Electronic Journal of Differential Equations, Vol. 2020 (2020), No. 96, pp. 1–26.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

TIME DISCRETIZATION OF AN ABSTRACT PROBLEM FROM LINEARIZED EQUATIONS OF A COUPLED SOUND AND HEAT FLOW

SHUNSUKE KURIMA

Abstract. Recently, a time discretization of simultaneous abstract evolution equations applied to parabolic-hyperbolic phase-field systems has been studied. This article focuses on a time discretization of an abstract problem that has application to linearized equations of coupled sound and heat flow. As examples, we also study some parabolic-hyperbolic phase-field systems.

1. Introduction

Matsubara-Yokota [10] established the existence, uniqueness, and regularity of solutions to the initial-boundary value problem for the linearized equations of coupled sound and heat flow

$$
\begin{gathered}
\theta_t + (\gamma-1)\varphi_t - \sigma\Delta\theta = 0 \quad \text{in } \Omega\times(0,\infty), \\
\varphi_{tt} - c^2\Delta\varphi - m^2\varphi = -c^2\Delta\theta \quad \text{in } \Omega\times(0,\infty), \\
\theta = \varphi = 0 \quad \text{on } \partial\Omega\times(0,\infty), \\
\theta(0)=\theta_0, \quad \varphi(0)=\varphi_0, \quad \varphi_t(0)=v_0 \quad \text{in } \Omega\,,
\end{gathered}
$$

by applying the Hille-Yosida theorem, where $c>0$, $\sigma>0$, $m\in\mathbb{R}$ and $\gamma>1$ are constants, $\Omega\subset\mathbb{R}^d$ $(d\in\mathbb{N})$ is a domain with smooth bounded boundary $\partial\Omega$, and θ_0, φ_0, v_0 are given functions.

Reference [9] presents the existence of solutions to the initial valued problem for the simultaneous abstract evolution equation

$$
\begin{gathered}
\frac{d\theta}{dt} + \frac{d\varphi}{dt} + A_1\theta = f \quad \text{in } (0,T), \\
L\frac{d^2\varphi}{dt^2} + B\frac{d\varphi}{dt} + A_2\varphi + \Phi\varphi + \mathcal{L}\varphi = \theta \quad \text{in } (0,T), \\
\theta(0)=\theta_0, \quad \varphi(0)=\varphi_0, \quad \frac{d\varphi}{dt}(0)=v_0
\end{gathered}
$$

where $T>0$, $L : H\to H$ is a bounded linear positive selfadjoint operator, $B : D(B)\subset H\to H$, $A_j : D(A_j)\subset H\to H$ $(j=1,2)$ are linear maximal monotone selfadjoint operators, H and V are real Hilbert spaces satisfying $V\subset H$, V_j $(j=$

2010 *Mathematics Subject Classification.* 35A35, 47N20, 35G30, 35L70.
Key words and phrases. Simultaneous evolution equations; linearized equations; coupled sound and heat flow; time discretization; error estimate.

Submitted January 20, 2020. Published September 19, 2020.

1, 2) are linear subspaces of V satisfying $D(A_j) \subset V_j$ $(j = 1, 2)$, $\Phi : D(\Phi) \subset H \to H$ is a maximal monotone operator, $\mathcal{L} : H \to H$ is a Lipschitz continuous operator, $f : (0, T) \to H$ and $\theta_0 \in V_1$, $\varphi_0, v_0 \in V_2$ are given. By employing a time discretization scheme in [3, 4], an error estimate for the difference between continuous and discrete solutions was presented.

Moreover in [9], assuming conditions from [3, Section 2] and [4, 5, 6, 7, 12, 13], some parabolic-hyperbolic phase-field systems are contained as examples under homogeneous Dirichlet–Dirichlet boundary conditions, homogeneous Dirichlet-Neumann boundary conditions, homogeneous Neumann-Dirichlet boundary conditions, or homogeneous Neumann-Neumann boundary conditions.

In this article we consider the existence and uniqueness of solutions of the abstract problem

$$
\begin{gathered}
\frac{d\theta}{dt} + \eta \frac{d\varphi}{dt} + A_1\theta = 0 \quad \text{in } (0, T), \\
L\frac{d^2\varphi}{dt^2} + B_1\frac{d\varphi}{dt} + A_2\varphi + \Phi\varphi + \mathcal{L}\varphi = B_2\theta \quad \text{in } (0, T), \\
\theta(0) = \theta_0,\ \varphi(0) = \varphi_0,\ \frac{d\varphi}{dt}(0) = v_0,
\end{gathered}
\tag{1.1}
$$

where $T > 0$, $\eta > 0$, $L : H \to H$ is a bounded linear positive selfadjoint operator, $B_j : D(B_j) \subset H \to H$, $A_j : D(A_j) \subset H \to H$ $(j = 1, 2)$ are linear maximal monotone selfadjoint operators, $D(A_j) \subset V$ $(j = 1, 2)$, $\Phi : D(\Phi) \subset H \to H$ is a maximal monotone operator, $\mathcal{L} : H \to H$ is a Lipschitz continuous operator, $\theta_0, \varphi_0, v_0 \in V$ are given. Moreover, we study the problem

$$
\begin{gathered}
\delta_h\theta_n + \eta\delta_h\varphi_n + A_1\theta_{n+1} = 0, \\
Lz_{n+1} + B_1v_{n+1} + A_2\varphi_{n+1} + \Phi\varphi_{n+1} + \mathcal{L}\varphi_{n+1} = B_2\theta_{n+1}, \\
z_0 = z_1,\ z_{n+1} = \delta_h v_n, \\
v_{n+1} = \delta_h\varphi_n
\end{gathered}
\tag{1.2}
$$

for $n = 0, \dots, N-1$, where $h = \frac{T}{N}$, $N \in \mathbb{N}$,

$$
\delta_h\theta_n := \frac{\theta_{n+1} - \theta_n}{h}, \quad \delta_h\varphi_n := \frac{\varphi_{n+1} - \varphi_n}{h}, \quad \delta_h v_n := \frac{v_{n+1} - v_n}{h}. \tag{1.3}
$$

Putting

$$
\widehat{\theta}_h(0) := \theta_0, \quad \frac{d\widehat{\theta}_h}{dt}(t) := \delta_h\theta_n, \quad \widehat{\varphi}_h(0) := \varphi_0, \quad \frac{d\widehat{\varphi}_h}{dt}(t) := \delta_h\varphi_n, \tag{1.4}
$$

$$
\widehat{v}_h(0) := v_0,\ \frac{d\widehat{v}_h}{dt}(t) := \delta_h v_n, \tag{1.5}
$$

$$
\overline{\theta}_h(t) := \theta_{n+1},\ \overline{z}_h(t) := z_{n+1}, \quad \overline{\varphi}_h(t) := \varphi_{n+1},\ \overline{v}_h(t) := v_{n+1} \tag{1.6}
$$

for a.a. $t \in (nh, (n+1)h)$, $n = 0, \dots, N-1$, we can rewrite (1.2) as

$$
\begin{gathered}
\frac{d\widehat{\theta}_h}{dt} + \eta\frac{d\widehat{\varphi}_h}{dt} + A_1\overline{\theta}_h = 0 \quad \text{in } (0, T), \\
L\overline{z}_h + B_1\overline{v}_h + A_2\overline{\varphi}_h + \Phi\overline{\varphi}_h + \mathcal{L}\overline{\varphi}_h = B_2\overline{\theta}_h \quad \text{in } (0, T), \\
\overline{z}_h = \frac{d\widehat{v}_h}{dt},\ \overline{v}_h = \frac{d\widehat{\varphi}_h}{dt} \quad \text{in } (0, T), \\
\widehat{\theta}_h(0) = \theta_0, \quad \widehat{\varphi}_h(0) = \varphi_0, \quad \widehat{v}_h(0) = v_0.
\end{gathered}
\tag{1.7}
$$

We will use the following assumptions:

(A1) V and H are real Hilbert spaces satisfying $V \subset H$ with dense, continuous and compact embedding. Moreover, the inclusions $V \subset H \subset V^*$ hold by identifying H with its dual space H^*, where V^* is the dual space of V.

(A2) $L : H \to H$ is a bounded linear operator fulfilling

$$(Lw, z)_H = (w, Lz)_H \text{ for all } w, z \in H, \quad (Lw, w)_H \geq c_L \|w\|_H^2$$

for all $w \in H$, where $c_L > 0$ is a constant.

