. To the best of authors' knowledge, this work is new in the literature and may be seen as an extension of the blow-up criterion for ideal MHD equations due to Caflisch et al [7] and that of ideal Boussinesq equations due to Manna et al [23].

We note that, in view of the recent work of Bourgain and Li [4] on the ill-posedness of the two and three dimensional Euler equations in $H^{n/2+1}$, $n = 2, 3$, it seems likely that the ideal magnetic Bénard problem is also ill-posed in $H^{n/2+1}$, $n = 2, 3$, although it still remains an open problem.

To be precise, in this work, we consider the ideal magnetic Bénard problem

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p_* = (\mathbf{b} \cdot \nabla)\mathbf{b} + \theta e_n, \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.5}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla)\theta = \mathbf{u} \cdot e_n, \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.6}$$

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{b} = (\mathbf{b} \cdot \nabla)\mathbf{u}, \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.7}$$

with

$$\nabla \cdot \mathbf{u} = 0 = \nabla \cdot \mathbf{b}, \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.8}$$

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \theta(x, 0) = \theta_0(x), \quad \mathbf{b}(x, 0) = \mathbf{b}_0(x) \quad \text{in } \mathbb{R}^n, \tag{1.9}$$

and prove the following main results.

First we state the result concerning the existence of strong local-in-time solutions.

Theorem 1.1. *Let $s \in \mathbb{R}$ be such that $s > \frac{n}{2} + 1, n = 2, 3$. Let*

$$(\mathbf{u_0}, \theta_0, \mathbf{b_0}) \in H^s_\sigma(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \times H^s_\sigma(\mathbb{R}^n).$$

Then there exists a unique strong solution $(\mathbf{u}, \theta, \mathbf{b})$ to the problem (1.5)-(1.9), with

$$\mathbf{u} \in C([0, T^*]; H^s_\sigma(\mathbb{R}^n)), \quad \theta \in C([0, T^*]; H^s(\mathbb{R}^n)) \quad \mathbf{b} \in C([0, T^*]; H^s_\sigma(\mathbb{R}^n))$$

for some finite time $T^* = T^*(s, \|\mathbf{u}_0\|_{H^s_\sigma}, \|\theta_0\|_{H^s}, \|\mathbf{b}_0\|_{H^s_\sigma}) > 0$.

To prove this result, we consider the Fourier truncated ideal magnetic Bénard problem on the whole of $\mathbb{R}^n$, $n = 2, 3$, and show that the solutions $(\mathbf{u}^R, \theta^R, \mathbf{b}^R)$ of some smoothed version of the ideal magnetic Bénard system exist. We then establish that the H^s-norm of $(\mathbf{u}^R, \theta^R, \mathbf{b}^R)$ are uniformly bounded up to a terminal time $\tilde{T}$, which is independent of R. We further show that up to the blowup time, the solution $(\mathbf{u}^R, \theta^R, \mathbf{b}^R)$ is a Cauchy sequence in the L^2-norm as $R \to \infty$, and by using Sobolev interpolation, $(\mathbf{u}^R, \theta^R, \mathbf{b}^R) \to (\mathbf{u}, \theta, \mathbf{b})$ in any $H^{s'}$ for $0 < s' < s$. Finally we provide the proof of Theorem 1.1 in Theorem 3.10.

Next, we establish that the BMO norms of the vorticity and electrical current control the breakdown of smooth solutions. Our main result concerning the blow-up criterion is as follows.

Theorem 1.2. *Let $(\mathbf{u_0}, \theta_0, \mathbf{b_0})$ have same regularity as above and $s > \frac{n}{2} + 1, n = 2, 3$. If $(\mathbf{u}, \theta, \mathbf{b})$ satisfy the condition*

$$\int_0^{T^*} \left(\|\nabla \times \mathbf{u}(\tau)\|_{BMO} + \|\nabla \theta(\tau)\|_{BMO} + \|\nabla \times \mathbf{b}(\tau)\|_{BMO}\right) d\tau < \infty,$$

then the solution $(\mathbf{u},\theta,\mathbf{b})$ *can be continuously extended to* $[0,T]$ *for some* $T>T^*$. *However, if* $\theta_0\in H^s(\mathbb{R}^n)\cap W^{1,p}(\mathbb{R}^n)$, $2\le p\le\infty$, *then the condition*

$$\int_0^{T^*}(\|\nabla\times\mathbf{u}(\tau)\|_{BMO}+\|\nabla\times\mathbf{b}(\tau)\|_{BMO})\,d\tau<\infty$$

is sufficient to ensure that the solution $(\mathbf{u},\theta,\mathbf{b})$ *can be extended continuously to* $[0,T]$ *for some* $T>T^*$.

Remark 1.3. The above result still holds if we replace BMO with the Besov space $B^0_{\infty,\infty}$ used in Kozono et al [19] or if we replace the condition by the one introduced in Planchon [25]. To be precise, the condition above can be weakened to

$$\begin{aligned}&\int_0^{T^*}\Big(\|\nabla\times\mathbf{u}(\tau)\|_{B^0_{\infty,\infty}}+\|\nabla\times\mathbf{b}(\tau)\|_{B^0_{\infty,\infty}}\Big)\,d\tau\\&=\int_0^{T^*}\Big(\sup_j\|\triangle_j(\nabla\times\mathbf{u}(\tau))\|_{L^\infty}+\sup_j\|\triangle_j(\nabla\times\mathbf{b}(\tau))\|_{L^\infty}\Big)\,d\tau<\infty,\end{aligned}$$

or to

$$\lim_{\delta\to0}\int_{T^*-\delta}^{T^*}\Big(\sup_j\|\triangle_j(\nabla\times\mathbf{u}(\tau))\|_{L^\infty}+\sup_j\|\triangle_j(\nabla\times\mathbf{b}(\tau))\|_{L^\infty}\Big)\,d\tau<\epsilon,$$

for some sufficiently small $\epsilon>0$.

The rest of the article is organized as follows. We define various operators, function spaces, and certain basic inequalities in Section 2. In Section 3, we start investigating about the ideal magnetic Bénard problem and prove results concerning energy estimates and convergence of the approximate solutions before proving Theorem 1.1 and 3.10. In section 4, we prove Theorems 1.2, Theorem 4.1 and 4.3.

2. Preliminaries

2.1. **Fractional derivative operator.** Let us define J^s (real $s>0$), which denotes the Bessel potential of order s, in terms of Fourier transform as follows:

$$\mathcal{F}[J^sf](\xi)=(1+|\xi|^2)^{s/2}\widehat{f}(\xi).$$

J^s is also equivalent to the operator $(I-\Delta)^{s/2}$.

Assume $0<s<\infty$ and $f\in L^2(\mathbb{R}^n)$. Then $f\in H^s(\mathbb{R}^n)$ if $(1+|\xi|^2)^{s/2}\widehat{f}(\xi)\in L^2(\mathbb{R}^n)$. The norm on $H^s(\mathbb{R}^n)$ is

$$\|f\|_{H^s}=\Big(\int_{\mathbb{R}^n}[(1+|\xi|^2)^{s/2}|\widehat{f}(\xi)|]^2\Big)^{1/2}=\|(1+|\xi|^2)^{s/2}\widehat{f}(\xi)\|_{L^2}=\|J^sf\|_{L^2}\tag{2.1}$$

and the inner product on $H^s(\mathbb{R}^n)$ is

$$\begin{aligned}(f,g)_{H^s}&=\Big((1+|\xi|^2)^{s/2}\widehat{f}(\xi),(1+|\xi|^2)^{s/2}\widehat{g}(\xi)\Big)_{L^2}\\&=(\mathcal{F}[J^sf](\xi),\mathcal{F}[J^sg](\xi))_{L^2}\\&=(J^sf,J^sg)_{L^2}.\end{aligned}$$

Remark 2.1. It is trivial to show that

$$\|\nabla f\|_{H^{s-1}}\le\|f\|_{H^s}.$$

2.2. **Fourier truncation operator.** Let us define the Fourier truncation operator $\mathcal{S}_R$ as follows:

$$\widehat{\mathcal{S}_R f}(\xi) := \mathbf{1}_{B_R}(\xi)\widehat{f}(\xi),$$

where B_R, a ball of radius R centered at the origin and $\mathbf{1}_{B_R}$ is the indicator function. Then we infer the following properties:

(1) $\|\mathcal{S}_R f\|_{H^s(\mathbb{R}^n)} \le C\|f\|_{H^s(\mathbb{R}^n)}$, where C is a constant independent of R;
(2) $\|\mathcal{S}_R f - f\|_{H^s(\mathbb{R}^n)} \le \frac{C}{R^k}\|f\|_{H^{s+k}(\mathbb{R}^n)}$;
(3) $\|(\mathcal{S}_R - \mathcal{S}_{R'})f\|_{H^s} \le C\max\{(\frac{1}{R})^k, (\frac{1}{R'})^k\}\|f\|_{H^{s+k}}$.

For the proofs of these properties see [13]. We define the function spaces

$$H^s_\sigma(\mathbb{R}^n) = \{f \in H^s(\mathbb{R}^n) : \nabla\cdot f = 0\}, \quad \mathbf{H}^s_\sigma(\mathbb{R}^n) = \big(H^s_\sigma(\mathbb{R}^n)\big)^n.$$

Remark 2.2. If $s > n/2$, then each $f \in H^s(\mathbb{R}^n)$ is bounded and continuous and hence

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le C\|f\|_{H^s(\mathbb{R}^n)}, \quad \text{for } s > n/2.$$

Also, note that H^s is an algebra for $s > n/2$, i.e., if $f, g \in H^s(\mathbb{R}^n)$, then $fg \in H^s(\mathbb{R}^n)$, for $s > n/2$. Hence, we have

$$\|fg\|_{H^s} \le C\|f\|_{H^s}\|g\|_{H^s}, \quad \text{for } s > n/2.$$

Lemma 2.3. *Fix $s > n/2$ and let $f \in \mathbf{H}^s_\sigma$ and $g \in H^s$. Then*

$$\|(f\cdot\nabla)g\|_{H^{s-1}} \le C\|f\|_{H^s}\|g\|_{H^s}.$$

Proof. We being in $\mathbf{H}^s_\sigma$, f is divergence free, and hence $(f\cdot\nabla)g = \nabla\cdot(f\otimes g)$. Rest of the proof is straightforward, since H^s is an algebra for $s > n/2$. □

Lemma 2.4 (Sobolev inequality). *For $f \in H^s(\mathbb{R}^n)$, we have*

$$\|f\|_{L^q(\mathbb{R}^n)} \le C_{n,s,q}\|f\|_{H^s(\mathbb{R}^n)}$$

provided that q lies in the following range

(i) *if $s < n/2$, then $2 \le q \le \frac{2n}{n-2s}$.*
(ii) *if $s = n/2$, then $2 \le q < \infty$.*
(iii) *if $s > n/2$, then $2 \le q \le \infty$.*

For details see Kesavan [18].

Remark 2.5. We deduce the following result using Lemma 2.4. For $n = 2$, we use Hölder's inequality with exponents $2/\epsilon$ and $2/(1-\epsilon)$, and Sobolev inequality for $0 < \epsilon < s-1$ to obtain

$$\|fg\|_{L^2} \le \|f\|_{L^{2/\epsilon}}\|g\|_{L^{2/1-\epsilon}} \le C\|f\|_{\dot{H}^{1-\epsilon}}\|g\|_{\dot{H}^\epsilon} \le C\|f\|_{H^1}\|g\|_{H^{s-1}}.$$

For $n = 3$, we use Hölder's inequality with exponents 6 and 3, and Sobolev inequality to obtain

$$\|fg\|_{L^2} \le \|f\|_{L^6}\|g\|_{L^3} \le C\|f\|_{\dot{H}^1}\|g\|_{\dot{H}^{1/2}} \le C\|f\|_{H^1}\|g\|_{H^{1/2}} \le C\|f\|_{H^1}\|g\|_{H^{s-1}}.$$

We note that for both 2D and 3D we have the same estimate.

Lemma 2.6 (Interpolation in Sobolev spaces). *Given $s > 0$, there exists a constant C depending on s, so that for all $f \in H^s(\mathbb{R}^n)$ and $0 < s' < s$,*

$$\|f\|_{H^{s'}} \le C\|f\|_{L^2}^{1-s'/s}\|f\|_{H^s}^{s'/s}.$$

For details see [2] and for a proof see [22, Theorem 9.6, Remark 9.1].

Lemma 2.7 (Gagliardo-Nirenberg interpolation inequality [24]). *Let $g \in L^q(\mathbb{R}^n)$ and its derivatives of order m, $D^m g \in L^r(\mathbb{R}^n)$, $1 \le q, r \le \infty$. For the derivatives $D^j g$, $0 \le j < m$, it holds*

$$\|D^j g\|_{L^p} \le C \|D^m g\|_{L^r}^a \|g\|_{L^q}^{1-a},$$

where

$$\frac{1}{p} = \frac{j}{n} + \Big(\frac{1}{r} - \frac{m}{n}\Big)a + \frac{1-a}{q},$$

for all a in the interval $j/m \le a < 1$. The constant C depends only on n, m, j, q, r, a.

2.3. **Commutator estimates.** Let f and g are Schwartz class functions. Then for $s \ge 0$ we define

$$[J^s, f]g = J^s(fg) - f(J^s g),$$

and

$$[J^s, f]\nabla g = J^s((f \cdot \nabla)g) - (f \cdot \nabla)J^s g. \tag{2.2}$$

where $[J^s, f] = J^s f - f J^s$ is the commutator, in which f is regarded as a multiplication operator.

Lemma 2.8. *For $s \ge 0$, and $1 < p < \infty$, we have a basic estimate*

$$\|[J^s, f]g\|_{L^p} \le C \left(\|\nabla f\|_{L^\infty} \|J^{s-1} g\|_{L^p} + \|J^s f\|_{L^p} \|g\|_{L^\infty} \right),$$

where C is a constant depending only on n, p, s.

For a proof of the above lemma see the appendix in [17].

2.4. BMO **space and logarithmic Sobolev inequality.**

Definition 2.9. The space BMO (Bounded Mean Oscillation) is the Banach space of all functions $f \in L^1_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{BMO} = \sup_Q \Big(\frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx \Big) < \infty,$$

where the sup ranges over all cubes $Q \subset \mathbb{R}^n$, and f_Q is the mean of f over Q. For more details see [12].

The space BMO has two distinct advantageous properties compared to L^∞. The first being the Riesz transforms are bounded in BMO and secondly the singular integral operators of the Calderon-Zygmund type are also bounded in BMO. Hence, one can show that $\|\nabla \mathbf{u}\|_{BMO} \le C \|\nabla \times \mathbf{u}\|_{BMO}$ (see [20]).