(A3) $A_j : D(A_j) \subset H \to H$ $(j = 1, 2)$ are linear maximal monotone selfadjoint operators, where $D(A_j)$ $(j = 1, 2)$ are linear subspaces of H and $D(A_j) \subset V$ $(j = 1, 2)$. Moreover, there exist bounded linear monotone operators $A_j^* : V \to V^*$ $(j = 1, 2)$ such that

$$\langle A_j^* w, z\rangle_{V^*,V} = \langle A_j^* z, w\rangle_{V^*,V} \quad \text{for all } w, z \in V,$$
$$A_j^* w = A_j w \quad \text{for all } w \in D(A_j).$$

Moreover, for all $\alpha > 0$ and for $j = 1, 2$ there exists $\omega_{j,\alpha} > 0$ such that

$$\langle A_j^* w, w\rangle_{V^*,V} + \alpha \|w\|_H^2 \geq \omega_{j,\alpha} \|w\|_V^2 \quad \text{for all } w \in V.$$

(A4) $B_j : D(B_j) \subset H \to H$ $(j = 1, 2)$ are linear maximal monotone selfadjoint operators, where $D(B_j)$ $(j = 1, 2)$ are linear subspaces of H, satisfying $D(A_1) \subset D(B_2)$ and

$$D(B_1) \cap D(A_2) \neq \emptyset,$$
$$(B_1 w, A_2 w)_H \geq 0 \quad \text{for all } w \in D(B_1) \cap D(A_2),$$
$$(B_2 w, A_1 w)_H \geq 0 \quad \text{for all } w \in D(A_1),$$
$$(B_1 w, A_2 z)_H = (B_1 z, A_2 w)_H \quad \text{for all } w, z \in D(B_1) \cap D(A_2).$$

(A5) There exists a constant $C_{A_1,B_2} > 0$ such that

$$\|B_2 \theta\|_H \leq C_{A_1,B_2} (\|A_1 \theta\|_H + \|\theta\|_H) \quad \text{for all } \theta \in D(A_1).$$

(A6) $\Phi : D(\Phi) \subset H \to H$ is a maximal monotone operator satisfying $\Phi(0) = 0$ and $V \subset D(\Phi)$. Moreover, there exist constants $p, q, C_\Phi > 0$ such that

$$\|\Phi w - \Phi z\|_H \leq C_\Phi (1 + \|w\|_V^p + \|z\|_V^q) \|w - z\|_V \quad \text{for all } w, z \in V.$$

(A7) There exists a lower semicontinuous convex function $i : V \to \{x \in \mathbb{R} \mid x \geq 0\}$ such that $(\Phi w, w - z)_H \geq i(w) - i(z)$ for all $w, z \in V$.

(A8) $\Phi_\lambda(0) = 0$, $(\Phi_\lambda w, B_1 w)_H \geq 0$ for all $w \in D(B_1)$, $(\Phi_\lambda w, A_2 w)_H \geq 0$ for all $w \in D(A_2)$, where $\lambda > 0$ and $\Phi_\lambda : H \to H$ is the Yosida approximation of Φ.

(A9) $B_j^* : V \to V^*$ $(j = 1, 2)$ are bounded linear monotone operators fulfilling

$$\langle B_j^* w, z\rangle_{V^*,V} = \langle B_j^* z, w\rangle_{V^*,V} \quad \text{for all } w, z \in V,$$
$$B_j^* w = B_j w \quad \text{for all } w \in D(B_j) \cap V.$$

(A10) For all $g \in H$, $a, b, c, d, d' > 0$, $\lambda > 0$, if there exists $\varphi_\lambda \in V$ such that

$$L\varphi_\lambda + aB_1^* \varphi_\lambda + bA_2^* \varphi_\lambda + c\Phi_\lambda \varphi_\lambda + d\mathcal{L}\varphi_\lambda + d' B_2 (I + hA_1)^{-1} \varphi_\lambda = g$$

in V^*, then it follows that $\varphi_\lambda \in D(B_1) \cap D(A_2)$ and

$$L\varphi_\lambda + aB_1 \varphi_\lambda + bA_2 \varphi_\lambda + c\Phi_\lambda \varphi_\lambda + d\mathcal{L}\varphi_\lambda + d' B_2 (I + hA_1)^{-1} \varphi_\lambda = g$$

in H.

(A11) $\mathcal{L} : H \to H$ is a Lipschitz continuous operator with Lipschitz constant $C_{\mathcal{L}} > 0$.

(A12) $\theta_0 \in D(A_1)$, $A_1\theta_0 \in V$, $\varphi_0 \in D(B_1) \cap D(A_2)$, $v_0 \in D(B_1) \cap V$.

We set conditions (A2) and (A3) as in [3, Section 2]. Condition (A10) is equivalent to the elliptic regularity in some cases (see Section 2). We set conditions (A6)–(A8) and (A11) keeping mind typical examples of not only linearized equations of coupled sound and heat flow, but also of parabolic-hyperbolic phase-field systems; see Section 2 and and assumptions in [4, 5, 6, 7, 12, 13].

Remark 1.1. Owing to (1.4)-(1.6), the reader can check directly the following identities:

$$\|\widehat{\varphi}_h\|_{L^\infty(0,T;V)} = \max\{\|\varphi_0\|_V, \|\overline{\varphi}_h\|_{L^\infty(0,T;V)}\}, \tag{1.8}$$

$$\|\widehat{v}_h\|_{L^\infty(0,T;V)} = \max\{\|v_0\|_V, \|\overline{v}_h\|_{L^\infty(0,T;V)}\}, \tag{1.9}$$

$$\|\widehat{\theta}_h\|_{L^\infty(0,T;V)} = \max\{\|\theta_0\|_V, \|\overline{\theta}_h\|_{L^\infty(0,T;V)}\}, \tag{1.10}$$

$$\|\overline{\varphi}_h - \widehat{\varphi}_h\|_{L^\infty(0,T;V)} = h\Big\|\frac{d\widehat{\varphi}_h}{dt}\Big\|_{L^\infty(0,T;V)} = h\|\overline{v}_h\|_{L^\infty(0,T;V)}, \tag{1.11}$$

$$\|\overline{v}_h - \widehat{v}_h\|_{L^\infty(0,T;H)} = h\Big\|\frac{d\widehat{v}_h}{dt}\Big\|_{L^\infty(0,T;H)} = h\|\overline{z}_h\|_{L^\infty(0,T;H)}, \tag{1.12}$$

$$\|\overline{\theta}_h - \widehat{\theta}_h\|^2_{L^2(0,T;V)} = \frac{h^2}{3}\Big\|\frac{d\widehat{\theta}_h}{dt}\Big\|^2_{L^2(0,T;V)}. \tag{1.13}$$

Definition 1.2. A pair (θ,φ) with

$$\theta \in H^1(0,T;V) \cap L^\infty(0,T;V) \cap L^\infty(0,T;D(A_1)),$$
$$\varphi \in W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V) \cap L^2(0,T;D(A_2)),$$
$$\frac{d\varphi}{dt} \in L^2(0,T;D(B_1)), \quad \Phi\varphi \in L^\infty(0,T;H)$$

is called a solution of (1.1) if (θ,φ) satisfies

$$\frac{d\theta}{dt} + \eta\frac{d\varphi}{dt} + A_1\theta = 0 \quad \text{in } H \quad \text{a.e. on } (0,T), \tag{1.14}$$

$$L\frac{d^2\varphi}{dt^2} + B_1\frac{d\varphi}{dt} + A_2\varphi + \Phi\varphi + \mathcal{L}\varphi = B_2\theta \quad \text{in } H \quad \text{a.e. on } (0,T), \tag{1.15}$$

$$\theta(0) = \theta_0, \quad \varphi(0) = \varphi_0, \quad \frac{d\varphi}{dt}(0) = v_0 \quad \text{in } H. \tag{1.16}$$

Our main results read as follows.

Theorem 1.3. *Assume that* (A1)–(A12) *hold. Then there exists $h_0 \in (0,1)$ such that for all $h \in (0,h_0)$ there exists a unique solution $(\theta_{n+1},\varphi_{n+1})$ of* (1.2) *satisfying*

$$\theta_{n+1} \in D(A_1), \quad \varphi_{n+1} \in D(B_1) \cap D(A_2) \quad \textit{for } n = 0,\dots,N-1.$$

Theorem 1.4. *Assume that* (A1)–(A12) *hold. Then there exists a unique solution (θ,φ) of* (1.1).

Theorem 1.5. *Let h_0 be as in Theorem 1.3, and assume that* (A1)–(A12) *hold. Then there exist constants $h_{00} \in (0,h_0)$ and $M = M(T) > 0$ such that*

$$\|L^{1/2}(\widehat{v}_h - v)\|_{L^\infty(0,T;H)} + \|B_1^{1/2}(\overline{v}_h - v)\|_{L^2(0,T;H)} + \|\widehat{\varphi}_h - \varphi\|_{L^\infty(0,T;V)}$$
$$+ \|\widehat{\theta}_h - \theta\|_{L^\infty(0,T;H)} + \|\overline{\theta}_h - \theta\|_{L^2(0,T;V)}$$

$$
\begin{aligned}
&+ \|B_2^{1/2}(\widehat{\theta}_h - \theta)\|_{L^\infty(0,T;H)} + \int_0^T (B_2(\overline{\theta}_h(t) - \theta(t)), A_1(\overline{\theta}_h(t) - \theta(t)))_H \, dt \\
&\le M h^{1/2}
\end{aligned}
$$

for all $h \in (0, h_{00})$*, where* $v = \frac{d\varphi}{dt}$.

This article is organized as follows. In Section 2 we give the linearized equations of coupled sound and heat flow and some parabolic-hyperbolic phase-field systems as examples. In Section 3 we derive existence of solutions to (1.2). In Section 4 we prove that there exists a solution of (1.1). In Section 5 we establish uniqueness for (1.1). In Section 6 we obtain error estimates between solutions of (1.1) and solutions of (1.7).

2. Examples

Example 2.1. We have the problem

$$
\begin{gathered}
\theta_t + (\gamma - 1)\varphi_t - \sigma\Delta\theta = 0 \quad \text{in } \Omega \times (0,T), \\
\varphi_{tt} - c^2\Delta\varphi - m^2\varphi = -c^2\Delta\theta \quad \text{in } \Omega \times (0,T), \\
\theta = \varphi = 0 \quad \text{on } \partial\Omega \times (0,T), \\
\theta(0) = \theta_0, \quad \varphi(0) = \varphi_0, \quad \varphi_t(0) = v_0 \quad \text{in } \Omega,
\end{gathered}
\tag{2.1}
$$

where $c > 0$, $\sigma > 0$, $m \in \mathbb{R}$, $\gamma > 1$, $T > 0$ are constants and $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, under the assumption that

$$
\theta_0 \in H^2(\Omega) \cap H_0^1(\Omega), -\Delta\theta_0 \in H_0^1(\Omega), \varphi_0 \in H^2(\Omega) \cap H_0^1(\Omega), v_0 \in H_0^1(\Omega).
$$

Indeed, putting

$$
\begin{gathered}
V := H_0^1(\Omega), \ H := L^2(\Omega), \quad L := I : H \to H, \\
A_1 := -\sigma\Delta : D(A_1) := H^2(\Omega) \cap H_0^1(\Omega) \subset H \to H, \\
B_1 := 0 : D(B_1) := H \to H, \\
A_2 := -c^2\Delta : D(A_2) := H^2(\Omega) \cap H_0^1(\Omega) \subset H \to H, \\
B_2 := -c^2\Delta : D(B_2) := H^2(\Omega) \cap H_0^1(\Omega) \subset H \to H,
\end{gathered}
$$

and defining the operators $A_1^* : V \to V^*$, $B_1^* : V \to V^*$, $A_2^* : V \to V^*$, $\Phi : D(\Phi) \subset H \to H$, $\mathcal{L} : H \to H$, $B_2^* : V \to V^*$ as

$$
\begin{gathered}
\langle A_1^* w, z \rangle_{V^*,V} := \sigma \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V, \\
\langle B_1^* w, z \rangle_{V^*,V} := 0 \quad \text{for } w, z \in V, \\
\langle A_2^* w, z \rangle_{V^*,V} := c^2 \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V, \\
\Phi z := 0 \quad \text{for } z \in D(\Phi) := H, \\
\mathcal{L} z := -m^2 z \quad \text{for } z \in H, \\
\langle B_2^* w, z \rangle_{V^*,V} := c^2 \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V,
\end{gathered}
$$

we can check that (A1)–(A12) hold. Similarly, we can confirm that the homogeneous Neumann-Neumann problem is an example.

Example 2.2. Now we have the problem

$$\begin{gathered} \theta_t + (\gamma - 1)\varphi_t - \sigma\Delta\theta = 0 \quad \text{in } \Omega \times (0,T), \\ \varphi_{tt} + \varepsilon\varphi_t - c^2\Delta\varphi + \beta(\varphi) + \pi(\varphi) = -c^2\Delta\theta \quad \text{in } \Omega \times (0,T), \\ \theta = \varphi = 0 \quad \text{on } \partial\Omega \times (0,T), \\ \theta(0) = \theta_0,\ \varphi(0) = \varphi_0,\ \varphi_t(0) = v_0 \quad \text{in } \Omega, \end{gathered} \tag{2.2}$$

where $c > 0$, $\sigma > 0$, $\varepsilon \geq 0$, $\gamma > 1$, $T > 0$ are constants and $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, under the following conditions:

- (A13) $\beta : \mathbb{R} \to \mathbb{R}$ is a single-valued maximal monotone function and there exists a proper differentiable (lower semicontinuous) convex function $\widehat{\beta} : \mathbb{R} \to [0, +\infty)$ such that $\widehat{\beta}(0) = 0$ and $\beta(r) = \widehat{\beta}\,'(r) = \partial\widehat{\beta}(r)$ for all $r \in \mathbb{R}$, where $\widehat{\beta}\,'$ and $\partial\widehat{\beta}$, respectively, are the differential and subdifferential of $\widehat{\beta}$.
- (A14) $\beta \in C^2(\mathbb{R})$. Moreover, there exists a constant $C_\beta > 0$ such that $|\beta''(r)| \leq C_\beta(1 + |r|)$ for all $r \in \mathbb{R}$.
- (A15) $\pi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function.
- (A16) $\theta_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $-\Delta\theta_0 \in H_0^1(\Omega)$, $\varphi_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $v_0 \in H_0^1(\Omega)$.

Indeed, putting

$$\begin{gathered} V := H_0^1(\Omega), \quad H := L^2(\Omega), \quad L := I : H \to H, \\ A_1 := -\sigma\Delta, \quad D(A_1) := H^2(\Omega) \cap H_0^1(\Omega) \subset H \to H, \\ B_1 := \varepsilon I, \quad D(B_1) := H \to H, \\ A_2 := -c^2\Delta, \quad D(A_2) := H^2(\Omega) \cap H_0^1(\Omega) \subset H \to H, \\ B_2 := -c^2\Delta, \quad D(B_2) := H^2(\Omega) \cap H_0^1(\Omega) \subset H \to H \end{gathered}$$

and defining the operators $A_1^* : V \to V^*$, $B_1^* : V \to V^*$, $A_2^* : V \to V^*$, $\Phi : D(\Phi) \subset H \to H$, $\mathcal{L} : H \to H$, $B_2^* : V \to V^*$ as

$$\begin{gathered} \langle A_1^* w, z\rangle_{V^*,V} := \sigma \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V, \\ \langle B_1^* w, z\rangle_{V^*,V} := \varepsilon(w,z)_H \quad \text{for } w, z \in V, \\ \langle A_2^* w, z\rangle_{V^*,V} := c^2 \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V, \\ \Phi z := \beta(z) \quad \text{for } z \in D(\Phi) := \{z \in H \mid \beta(z) \in H\}, \\ \mathcal{L}z := \pi(z) \quad \text{for } z \in H, \\ \langle B_2^* w, z\rangle_{V^*,V} := c^2 \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V, \end{gathered}$$

we can confirm that (A1)–(A12) hold, see [9]. Similarly, we can verify that the homogeneous Neumann–Neumann problem is an example.

Example 2.3. We have the problem

$$\begin{gathered} \theta_t + (\gamma - 1)\varphi_t - \sigma\Delta\theta = 0 \quad \text{in } \Omega \times (0,T), \\ \varphi_{tt} - \varepsilon\Delta\varphi_t - c^2\Delta\varphi + \beta(\varphi) + \pi(\varphi) = -c^2\Delta\theta \quad \text{in } \Omega \times (0,T), \\ \theta = \varphi = 0 \quad \text{on } \partial\Omega \times (0,T), \\ \theta(0) = \theta_0,\ \varphi(0) = \varphi_0,\ \varphi_t(0) = v_0 \text{in } \Omega, \end{gathered} \tag{2.3}$$

where $c > 0$, $\sigma > 0$, $\varepsilon \geq 0$, $\gamma > 1$, $T > 0$ are constants and $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, under the three conditions (A13)–(A15) and the condition

(A17) $\theta_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $-\Delta\theta_0 \in H^1_0(\Omega)$, $\varphi_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $v_0 \in H^2(\Omega) \cap H^1_0(\Omega)$.

Indeed, putting

$$\begin{gathered}
V := H^1_0(\Omega),\ H := L^2(\Omega), \quad L := I : H \to H, \\
A_1 := -\sigma\Delta, \quad D(A_1) := H^2(\Omega) \cap H^1_0(\Omega) \subset H \to H, \\
B_1 := -\varepsilon\Delta, \quad D(B_1) := H^2(\Omega) \cap H^1_0(\Omega) \subset H \to H, \\
A_2 := -c^2\Delta, \quad D(A_2) := H^2(\Omega) \cap H^1_0(\Omega) \subset H \to H, \\
B_2 := -c^2\Delta, \quad D(B_2) := H^2(\Omega) \cap H^1_0(\Omega) \subset H \to H
\end{gathered}$$

and defining the operators $A_1^* : V \to V^*$, $B_1^* : V \to V^*$, $A_2^* : V \to V^*$, $\Phi : D(\Phi) \subset H \to H$, $\mathcal{L} : H \to H$, $B_2^* : V \to V^*$ as

$$\begin{gathered}
\langle A_1^* w, z\rangle_{V^*,V} := \sigma \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V, \\
\langle B_1^* w, z\rangle_{V^*,V} := \varepsilon \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V, \\
\langle A_2^* w, z\rangle_{V^*,V} := c^2 \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V, \\
\Phi z := \beta(z) \quad \text{for } z \in D(\Phi) := \{z \in H \mid \beta(z) \in H\}, \\
\mathcal{L} z := \pi(z) \quad \text{for } z \in H, \\
\langle B_2^* w, z\rangle_{V^*,V} := c^2 \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V,
\end{gathered}$$

we can verify that (A1)–(A12) hold, see [9]. Similarly, we can check that the homogeneous Neumann-Neumann problem is an example.