It is well known that the Sobolev space $W^{s,p}$ is embedded continuously into L^∞ for $sp > n$. However this embedding is false in the space $W^{k,r}$ when $kr = n$. Brezis-Gallouet [5] and Brezis-Wainger [6] provided the following inequality which relates the function spaces L^∞ and $W^{s,p}$ at the critical value and was used to prove the existence of global solutions to the nonlinear Schrödinger equations.

Lemma 2.10. *Let $sp > n$. Then*

$$\|f\|_{L^\infty} \le C\Big(1 + \log^{\frac{r-1}{r}}(1 + \|f\|_{W^{s,p}})\Big),$$

provided $\|f\|_{W^{k,r}} \le 1$ for $kr = n$.

Similar embedding was investigated by Beale-Kato-Majda [3] for vector functions to obtain the blow-up criterion of the solutions to the Euler equations.

Lemma 2.11. *Let $s > \frac{n}{p} + 1$, then we have*
$$\|\nabla f\|_{L^\infty} \le C\Big(1 + \|\nabla\cdot f\|_{L^\infty} + \|\nabla\times f\|_{L^\infty}\left(1 + log(e + \|f\|_{W^{s,p}})\right)\Big),$$
for all $f \in W^{s,p}(\mathbb{R}^n)$.

Kozono and Taniuchi improved the above logarithmic Sobolev inequality in BMO space, and applied the result to the three-dimensional Euler equations to prove that BMO-norm of the vorticity controls breakdown of smooth solutions.

Lemma 2.12. *Let $1 < p < \infty$ and let $s > \frac{n}{p}$, then we have*
$$\|f\|_{L^\infty} \le C\Big(1 + \|f\|_{BMO}(1 + \log^+\|f\|_{W^{s,p}})\Big),$$
for all $f \in W^{s,p}$, where $\log^+ a = \log a$ if $a \ge 1$ and zero otherwise.

For a proof of the above lemma, see [20, Theorem 1]. Throughout the following sections, C denotes a generic constant.

3. Energy estimates, local existence and uniqueness for the magnetic Bénard problem

We consider the following truncated ideal magnetic Bénard problem on the whole of $\mathbb{R}^n$, for $n = 2, 3$:
$$\frac{\partial \mathbf{u}^R}{\partial t} + \mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R] + \nabla p^R = \theta^R e_n + \mathcal{S}_R[(\mathbf{b}^R\cdot\nabla)\mathbf{b}^R], \tag{3.1}$$
$$\frac{\partial \theta^R}{\partial t} + \mathcal{S}_R\big[(\mathbf{u}^R\cdot\nabla)\theta^R\big] = \mathbf{u}^R\cdot e_n, \tag{3.2}$$
$$\frac{\partial \mathbf{b}^R}{\partial t} + \mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{b}^R] = \mathcal{S}_R[(\mathbf{b}^R\cdot\nabla)\mathbf{u}^R], \tag{3.3}$$
$$\nabla\cdot\mathbf{u}^R = 0 = \nabla\cdot\mathbf{b}^R, \tag{3.4}$$
$$\mathbf{u}^R(0) = \mathcal{S}_R\mathbf{u}_0, \theta^R(0) = \mathcal{S}_R\theta_0, \mathbf{b}^R(0) = \mathcal{S}_R\mathbf{b}_0. \tag{3.5}$$

As the truncations are invariant under the flow of the equation, by taking the truncated initial data we ensure that $\mathbf{u}^R$, $\mathbf{b}^R$ lie in the space
$$V_R^\sigma := \{g \in L^2(\mathbb{R}^n) : \text{supp}(\widehat{g}) \subset B_R, \nabla\cdot g = 0\}$$
and θ^R lies in the space
$$V_R := \{g \in L^2(\mathbb{R}^n) : \text{supp}(\widehat{g}) \subset B_R\}.$$
The divergence free condition for $\mathbf{u}^R$ can be obtained easily as
$$\widehat{\nabla\cdot\mathbf{u}^R}(\xi) = i\xi\cdot\mathbf{1}_{B_R}(\xi)\widehat{\mathbf{u}}(\xi) = \mathbf{1}_{B_R}(\xi)i\xi\cdot\widehat{\mathbf{u}}(\xi) = \mathbf{1}_{B_R}(\xi)\widehat{\nabla\cdot\mathbf{u}}(\xi) = 0.$$
Similarly we obtain divergence free condition for $\mathbf{b}^R$.

Proposition 3.1. *Let $(\mathbf{u}^R, \mathbf{b}^R) \in H^s_\sigma(\mathbb{R}^n)\times H^s_\sigma(\mathbb{R}^n)$, for $s > n/2 + 1$. Then the nonlinear operator $F(\mathbf{u}^R, \mathbf{b}^R) := \mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{b}^R]$ is locally Lipschitz in $\mathbf{u}^R$ and $\mathbf{b}^R$ on the space V_R^σ.*

Proof. Let $\mathbf{b}^R \in H^s_\sigma(\mathbb{R}^n)$, for $s > n/2 + 1$. Then for proving $F(\cdot,\cdot)$ to be locally Lipschitz in $\mathbf{u}^R$, we use integration by parts, Hölder's inequality and Lemma 2.4 to obtain
$$|\big(F(\mathbf{u}_1^R, \mathbf{b}^R) - F(\mathbf{u}_2^R, \mathbf{b}^R), \mathbf{u}_1^R - \mathbf{u}_2^R\big)_{L^2}|$$

$$
\begin{aligned}
&= |\big(\mathcal{S}_R[(\mathbf{u}_1^R - \mathbf{u}_2^R)\cdot\nabla\mathbf{b}^R], \mathbf{u}_1^R - \mathbf{u}_2^R\big)_{L^2}| \\
&= |\big(\big((\mathbf{u}_1^R - \mathbf{u}_2^R)\cdot\nabla\big)\mathbf{b}^R, \mathcal{S}_R(\mathbf{u}_1^R - \mathbf{u}_2^R)\big)_{L^2}| \\
&= | - \big(\big((\mathbf{u}_1^R - \mathbf{u}_2^R)\cdot\nabla\big)(\mathbf{u}_1^R - \mathbf{u}_2^R), \mathcal{S}_R\mathbf{b}^R\big)_{L^2} \\
&\le \|\mathbf{u}_1^R - \mathbf{u}_2^R\|_{L^2_\sigma}\|\nabla(\mathbf{u}_1^R - \mathbf{u}_2^R)\|_{L^2_\sigma}\|\mathcal{S}_R\mathbf{b}^R\|_{L^\infty} \\
&\le \|\mathbf{u}_1^R - \mathbf{u}_2^R\|_{L^2_\sigma}\|\mathbf{u}_1^R - \mathbf{u}_2^R\|_{H^1_\sigma}\|\mathbf{b}^R\|_{L^\infty} \\
&\le C\|\mathbf{u}_1^R - \mathbf{u}_2^R\|_{H^s_\sigma}\|\mathbf{b}^R\|_{H^s_\sigma}\|\mathbf{u}_1^R - \mathbf{u}_2^R\|_{L^2_\sigma}.
\end{aligned}
$$

For $\mathbf{b}^R \in H^s_\sigma(\mathbb{R}^n)$, this gives

$$
\|F(\mathbf{u}_1^R, \mathbf{b}^R) - F(\mathbf{u}_2^R, \mathbf{b}^R)\|_{L^2} \le C\|\mathbf{b}^R\|_{H^s_\sigma}\|\mathbf{u}_1^R - \mathbf{u}_2^R\|_{H^s_\sigma}
$$

And hence $F(\cdot,\cdot)$ is locally Lipschitz in $\mathbf{u}^R$. To prove F to be locally Lipschitz in $\mathbf{b}^R$, we use Remark 2.3. For $s > n/2 + 1$ and $\mathbf{u}^R \in H^s_\sigma(\mathbb{R}^n)$, we have

$$
\begin{aligned}
&|\big(F(\mathbf{u}^R, \mathbf{b}_1^R) - F(\mathbf{u}^R, \mathbf{b}_2^R), \mathbf{b}_1^R - \mathbf{b}_2^R\big)_{L^2}| \\
&= |\big(\mathcal{S}_R(\mathbf{u}^R\cdot\nabla)(\mathbf{b}_1^R - \mathbf{b}_2^R), \mathbf{b}_1^R - \mathbf{b}_2^R\big)_{L^2}| \\
&= |\big((\mathbf{u}^R\cdot\nabla)(\mathbf{b}_1^R - \mathbf{b}_2^R), \mathcal{S}_R(\mathbf{b}_1^R - \mathbf{b}_2^R)\big)_{L^2}| \\
&\le \|(\mathbf{u}^R\cdot\nabla)(\mathbf{b}_1^R - \mathbf{b}_2^R)\|_{L^2_\sigma}\|\mathcal{S}_R(\mathbf{b}_1^R - \mathbf{b}_2^R)\|_{L^2_\sigma} \\
&\le C\|\mathbf{u}^R\|_{H^1_\sigma}\|\nabla(\mathbf{b}_1^R - \mathbf{b}_2^R)\|_{H^{s-1}_\sigma}\|\mathbf{b}_1^R - \mathbf{b}_2^R\|_{L^2_\sigma} \\
&\le C\|\mathbf{u}^R\|_{H^s_\sigma}\|\mathbf{b}_1^R - \mathbf{b}_2^R\|_{H^s_\sigma}\|\mathbf{b}_1^R - \mathbf{b}_2^R\|_{L^2_\sigma}
\end{aligned}
$$

Hence for $\mathbf{u}^R \in H^s_\sigma(\mathbb{R}^n)$, we have

$$
\|\big(F(\mathbf{u}^R, \mathbf{b}_1^R) - F(\mathbf{u}^R, \mathbf{b}_2^R)\|_{L^2} \le C\|\mathbf{u}^R\|_{H^s_\sigma}\|\mathbf{b}_1^R - \mathbf{b}_2^R\|_{H^s_\sigma}
$$

And hence $F(\cdot,\cdot)$ is locally Lipschitz in $\mathbf{b}^R$. □

Similarly one can show that $F(\mathbf{b}^R, \mathbf{u}^R)$ is locally Lipschitz in $\mathbf{b}^R$ and $\mathbf{u}^R$ on the space $V_R^\sigma \times V_R^\sigma$ and $F(\mathbf{u}^R, \theta^R)$ is locally Lipschitz in $\mathbf{u}^R$ and θ^R on the space $V_R^\sigma \times V_R$.

Hence by Picard's theorem for infinite dimensional ordinary differential equations, there exist a solution $(\mathbf{u}^R, \theta^R, \mathbf{b}^R)$ in $V_R^\sigma \times V^R \times V_R^\sigma$ for some interval $[0, T]$, where T depends on R. Moreover, the solution will exist as long as $\|\mathbf{u}^R\|_{H^s_\sigma}$, $\|\theta^R\|_{H^s}$ and $\|\mathbf{b}^R\|_{H^s_\sigma}$ remain finite.

3.1. **Energy estimates.** In this section we obtain L^2 and $H^s, s > n/2+1$, energy estimates for $\mathbf{u}^R, \theta^R$ and $\mathbf{b}^R$. In the course of proving the $\|\mathbf{u}^R\|_{H^s_\sigma}$, $\|\theta^R\|_{H^s}$ and $\|\mathbf{b}^R\|_{H^s_\sigma}$ are uniformly bounded, we will pick up a blow-up time T^*.

Proposition 3.2 (L^2-Energy Estimate). *Given $(\mathbf{u}_0, \theta_0, \mathbf{b}_0) \in L^2_\sigma(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times L^2_\sigma(\mathbb{R}^n)$ with $s > n/2 + 1$, then for any $t \in [0, T]$, where $0 < T < \infty$, we have*

$$
\sup_{t\in[0,T]}\Big(\|\mathbf{u}^R(t)\|^2_{L^2_\sigma} + \|\theta^R(t)\|^2_{L^2} + \|\mathbf{b}^R(t)\|^2_{L^2_\sigma}\Big) < C
$$

where C depends only on $\|\mathbf{u}_0\|_{L^2_\sigma}, \|\theta_0\|_{L^2}, \|\mathbf{b}_0\|_{L^2_\sigma}$ and T.

Proof. Consider the equations (3.1)-(3.3). Taking L^2-inner product of (3.1), (3.2) and (3.3) with $\mathbf{u}^R, \theta^R$ and $\mathbf{b}^R$ respectively, and adding we obtain

$$
\frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{u}^R\|^2_{L^2_\sigma} + \|\theta^R\|^2_{L^2} + \|\mathbf{b}^R\|^2_{L^2_\sigma}\Big) = \big(\theta^R e_n, \mathbf{u}^R\big)_{L^2} + \big((\mathbf{u}^R\cdot e_n), \theta^R\big)_{L^2}. \quad (3.6)
$$

In the above calculation, we have used that $\big((\mathbf{u}^R\cdot\nabla)\mathbf{u}^R,\mathbf{u}^R\big)_{L^2}$, $\big((\mathbf{u}^R\cdot\nabla)\theta^R,\theta^R\big)_{L^2}$ and $\big((\mathbf{u}^R\cdot\nabla)\mathbf{b}^R,\mathbf{b}^R\big)_{L^2}$ vanish and $\big((\mathbf{b}^R\cdot\nabla)\mathbf{b}^R,\mathbf{u}^R\big)_{L^2}=-\big((\mathbf{b}^R\cdot\nabla)\mathbf{u}^R,\mathbf{b}^R\big)_{L^2}$. It is easy to see that

$$\begin{aligned}|(\theta^R e_n,\mathbf{u}^R)_{L^2}| &\le \|\theta^R e_n\|_{L^2}\|\mathbf{u}^R\|_{L^2}\le \|\theta^R\|_{L^2}\|\mathbf{u}^R\|_{L^2}\\ &\le \frac{1}{2}\big(\|\mathbf{u}^R\|_{L^2}^2+\|\theta^R\|_{L^2}^2+\|\mathbf{b}^R\|_{L^2}^2\big),\end{aligned}$$

and

$$|\big((\mathbf{u}^R\cdot e_n),\theta^R\big)_{L^2}|\le \|\mathbf{u}^R\|_{L^2}\|\theta^R\|_{L^2}\le\frac{1}{2}\left(\|\mathbf{u}^R\|_{L^2}^2+\|\theta^R\|_{L^2}^2+\|\mathbf{b}^R\|_{L^2}^2\right).$$

Using the above estimates in (3.6) and letting $Y(t)=\|\mathbf{u}^R(t)\|_{L^2_\sigma}^2+\|\theta^R(t)\|_{L^2}^2+\|\mathbf{b}^R(t)\|_{L^2_\sigma}^2$, we obtain

$$\frac{dY(t)}{dt}\le 2Y(t).$$

Straightforward integration and the fact that $\|\mathbf{u}^R(0)\|_{L^2_\sigma}\le\|u_0\|_{L^2_\sigma}$, $\|\theta^R(0)\|_{L^2}\le\|\theta_0\|_{L^2}$ and $\|\mathbf{b}^R(0)\|_{L^2_\sigma}\le\|\mathbf{b}_0\|_{L^2_\sigma}$ yield

$$\sup_{t\in[0,T]}Y(t)\le C(\|\mathbf{u}_0\|_{L^2_\sigma},\|\theta_0\|_{L^2},\|\mathbf{b}_0\|_{L^2_\sigma},T)$$

So we have the desired result. □

Proposition 3.3. *Let* $(\mathbf{u}_0,\theta_0,\mathbf{b}_0)\in H^s_\sigma(\mathbb{R}^n)\times H^s(\mathbb{R}^n)\times H^s_\sigma(\mathbb{R}^n)$ *with* $s>n/2+1$. *Then there exists a time* $T^*=T^*(s,\|u_0\|_{H^s_\sigma},\|\theta_0\|_{H^s},\|\mathbf{b}_0\|_{H^s_\sigma})>0$ *such that*

$$\sup_{t\in[0,T^*]}\|\mathbf{u}^R(t)\|_{H^s_\sigma},\quad \sup_{t\in[0,T^*]}\|\theta^R(t)\|_{H^s},\quad \sup_{t\in[0,T^*]}\|\mathbf{b}^R(t)\|_{H^s_\sigma}$$

are bounded uniformly in R.