Example 2.4. We have the problem

$$\begin{gathered}
\theta_t + \varphi_t - \Delta\theta = 0 \quad \text{in } \Omega \times (0, T), \\
\varphi_{tt} + \varphi_t - \Delta\varphi + \beta(\varphi) + \pi(\varphi) = \theta \quad \text{in } \Omega \times (0, T), \\
\theta = \varphi = 0 \quad \text{on } \partial\Omega \times (0, T), \\
\theta(0) = \theta_0, \quad \varphi(0) = \varphi_0, \quad \varphi_t(0) = v_0 \quad \text{in } \Omega,
\end{gathered} \tag{2.4}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $T > 0$, under the four conditions (A13)–(A16). Indeed, putting

$$\begin{gathered}
V := H^1_0(\Omega), \quad H := L^2(\Omega), \quad L := I : H \to H, \\
A_1 := -\Delta, \quad D(A_1) := H^2(\Omega) \cap H^1_0(\Omega) \subset H \to H, \\
B_1 := I, \quad D(B_1) := H \to H, \\
A_2 := -\Delta, \quad D(A_2) := H^2(\Omega) \cap H^1_0(\Omega) \subset H \to H, \\
B_2 := I, \quad D(B_2) := H \to H
\end{gathered}$$

and defining the operators $A_1^*: V \to V^*$, $B_1^*: V \to V^*$, $A_2^*: V \to V^*$, $\Phi: D(\Phi) \subset H \to H$, $\mathcal{L}: H \to H$, $B_2^*: V \to V^*$ as

$$
\begin{gathered}
\langle A_1^* w, z\rangle_{V^*,V} := \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V,\\
\langle B_1^* w, z\rangle_{V^*,V} := (w, z)_H \quad \text{for } w, z \in V,\\
\langle A_2^* w, z\rangle_{V^*,V} := \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V,\\
\Phi z := \beta(z) \quad \text{for } z \in D(\Phi) := \{z \in H \mid \beta(z) \in H\},\\
\mathcal{L} z := \pi(z) \quad \text{for } z \in H,\\
\langle B_2^* w, z\rangle_{V^*,V} := (w, z)_H \quad \text{for } w, z \in V,
\end{gathered}
$$

we can confirm that (A1)–(A12) hold, see [9]. Similarly, we can show that the homogeneous Neumann–Neumann problem is an example.

Example 2.5. We have the problem

$$
\begin{gathered}
\theta_t + \varphi_t - \Delta\theta = 0 \quad \text{in } \Omega \times (0,T),\\
\varphi_{tt} - \Delta\varphi_t - \Delta\varphi + \beta(\varphi) + \pi(\varphi) = \theta \quad \text{in } \Omega \times (0,T),\\
\theta = \varphi = 0 \quad \text{on } \partial\Omega \times (0,T),\\
\theta(0) = \theta_0,\ \varphi(0) = \varphi_0,\ \varphi_t(0) = v_0 \quad \text{in } \Omega
\end{gathered} \tag{2.5}
$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $T > 0$, under the four conditions (A13)-(A15), (A17). Indeed, putting

$$
\begin{gathered}
V := H_0^1(\Omega), \quad H := L^2(\Omega), \quad L := I : H \to H,\\
A_1 := -\Delta, \quad D(A_1) := H^2(\Omega) \cap H_0^1(\Omega) \subset H \to H,\\
B_1 := -\Delta, \quad D(B_1) := H^2(\Omega) \cap H_0^1(\Omega) \subset H \to H,\\
A_2 := -\Delta, \quad D(A_2) := H^2(\Omega) \cap H_0^1(\Omega) \subset H \to H,\\
B_2 := I, \quad D(B_2) := H \to H
\end{gathered}
$$

and defining the operators $A_1^*: V \to V^*$, $B_1^*: V \to V^*$, $A_2^*: V \to V^*$, $\Phi: D(\Phi) \subset H \to H$, $\mathcal{L}: H \to H$, $B_2^*: V \to V^*$ as

$$
\begin{gathered}
\langle A_1^* w, z\rangle_{V^*,V} := \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V,\\
\langle B_1^* w, z\rangle_{V^*,V} := \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V,\\
\langle A_2^* w, z\rangle_{V^*,V} := \int_\Omega \nabla w \cdot \nabla z \quad \text{for } w, z \in V,\\
\Phi z := \beta(z) \quad \text{for } z \in D(\Phi) := \{z \in H \mid \beta(z) \in H\},\\
\mathcal{L} z := \pi(z) \quad \text{for } z \in H,\\
\langle B_2^* w, z\rangle_{V^*,V} := (w, z)_H \quad \text{for } w, z \in V,
\end{gathered}
$$

we can check that (A1)–(A12) hold, see [9]. Similarly, we can show that the homogeneous Neumann-Neumann problem is an example.

3. Existence of discrete solutions

In this section we prove Theorem 1.3.

Lemma 3.1. *There exists $h_1 \in (0,1)$ such that $0 < h_1 < \widetilde{h}$, where*

$$\widetilde{h} := \Big(\frac{c_L}{1 + C_{\mathcal{L}} + \eta C_{A_1,B_2}} + \frac{\eta^2 C_{A_1,B_2}^2}{4(1 + C_{\mathcal{L}} + \eta C_{A_1,B_2})}\Big)^{1/2} - \frac{\eta C_{A_1,B_2}}{2(1 + C_{\mathcal{L}} + \eta C_{A_1,B_2})}$$

and for all $g \in H$ and all $h \in (0, h_1)$ there exists a unique solution $\varphi \in D(B_1) \cap D(A_2)$ of the equation

$$L\varphi + hB_1\varphi + h^2A_2\varphi + h^2\Phi\varphi + h^2\mathcal{L}\varphi + \eta h^2 B_2(I + hA_1)^{-1}\varphi = g \quad \text{in } H.$$

Proof. We define the operator $\Psi : V \to V^*$ as

$$\begin{aligned}\langle \Psi\varphi, w\rangle_{V^*,V} :=&(L\varphi, w)_H + h\langle B_1^*\varphi, w\rangle_{V^*,V} + h^2\langle A_2^*\varphi, w\rangle_{V^*,V} + h^2(\Phi_\lambda\varphi, w)_H \\ &+ h^2(\mathcal{L}\varphi, w)_H + \eta h^2(B_2(I + hA_1)^{-1}\varphi, w)_H \quad \text{for } \varphi, w \in V.\end{aligned}$$

Then the operator $\Psi : V \to V^*$ is monotone, continuous and coercive for all $h \in (0, \widetilde{h})$. Indeed, since the condition (A5) yields that

$$\begin{aligned}\|B_2(I + hA_1)^{-1}\varphi\|_H &\le C_{A_1,B_2}(\|(I + hA_1)^{-1}\varphi\|_H + \|A_1(I + hA_1)^{-1}\varphi\|_H) \\ &\le C_{A_1,B_2}(1 + h^{-1})\|\varphi\|_H\end{aligned} \tag{3.1}$$

for all $\varphi \in H$, we derive, from (A2), (A3), (A11), the monotonicity of B_1^* and Φ_λ, and (3.1) that

$$\begin{aligned}&\langle \Psi\varphi - \Psi\overline{\varphi}, \varphi - \overline{\varphi}\rangle_{V^*,V} \\ &= (L(\varphi - \overline{\varphi}), \varphi - \overline{\varphi})_H + h\langle B_1^*(\varphi - \overline{\varphi}), \varphi - \overline{\varphi}\rangle_{V^*,V} + h^2\langle A_2^*(\varphi - \overline{\varphi}), \varphi - \overline{\varphi}\rangle_{V^*,V} \\ &\quad + h^2(\Phi_\lambda\varphi - \Phi_\lambda\overline{\varphi}, \varphi - \overline{\varphi})_H + h^2(\mathcal{L}\varphi - \mathcal{L}\overline{\varphi}, \varphi - \overline{\varphi})_H \\ &\quad + \eta h^2(B_2(I + hA_1)^{-1}(\varphi - \overline{\varphi}), (\varphi - \overline{\varphi}))_H \\ &\ge c_L\|\varphi - \overline{\varphi}\|_H^2 + \omega_{2,1}h^2\|\varphi - \overline{\varphi}\|_V^2 - h^2\|\varphi - \overline{\varphi}\|_H^2 - C_{\mathcal{L}}h^2\|\varphi - \overline{\varphi}\|_H^2 \\ &\quad - \eta C_{A_1,B_2}(h + h^2)\|\varphi - \overline{\varphi}\|_H^2 \\ &\ge \omega_{2,1}h^2\|\varphi - \overline{\varphi}\|_V^2\end{aligned}$$

for all $\varphi, \overline{\varphi} \in V$ and all $h \in (0, \widetilde{h})$. It follows from the boundedness of the operators $L : H \to H$, $B_1^* : V \to V^*$, $A_2^* : V \to V^*$, the Lipschitz continuity of $\Phi_\lambda : H \to H$, the condition (A11), (3.1) and the continuity of the embedding $V \hookrightarrow H$ that there exists a constant $C_1 = C_1(\lambda) > 0$ such that

$$\begin{aligned}&|\langle \Psi\varphi - \Psi\overline{\varphi}, w\rangle_{V^*,V}| \\ &\le |(L(\varphi - \overline{\varphi}), w)_H| + h|\langle B_1^*(\varphi - \overline{\varphi}), w\rangle_{V^*,V}| + h^2|\langle A_2^*(\varphi - \overline{\varphi}), w\rangle_{V^*,V}| \\ &\quad + h^2|(\Phi_\lambda\varphi - \Phi_\lambda\overline{\varphi}, w)_H| + h^2|(\mathcal{L}\varphi - \mathcal{L}\overline{\varphi}, w)_H| \\ &\quad + \eta h^2|(B_2(I + hA_1)^{-1}(\varphi - \overline{\varphi}), w)_H| \\ &\le C_1(1 + h + h^2)\|\varphi - \overline{\varphi}\|_V\|w\|_V\end{aligned}$$

for all $\varphi, \overline{\varphi} \in V$ and all $h > 0$. Moreover, the inequality $\langle \Psi\varphi - \mathcal{L}0, \varphi\rangle_{V^*,V} \ge \omega_{2,1}h^2\|\varphi\|_V^2$ holds for all $\varphi \in V$ and all $h \in (0, \widetilde{h})$. Therefore the operator $\Psi : V \to V^*$ is surjective for all $h \in (0, \widetilde{h})$ (see e.g., [2, p. 37]) and then we see from (A10) that for all $g \in H$ and all $h \in (0, \widetilde{h})$ there exists a unique solution $\varphi_\lambda \in D(B_1) \cap D(A_2)$ of the equation

$$L\varphi_\lambda + hB_1\varphi_\lambda + h^2A_2\varphi_\lambda + h^2\Phi_\lambda\varphi_\lambda + h^2\mathcal{L}\varphi_\lambda + \eta h^2B_2(I + hA_1)^{-1}\varphi_\lambda = g \quad (3.2)$$

in H. Here, multiplying (3.2) by φ_λ and using the Young inequality, (A11), (3.1), we infer that

$$\begin{aligned}
&(L\varphi_\lambda, \varphi_\lambda)_H + h(B_1\varphi_\lambda, \varphi_\lambda)_H + h^2\langle A_2^*\varphi_\lambda, \varphi_\lambda\rangle_{V^*,V} + h^2(\Phi_\lambda\varphi_\lambda, \varphi_\lambda)_H \\
&= (g, \varphi_\lambda)_H - h^2(\mathcal{L}\varphi_\lambda - \mathcal{L}0, \varphi_\lambda)_H - h^2(\mathcal{L}0, \varphi_\lambda)_H - \eta h^2(B_2(I + hA_1)^{-1}\varphi_\lambda, \varphi_\lambda)_H \\
&\le \frac{c_L}{2}\|\varphi_\lambda\|_H^2 + \frac{1}{2c_L}\|g\|_H^2 + C_{\mathcal{L}}h^2\|\varphi_\lambda\|_H^2 + \frac{\|\mathcal{L}0\|_H^2}{2}h^2 + \frac{1}{2}h^2\|\varphi_\lambda\|_H^2 \\
&\quad + \eta C_{A_1,B_2}(h + h^2)\|\varphi_\lambda\|_H^2,
\end{aligned}$$

whence the conditions (A2) and (A3), the monotonicity of B_1 and Φ_λ imply that there exists $h_1 \in (0, \min\{1, \widetilde{h}\})$ such that for all $h \in (0, h_1)$ there exists a constant $C_2 = C_2(h) > 0$ satisfying

$$\|\varphi_\lambda\|_V^2 \le C_2 \quad (3.3)$$

for all $\lambda > 0$. We have from (3.2), (A8), (3.1) and the Young inequality that

$$\begin{aligned}
&h^2\|\Phi_\lambda\varphi_\lambda\|_H^2 \\
&= (g, \Phi_\lambda\varphi_\lambda)_H - (L\varphi_\lambda, \Phi_\lambda\varphi_\lambda)_H - h(B_1\varphi_\lambda, \Phi_\lambda\varphi_\lambda)_H - h^2(A_2\varphi_\lambda, \Phi_\lambda\varphi_\lambda)_H \\
&\quad - h^2(\mathcal{L}\varphi_\lambda, \Phi_\lambda\varphi_\lambda)_H - \eta h^2(B_2(I + hA_1)^{-1}\varphi_\lambda, \Phi_\lambda\varphi_\lambda)_H \\
&\le \frac{2}{h^2}\|g\|_H^2 + \frac{2}{h^2}\|L\varphi_\lambda\|_H^2 + 2h^2\|\mathcal{L}\varphi_\lambda\|_H^2 + 2\eta^2C_{A_1,B_2}^2(1 + h)^2\|\varphi_\lambda\|_H^2 \\
&\quad + \frac{1}{2}h^2\|\Phi_\lambda\varphi_\lambda\|_H^2.
\end{aligned}$$

Thus, owing to the boundedness of the operator $L : H \to H$, (A11) and (3.3), it holds that for all $h \in (0, h_1)$ there exists a constant $C_3 = C_3(h) > 0$ such that

$$\|\Phi_\lambda\varphi_\lambda\|_H^2 \le C_3 \quad (3.4)$$

for all $\lambda > 0$. Then equation (3.2) yields

$$\begin{aligned}
&h\|B_1\varphi_\lambda\|_H^2 \\
&= (g, B_1\varphi_\lambda)_H - (L\varphi_\lambda, B_1\varphi_\lambda)_H - h^2(A_2\varphi_\lambda, B_1\varphi_\lambda)_H - h^2(\Phi_\lambda\varphi_\lambda, B_1\varphi_\lambda)_H \\
&\quad - h^2(\mathcal{L}\varphi_\lambda, B_1\varphi_\lambda)_H - \eta h^2(B_2(I + hA_1)^{-1}\varphi_\lambda, B_1\varphi_\lambda)_H,
\end{aligned}$$

and hence we deduce from the boundedness of the operator $L : H \to H$, (A4), (A8), (A11), (3.1), the Young inequality and (3.3) that for all $h \in (0, h_1)$ there exists a constant $C_4 = C_4(h) > 0$ satisfying

$$\|B_1\varphi_\lambda\|_H^2 \le C_4(h) \quad (3.5)$$

for all $\lambda > 0$. We derive from (3.1)-(3.5) that for all $h \in (0, h_1)$ there exists a constant $C_5 = C_5(h) > 0$ such that

$$\|A_2\varphi_\lambda\|_H^2 \le C_5(h) \quad (3.6)$$

for all $\lambda > 0$. Hence the inequalities (3.3)-(3.6) mean that there exist $\varphi \in D(B_1) \cap D(A_2)$ and $\xi \in H$ such that

$$\varphi_\lambda \to \varphi \quad \text{weakly in } V, \quad (3.7)$$

$$L\varphi_\lambda \to L\varphi \quad \text{weakly in } H, \tag{3.8}$$
$$\Phi_\lambda(\varphi_\lambda) \to \xi \quad \text{weakly in } H, \tag{3.9}$$
$$B_1\varphi_\lambda \to B_1\varphi \quad \text{weakly in } H, \tag{3.10}$$
$$A_2\varphi_\lambda \to A_2\varphi \quad \text{weakly in } H \tag{3.11}$$

as $\lambda = \lambda_j \to +0$. Here it follows from (3.3), (3.7), the compact of the embedding $V \hookrightarrow H$ that

$$\varphi_\lambda \to \varphi \quad \text{strongly in } H \tag{3.12}$$

as $\lambda = \lambda_j \to +0$. Also, we see from (3.9) and (3.12) that $(\Phi_\lambda\varphi_\lambda, \varphi_\lambda)_H \to (\xi, \varphi)_H$ as $\lambda = \lambda_j \to +0$. Thus the inclusion and the identity

$$\varphi \in D(\Phi), \quad \xi = \Phi\varphi \tag{3.13}$$

hold (see e.g., [1, Lemma 1.3, p. 42]).