Proof. Let J^s denote the fractional derivative operator as defined earlier. Now for $s>n/2+1$, apply J^s to all the equations (3.1)-(3.3):

$$\frac{\partial(J^s\mathbf{u}^R)}{\partial t}+\mathcal{S}_RJ^s[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R]+\nabla J^sp^R=J^s(\theta^Re_n)+\mathcal{S}_RJ^s[(\mathbf{b}^R\cdot\nabla)\mathbf{b}^R], \tag{3.7}$$

$$\frac{\partial(J^s\theta^R)}{\partial t}+\mathcal{S}_RJ^s[(\mathbf{u}^R\cdot\nabla)\theta^R]=J^s(\mathbf{u}^R\cdot e_n), \tag{3.8}$$

$$\frac{\partial(J^s\mathbf{b}^R)}{\partial t}+\mathcal{S}_RJ^s[(\mathbf{u}^R\cdot\nabla)\mathbf{b}^R]=\mathcal{S}_RJ^s[(\mathbf{b}^R\cdot\nabla)\mathbf{u}^R] \tag{3.9}$$

Taking the L^2-inner product of (3.7), (3.8) and (3.9) with $J^s\mathbf{u}^R$, $J^s\theta^R$ and $J^s\mathbf{b}^R$ respectively, we obtain

$$\begin{aligned}&\Big(\frac{\partial(J^s\mathbf{u}^R)}{\partial t},J^s\mathbf{u}^R\Big)_{L^2}+\big(\mathcal{S}_RJ^s[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R],J^s\mathbf{u}^R\big)_{L^2}+\big(\nabla J^sp^R,J^s\mathbf{u}^R\big)_{L^2}\\ &=\big(J^s(\theta^Re_n),J^s\mathbf{u}^R\big)_{L^2}+\big(\mathcal{S}_RJ^s[(\mathbf{b}^R\cdot\nabla)\mathbf{b}^R],J^s\mathbf{u}^R\big)_{L^2},\end{aligned} \tag{3.10}$$

$$\begin{aligned}&\big(\frac{\partial(J^s\theta^R)}{\partial t},J^s\theta^R\big)_{L^2}+\big(\mathcal{S}_RJ^s[(\mathbf{u}^R\cdot\nabla)\theta^R],J^s\theta^R\big)_{L^2}\\ &=\big(J^s(\mathbf{u}^R\cdot e_n),J^s\theta^R\big)_{L^2},\end{aligned} \tag{3.11}$$

$$\begin{aligned}&\big(\frac{\partial(J^s\mathbf{b}^R)}{\partial t},J^s\mathbf{b}^R\big)_{L^2}+\big(\mathcal{S}_RJ^s[(\mathbf{u}^R\cdot\nabla)\mathbf{b}^R],J^s\mathbf{b}^R\big)_{L^2}\\ &=\big(\mathcal{S}_RJ^s[(\mathbf{b}^R\cdot\nabla)\mathbf{u}^R],J^s\mathbf{b}^R\big)_{L^2}.\end{aligned} \tag{3.12}$$

We estimate each term of (3.10), (3.11) and (3.12) separately. (1)

$$
\begin{aligned}
\big(\frac{\partial(J^s\mathbf{u}^R)}{\partial t}, J^s\mathbf{u}^R\big)_{L^2} &= \int_{B_R}\frac{\partial J^s\mathbf{u}^R}{\partial t}J^s\mathbf{u}^R\,dx = \frac{1}{2}\int_{B_R}\frac{\partial|J^s\mathbf{u}^R|^2}{\partial t}\\
&= \frac{1}{2}\frac{d}{dt}\|J^s\mathbf{u}^R\|^2_{L^2_\sigma} = \frac{1}{2}\frac{d}{dt}\|\mathbf{u}^R\|^2_{H^s_\sigma}\,.
\end{aligned}
$$

(2) Applying weak Parseval's identity and using the fact that $\mathcal{S}_R\mathbf{u}^R = \mathbf{u}^R$, since $\mathbf{u}^R \in V_R^\sigma$ we obtain

$$
\big(\mathcal{S}_R J^s[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R], J^s\mathbf{u}^R\big)_{L^2} = \big(J^s[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R], J^s\mathbf{u}^R\big)_{L^2}\ .
$$

(3) Using the definition of commutator and incompressibility of $\mathbf{u}^R$, we obtain

$$
\begin{aligned}
\big([J^s,\mathbf{u}^R]\nabla\mathbf{u}^R, J^s\mathbf{u}^R\big)_{L^2} &= \big(J^s[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R] - \big(\mathbf{u}^R\cdot\nabla\big)\,J^s\mathbf{u}^R, J^s\mathbf{u}^R\big)_{L^2}\\
&= \big(J^s[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R], J^s\mathbf{u}^R\big)_{L^2}\ .
\end{aligned}
$$

Now using Lemma 2.8 and Hölder's inequality we obtain

$$
\begin{aligned}
&\big|\big([J^s,\mathbf{u}^R]\nabla\mathbf{u}^R, J^s\mathbf{u}^R\big)_{L^2}\big|\\
&\le \|[J^s,\mathbf{u}^R]\nabla\mathbf{u}^R\|_{L^2}\|J^s\mathbf{u}^R\|_{L^2}\\
&\le C\left(\|\nabla\mathbf{u}^R\|_{L^\infty}\|J^{s-1}\nabla\mathbf{u}^R\|_{L^2_\sigma} + \|J^s\mathbf{u}^R\|_{L^2_\sigma}\|\nabla\mathbf{u}^R\|_{L^\infty}\right)\|\mathbf{u}^R\|_{H^s_\sigma}\\
&\le C\Big(\|\nabla\mathbf{u}^R\|_{H^{s-1}_\sigma}\|\nabla\mathbf{u}^R\|_{H^{s-1}_\sigma} + \|\mathbf{u}^R\|_{H^s_\sigma}\|\nabla\mathbf{u}^R\|_{H^{s-1}_\sigma}\Big)\|\mathbf{u}^R\|_{H^s_\sigma}\\
&\le C\Big(\|\mathbf{u}^R\|^2_{H^s_\sigma} + \|\mathbf{u}^R\|^2_{H^s_\sigma}\Big)\|\mathbf{u}^R\|_{H^s_\sigma}\\
&\le C\Big(\|\mathbf{u}^R\|^2_{H^s_\sigma} + \|\theta^R\|^2_{H^s} + \|\mathbf{b}^R\|^2_{H^s_\sigma}\Big)\|\mathbf{u}^R\|_{H^s_\sigma}\,.
\end{aligned}
$$

(3) Using integration by parts we infer

$$
\big(\nabla J^s p^R, J^s\mathbf{u}^R\big)_{L^2} = \big(J^s p^R, J^s\nabla\cdot\mathbf{u}^R\big)_{L^2} = 0.
$$

(4) Using the Hölder's inequality and then Young's inequality we infer

$$
\begin{aligned}
\big|\big(J^s(\theta^R e_n), J^s\mathbf{u}^R\big)_{L^2}\big| &\le \|J^s(\theta^R e_n)\|_{L^2}\|J^s\mathbf{u}^R\|_{L^2_\sigma}\\
&\le \|\theta^R e_n\|_{H^s}\|\mathbf{u}^R\|_{H^s_\sigma}\\
&\le C\Big(\|\mathbf{u}^R\|^2_{H^s_\sigma} + \|\theta^R\|^2_{H^s} + \|\mathbf{b}^R\|^2_{H^s_\sigma}\Big).
\end{aligned}
$$

(5) Using the property of the bilinear operator, we have

$$
\begin{aligned}
\big(\mathcal{S}_R J^s[(\mathbf{b}^R\cdot\nabla)\mathbf{b}^R], J^s\mathbf{u}^R\big)_{L^2} &= \big(J^s[(\mathbf{b}^R\cdot\nabla)\mathbf{b}^R], J^s\mathcal{S}_R\mathbf{u}^R\big)_{L^2}\\
&= \big(J^s[(\mathbf{b}^R\cdot\nabla)\mathbf{b}^R], J^s\mathbf{u}^R\big)_{L^2}\\
&= -\big(J^s[(\mathbf{b}^R\cdot\nabla)\mathbf{u}^R], J^s\mathbf{b}^R\big)_{L^2}\ .
\end{aligned}
$$

(6) Similarly referring to (1) we infer

$$
\left(\frac{\partial(J^s\theta^R)}{\partial t}, J^s\theta^R\right)_{L^2} = \frac{1}{2}\frac{d}{dt}\|\theta^R\|^2_{H^s}\,.
$$

(7) Similar calculations as in (2) and using Lemma 2.8 we obtain

$$
\big(\mathcal{S}_R J^s[(\mathbf{u}^R\cdot\nabla)\theta^R], J^s\theta^R\big)_{L^2}
$$

and

$$
\big|\big([J^s,\mathbf{u}^R]\nabla\theta^R, J^s\theta^R\big)_{L^2}\big| \le \|[J^s,\mathbf{u}^R]\nabla\theta^R\|_{L^2}\|J^s\theta^R\|_{L^2}
$$

$$
\begin{aligned}
&\le C\left(\|\nabla \mathbf{u}^R\|_{L^\infty}\|J^{s-1}\nabla\theta^R\|_{L^2} + \|J^s\mathbf{u}^R\|_{L^2_\sigma}\|\nabla\theta^R\|_{L^\infty}\right)\|\theta^R\|_{H^s}\\
&\le C\left(\|\nabla \mathbf{u}^R\|_{H^{s-1}_\sigma}\|\nabla\theta^R\|_{H^{s-1}} + \|\mathbf{u}^R\|_{H^s_\sigma}\|\nabla\theta^R\|_{H^{s-1}}\right)\|\theta^R\|_{H^s}\\
&\le C\left(\|\mathbf{u}^R\|^2_{H^s_\sigma} + \|\theta^R\|^2_{H^s} + \|\mathbf{b}^R\|^2_{H^s_\sigma}\right)\|\theta^R\|_{H^s}\,.
\end{aligned}
$$

(8) Using the Hölder's inequality and the Young's inequality we obtain

$$
\begin{aligned}
\left|\left(J^s(\mathbf{u}^R\cdot e_n), J^s\theta^R\right)_{L^2}\right| &\le \|J^s(\mathbf{u}^R\cdot e_n)\|_{L^2_\sigma}\|J^s\theta^R\|_{L^2}\\
&\le \|\mathbf{u}^R\|_{H^s_\sigma}\|\theta^R\|_{H^s}\\
&\le C\left(\|\mathbf{u}^R\|^2_{H^s_\sigma} + \|\theta^R\|^2_{H^s} + \|\mathbf{b}^R\|^2_{H^s_\sigma}\right).
\end{aligned}
$$

(9) Similarly,

$$
\Big(\frac{\partial(J^s\mathbf{b}^R)}{\partial t}, J^s\mathbf{b}^R\Big)_{L^2} = \frac{1}{2}\frac{d}{dt}\|\mathbf{b}^R\|^2_{H^s_\sigma}\,.
$$

(10) Following similar steps as in (7), replacing θ^R by $\mathbf{b}^R$ we obtain

$$
\left|\left(\mathcal{S}_R J^s[(\mathbf{u}^R\cdot\nabla)\mathbf{b}^R], J^s\mathbf{b}^R\right)_{L^2}\right| \le C\left(\|\mathbf{u}^R\|^2_{H^s_\sigma} + \|\theta^R\|^2_{H^s} + \|\mathbf{b}^R\|^2_{H^s_\sigma}\right)\|\mathbf{b}^R\|_{H^s_\sigma}\,.
$$

(11) Weak Parseval's identity gives

$$
\left(\mathcal{S}_R J^s[(\mathbf{b}^R\cdot\nabla)\mathbf{u}^R], J^s\mathbf{b}^R\right)_{L^2} = \left(J^s[(\mathbf{b}^R\cdot\nabla)\mathbf{u}^R], J^s\mathbf{b}^R\right)_{L^2}\,.
$$

Now adding (3.10), (3.11) and (3.12) (using the estimates obtained through (1) to (11)) we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}^R\|^2_{H^s_\sigma} + \|\theta^R\|^2_{H^s} + \|\mathbf{b}^R\|^2_{H^s_\sigma}\right)\\
&\le C\left(\|\mathbf{u}^R\|^2_{H^s_\sigma} + \|\theta^R\|^2_{H^s} + \|\mathbf{b}^R\|^2_{H^s_\sigma}\right)\left(\|\mathbf{u}^R\|_{H^s_\sigma} + \|\theta^R\|_{H^s} + \|\mathbf{b}^R\|_{H^s_\sigma}\right)\\
&\le \frac{C}{2}\left(\|\mathbf{u}^R\|^2_{H^s_\sigma} + \|\theta^R\|^2_{H^s} + \|\mathbf{b}^R\|^2_{H^s_\sigma}\right)^2 + \frac{3C}{2}\left(\|\mathbf{u}^R\|^2_{H^s_\sigma} + \|\theta^R\|^2_{H^s} + \|\mathbf{b}^R\|^2_{H^s_\sigma}\right).
\end{aligned}
$$

Now letting $X(t) = \|\mathbf{u}^R(t)\|^2_{H^s_\sigma} + \|\theta^R(t)\|^2_{H^s} + \|\mathbf{b}^R(t)\|^2_{H^s_\sigma}$ we obtain

$$
\frac{d}{dt}X(t) \le 3CX(t) + X(t)^2 \le \frac{3}{2}C^2 + \Big(\frac{3}{2} + C\Big)X(t)^2.
$$

So for all $0 \le t \le T$,

$$
X(t) \le X_0 + \frac{3}{2}C^2 + \Big(\frac{3}{2} + C\Big)\int_0^t X(s)^2\,ds.
$$

Now applying Bihari's inequality [10], we have

$$
X(t) \le \frac{\frac{3}{2}C^2 + X_0}{1 - (\frac{3}{2}C^2 + X_0)(\frac{3}{2} + C)T}\,.
$$

Note that $\|\mathbf{u}^R(0)\|_{H^s_\sigma} \le \|u_0\|_{H^s_\sigma}$, $\|\theta^R(0)\|_{H^s} \le \|\theta_0\|_{H^s}$ and $\|\mathbf{b}^R(0)\|_{H^s_\sigma} \le \|\mathbf{b}_0\|_{H^s_\sigma}$. So provided we choose

$$
T^* < \frac{1}{(\frac{3}{2}C^2 + X_0)(\frac{3}{2} + C)}\,,
$$

then the norms $\|\mathbf{u}^R\|_{H^s_\sigma}, \|\theta^R\|_{H^s}$ and $\|\mathbf{b}^R\|_{H^s_\sigma}$ remain bounded on $[0, T^*]$ independent of R. $\square$

3.2. **Local existence and uniqueness.** In this subsection, we prove existence and uniqueness of the local-in time strong solution of the magnetic Bénard problem (1.5)-(1.8).

At first, we show that the family $(\mathbf{u}^R, \theta^R, \mathbf{b}^R)$ is Cauchy in a suitable space.

Proposition 3.4. *The family* $(\mathbf{u}^R, \theta^R, \mathbf{b}^R)$ *of solutions of the magnetic Bénard problem* (3.1)-(3.5) *are Cauchy in the space*

$$L^\infty\left([0,T^*]; L^2_\sigma(\mathbb{R}^n)\right) \times L^\infty\left([0,T^*]; L^2(\mathbb{R}^n)\right) \times L^\infty\left([0,T^*]; L^2_\sigma(\mathbb{R}^n)\right),$$

as $R \to \infty$.

Proof. We consider equations (3.1), (3.2) and (3.3). Then taking the difference between the equations for R and R' with $R' > R$ we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\big(\mathbf{u}^R - \mathbf{u}^{R'}\big) + \nabla\big(p^R - p^{R'}\big) \\ &= \theta^R e_n - \theta^{R'} e_n - \mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R] + \mathcal{S}_{R'}[(\mathbf{u}^{R'}\cdot\nabla)\mathbf{u}^{R'}] \\ &\quad + \mathcal{S}_R[(\mathbf{b}^R\cdot\nabla)\mathbf{b}^R] - \mathcal{S}_{R'}[(\mathbf{b}^{R'}\cdot\nabla)\mathbf{b}^{R'}], \end{aligned} \tag{3.13}$$

$$\frac{\partial}{\partial t}\big(\theta^R - \theta^{R'}\big) + \mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\theta^R] - \mathcal{S}_{R'}[(\mathbf{u}^{R'}\cdot\nabla)\theta^{R'}] = \mathbf{u}^R\cdot e_n - \mathbf{u}^{R'}\cdot e_n, \tag{3.14}$$

$$\begin{aligned} &\frac{\partial}{\partial t}\big(\mathbf{b}^R - \mathbf{b}^{R'}\big) + \mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{b}^R] - \mathcal{S}_{R'}[(\mathbf{u}^{R'}\cdot\nabla)\mathbf{b}^{R'}] \\ &= \mathcal{S}_R[(\mathbf{b}^R\cdot\nabla)\mathbf{u}^R] - \mathcal{S}_{R'}[(\mathbf{b}^{R'}\cdot\nabla)\mathbf{u}^{R'}]. \end{aligned} \tag{3.15}$$

Taking the inner product of (3.13), (3.14) and (3.15) with $\mathbf{u}^R - \mathbf{u}^{R'}$, $\theta^R - \theta^{R'}$ and $\mathbf{b}^R - \mathbf{b}^{R'}$ respectively, and then adding we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma} + \|\theta^R - \theta^{R'}\|^2_{L^2} + \|\mathbf{b}^R - \mathbf{b}^{R'}\|^2_{L^2_\sigma}\Big) \\ &= \big(\theta^R e_n - \theta^{R'} e_n, \mathbf{u}^R - \mathbf{u}^{R'}\big) - \big(\mathbf{u}^R\cdot e_n - \mathbf{u}^{R'}\cdot e_n, \theta^R - \theta^{R'}\big) \\ &\quad - \underbrace{\big(\mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R] - \mathcal{S}_{R'}[(\mathbf{u}^{R'}\cdot\nabla)\mathbf{u}^{R'}], \mathbf{u}^R - \mathbf{u}^{R'}\big)}_{I_1} \\ &\quad + \underbrace{\big(\mathcal{S}_R[(\mathbf{b}^R\cdot\nabla)\mathbf{b}^R] - \mathcal{S}_{R'}[(\mathbf{b}^{R'}\cdot\nabla)\mathbf{b}^{R'}], \mathbf{u}^R - \mathbf{u}^{R'}\big)}_{I_2} \\ &\quad - \underbrace{\big(\mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\theta^R] - \mathcal{S}_{R'}[(\mathbf{u}^{R'}\cdot\nabla)\theta^{R'}], \theta^R - \theta^{R'}\big)}_{I_3} \\ &\quad + \underbrace{\big(\mathcal{S}_R[(\mathbf{b}^R\cdot\nabla)\mathbf{u}^R] - \mathcal{S}_{R'}[(\mathbf{b}^{R'}\cdot\nabla)\mathbf{u}^{R'}], \mathbf{b}^R - \mathbf{b}^{R'}\big)}_{I_4} \\ &\quad - \underbrace{\big(\mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{b}^R] - \mathcal{S}_{R'}[(\mathbf{u}^{R'}\cdot\nabla)\mathbf{b}^{R'}], \mathbf{b}^R - \mathbf{b}^{R'}\big)}_{I_5} \end{aligned} \tag{3.16}$$

We will calculate each term on the right hand side of (3.16) separately. First observe that

$$\begin{aligned} \Big|\big(\theta^R e_n - \theta^{R'} e_n, \mathbf{u}^R - \mathbf{u}^{R'}\big)\Big| &\le \|\theta^R e_n - \theta^{R'} e_n\|_{L^2}\|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} \\ &\le \|\theta^R - \theta^{R'}\|_{L^2}\|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma}, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\Big|\big(\mathbf{u}^R\cdot e_n-\mathbf{u}^{R'}\cdot e_n,\theta^R-\theta^{R'}\big)\Big| &\le \|\mathbf{u}^R\cdot e_n-\mathbf{u}^{R'}\cdot e_n\|_{L^2_\sigma}\|\theta^R-\theta^{R'}\|_{L^2}\\ &\le \|\mathbf{u}^R-\mathbf{u}^{R'}\|_{L^2_\sigma}\|\theta^R-\theta^{R'}\|_{L^2}.\end{aligned} \tag{3.18}$$

We split $I_1=\big(\mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R]-\mathcal{S}_{R'}[(\mathbf{u}^{R'}\cdot\nabla)\mathbf{u}^{R'}],\mathbf{u}^R-\mathbf{u}^{R'}\big)$ in to three parts:

$$\begin{aligned}I_1*&=\Big((\mathcal{S}_R-\mathcal{S}_{R'})[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R],\mathbf{u}^R-\mathbf{u}^{R'}\Big)\\ &\quad+\Big(\mathcal{S}_{R'}[((\mathbf{u}^R-\mathbf{u}^{R'})\cdot\nabla)\mathbf{u}^R],\mathbf{u}^R-\mathbf{u}^{R'}\Big)\\ &\quad+\Big(\mathcal{S}_{R'}[(\mathbf{u}^{R'}\cdot\nabla)(\mathbf{u}^R-\mathbf{u}^{R'})],\mathbf{u}^R-\mathbf{u}^{R'}\Big).\end{aligned} \tag{3.19}$$

For $R'>R$, using the property of Fourier truncation operator provided $0<\epsilon<s-1$, the first term of (3.19) becomes

$$\begin{aligned}&\Big|\big((\mathcal{S}_R-\mathcal{S}_{R'})[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R],\mathbf{u}^R-\mathbf{u}^{R'}\big)\Big|\\ &\le \|(\mathcal{S}_R-\mathcal{S}_{R'})[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R]\|_{L^2_\sigma}\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{L^2_\sigma}\\ &\le \frac{C}{R^\epsilon}\|(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R\|_{H^\epsilon_\sigma}\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{L^2_\sigma}\\ &= \frac{C}{R^\epsilon}\|\nabla\cdot(\mathbf{u}^R\otimes\mathbf{u}^R)\|_{H^\epsilon_\sigma}\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{L^2_\sigma}\\ &\le \frac{C}{R^\epsilon}\|\mathbf{u}^R\otimes\mathbf{u}^R\|_{H^s_\sigma}\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{L^2_\sigma}\\ &\le \frac{C}{R^\epsilon}\|\mathbf{u}^R\|^2_{H^s_\sigma}\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{L^2_\sigma}.\end{aligned} \tag{3.20}$$

Now for $s>n/2+1$, the second term of (3.19),

$$\begin{aligned}&\Big|\big(\mathcal{S}_{R'}[((\mathbf{u}^R-\mathbf{u}^{R'})\cdot\nabla)\mathbf{u}^R],\mathbf{u}^R-\mathbf{u}^{R'}\big)\Big|\\ &\le \|((\mathbf{u}^R-\mathbf{u}^{R'})\cdot\nabla)\mathbf{u}^R\|_{L^2_\sigma}\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{L^2_\sigma}\\ &\le \|\mathbf{u}^R-\mathbf{u}^{R'}\|_{L^2_\sigma}\|\nabla\mathbf{u}^R\|_{L^\infty}\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{L^2_\sigma}\\ &\le \|\nabla\mathbf{u}^R\|_{H^{s-1}_\sigma}\|\mathbf{u}^R-\mathbf{u}^{R'}\|^2_{L^2_\sigma}\\ &\le \|\mathbf{u}^R\|_{H^s_\sigma}\|\mathbf{u}^R-\mathbf{u}^{R'}\|^2_{L^2_\sigma}.\end{aligned} \tag{3.21}$$

Using weak Parseval's identity, integration by parts and divergence free condition on $\mathbf{u}^R$ and $\mathbf{u}^{R'}$ to the third term of (3.19) we obtain

$$\big(\mathcal{S}_{R'}[(\mathbf{u}^{R'}\cdot\nabla)(\mathbf{u}^R-\mathbf{u}^{R'})],\mathbf{u}^R-\mathbf{u}^{R'}\big)=0.$$

Therefore, using (3.20), (3.21) in (3.19), we obtain

$$|I_1|\le\frac{C}{R^\epsilon}\|\mathbf{u}^R\|^2_{H^s_\sigma}\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{L^2_\sigma}+\|\mathbf{u}^R\|_{H^s_\sigma}\|\mathbf{u}^R-\mathbf{u}^{R'}\|^2_{L^2_\sigma}. \tag{3.22}$$

Similarly we split I_3 and I_5 to obtain

$$|I_3|\le\frac{C}{R^\epsilon}\|\mathbf{u}^R\|_{H^s_\sigma}\|\theta^R\|_{H^s}\|\theta^R-\theta^{R'}\|_{L^2}+\|\mathbf{u}^R-\mathbf{u}^{R'}\|_{L^2_\sigma}\|\theta^R\|_{H^s}\|\theta^R-\theta^{R'}\|_{L^2}. \tag{3.23}$$

and

$$\begin{aligned} |I_5| \le \frac{C}{R^{\epsilon}} \|\mathbf{u}^R\|_{H^s_\sigma} \|\mathbf{b}^R\|_{H^s_\sigma} \|\mathbf{b}^R - \mathbf{b}^{R'}\|_{L^2_\sigma} \\ + \|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} \|\mathbf{b}^R\|_{H^s_\sigma} \|\mathbf{b}^R - \mathbf{b}^{R'}\|_{L^2_\sigma} . \end{aligned} \tag{3.24}$$

We spilt I_2 and I_4 in the similar manner. However, note that one term of I_2 will cancel with one term of I_4 because

$$\big(\mathcal{S}_{R'}[(\mathbf{b}^{R'} \cdot \nabla)(\mathbf{b}^R - \mathbf{b}^{R'})], \mathbf{u}^R - \mathbf{u}^{R'}\big) = -\big((\mathbf{b}^{R'} \cdot \nabla)(\mathbf{u}^R - \mathbf{u}^{R'}), \mathbf{b}^R - \mathbf{b}^{R'}\big).$$

Therefore,

$$I_2 \le \frac{C}{R^{\epsilon}} \|\mathbf{b}^R\|^2_{H^s_\sigma} \|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} + \|\mathbf{b}^R\|_{H^s_\sigma} \|\mathbf{b}^R - \mathbf{b}^{R'}\|_{L^2_\sigma} \|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma}, \tag{3.25}$$

$$I_4 \le \frac{C}{R^{\epsilon}} \|\mathbf{b}^R\|_{H^s_\sigma} \|\mathbf{u}^R\|_{H^s_\sigma} \|\mathbf{b}^R - \mathbf{b}^{R'}\|_{L^2_\sigma} + \|\mathbf{u}^R\|_{H^s_\sigma} \|\mathbf{b}^R - \mathbf{b}^{R'}\|^2_{L^2_\sigma} . \tag{3.26}$$