Thanks to (3.2), (3.8)-(3.13) and (A11), we can verify that there exists a solution $\varphi \in D(B_1) \cap D(A_2)$ of the equation

$$L\varphi + hB_1\varphi + h^2A_2\varphi + h^2\Phi\varphi + h^2\mathcal{L}\varphi + \eta h^2 B_2(I + hA_1)^{-1}\varphi = g \quad \text{in } H.$$

Moreover, the solution φ of this problem is unique by (A2), (A3), the monotonicity of B_1 and Φ, (A11) and (3.1). $\square$

Proof of Theorem 1.3. Let h_1 be as in Lemma 3.1 and let $h \in (0, h_1)$. Then we infer from (1.3), the linearity of the operators A_1, L, B_1, B_2 and A_2 that problem (1.2) can be written as

$$\begin{aligned}
&\theta_{n+1} + hA_1\theta_{n+1} = \theta_n + \eta(\varphi_n - \varphi_{n+1}),\\
&L\varphi_{n+1} + hB_1\varphi_{n+1} + h^2A_2\varphi_{n+1} + h^2\Phi\varphi_{n+1} + h^2\mathcal{L}\varphi_{n+1}\\
&+ \eta h^2 B_2(I + hA_1)^{-1}\varphi_{n+1}\\
&= L\varphi_n + hLv_n + hB_1\varphi_n + h^2B_2(I + hA_1)^{-1}(\eta\varphi_n + \theta_n)
\end{aligned} \tag{3.14}$$

and then proving Theorem 1.3 is equivalent to show existence and uniqueness of solutions to (3.14) for $n = 0, \dots, N-1$. It suffices to consider the case that $n = 0$. Owing to Lemma 3.1, there exists a unique solution $\varphi_1 \in D(B_1) \cap D(A_2)$ of the equation

$$\begin{aligned}
&L\varphi_1 + hB_1\varphi_1 + h^2A_2\varphi_1 + h^2\Phi\varphi_1 + h^2\mathcal{L}\varphi_1 + \eta h^2B_2(I + hA_1)^{-1}\varphi_1\\
&= L\varphi_0 + hLv_0 + hB_1\varphi_0 + h^2B_2(I + hA_1)^{-1}(\eta\varphi_0 + \theta_0).
\end{aligned}$$

Therefore, putting $\theta_1 := (I + hA_1)^{-1}(\theta_0 + \eta(\varphi_0 - \varphi_1))$, we can conclude that there exists a unique solution (θ_1, φ_1) of (3.14) in the case that $n = 0$. $\square$

4. Uniform estimates for (1.7) and passage to the limit

In this section we will derive a priori estimates for (1.7) and will show Theorem 1.4 by passing to the limit in (1.7) as $h \to +0$.

Lemma 4.1. *Let h_0 be as in Theorem 1.3. Then there exist constants $h_2 \in (0, h_0)$ and $C = C(T) > 0$ such that*

$$\begin{aligned}
&\|\overline{v}_h\|^2_{L^\infty(0,T;H)} + h\|\overline{z}_h\|^2_{L^2(0,T;H)} + \|B_1^{1/2}\overline{v}_h\|^2_{L^2(0,T;H)} + \|\overline{\varphi}_h\|^2_{L^\infty(0,T;V)}\\
&+ h\|\overline{v}_h\|^2_{L^2(0,T;V)} + \|B_2^{1/2}\overline{\theta}_h\|^2_{L^\infty(0,T;H)} + h\Big\|B_2^{1/2}\frac{d\widehat{\theta}_h}{dt}\Big\|^2_{L^2(0,T;H)} \le C
\end{aligned}$$

for all $h \in (0, h_2)$.

Proof. We test the second equation in (1.2) by hv_{n+1} $(= \varphi_{n+1} - \varphi_n)$ and recall (1.3) to obtain that

$$\begin{aligned} &(L(v_{n+1} - v_n), v_{n+1})_H + h\|B_1^{1/2} v_{n+1}\|_H^2 + \langle A_2^* \varphi_{n+1}, \varphi_{n+1} - \varphi_n \rangle_{V^*,V} \\ &+ (\varphi_{n+1}, \varphi_{n+1} - \varphi_n)_H + (\Phi\varphi_{n+1}, \varphi_{n+1} - \varphi_n)_H \\ &= h(B_2\theta_{n+1}, v_{n+1})_H - h(\mathcal{L}\varphi_{n+1}, v_{n+1})_H + h(\varphi_{n+1}, v_{n+1})_H. \end{aligned} \tag{4.1}$$

Here it holds

$$\begin{aligned} &(L(v_{n+1} - v_n), v_{n+1})_H \\ &= (L^{1/2}(v_{n+1} - v_n), L^{1/2} v_{n+1})_H \\ &= \frac{1}{2}\|L^{1/2} v_{n+1}\|_H^2 - \frac{1}{2}\|L^{1/2} v_n\|_H^2 + \frac{1}{2}\|L^{1/2}(v_{n+1} - v_n)\|_H^2 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\langle A_2^* \varphi_{n+1}, \varphi_{n+1} - \varphi_n \rangle_{V^*,V} + (\varphi_{n+1}, \varphi_{n+1} - \varphi_n)_H \\ &= \frac{1}{2}\langle A_2^* \varphi_{n+1}, \varphi_{n+1} \rangle_{V^*,V} - \frac{1}{2}\langle A_2^* \varphi_n, \varphi_n \rangle_{V^*,V} \\ &\quad + \frac{1}{2}\langle A_2^*(\varphi_{n+1} - \varphi_n), \varphi_{n+1} - \varphi_n \rangle_{V^*,V} \\ &\quad + \frac{1}{2}\|\varphi_{n+1}\|_H^2 - \frac{1}{2}\|\varphi_n\|_H^2 + \frac{1}{2}\|\varphi_{n+1} - \varphi_n\|_H^2. \end{aligned} \tag{4.3}$$

The first equation in (1.2) yields

$$\begin{aligned} &h(B_2\theta_{n+1}, v_{n+1})_H \\ &= \frac{h}{\eta}\Big(B_2\theta_{n+1}, -\frac{\theta_{n+1} - \theta_n}{h} - A_1\theta_{n+1}\Big)_H \\ &= -\frac{1}{2\eta}\Big(\|B_2^{1/2}\theta_{n+1}\|_H^2 - \|B_2^{1/2}\theta_n\|_H^2 + \|B_2^{1/2}(\theta_{n+1} - \theta_n)\|_H^2\Big) \\ &\quad - \frac{h}{\eta}(B_2\theta_{n+1}, A_1\theta_{n+1})_H. \end{aligned} \tag{4.4}$$

From (4.1)–(4.4), (A4), (A7), (A11), the continuity of the embedding $V \hookrightarrow H$, and Young's inequality, we have that there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned} &\frac{1}{2}\|L^{1/2} v_{n+1}\|_H^2 - \frac{1}{2}\|L^{1/2} v_n\|_H^2 + \frac{1}{2}\|L^{1/2}(v_{n+1} - v_n)\|_H^2 + h\|B_1^{1/2} v_{n+1}\|_H^2 \\ &+ \frac{1}{2}\langle A_2^* \varphi_{n+1}, \varphi_{n+1} \rangle_{V^*,V} - \frac{1}{2}\langle A_2^* \varphi_n, \varphi_n \rangle_{V^*,V} \\ &+ \frac{1}{2}\langle A_2^*(\varphi_{n+1} - \varphi_n), \varphi_{n+1} - \varphi_n \rangle_{V^*,V} + \frac{1}{2}\|\varphi_{n+1}\|_H^2 - \frac{1}{2}\|\varphi_n\|_H^2 \\ &+ \frac{1}{2}\|\varphi_{n+1} - \varphi_n\|_H^2 + i(\varphi_{n+1}) - i(\varphi_n) \\ &+ \frac{1}{2\eta}\|B_2^{1/2}\theta_{n+1}\|_H^2 - \frac{1}{2\eta}\|B_2^{1/2}\theta_n\|_H^2 + \frac{1}{2\eta}\|B_2^{1/2}(\theta_{n+1} - \theta_n)\|_H^2 \\ &\le h\|v_{n+1}\|_H^2 + C_1 h\|\varphi_{n+1}\|_V^2 + C_2 h \end{aligned} \tag{4.5}$$

for all $h \in (0, h_0)$. Moreover, summing (4.5) over $n = 0, \dots, m-1$ with $1 \le m \le N$ leads to the inequality

$$
\begin{aligned}
&\frac{1}{2}\|L^{1/2}v_m\|_H^2 + \frac{1}{2}\sum_{n=0}^{m-1}\|L^{1/2}(v_{n+1}-v_n)\|_H^2 + h\sum_{n=0}^{m-1}\|B_1^{1/2}v_{n+1}\|_H^2 \\
&+ \frac{1}{2}\langle A_2^*\varphi_m, \varphi_m\rangle_{V^*,V} + \frac{1}{2}\|\varphi_m\|_H^2 + \frac{1}{2}\sum_{n=0}^{m-1}\langle A_2^*(\varphi_{n+1}-\varphi_n), \varphi_{n+1}-\varphi_n\rangle_{V^*,V} \\
&+ \frac{1}{2}\sum_{n=0}^{m-1}\|\varphi_{n+1}-\varphi_n\|_H^2 + i(\varphi_m) + \frac{1}{2\eta}\|B_2^{1/2}\theta_m\|_H^2 \\
&+ \frac{1}{2\eta}\sum_{n=0}^{m-1}\|B_2^{1/2}(\theta_{n+1}-\theta_n)\|_H^2 \\
&\le \frac{1}{2}\|L^{1/2}v_0\|_H^2 + \frac{1}{2}\langle A_2^*\varphi_0, \varphi_0\rangle_{V^*,V} + \frac{1}{2}\|\varphi_0\|_H^2 + i(\varphi_0) + \frac{1}{2\eta}\|B_2^{1/2}\theta_0\|_H^2 \\
&\quad + h\sum_{n=0}^{m-1}\|v_{n+1}\|_H^2 + C_1 h\sum_{n=0}^{m-1}\|\varphi_{n+1}\|_V^2 + C_2 T
\end{aligned}
\tag{4.6}
$$

for all $h \in (0, h_0)$. We see from (A3) that

$$
\frac{1}{2}\langle A_2^*\varphi_m, \varphi_m\rangle_{V^*,V} + \frac{1}{2}\|\varphi_m\|_H^2 \ge \frac{\omega_{2,1}}{2}\|\varphi_m\|_V^2 \tag{4.7}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\sum_{n=0}^{m-1}\langle A_2^*(\varphi_{n+1}-\varphi_n), \varphi_{n+1}-\varphi_n\rangle_{V^*,V} + \frac{1}{2}\sum_{n=0}^{m-1}\|\varphi_{n+1}-\varphi_n\|_H^2 \\
&\ge \frac{\omega_{2,1}}{2}\sum_{n=0}^{m-1}\|\varphi_{n+1}-\varphi_n\|_V^2 = \frac{\omega_{2,1}}{2}h^2\sum_{n=0}^{m-1}\|v_{n+1}\|_V^2.
\end{aligned}
\tag{4.8}
$$

Thus from (4.6)-(4.8) and (A2) we obtain

$$
\begin{aligned}
&\Big(\frac{c_L}{2} - h\Big)\|v_m\|_H^2 + \frac{c_L}{2}h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_H^2 + h\sum_{n=0}^{m-1}\|B_1^{1/2}v_{n+1}\|_H^2 \\
&+ \Big(\frac{\omega_{2,1}}{2} - C_1 h\Big)\|\varphi_m\|_V^2 + \frac{\omega_{2,1}}{2}h^2\sum_{n=0}^{m-1}\|v_{n+1}\|_V^2 + \frac{1}{2\eta}\|B_2^{1/2}\theta_m\|_H^2 \\
&+ \frac{1}{2\eta}h^2\sum_{n=0}^{m-1}\|B_2^{1/2}\delta_h\theta_n\|_H^2 \\
&\le \frac{1}{2}\|L^{1/2}v_0\|_H^2 + \frac{1}{2}\langle A_2^*\varphi_0, \varphi_0\rangle_{V^*,V} + \frac{1}{2}\|\varphi_0\|_H^2 + i(\varphi_0) + \frac{1}{2\eta}\|B_2^{1/2}\theta_0\|_H^2 \\
&\quad + h\sum_{j=0}^{m-1}\|v_j\|_H^2 + C_1 h\sum_{j=0}^{m-1}\|\varphi_j\|_V^2 + C_2 T,
\end{aligned}
$$

whence there exist constants $h_2 \in (0, h_0)$ and $C_3 = C_3(T) > 0$ such that

$$\begin{aligned} &\|v_m\|_H^2 + h^2 \sum_{n=0}^{m-1} \|z_{n+1}\|_H^2 + h \sum_{n=0}^{m-1} \|B_1^{1/2} v_{n+1}\|_H^2 \\ &+ \|\varphi_m\|_V^2 + h^2 \sum_{n=0}^{m-1} \|v_{n+1}\|_V^2 + \|B_2^{1/2}\theta_m\|_H^2 + h^2 \sum_{n=0}^{m-1} \|B_2^{1/2}\delta_h\theta_n\|_H^2 \\ &\le C_3 h \sum_{j=0}^{m-1} \|v_j\|_H^2 + C_3 h \sum_{j=0}^{m-1} \|\varphi_j\|_V^2 + C_3 \end{aligned} \tag{4.9}$$

for all $h \in (0, h_2)$. Therefore from inequality (4.9) and the discrete Gronwall lemma (see e.g., [8, Prop. 2.2.1]) there exists a constant $C_4 = C_4(T) > 0$ such that

$$\begin{aligned} &\|v_m\|_H^2 + h^2 \sum_{n=0}^{m-1} \|z_{n+1}\|_H^2 + h \sum_{n=0}^{m-1} \|B_1^{1/2} v_{n+1}\|_H^2 \\ &+ \|\varphi_m\|_V^2 + h^2 \sum_{n=0}^{m-1} \|v_{n+1}\|_V^2 + \|B_2^{1/2}\theta_m\|_H^2 + h^2 \sum_{n=0}^{m-1} \|B_2^{1/2}\delta_h\theta_n\|_H^2 \le C_4 \end{aligned}$$

for all $h \in (0, h_2)$ and $m = 1, \dots, N$. □

Lemma 4.2. *Let h_2 be as in Lemma 4.1. Then there exists a constant $C = C(T) > 0$ such that*

$$\|z_1\|_H^2 + h\|B_1^{1/2} z_1\|_H^2 + \|v_1\|_V^2 + h^2\|z_1\|_V^2 + \langle B_2^*(\eta v_1 + A_1\theta_1), \eta v_1 + A_1\theta_1\rangle_{V^*,V} \le C$$

for all $h \in (0, h_2)$.

Proof. The second equation in (1.2), the identities $v_1 = v_0 + hz_1$ and $\varphi_1 = \varphi_0 + hv_1$ yield that

$$Lz_1 + B_1 v_0 + hB_1 z_1 + A_2\varphi_0 + hA_2 v_1 + \Phi\varphi_1 + \mathcal{L}\varphi_1 = B_2\theta_1. \tag{4.10}$$

Then we test (4.10) by z_1 to infer that

$$\begin{aligned} &\|L^{1/2} z_1\|_H^2 + (B_1 v_0, z_1)_H + h(B_1 z_1, z_1)_H + (A_2\varphi_0, z_1)_H + h(A_2 v_1, z_1)_H \\ &+ (\Phi\varphi_1, z_1)_H + (\mathcal{L}\varphi_1, z_1)_H \\ &= (B_2\theta_1, z_1)_H. \end{aligned} \tag{4.11}$$

From (A3) we obtain

$$\begin{aligned} &h(A_2 v_1, z_1)_H \\ &= (A_2 v_1, v_1 - v_0)_H \\ &= \langle A_2^* v_1, v_1 - v_0\rangle_{V^*,V} \\ &= \frac{1}{2}\langle A_2^* v_1, v_1\rangle_{V^*,V} - \frac{1}{2}\langle A_2^* v_0, v_0\rangle_{V^*,V} + \frac{1}{2}\langle A_2^*(v_1 - v_0), v_1 - v_0\rangle_{V^*,V} \\ &\ge \frac{\omega_{2,1}}{2}\|v_1\|_V^2 - \frac{1}{2}\|v_1\|_H^2 - \frac{1}{2}\langle A_2^* v_0, v_0\rangle_{V^*,V} + \frac{\omega_{2,1}}{2}\|v_1 - v_0\|_V^2 \\ &\quad - \frac{1}{2}\|v_1 - v_0\|_H^2. \end{aligned} \tag{4.12}$$

We see from (A6) and Lemma 4.1 that there exists a constant $C_1 = C_1(T) > 0$ such that

$$|(\Phi\varphi_1, z_1)_H| \le C_\Phi(1 + \|\varphi_1\|_V^p)\|\varphi_1\|_V\|z_1\|_H \le C_1\|z_1\|_H. \tag{4.13}$$

Also, the first equation in (1.2) and the identity $v_1 - v_0 = hz_1$ imply that

$$\begin{aligned} &\frac{1}{2\eta}\langle B_2^*(\eta v_1 + A_1\theta_1), \eta v_1 + A_1\theta_1\rangle_{V^*,V} \\ &- \frac{1}{2\eta}\langle B_2^*(\eta v_0 + A_1\theta_0), \eta v_0 + A_1\theta_0\rangle_{V^*,V} \\ &+ \frac{1}{2\eta}\langle B_2^*(\eta(v_1 - v_0) + A_1(\theta_1 - \theta_0)), \eta(v_1 - v_0) + A_1(\theta_1 - \theta_0)\rangle_{V^*,V} \\ &= \frac{1}{\eta}\langle B_2^*(\eta v_1 + A_1\theta_1), \eta(v_1 - v_0) + A_1(\theta_1 - \theta_0)\rangle_{V^*,V} \\ &= -(B_2\theta_1, z_1)_H + (B_2\theta_0, z_1)_H - \frac{1}{\eta h}(B_2(\theta_1 - \theta_0), A_1(\theta_1 - \theta_0))_H. \end{aligned} \tag{4.14}$$

It follows from (4.11)-(4.14), (A2), (A4) and the monotonicity of $B_2^* : V \to V^*$ that

$$\begin{aligned} &c_L\|z_1\|_H^2 + h\|B_1^{1/2}z_1\|_H^2 + \frac{\omega_{2,1}}{2}\|v_1\|_V^2 + \frac{\omega_{2,1}}{2}h^2\|z_1\|_V^2 \\ &+ \frac{1}{2\eta}\langle B_2^*(\eta v_1 + A_1\theta_1), \eta v_1 + A_1\theta_1\rangle_{V^*,V} \\ &\le -(B_1v_0, z_1)_H - (A_2\varphi_0, z_1)_H + \frac{1}{2}\|v_1\|_H^2 + \frac{1}{2}\langle A_2^*v_0, v_0\rangle_{V^*,V} \\ &\quad + \frac{1}{2}\|v_1 - v_0\|_H^2 + C_1\|z_1\|_H - (\mathcal{L}\varphi_1, z_1)_H + (B_2\theta_0, z_1)_H \\ &\quad + \frac{1}{2\eta}\langle B_2^*(\eta v_0 + A_1\theta_0), \eta v_0 + A_1\theta_0\rangle_{V^*,V}. \end{aligned} \tag{4.15}$$

Thus we deduce from (4.15), (A11), the Young inequality and Lemma 4.1 that Lemma 4.2 holds. □

Lemma 4.3. *Let h_2 be as in Lemma 4.1. Then there exist constants $h_3 \in (0, h_2)$ and $C = C(T) > 0$ such that*

$$\|\overline{z}_h\|_{L^\infty(0,T;H)}^2 + \|B_1^{1/2}\overline{z}_h\|_{L^2(0,T;H)}^2 + \|\overline{v}_h\|_{L^\infty(0,T;V)}^2 + h\|\overline{z}_h\|_{L^2(0,T;V)}^2 \le C$$

for all $h \in (0, h_3)$.