Using the estimates obtained in (3.17), (3.18), (3.22)-(3.26) in (3.16), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma} + \|\theta^R - \theta^{R'}\|^2_{L^2} + \|\mathbf{b}^R - \mathbf{b}^{R'}\|^2_{L^2_\sigma}\Big) \\ &\le \frac{C}{R^{\epsilon}} \|\mathbf{u}^R\|^2_{H^s_\sigma} \|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} + \|\mathbf{u}^R\|_{H^s_\sigma} \|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma} \\ &\quad + \frac{C}{R^{\epsilon}} \|\mathbf{u}^R\|_{H^s_\sigma} \|\theta^R\|_{H^s} \|\theta^R - \theta^{R'}\|_{L^2} + \|\mathbf{b}^R\|_{H^s_\sigma} \|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} \|\mathbf{b}^R - \mathbf{b}^{R'}\|_{L^2_\sigma} \\ &\quad + \|\theta^R\|_{H^s} \|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} \|\theta^R - \theta^{R'}\|_{L^2} + \frac{C}{R^{\epsilon}} \|\mathbf{b}^R\|_{H^s_\sigma} \|\mathbf{u}^R\|_{H^s_\sigma} \|\mathbf{b}^R - \mathbf{b}^{R'}\|_{L^2_\sigma} \\ &\quad + \frac{C}{R^{\epsilon}} \|\mathbf{b}^R\|^2_{H^s_\sigma} \|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} + \|\mathbf{b}^R\|_{H^s_\sigma} \|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} \|\mathbf{b}^R - \mathbf{b}^{R'}\|_{L^2_\sigma} \\ &\quad + \frac{C}{R^{\epsilon}} \|\mathbf{b}^R\|_{H^s_\sigma} \|\mathbf{u}^R\|_{H^s_\sigma} \|\mathbf{b}^R - \mathbf{b}^{R'}\|_{L^2_\sigma} + \|\mathbf{u}^R\|_{H^s_\sigma} \|\mathbf{b}^R - \mathbf{b}^{R'}\|^2_{L^2_\sigma} \\ &\quad + 2\|\theta^R - \theta^{R'}\|_{L^2} \|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} . \end{aligned}$$

Applying Proposition 3.3 and Young's inequality and rearranging the terms we obtain

$$\begin{aligned} &\frac{d}{dt}\Big(\|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma} + \|\theta^R - \theta^{R'}\|^2_{L^2} + \|\mathbf{b}^R - \mathbf{b}^{R'}\|^2_{L^2_\sigma}\Big) \\ &\le \frac{C_1}{R^{\epsilon}} \|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} + C_2 \|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma} + 2\Big(\|\theta^R - \theta^{R'}\|^2_{L^2} + \|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma}\Big) \\ &\quad + \frac{C_3}{R^{\epsilon}} \|\theta^R - \theta^{R'}\|_{L^2} + C_4\Big(\|\theta^R - \theta^{R'}\|^2_{L^2} + \|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma}\Big) \\ &\quad + \frac{C_5}{R^{\epsilon}} \|\mathbf{b}^R - \mathbf{b}^{R'}\|_{L^2_\sigma} + C_6\Big(\|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma} + \|\mathbf{b}^R - \mathbf{b}^{R'}\|^2_{L^2_\sigma}\Big) \\ &\quad + \frac{C_7}{R^{\epsilon}} \|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} + C_8\Big(\|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma} + \|\mathbf{b}^R - \mathbf{b}^{R'}\|^2_{L^2_\sigma}\Big) \\ &\quad + C_{10} \|\mathbf{b}^R - \mathbf{b}^{R'}\|^2_{L^2_\sigma} \\ &\le \frac{M}{R^{\epsilon}}\Big(\|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} + \|\theta^R - \theta^{R'}\|_{L^2} + \|\mathbf{b}^R - \mathbf{b}^{R'}\|_{L^2_\sigma}\Big) \\ &\quad + M\Big(\|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma} + \|\theta^R - \theta^{R'}\|^2_{L^2} + \|\mathbf{b}^R - \mathbf{b}^{R'}\|^2_{L^2_\sigma}\Big). \end{aligned} \tag{3.27}$$

Let $Y(t) = \|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} + \|\theta^R - \theta^{R'}\|_{L^2} + \|\mathbf{b}^R - \mathbf{b}^{R'}\|_{L^2_\sigma}$, then we infer

$$\begin{aligned}&\|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma} + \|\theta^R - \theta^{R'}\|^2_{L^2} + \|\mathbf{b}^R - \mathbf{b}^{R'}\|^2_{L^2_\sigma}\\ &\le Y(t)^2 \le 3\left(\|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma} + \|\theta^R - \theta^{R'}\|^2_{L^2} + \|\mathbf{b}^R - \mathbf{b}^{R'}\|^2_{L^2_\sigma}\right).\end{aligned}$$

Thus, we obtain

$$\frac{d}{dt}\left(Y(t)^2\right) \le 3\frac{d}{dt}\left(\|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma} + \|\theta^R - \theta^{R'}\|^2_{L^2} + \|\mathbf{b}^R - \mathbf{b}^{R'}\|^2_{L^2_\sigma}\right),$$

which implies

$$2Y\frac{dY}{dt} \le 3\frac{d}{dt}\left(\|\mathbf{u}^R - \mathbf{u}^{R'}\|^2_{L^2_\sigma} + \|\theta^R - \theta^{R'}\|^2_{L^2} + \|\mathbf{b}^R - \mathbf{b}^{R'}\|^2_{L^2_\sigma}\right)$$

From (3.27), we obtain

$$\frac{dY}{dt} \le MY + \frac{M}{R^\epsilon}.$$

Finally, applying the Gronwall's lemma we infer

$$\sup_{t\in[0,T^*]} Y(t) \le \frac{C(M,T^*)}{R^\epsilon} \to 0, \tag{3.28}$$

as $R \to \infty$ (as $R' > R$, $R' \to \infty$ as well), concluding that $(\mathbf{u}^R, \theta^R, \mathbf{b}^R)$ are Cauchy in $L^\infty\left([0,T^*]; L^2_\sigma(\mathbb{R}^n)\right) \times L^\infty\left([0,T^*]; L^2(\mathbb{R}^n)\right) \times L^\infty\left([0,T^*]; L^2_\sigma(\mathbb{R}^n)\right)$ as $R \to \infty$. □

Now we show the following convergence result.

Proposition 3.5. *For any $s' > n/2 + 1$ with $s' < s$, $(\mathbf{u}^R, \theta^R, \mathbf{b}^R) \to (\mathbf{u}, \theta, \mathbf{b})$ in $L^\infty\big([0,T^*]; H^{s'}_\sigma(\mathbb{R}^n)\big) \times L^\infty\big([0,T^*]; H^{s'}(\mathbb{R}^n)\big) \times L^\infty\big([0,T^*]; H^{s'}_\sigma(\mathbb{R}^n)\big)$.*

Proof. From Proposition 3.4 we conclude that $(\mathbf{u}^R, \theta^R, \mathbf{b}^R) \to (\mathbf{u}, \theta, \mathbf{b})$ strongly in $L^\infty\left([0,T^*]; L^2_\sigma(\mathbb{R}^n)\right) \times L^\infty\left([0,T^*]; L^2(\mathbb{R}^n)\right) \times L^\infty\left([0,T^*]; L^2_\sigma(\mathbb{R}^n)\right)$.

Using Lemma 2.6 for $s' < s$ and $s' > n/2 + 1$,

$$\begin{aligned}\sup_{t\in[0,T^*]} \|\mathbf{b}^R - \mathbf{b}\|_{H^{s'}_\sigma} &\le C \sup_{t\in[0,T^*]}\left(\|\mathbf{b}^R - \mathbf{b}\|^{1-s'/s}_{L^2_\sigma} \|\mathbf{b}^R - \mathbf{b}\|^{s'/s}_{H^s_\sigma}\right)\\ &\le C\Big(\sup_{t\in[0,T^*]} \|\mathbf{b}^R - \mathbf{b}\|_{L^2_\sigma}\Big)^{\frac{1-s'}{s}} \Big(\sup_{t\in[0,T^*]} \|\mathbf{b}^R - \mathbf{b}\|_{H^s_\sigma}\Big)^{s'/s}.\end{aligned}$$

From Propositions 3.3 and 3.4 we obtain

$$\sup_{t\in[0,T^*]} \|\mathbf{b}^R - \mathbf{b}\|_{H^{s'}_\sigma} \le M\Big(\sup_{t\in[0,T^*]} \|\mathbf{b}^R - \mathbf{b}\|_{L^2_\sigma}\Big)^{1-s'/s} \to 0 \quad \text{as } R \to \infty.$$

So we obtain

$$\mathbf{b}^R \to \mathbf{b} \quad \text{in } L^\infty\left([0,T^*]; H^{s'}_\sigma(\mathbb{R}^n)\right), \tag{3.29}$$

Similarly we can show that

$$\theta^R \to \theta \quad \text{in } L^\infty\left([0,T^*]; H^{s'}(\mathbb{R}^n)\right), \quad \mathbf{u}^R \to \mathbf{u} \quad \text{in } L^\infty\left([0,T^*]; H^{s'}_\sigma(\mathbb{R}^n)\right),$$

which gives the desired result. □

Now we use the above result to show the term-wise convergence of the non-linear terms.

Proposition 3.6. *For any $s' > n/2+1$, we have*

$$\begin{gathered}\mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R] \to (\mathbf{u}\cdot\nabla)\mathbf{u},\\ \mathcal{S}_R[(\mathbf{b}^R\cdot\nabla)\mathbf{b}^R] \to (\mathbf{b}\cdot\nabla)\mathbf{b},\\ \mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{b}^R] \to (\mathbf{u}\cdot\nabla)\mathbf{b},\\ \mathcal{S}_R[(\mathbf{b}^R\cdot\nabla)\mathbf{u}^R] \to (\mathbf{b}\cdot\nabla)\mathbf{u}, \quad \textit{strongly in } L^\infty\Big([0,T^*];H^{s'-1}_\sigma(\mathbb{R}^n)\Big),\\ \mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\theta^R] \to (\mathbf{u}\cdot\nabla)\theta, \quad \textit{strongly in } L^\infty\Big([0,T^*];H^{s'-1}(\mathbb{R}^n)\Big),\end{gathered}$$

as $R\to\infty$.

Proof. We prove the result for the non-linear terms $\mathcal{S}_R[(\mathbf{b}^R\cdot\nabla)\mathbf{b}^R]$, when $s' > n/2+1$. One can follow the similar steps to show the convergence of the other non-linear terms in the respective spaces as claimed.

Using the properties of Fourier truncation operator and Remark 2.2 we have

$$\begin{aligned}&\sup_{t\in[0,T^*]}\|\mathcal{S}_R[(\mathbf{b}^R\cdot\nabla)\mathbf{b}^R]-(\mathbf{b}\cdot\nabla)\mathbf{b}\|_{H^{s'-1}_\sigma}\\ &\le \sup_{t\in[0,T^*]}\Big(\|\mathcal{S}_R[(\mathbf{b}^R-\mathbf{b})\cdot\nabla)\mathbf{b}^R]\|_{H^{s'-1}_\sigma}+\|\mathcal{S}_R[(\mathbf{b}\cdot\nabla)(\mathbf{b}^R-\mathbf{b})]\|_{H^{s'-1}_\sigma}\Big)\\ &\le \sup_{t\in[0,T^*]}\Big(C\|[(\mathbf{b}^R-\mathbf{b})\cdot\nabla)\mathbf{b}^R]\|_{H^{s'-1}_\sigma}+C\|[(\mathbf{b}\cdot\nabla)(\mathbf{b}^R-\mathbf{b})]\|_{H^{s'-1}_\sigma}\Big)\\ &\le \sup_{t\in[0,T^*]}\Big(C\|\mathbf{b}^R-\mathbf{b}\|_{H^{s'}_\sigma}\|\mathbf{b}^R\|_{H^{s'}_\sigma}+C\|\mathbf{b}\|_{H^{s'}_\sigma}\|\mathbf{b}^R-\mathbf{b}\|_{H^{s'}_\sigma}\Big)\end{aligned}$$

Clearly from (3.29), Propositions 3.3 and 3.4, the right-hand side tends to 0 as $R\to\infty$. □

Next we show the convergence of time derivatives.

Proposition 3.7. *For any $s' > n/2+1$,*

$$\frac{\partial\mathbf{u}^R}{\partial t}\to\frac{\partial\mathbf{u}}{\partial t} \textit{ and } \frac{\partial\mathbf{b}^R}{\partial t}\to\frac{\partial\mathbf{b}}{\partial t} \quad \textit{strongly in the space } L^\infty\big([0,T^*];H^{s'-1}_\sigma(\mathbb{R}^n)\big)$$

and

$$\frac{\partial\theta^R}{\partial t}\to\frac{\partial\theta}{\partial t} \quad \textit{strongly in } L^\infty\big([0,T^*];H^{s'-1}(\mathbb{R}^n)\big)$$

as $R\to\infty$.

Proof. Takingthe $H^{s'-1}$-norm on both sides of (3.1)-(3.3) and using the properties of the Fourier truncation operator, and Remarks 2.2 and 2.3, we obtain for $s' > n/2+1$,

$$\begin{aligned}\|\frac{\partial\mathbf{u}^R}{\partial t}\|_{H^{s'-1}_\sigma} &\le \|\theta^R e_n\|_{H^{s'-1}}+\|\mathcal{S}_R[(\mathbf{b}^R\cdot\nabla)\mathbf{b}^R]\|_{H^{s'-1}_\sigma}+\|\mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R]\|_{H^{s'-1}_\sigma}\\ &\le C\Big(\|\theta^R\|_{H^{s'}}+\|\mathbf{b}^R\|^2_{H^{s'}_\sigma}+\|\mathbf{u}^R\|^2_{H^{s'}_\sigma}\Big),\end{aligned}$$

$$\begin{aligned}\|\frac{\partial\theta^R}{\partial t}\|_{H^{s'-1}_\sigma} &\le \|\mathbf{u}^R\cdot e_n\|_{H^{s'-1}_\sigma}+\|\mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\theta^R]\|_{H^{s'-1}_\sigma}\\ &\le C\Big(\|\mathbf{u}^R\|_{H^{s'}_\sigma}+\|\mathbf{u}^R\|_{H^{s'}_\sigma}\|\theta^R\|_{H^{s'}}\Big)\end{aligned}$$

and

$$\begin{aligned}\|\frac{\partial \mathbf{b}^R}{\partial t}\|_{H^{s'-1}_\sigma} &\le \|\mathcal{S}_R[(\mathbf{b}^R\cdot\nabla)\mathbf{u}^R]\|_{H^{s'-1}_\sigma} + \|\mathcal{S}_R[(\mathbf{u}^R\cdot\nabla)\mathbf{b}^R]\|_{H^{s'-1}_\sigma}\\ &\le C\|\mathbf{b}^R\|_{H^{s'}_\sigma}\|\mathbf{u}^R\|_{H^{s'}_\sigma}\end{aligned}$$

After adding these inequalities

$$\begin{aligned}&\|\frac{\partial \mathbf{u}^R}{\partial t}\|_{H^{s'-1}_\sigma} + \|\frac{\partial \theta^R}{\partial t}\|_{H^{s'-1}_\sigma} + \|\frac{\partial \mathbf{b}^R}{\partial t}\|_{H^{s'-1}_\sigma}\\ &\le C(\|\mathbf{u}^R\|_{H^{s'}_\sigma} + \|\theta^R\|_{H^{s'}} + \|\mathbf{u}^R\|^2_{H^{s'}_\sigma} + \|\mathbf{b}^R\|^2_{H^{s'}_\sigma}\\ &\quad + \|\mathbf{u}^R\|_{H^{s'}_\sigma}\|\theta^R\|_{H^{s'}} + \|\mathbf{b}^R\|_{H^{s'}_\sigma}\|\mathbf{u}^R\|_{H^{s'}_\sigma})\end{aligned} \tag{3.30}$$

Now taking supremum in both side over $t \in [0,T^*]$, then using Proposition 3.3 and dropping the first two terms of left hand side we obtain

$$\sup_{t\in[0,T^*]}\|\frac{\partial \mathbf{b}^R}{\partial t}\|_{H^{s'-1}_\sigma} \le C(T^*) < \infty.$$

Using the Banach-Alaoglu Theorem (see Robinson [27], Yosida [32]) we can extract a subsequence $R_m \to +\infty$ such that

$$\frac{\partial \mathbf{b}^{R_m}}{\partial t} \overset{*}{\rightharpoonup} \frac{\partial \mathbf{b}}{\partial t} \quad \text{in } L^\infty\big([0,T^*]; H^{s'-1}_\sigma(\mathbb{R}^n)\big). \tag{3.31}$$

Similar argument works for $\frac{\partial \mathbf{u}^R}{\partial t}$ and $\frac{\partial \theta^R}{\partial t}$ as well. Note that $\|\mathbf{u}^R\|_{H^s_\sigma}\|\theta^R\|_{H^s} \to \|\mathbf{u}\|_{H^s_\sigma}\|\theta\|_{H^s}$ and $\|\mathbf{b}^R\|_{H^{s'}_\sigma}\|\mathbf{u}^R\|_{H^{s'}_\sigma} \to \|\mathbf{b}\|_{H^{s'}_\sigma}\|\mathbf{u}\|_{H^{s'}_\sigma}$ because of the strong convergences of $(\mathbf{u}^R,\theta^R,\mathbf{b}^R)$ to $(\mathbf{u},\theta,\mathbf{b})$ in $L^\infty\big([0,T^*]; H^{s'}_\sigma(\mathbb{R}^n)\big) \times L^\infty\big([0,T^*]; H^{s'}(\mathbb{R}^n)\big) \times L^\infty\big([0,T^*]; H^{s'}_\sigma(\mathbb{R}^n)\big)$. Hence all the terms on the right-hand side of (3.30) converge strongly (from Proposition 3.4), we observe that the convergence of the time derivatives are strong. □

Finally, we are ready to prove the solution lies in the desired space.

Proposition 3.8. *For $s > n/2+1$, $(\mathbf{u},\theta,\mathbf{b})$ lie in the space*

$$L^\infty([0,T^*]; H^s_\sigma(\mathbb{R}^n)) \times L^\infty([0,T^*]; H^s(\mathbb{R}^n)) \times L^\infty([0,T^*]; H^s_\sigma(\mathbb{R}^n)).$$

Proof. By the Banach-Alaoglu Theorem, the uniform bounds in Proposition 3.3 guarantee the existence of a subsequence such that

$$\mathbf{u}^{R_m} \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in } L^\infty([0,T^*]; H^s_\sigma(\mathbb{R}^n)); \tag{3.32}$$

$$\theta^{R_m} \overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty([0,T^*]; H^s(\mathbb{R}^n)); \tag{3.33}$$

$$\mathbf{b}^{R_m} \overset{*}{\rightharpoonup} \mathbf{b} \quad \text{in } L^\infty([0,T^*]; H^s_\sigma(\mathbb{R}^n)); \tag{3.34}$$

which guarantees that the limit satisfies

$$\mathbf{u} \in L^\infty([0,T^*]; H^s_\sigma(\mathbb{R}^n)), \quad \theta \in L^\infty([0,T^*]; H^s(\mathbb{R}^n)), \tag{3.35}$$

$$\mathbf{b} \in L^\infty([0,T^*]; H^s_\sigma(\mathbb{R}^n)), \tag{3.36}$$

which is the desired result. □

Now we prove the uniqueness of solutions in the suitable space.

Proposition 3.9. *Let* $(\mathbf{u}_0, \theta_0, \mathbf{b}_0) \in H^s_\sigma(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \times H^s_\sigma(\mathbb{R}^n)$ *for* $s > n/2 + 1$. *Let the solutions* $(\mathbf{u}, \theta, \mathbf{b})$ *of the ideal magnetic Bénard problem* (1.5)-(1.8) *have the regularity*

$$\mathbf{u} \in L^\infty([0,T^*]; H^s_\sigma(\mathbb{R}^n)), \theta \in L^\infty([0,T^*]; H^s(\mathbb{R}^n)), \mathbf{b} \in L^\infty([0,T^*]; H^s_\sigma(\mathbb{R}^n)).$$

Then the solution $(\mathbf{u}, \theta, \mathbf{b})$ *is unique in* $[0, T^*]$.

Proof. The proof of the uniqueness is very similar to the proof of Proposition 3.4. Let $(\mathbf{u}^R, \theta^R, \mathbf{b}^R)$ and $(\mathbf{u}^{R'}, \theta^{R'}, \mathbf{b}^{R'})$ be two solutions of the truncated ideal magnetic Bénard problem (3.1)-(3.3) for $R' > R$. Then from (3.28), we have

$$\sup_{t\in[0,T^*]} \Big(\|\mathbf{u}^R - \mathbf{u}^{R'}\|_{L^2_\sigma} + \|\theta^R - \theta^{R'}\|_{L^2} + \|\mathbf{b}^R - \mathbf{b}^{R'}\|_{L^2_\sigma} \Big) \le \frac{C}{R^\epsilon}.$$

Now letting $R \to R'$ then letting $R \to \infty$ we observe that

$$\mathbf{u}^R \to \mathbf{u}^{R'}, \quad \theta^R \to \theta^{R'} \quad \textit{and} \quad \mathbf{b}^R \to \mathbf{b}^{R'}.$$

Thus we have the uniqueness of the limits $(\mathbf{u}, \theta, \mathbf{b})$. □

We finally prove that the solutions $(\mathbf{u}, \theta, \mathbf{b})$ are continuous in time.

Theorem 3.10. *Let* $s > \frac{n}{2} + 1$, $\mathbf{u}_0 \in H^s_\sigma(\mathbb{R}^n)$, $\theta_0 \in H^s(\mathbb{R}^n)$ *and* $\mathbf{b}_0 \in H^s_\sigma(\mathbb{R}^n)$. *Then there exists a unique strong solution*

$$(\mathbf{u}, \theta, \mathbf{b}) \in C([0,T^*]; H^s_\sigma(\mathbb{R}^n)) \times C([0,T^*]; H^s(\mathbb{R}^n)) \times C([0,T^*]; H^s_\sigma(\mathbb{R}^n))$$

to the system (1.5)-(1.8).

Proof. We shall prove $\mathbf{u} \in C([0,T^*]; H^s_\sigma(\mathbb{R}^n))$. Proofs for θ and $\mathbf{b}$ will follow in the similar manner. Let us first recall that for $s \in \mathbb{R}$, $1 \le p, q < \infty$, the inhomogeneous Besov space $B^s_{p,q}$ is defined as the space of all tempered distributions $f \in S'(\mathbb{R}^n)$ such that

$$B^s_{p,q} = \{ f \in S'(\mathbb{R}^n) : \|f\|_{B^s_{p,q}} < \infty \},$$

where

$$\|f\|_{B^s_{p,q}} = \Big(\sum_{j\ge -1} 2^{jqs} \|\Delta_j f\|^q_{L^p} \Big)^{\frac{1}{q}},$$

where Δ_j is the inhomogeneous Littlewood-Paley operator. We note that $\|f\|_{B^s_{2,2}} \approx \|f\|_{H^s}$. For details see Chapter 3 of [21].

We consider $t_1, t_2 \in [0, T^*]$ such that $0 \le t_1 < t_2 \le T^*$. Then

$$\|\mathbf{u}(t_2) - \mathbf{u}(t_1)\|_{H^s_\sigma} \approx \|\mathbf{u}(t_2) - \mathbf{u}(t_1)\|_{B^s_{2,2}} = \Big\{ \sum_{j\in\mathbb{Z}} \big(2^{js} \|\Delta_j \mathbf{u}(t_2) - \Delta_j \mathbf{u}(t_1)\|_{L^2_\sigma} \big)^2 \Big\}^{1/2}.$$

Let $\epsilon > 0$ be arbitrarily small. As $\mathbf{u} \in L^\infty([0,T^*]; H^s(\mathbb{R}^n))$, there exists an integer $N > 0$ such that

$$\Big\{ \sum_{j\ge N} \Big(2^{js} \|\Delta_j \mathbf{u}(t_2) - \Delta_j \mathbf{u}(t_1)\|_{L^2_\sigma} \Big)^2 \Big\}^{1/2} < \frac{\epsilon}{2}. \tag{3.37}$$

But we have

$$\begin{aligned} &\Big\{ \sum_{j\in\mathbb{Z}} \big(2^{js} \|\Delta_j \mathbf{u}(t_2) - \Delta_j \mathbf{u}(t_1)\|_{L^2_\sigma} \big)^2 \Big\}^{1/2} \\ &= \Big\{ \Big(\sum_{j<N} + \sum_{j\ge N} \Big) \big(2^{js} \|\Delta_j \mathbf{u}(t_2) - \Delta_j \mathbf{u}(t_1)\|_{L^2_\sigma} \big)^2 \Big\}^{1/2}. \end{aligned}$$

Now for $0 \le t_1 < t_2 \le T^*$ we have

$$\begin{aligned}\Delta_j \mathbf{u}(t_2) - \Delta_j \mathbf{u}(t_1) &= \int_{t_1}^{t_2} \frac{\partial}{\partial \tau} \Delta_j \mathbf{u}(\tau)\, d\tau \\ &= \int_{t_1}^{t_2} \Delta_j \mathcal{P}[(\mathbf{b}\cdot\nabla)\mathbf{b} + \theta e_n - (\mathbf{u}\cdot\nabla)\mathbf{u}](\tau)\, d\tau.\end{aligned}$$

So we obtain

$$\begin{aligned}&\sum_{j<N} 2^{2js} \big\|\Delta_j \mathbf{u}(t_2) - \Delta_j \mathbf{u}(t_1)\big\|_{L^2_\sigma}^2 \\ &= \sum_{j<N} 2^{2js} \big\| \int_{t_1}^{t_2} \Delta_j \mathcal{P}[(\mathbf{b}\cdot\nabla)\mathbf{b} + \theta e_n - (\mathbf{u}\cdot\nabla)\mathbf{u}](\tau)\, d\tau \big\|_{L^2}^2 \\ &\le \sum_{j<N} 2^{2js} \Big(\int_{t_1}^{t_2} [\|\Delta_j(\mathbf{b}\cdot\nabla\mathbf{b})\|_{L^2_\sigma} + \|\Delta_j\theta\|_{L^2} + \|\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2_\sigma}]\, d\tau\Big)^2 \qquad (3.38) \\ &= \sum_{j<N} 2^{2j} \Big(\int_{t_1}^{t_2} 2^{j(s-1)} \big[\|\Delta_j(\mathbf{b}\cdot\nabla\mathbf{b})\|_{L^2_\sigma} + \|\Delta_j\theta\|_{L^2} + \|\Delta_j(\mathbf{u}\cdot\nabla\mathbf{u})\|_{L^2_\sigma}\big]\, d\tau\Big)^2 \\ &\le \sum_{j<N} 2^{2j} \int_{t_1}^{t_2} \Big(\|(\mathbf{b}\cdot\nabla)\mathbf{b}\|_{H^{s-1}_\sigma}^2 + \|\theta\|_{H^{s-1}}^2 + \|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{H^{s-1}_\sigma}^2 \Big)\, d\tau\,.\end{aligned}$$

Clearly the individual terms of right-hand side of (3.38) are far less than their $L^\infty([0,T^*]; H^{s-1})$-norm.

As $(\mathbf{u}, \theta, \mathbf{b}) \in L^\infty([0,T^*]; H^s_\sigma) \times L^\infty([0,T^*]; H^s) \times L^\infty([0,T^*]; H^s_\sigma)$ and from Remark 2.2 and Remark 2.3 we obtain

$$\begin{aligned}\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L^\infty([0,T^*];H^{s-1}_\sigma)}^2 &= \Big(\sup_{t\in[0,T^*]} \|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{H^{s-1}_\sigma}\Big)^2 \\ &\le \Big(\sup_{t\in[0,T^*]} \|\mathbf{u}\|_{H^s_\sigma} \cdot \sup_{t\in[0,T^*]} \|\mathbf{u}\|_{H^s_\sigma}\Big)^2 < C_1 < \infty.\end{aligned}$$

Similarly, $\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{L^\infty([0,T^*];H^{s-1}_\sigma)}^2 < C_2$ and from Proposition 3.3, we have $\|\theta\|_{L^\infty([0,T^*];H^{s-1}_\sigma)}^2 < C_3$. Choosing $M = C\cdot\max\{C_1, C_2, C_3\}$, for $|t_2 - t_1| < \frac{\epsilon}{M 2^{2N+1}}$, from (3.38) we obtain

$$\sum_{j<N} 2^{2js} \|\Delta_j \mathbf{u}(t_2) - \Delta_j \mathbf{u}(t_1)\|_{L^2_\sigma}^2 \le M \sum_{j<N} 2^{2j} |t_2 - t_1| \le M 2^{2N} |t_2 - t_1| < \frac{\epsilon}{2}. \quad (3.39)$$

Finally combining (3.37) and (3.39), we conclude that $\mathbf{u} \in C([0,T^*]; H^s_\sigma(\mathbb{R}^n))$. $\square$

4. Blow-up criterion

In this section, we will establish the Blow-up criterion of the local-in-time solution obtained in the previous section. We show that the BMO norms of the vorticity and electrical current inhibit the breakdown of smooth solutions, relaxing the condition on the gradient of temperature, under suitable assumption on the regularity of the initial data.

Theorem 4.1. *Let* $(\mathbf{u}_0, \theta_0, \mathbf{b}_0) \in H^s_\sigma(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \times H^s_\sigma(\mathbb{R}^n)$, $s > \frac{n}{2} + 1$, $n = 2, 3$. *Let* $(\mathbf{u}, \theta, \mathbf{b}) \in C([0,T^*]; H^s_\sigma(\mathbb{R}^n)) \times C([0,T^*]; H^s(\mathbb{R}^n)) \times C([0,T^*]; H^s_\sigma(\mathbb{R}^n))$ *be a* *strong solution of the magnetic Bénard problem* (1.5)-(1.8). *If* $(\mathbf{u},\theta,\mathbf{b})$ *satisfies*

$$\int_0^{T^*} \left(\|\nabla\times\mathbf{u}(\tau)\|_{BMO} + \|\nabla\theta(\tau)\|_{BMO} + \|\nabla\times\mathbf{b}(\tau)\|_{BMO}\right) d\tau < \infty, \tag{4.1}$$

then the solution $(\mathbf{u},\theta,\mathbf{b})$ *can be continuously extended to* $[0,T]$ *for some* $T > T^*$.

Proof. Applying J^s to (1.5)-(1.7) and then taking L^2-inner product with $J^s\mathbf{u}, J^s\theta$ and $J^s\mathbf{b}$ respectively, we obtain, for $s > \frac{n}{2}+1$,

$$\begin{aligned}\Big(\frac{\partial(J^s\mathbf{u})}{\partial t}, J^s\mathbf{u}\Big)_{L^2} &= (J^s[(\mathbf{b}\cdot\nabla)\mathbf{b}], J^s\mathbf{u})_{L^2} - (J^s[(\mathbf{u}\cdot\nabla)\mathbf{u}], J^s\mathbf{u})_{L^2}\\ &\quad - (\nabla J^s p_*, J^s\mathbf{u})_{L^2} + (J^s(\theta e_n), J^s\mathbf{u})_{L^2}\,,\end{aligned} \tag{4.2}$$

$$\Big(\frac{\partial(J^s\theta)}{\partial t}, J^s\theta\Big)_{L^2} = -\,(J^s[(\mathbf{u}\cdot\nabla)\theta], J^s\theta)_{L^2} + (J^s u_n, J^s\theta)_{L^2}\,, \tag{4.3}$$

$$\Big(\frac{\partial(J^s\mathbf{b})}{\partial t}, J^s\mathbf{b}\Big)_{L^2} = (J^s[(\mathbf{b}\cdot\nabla)\mathbf{u}], J^s\mathbf{b})_{L^2} - (J^s[(\mathbf{u}\cdot\nabla)\mathbf{b}], J^s\mathbf{b})_{L^2}\,. \tag{4.4}$$

Using the definition of commutator, (4.2)-(4.4) become

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|J^s\mathbf{u}\|^2_{L^2_\sigma} &= ([J^s,\mathbf{b}]\nabla\mathbf{b}, J^s\mathbf{u})_{L^2} + ((\mathbf{b}\cdot\nabla)J^s\mathbf{b}, J^s\mathbf{u})_{L^2}\\ &\quad - ([J^s,\mathbf{u}]\nabla\mathbf{u}, J^s\mathbf{u})_{L^2} - ((\mathbf{u}\cdot\nabla)J^s\mathbf{u}, J^s\mathbf{u})_{L^2}\\ &\quad - (J^s p_*, J^s\nabla\cdot\mathbf{u})_{L^2} + (J^s(\theta e_n), J^s\mathbf{u})_{L^2}\,,\end{aligned} \tag{4.5}$$

$$\frac{1}{2}\frac{d}{dt}\|J^s\theta\|^2_{L^2} = -\,([J^s,\mathbf{u}]\nabla\theta, J^s\theta)_{L^2} - ((\mathbf{u}\cdot\nabla)J^s\theta, J^s\theta)_{L^2} + (J^s u_n, J^s\theta)_{L^2}\,, \tag{4.6}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|J^s\mathbf{b}\|^2_{L^2_\sigma}\\ &= ([J^s,\mathbf{b}]\nabla\mathbf{u}, J^s\mathbf{b})_{L^2} + ((\mathbf{b}\cdot\nabla)J^s\mathbf{u}, J^s\mathbf{b})_{L^2}\\ &\quad - ([J^s,\mathbf{u}]\nabla\mathbf{b}, J^s\mathbf{b})_{L^2} - ((\mathbf{u}\cdot\nabla)J^s\mathbf{b}, J^s\mathbf{b})_{L^2}\,.\end{aligned} \tag{4.7}$$

Now adding (4.5), (4.6) and (4.7), then applying integration by parts, divergence free condition on $\mathbf{u}$ and $\mathbf{b}$ and the fact $((\mathbf{b}\cdot\nabla)J^s\mathbf{b}, J^s\mathbf{u})_{L^2} = -\,((\mathbf{b}\cdot\nabla)J^s\mathbf{u}, J^s\mathbf{b})_{L^2}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{u}\|^2_{H^s_\sigma} + \|\theta\|^2_{H^s} + \|\mathbf{b}\|^2_{H^s_\sigma}\Big)\\ &= ([J^s,\mathbf{b}]\nabla\mathbf{b}, J^s\mathbf{u})_{L^2} - ([J^s,\mathbf{u}]\nabla\mathbf{u}, J^s\mathbf{u})_{L^2} + (J^s(\theta e_n), J^s\mathbf{u})_{L^2}\\ &\quad - ([J^s,\mathbf{u}]\nabla\theta, J^s\theta)_{L^2} + (J^s u_n, J^s\theta)_{L^2} + ([J^s,\mathbf{b}]\nabla\mathbf{u}, J^s\mathbf{b})_{L^2}\\ &\quad - ([J^s,\mathbf{u}]\nabla\mathbf{b}, J^s\mathbf{b})_{L^2}\,.\end{aligned} \tag{4.8}$$

We estimate each term on the right-hand side of (4.8) separately. Using Lemma 2.8, Remark 2.1, Young's inequality and finally rearranging, we obtain

$$\begin{aligned}&|\,([J^s,\mathbf{b}]\nabla\mathbf{b}, J^s\mathbf{u})_{L^2}\,|\\ &\le \|[J^s,\mathbf{b}]\nabla\mathbf{b}\|_{L^2_\sigma}\|J^s\mathbf{u}\|_{L^2_\sigma}\\ &\le C\Big(\|\nabla\mathbf{b}\|_{L^\infty}\|J^{s-1}\nabla\mathbf{b}\|_{L^2_\sigma} + \|J^s\mathbf{b}\|_{L^2_\sigma}\|\nabla\mathbf{b}\|_{L^\infty}\Big)\|J^s\mathbf{u}\|_{L^2_\sigma}\\ &\le C\Big(\|\nabla\mathbf{b}\|_{L^\infty}\|\nabla\mathbf{b}\|_{H^{s-1}_\sigma} + \|\mathbf{b}\|_{H^s_\sigma}\|\nabla\mathbf{b}\|_{L^\infty}\Big)\|\mathbf{u}\|_{H^s_\sigma}\\ &\le C\left(\|\nabla\mathbf{b}\|_{L^\infty}\|\mathbf{b}\|_{H^s_\sigma}\|\mathbf{u}\|_{H^s_\sigma} + \|\mathbf{b}\|_{H^s_\sigma}\|\mathbf{u}\|_{H^s_\sigma}\|\nabla\mathbf{b}\|_{L^\infty}\right)\\ &\quad \le C\|\nabla\mathbf{b}\|_{L^\infty}(\|\mathbf{u}\|_{H^s_\sigma}\|\mathbf{b}\|_{H^s_\sigma})\end{aligned}$$

$$\le C(\|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla\theta\|_{L^\infty} + \|\nabla \mathbf{b}\|_{L^\infty})(\|\mathbf{u}\|^2_{H^s_\sigma} + \|\theta\|^2_{H^s} + \|\mathbf{b}\|^2_{H^s_\sigma}).$$

Similarly the other commutator terms on the right hand side of (4.8) can be estimated.

$$\begin{aligned}
&|([J^s, \mathbf{u}]\nabla \mathbf{u}, J^s\mathbf{u})_{L^2}| \\
&\le C(\|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla\theta\|_{L^\infty} + \|\nabla \mathbf{b}\|_{L^\infty})(\|\mathbf{u}\|^2_{H^s_\sigma} + \|\theta\|^2_{H^s} + \|\mathbf{b}\|^2_{H^s_\sigma}),\\
&|([J^s, \mathbf{u}]\nabla \theta, J^s\theta)_{L^2}| \\
&\le C(\|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla\theta\|_{L^\infty} + \|\nabla \mathbf{b}\|_{L^\infty})(\|\mathbf{u}\|^2_{H^s_\sigma} + \|\theta\|^2_{H^s} + \|\mathbf{b}\|^2_{H^s_\sigma}),\\
&|([J^s, \mathbf{b}]\nabla \mathbf{u}, J^s\mathbf{b})_{L^2}| \\
&\le C(\|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla\theta\|_{L^\infty} + \|\nabla \mathbf{b}\|_{L^\infty})(\|\mathbf{u}\|^2_{H^s_\sigma} + \|\theta\|^2_{H^s} + \|\mathbf{b}\|^2_{H^s_\sigma}),\\
&|([J^s, \mathbf{u}]\nabla \mathbf{b}, J^s\mathbf{b})_{L^2}| \\
&\le C(\|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla\theta\|_{L^\infty} + \|\nabla \mathbf{b}\|_{L^\infty})(\|\mathbf{u}\|^2_{H^s_\sigma} + \|\theta\|^2_{H^s} + \|\mathbf{b}\|^2_{H^s_\sigma}).
\end{aligned}$$

Now

$$\begin{aligned}
|\,(J^s(\theta e_n), J^s\mathbf{u})_{L^2}\,| &\le \|J^s(\theta e_n)\|_{L^2_\sigma}\|J^s\mathbf{u}\|_{L^2_\sigma} \le \|\theta\|_{H^s}\|\mathbf{u}\|_{H^s_\sigma}\\
&\le C(\|\mathbf{u}\|^2_{H^s_\sigma} + \|\theta\|^2_{H^s} + \|\mathbf{b}\|^2_{H^s_\sigma}),
\end{aligned}$$

and

$$|\,(J^s\mathbf{u}_n, J^s\theta)_{L^2}\,| \le C(\|\mathbf{u}\|^2_{H^s_\sigma} + \|\theta\|^2_{H^s} + \|\mathbf{b}\|^2_{H^s_\sigma}).$$

Combining all the estimates above, from (4.8) after taking $X(t) = \|\mathbf{u}(t)\|^2_{H^s_\sigma} + \|\theta(t)\|^2_{H^s} + \|\mathbf{b}(t)\|^2_{H^s_\sigma}$, for $t \in [0, T^*]$, we obtain

$$\frac{d}{dt}X(t) \le C(\|\nabla \mathbf{u}\|_{L^\infty} + \|\nabla\theta\|_{L^\infty} + \|\nabla \mathbf{b}\|_{L^\infty} + 2)X(t).$$

Standard Gronwall's inequality yields

$$X(t) \le X(0)\exp\Big(C\int_0^t (\|\nabla \mathbf{u}(\tau)\|_{L^\infty} + \|\nabla\theta(\tau)\|_{L^\infty} + \|\nabla \mathbf{b}(\tau)\|_{L^\infty} + 2)\,d\tau\Big).$$

Hence

$$X(t) \le X(0)\exp\Big(C\int_0^t (\|\nabla \mathbf{u}(\tau)\|_{L^\infty} + \|\nabla\theta(\tau)\|_{L^\infty} + \|\nabla \mathbf{b}(\tau)\|_{L^\infty} + 2)\,d\tau\Big). \tag{4.9}$$

By the logarithmic Sobolev inequality in Lemma 2.12, and the fact that singular integral operators of Calderon-Zygmund type are bounded in BMO (i.e. $\|\nabla \mathbf{u}\|_{BMO} \le \|\nabla\times\mathbf{u}\|_{BMO}$), for $s > \frac{n}{2} + 1$ we obtain

$$\begin{aligned}
\|\nabla \mathbf{u}\|_{L^\infty} &\le C\big[1 + \|\nabla\mathbf{u}\|_{BMO}\big(1 + \log^+\|\nabla\mathbf{u}\|_{H^{s-1}_\sigma}\big)\big]\\
&\le C\big[1 + \|\nabla\times\mathbf{u}\|_{BMO}\big(1 + \log^+\|\mathbf{u}\|_{H^s_\sigma}\big)\big]\\
&\le C\big[1 + \|\nabla\times\mathbf{u}\|_{BMO}\big(1 + \frac{1}{2}\log^+\|\mathbf{u}\|^2_{H^s_\sigma}\big)\big]\\
&\le C\big[1 + \|\nabla\times\mathbf{u}\|_{BMO}\big(1 + \frac{1}{2}\log^+\big(\|\mathbf{u}\|^2_{H^s_\sigma} + \|\theta\|^2_{H^s} + \|\mathbf{b}\|^2_{H^s_\sigma}\big)\big)\big]\\
&\le C\big[1 + \|\nabla\times\mathbf{u}\|_{BMO}\big(1 + \log^+\big(\|\mathbf{u}\|^2_{H^s_\sigma} + \|\theta\|^2_{H^s} + \|\mathbf{b}\|^2_{H^s_\sigma}\big)\big)\big].
\end{aligned} \tag{4.10}$$

Similarly we obtain

$$\|\nabla\theta\|_{L^\infty} \le C\big[1 + \|\nabla\theta\|_{BMO}\big(1 + \log^+\big(\|\mathbf{u}\|^2_{H^s_\sigma} + \|\theta\|^2_{H^s} + \|\mathbf{b}\|^2_{H^s_\sigma}\big)\big)\big], \tag{4.11}$$

and

$$\|\nabla \mathbf{b}\|_{L^\infty} \le C\big[1 + \|\nabla\times\mathbf{b}\|_{BMO}\big(1 + \log^+\big(\|\mathbf{u}\|^2_{H^s_\sigma} + \|\theta\|^2_{H^s} + \|\mathbf{b}\|^2_{H^s_\sigma}\big)\big)\big]. \tag{4.12}$$

Now using (4.10), (4.11) and (4.12) in (4.9), for all $t \in [0, T^*]$ we obtain

$$\begin{aligned} X(t) \le X(0)\exp\Big[C\int_0^t \Big\{5 + (\|\nabla\times\mathbf{u}(\tau)\|_{BMO} + \|\nabla\theta(\tau)\|_{BMO} \\ + \|\nabla\times\mathbf{b}(\tau)\|_{BMO})\big(1 + \log^+ X(\tau)\big)\Big\}\,d\tau\Big]. \end{aligned}$$

Taking "log" on both sides we obtain for all $t \in [0, T^*]$,

$$\begin{aligned} \log X(t) \le \log X(0) + C\int_0^t \Big\{5 + (\|\nabla\times\mathbf{u}(\tau)\|_{BMO} + \|\nabla\theta(\tau)\|_{BMO} \\ + \|\nabla\times\mathbf{b}(\tau)\|_{BMO})(1 + \log^+ X(\tau))\Big\}\,d\tau. \end{aligned}$$

Rearranging the terms we have

$$\begin{aligned} \log(eX(t)) \le \log(eX(0)) + CT^* + \int_0^t \Big\{(\|\nabla\times\mathbf{u}(\tau)\|_{BMO} + \|\nabla\theta(\tau)\|_{BMO} \\ + \|\nabla\times\mathbf{b}(\tau)\|_{BMO})(\log(eX(\tau)))\Big\}\,d\tau. \end{aligned}$$

Now Gronwall's inequality yields

$$\begin{aligned} \log(eX(t)) \le \Big\{(\log(eX(0)) + CT^*)\exp\Big(C\int_0^t (\|\nabla\times\mathbf{u}(\tau)\|_{BMO} \\ + \|\nabla\theta(\tau)\|_{BMO} + \|\nabla\times\mathbf{b}(\tau)\|_{BMO})\,d\tau\Big)\Big\}. \end{aligned}$$

Taking supremum over all $t \in [0, T^*]$ we obtain

$$\begin{aligned} \sup_{t\in[0,T^*]} \log X(t) &\le \sup_{t\in[0,T^*]} \log(eX(t)) \\ &\le (\log(eX(0)) + CT^*)\exp\Big(C\int_0^{T^*} (\|\nabla\times\mathbf{u}(\tau)\|_{BMO} \\ &\quad + \|\nabla\theta(\tau)\|_{BMO} + \|\nabla\times\mathbf{b}(\tau)\|_{BMO})\,d\tau\Big). \end{aligned}$$

So finally we acquire

$$\begin{aligned} \sup_{t\in[0,T^*]} X(t) \le e^{(1+CT^*)}X(0)\exp\Big\{\exp\Big(C\int_0^{T^*} (\|\nabla\times\mathbf{u}(\tau)\|_{BMO} \\ + \|\nabla\theta(\tau)\|_{BMO} + \|\nabla\times\mathbf{b}(\tau)\|_{BMO})\,d\tau\Big)\Big\}. \end{aligned}$$

This implies that if

$$\int_0^{T^*} (\|\nabla\times\mathbf{u}(\tau)\|_{BMO} + \|\nabla\theta(\tau)\|_{BMO} + \|\nabla\times\mathbf{b}(\tau)\|_{BMO})\,d\tau < \infty,$$

then by continuation of local solutions, we can extend the solution to $[0, T]$ for some $T > T^*$. □

We now show that the assumption we made in Theorem 4.1 for $\nabla\theta$ can be relaxed completely. In other words, provided $\theta_0 \in H^s(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n), p \ge 2$, the bound on curl of $\mathbf{u}$ and curl of $\mathbf{b}$ are enough to extend the solution continuously to some time $T > T^*$. Before proving this result let us note the following vector identity.

Remark 4.2. Using the vector product we have the following identity:

$$\begin{aligned}\nabla(\mathbf{u}\cdot\nabla\theta) &= (\mathbf{u}\cdot\nabla)\nabla\theta + (\nabla\theta\cdot\nabla)\mathbf{u} + \mathbf{u}\times(\nabla\times\nabla\theta) + \nabla\theta\times(\nabla\times\mathbf{u})\\ &= (\mathbf{u}\cdot\nabla)\nabla\theta + (\nabla\theta\cdot\nabla)\mathbf{u} + \nabla\theta\times(\nabla\times\mathbf{u})\\ &= (\mathbf{u}\cdot\nabla)\nabla\theta + (\nabla\mathbf{u})^t\cdot\nabla\theta\end{aligned}$$

where we have that the curl of the gradient of a scalar function is zero (i.e., $\mathbf{u}\times(\nabla\times\nabla\theta) = 0$) and $(\nabla\mathbf{u})^t\cdot\nabla\theta = (\nabla\theta\cdot\nabla)\mathbf{u} + \nabla\theta\times(\nabla\times\mathbf{u})$.

Theorem 4.3. *Let $s > \frac{n}{2}+1$, $\mathbf{u}_0 \in H^s_\sigma(\mathbb{R}^n)$, $\mathbf{b}_0 \in H^s_\sigma(\mathbb{R}^n)$ and $\theta_0 \in H^s(\mathbb{R}^n)\cap W^{1,p}(\mathbb{R}^n)$, for $2 \le p \le \infty$, $n = 2,3$. Let $(\mathbf{u},\theta,\mathbf{b}) \in C([0,T^*];H^s_\sigma(\mathbb{R}^n)) \times C([0,T^*];H^s(\mathbb{R}^n)) \times C([0,T^*];H^s(\mathbb{R}^n))$ be a strong solution of the ideal magnetic Bénard problem* (1.5)-(1.8)*. Then*

$$\int_0^{T^*} \|\nabla\times\mathbf{u}(\tau)\|_{BMO} + \|\nabla\times\mathbf{b}(\tau)\|_{BMO}\,d\tau < \infty$$

guarantees that the solution can be extended continuously to $[0,T]$ for some $T > T^*$.

Proof. We rewrite equation (1.6) as

$$\frac{\partial\theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta = \mathbf{u}_n,$$

and apply the gradient operator $\nabla = (\partial_{x_1},\dots,\partial_{x_n})$ on both sides and take L^2-inner product with $\nabla\theta|\nabla\theta|^{p-2}$ to obtain

$$\Big(\frac{\partial}{\partial t}(\nabla\theta), \nabla\theta|\nabla\theta|^{p-2}\Big) + \big(\nabla(\mathbf{u}\cdot\nabla\theta), \nabla\theta|\nabla\theta|^{p-2}\big) = \big(\nabla\mathbf{u}_n, \nabla\theta|\nabla\theta|^{p-2}\big).$$

Using the vector identity in Remark 4.2 we obtain

$$\begin{aligned}&\Big(\frac{\partial}{\partial t}(\nabla\theta), \nabla\theta|\nabla\theta|^{p-2}\Big) + \big((\nabla\mathbf{u})^t\cdot\nabla\theta, \nabla\theta|\nabla\theta|^{p-2}\big) + \big((\mathbf{u}\cdot\nabla)\nabla\theta, \nabla\theta|\nabla\theta|^{p-2}\big)\\ &= \big(\nabla\mathbf{u}_n, \nabla\theta|\nabla\theta|^{p-2}\big).\end{aligned} \tag{4.13}$$

We calculate each term separately. The first term of left-hand side of (4.13) gives

$$\Big(\frac{\partial}{\partial t}(\nabla\theta), \nabla\theta|\nabla\theta|^{p-2}\Big) = \frac{1}{p}\int_{\mathbb{R}^n}\frac{\partial}{\partial t}|\nabla\theta|^p\,dx = \frac{1}{p}\frac{d}{dt}\|\nabla\theta\|^p_{L^p},$$

and

$$\begin{aligned}\big((\nabla\mathbf{u})^t\cdot\nabla\theta, \nabla\theta|\nabla\theta|^{p-2}\big) &= \int_{\mathbb{R}^n}(\nabla\mathbf{u})^t\cdot\nabla\theta\cdot\nabla\theta|\nabla\theta|^{p-2}\,dx\\ &\le \int_{\mathbb{R}^n}(\nabla\mathbf{u})^t\cdot|\nabla\theta|^p \le \|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\theta\|^p_{L^p}.\end{aligned}$$

Integrating by parts and the divergence free condition of $\mathbf{u}$, from the third term of (4.13), we have

$$\begin{aligned}\big((\mathbf{u}\cdot\nabla)\nabla\theta, \nabla\theta|\nabla\theta|^{p-2}\big) &= \int_{\mathbb{R}^n}(\mathbf{u}\cdot\nabla)\nabla\theta\cdot\nabla\theta|\nabla\theta|^{p-2}\,dx\\ &= \frac{1}{p}\int_{\mathbb{R}^n}\mathbf{u}\cdot\nabla|\nabla\theta|^p\,dx\\ &= -\frac{1}{p}\int_{\mathbb{R}^n}(\nabla\cdot\mathbf{u})\cdot|\nabla\theta|^p\,dx = 0.\end{aligned}$$

Now

$$\begin{aligned} |(\nabla \mathbf{u}_n, \nabla\theta|\nabla\theta|^{p-2})| &= \Big|\int_{\mathbb{R}^n} \nabla \mathbf{u}_n \cdot \nabla\theta|\nabla\theta|^{p-2}\,dx\Big| \le \int_{\mathbb{R}^n} |\nabla \mathbf{u}_n||\nabla\theta|^{p-1}\,dx \\ &\le \Big(\int_{\mathbb{R}^n} |\nabla \mathbf{u}_n|^p\Big)^{1/p}\Big(\int_{\mathbb{R}^n} (|\nabla\theta|^{p-1})^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} \\ &\le \|\nabla \mathbf{u}_n\|_{L^p}\|\nabla\theta\|_{L^p}^{p-1} \\ &\le \frac{1}{p}\big(\|\nabla \mathbf{u}_n\|_{L^p}^p + (p-1)\|\nabla\theta\|_{L^p}^p\big). \end{aligned}$$

From the Gagliardo-Nirenberg interpolation inequality (see Lemma 2.7), while $j = 1$, $m = 3$, $r = 2$, $q = 2$, we have: for $n = 2$,

$$\|\nabla \mathbf{u}_2\|_{L^p} \le C\|\mathbf{u}_2\|_{L^2}^{\frac{2+p}{3p}}\|\mathbf{u}_2\|_{H^3}^{\frac{2p-2}{3p}}, \quad p \ge 2, \tag{4.14}$$

and for $n = 3$,

$$\|\nabla \mathbf{u}_3\|_{L^p} \le C\|\mathbf{u}_3\|_{L^2}^{\frac{6+p}{6p}}\|\mathbf{u}_3\|_{H^3}^{\frac{5p-6}{6p}}, \quad p \ge 2. \tag{4.15}$$

From the term-wise estimates of (4.13), when $n = 3$, we obtain

$$\frac{d}{dt}\|\nabla\theta\|_{L^p}^p \le p\|\nabla \mathbf{u}\|_{L^\infty}\|\nabla\theta\|_{L^p}^p + C^p\|\mathbf{u}_3\|_{L^2}^{\frac{6+p}{6}}\|\mathbf{u}_3\|_{H^3}^{\frac{5p-6}{6}} + (p-1)\|\nabla\theta\|_{L^p}^p,$$

which further gives by Gronwall's inequality,

$$\begin{aligned} \|\nabla\theta\|_{L^p}^p &\le \Big(\|\nabla\theta_0\|_{L^p}^p + C^p\int_0^t \|\mathbf{u}_3(\tau)\|_{L^2}^{\frac{6+p}{6}}\|\mathbf{u}_3(\tau)\|_{H^3}^{\frac{5p-6}{6}}\,d\tau\Big) \\ &\quad\times \exp\Big(\int_0^t (p\|\nabla \mathbf{u}(\tau)\|_{L^\infty} + p - 1)\,d\tau\Big) \\ &\le \Big[\|\nabla\theta_0\|_{L^p}^p + C^p\,T^*\Big(\sup_{t\in[0,T^*]}\|\mathbf{u}_3\|_{L^2}\Big)^{\frac{6+p}{6}}\Big(\sup_{t\in[0,T^*]}\|\mathbf{u}_3\|_{H^3}\Big)^{\frac{5p-6}{6}}\Big] \\ &\quad\times \exp(pT^*)\exp\Big(p\int_0^t \|\nabla \mathbf{u}(\tau)\|_{L^\infty}\,d\tau\Big). \end{aligned}$$

Therefore, when $n = 3$, we obtain

$$\begin{aligned} \|\nabla\theta\|_{L^p} &\le \Big[\|\nabla\theta_0\|_{L^p}^p + C^p\,T^*\Big(\sup_{t\in[0,T^*]}\|\mathbf{u}_3\|_{L^2}\Big)^{\frac{6+p}{6}}\Big(\sup_{t\in[0,T^*]}\|\mathbf{u}_3\|_{H^3}\Big)^{\frac{5p-6}{6}}\Big]^{1/p} \\ &\quad\times \exp(T^*)\exp\Big(\int_0^t \|\nabla \mathbf{u}(\tau)\|_{L^\infty}\,d\tau\Big) \\ &\le \Big[\|\nabla\theta_0\|_{L^p} + C{T^*}^{1/p}\Big(\sup_{t\in[0,T^*]}\|\mathbf{u}_3\|_{L^2}\Big)^{\frac{6+p}{6p}}\Big(\sup_{t\in[0,T^*]}\|\mathbf{u}_3\|_{H^3}\Big)^{\frac{5p-6}{6p}}\Big] \\ &\quad\times \exp(T^*)\exp\Big(\int_0^t \|\nabla \mathbf{u}(\tau)\|_{L^\infty}\,d\tau\Big). \end{aligned}$$

Since the L^2-energy estimate and H^3-energy estimate of $\mathbf{u}$ are finite by Propositions 3.2 and 3.3, letting $p \to \infty$, we finally obtain

$$\|\nabla\theta\|_{L^\infty} \le C(T^*)\|\nabla\theta_0\|_{L^\infty}\exp\Big(\int_0^t \|\nabla \mathbf{u}(\tau)\|_{L^\infty}\,d\tau\Big).$$

Similarly, the case $n = 2$ (from (4.14)) will also yield the same above estimate. Note that, by Lemma 2.12, and properties of the BMO space, we further have

$$\|\nabla\theta\|_{L^\infty} \le \|\nabla\theta_0\|_{L^\infty} \exp\Big(C\int_0^t \big(1 + \|\nabla\times\mathbf{u}(\tau)\|_{BMO}\big(1+\log^+\|\mathbf{u}(\tau)\|_{H^s_\sigma}\big)\big)\,d\tau\Big).$$

As $\theta_0 \in H^s(\mathbb{R}^n)\cap W^{1,p}(\mathbb{R}^n)$, $2 \le p \le \infty$ and $\sup_{t\in[0,T^*]}\|\mathbf{u}\|_{H^s_\sigma}$ is bounded for $s > n/2+1$, we have

$$\|\nabla\theta\|_{L^\infty} \le C\exp\Big(\int_0^{T^*}\|\nabla\times\mathbf{u}(\tau)\|_{BMO}\,d\tau\Big), \tag{4.16}$$

where $C = C(\|\nabla\theta_0\|_{L^\infty}, \|\mathbf{u}\|_{H^s_\sigma}, T^*)$. From the assumption

$$\int_0^{T^*}\|\nabla\times\mathbf{u}(\tau)\|_{BMO}\,d\tau < \infty,$$

the estimate in (4.16) is bounded. Hence,

$$\|\nabla\theta\|_{BMO} \le 2\|\nabla\theta\|_{L^\infty} \le C < \infty.$$

So the bound on BMO norms of vorticity and electrical current are enough to guarantee that the solution can be extended to $[0,T]$ for some $T > T^*$ provided

$$\theta_0 \in H^s(\mathbb{R}^n)\cap W^{1,p}(\mathbb{R}^n).$$

This proves the desired result. □

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