Proof. Let $n \in \{1, \dots, N-1\}$. Then we have from the second equation in (1.2) that

$$\begin{aligned} &L(z_{n+1} - z_n) + hB_1z_{n+1} + hA_2v_{n+1} + \Phi\varphi_{n+1} - \Phi\varphi_n + \mathcal{L}\varphi_{n+1} - \mathcal{L}\varphi_n \\ &= B_2(\theta_{n+1} - \theta_n). \end{aligned}$$

Since

$$\begin{aligned} &(L(z_{n+1} - z_n), z_{n+1})_H = (L^{1/2}(z_{n+1} - z_n), L^{1/2}z_{n+1})_H \\ &= \frac{1}{2}\|L^{1/2}z_{n+1}\|_H^2 - \frac{1}{2}\|L^{1/2}z_n\|_H^2 + \frac{1}{2}\|L^{1/2}(z_{n+1} - z_n)\|_H^2, \end{aligned}$$

it follows that

$$\begin{aligned} &\frac{1}{2}\|L^{1/2}z_{n+1}\|_H^2 - \frac{1}{2}\|L^{1/2}z_n\|_H^2 + \frac{1}{2}\|L^{1/2}(z_{n+1} - z_n)\|_H^2 + h\|B_1^{1/2}z_{n+1}\|_H^2 \\ &+ \langle A_2^*v_{n+1}, v_{n+1} - v_n\rangle_{V^*,V} + (v_{n+1}, v_{n+1} - v_n)_H \\ &= -h\Big(\frac{\Phi\varphi_{n+1} - \Phi\varphi_n}{h}, z_{n+1}\Big)_H - h\Big(\frac{\mathcal{L}\varphi_{n+1} - \mathcal{L}\varphi_n}{h}, z_{n+1}\Big)_H \\ &\quad + (B_2(\theta_{n+1} - \theta_n), z_{n+1})_H + h(v_{n+1}, z_{n+1})_H. \end{aligned} \tag{4.16}$$

On the other hand,

$$\begin{aligned}&\langle A_2^*v_{n+1}, v_{n+1}-v_n\rangle_{V^*,V} + (v_{n+1}, v_{n+1}-v_n)_H \\&= \frac{1}{2}\langle A_2^*v_{n+1}, v_{n+1}\rangle_{V^*,V} - \frac{1}{2}\langle A_2^*v_n, v_n\rangle_{V^*,V} \\&\quad + \frac{1}{2}\langle A_2^*(v_{n+1}-v_n), v_{n+1}-v_n\rangle_{V^*,V} + \frac{1}{2}\|v_{n+1}\|_H^2 - \frac{1}{2}\|v_n\|_H^2 \\&\quad + \frac{1}{2}\|v_{n+1}-v_n\|_H^2 .\end{aligned} \tag{4.17}$$

Condition (A6) and Lemma 4.1 mean that there exists a constant $C_1 = C_1(T) > 0$ such that

$$\begin{aligned}&- h\Big(\frac{\Phi\varphi_{n+1} - \Phi\varphi_n}{h}, z_{n+1}\Big)_H \\&\le C_\Phi h(1 + \|\varphi_{n+1}\|_V^p + \|\varphi_n\|_V^q)\|v_{n+1}\|_V\|z_{n+1}\|_H \\&\le C_1 h\|v_{n+1}\|_V\|z_{n+1}\|_H\end{aligned} \tag{4.18}$$

for all $h \in (0, h_2)$. Also, the first equation in (1.2) and the identity $v_{n+1} - v_n = hz_{n+1}$ yield

$$\begin{aligned}&\frac{1}{2\eta}\langle B_2^*(\eta v_{n+1} + A_1\theta_{n+1}), \eta v_{n+1} + A_1\theta_{n+1}\rangle_{V^*,V} \\&- \frac{1}{2\eta}\langle B_2^*(\eta v_n + A_1\theta_n), \eta v_n + A_1\theta_n\rangle_{V^*,V} \\&+ \frac{1}{2\eta}\langle B_2^*(\eta(v_{n+1}-v_n) + A_1(\theta_{n+1}-\theta_n)), \eta(v_{n+1}-v_n) \\&+ A_1(\theta_{n+1}-\theta_n)\rangle_{V^*,V} \\&= \frac{1}{\eta}\langle B_2^*(\eta v_{n+1} + A_1\theta_{n+1}), \eta(v_{n+1}-v_n) + A_1(\theta_{n+1}-\theta_n)\rangle_{V^*,V} \\&= -(B_2(\theta_{n+1}-\theta_n), z_{n+1})_H - \frac{1}{\eta h}(B_2(\theta_{n+1}-\theta_n), A_1(\theta_{n+1}-\theta_n))_H.\end{aligned} \tag{4.19}$$

It follows from (4.16)-(4.19), (A4) and (A11) that there exists a constant $C_2 = C_2(T) > 0$ such that

$$\begin{aligned}&\frac{1}{2}\|L^{1/2}z_{n+1}\|_H^2 - \frac{1}{2}\|L^{1/2}z_n\|_H^2 + \frac{1}{2}\|L^{1/2}(z_{n+1}-z_n)\|_H^2 + h\|B_1^{1/2}z_{n+1}\|_H^2 \\&+ \frac{1}{2}\langle A_2^*v_{n+1}, v_{n+1}\rangle_{V^*,V} - \frac{1}{2}\langle A_2^*v_n, v_n\rangle_{V^*,V} \\&+ \frac{1}{2}\langle A_2^*(v_{n+1}-v_n), v_{n+1}-v_n\rangle_{V^*,V} \\&+ \frac{1}{2}\|v_{n+1}\|_H^2 - \frac{1}{2}\|v_n\|_H^2 + \frac{1}{2}\|v_{n+1}-v_n\|_H^2 \\&+ \frac{1}{2\eta}\langle B_2^*(\eta v_{n+1} + A_1\theta_{n+1}), \eta v_{n+1} + A_1\theta_{n+1}\rangle_{V^*,V} \\&- \frac{1}{2\eta}\langle B_2^*(\eta v_n + A_1\theta_n), \eta v_n + A_1\theta_n\rangle_{V^*,V} \\&+ \frac{1}{2\eta}\big\langle B_2^*(\eta(v_{n+1}-v_n) + A_1(\theta_{n+1}-\theta_n)), \eta(v_{n+1}-v_n) \\&+ A_1(\theta_{n+1}-\theta_n)\big\rangle_{V^*,V} \\&\le C_2 h\|v_{n+1}\|_V\|z_{n+1}\|_H\end{aligned} \tag{4.20}$$

for all $h \in (0, h_2)$. Then we sum (4.20) over $n = 1, \dots, \ell - 1$ with $2 \le \ell \le N$ to obtain

$$
\begin{aligned}
&\frac{1}{2}\|L^{1/2} z_\ell\|_H^2 + \frac{1}{2}\sum_{n=1}^{\ell-1} \|L^{1/2}(z_{n+1} - z_n)\|_H^2 + h\sum_{n=1}^{\ell-1} \|B_1^{1/2} z_{n+1}\|_H^2 \\
&+ \frac{1}{2}\langle A_2^* v_\ell, v_\ell\rangle_{V^*,V} + \frac{1}{2}\sum_{n=1}^{\ell-1} \langle A_2^*(v_{n+1} - v_n), v_{n+1} - v_n\rangle_{V^*,V} \\
&+ \frac{1}{2}\|v_\ell\|_H^2 + \frac{1}{2}\sum_{n=1}^{\ell-1}\|v_{n+1} - v_n\|_H^2 + \frac{1}{2\eta}\langle B_2^*(\eta v_\ell + A_1\theta_\ell), \eta v_\ell + A_1\theta_\ell\rangle_{V^*,V} \\
&\le \frac{1}{2}\|L^{1/2} z_1\|_H^2 + \frac{1}{2}\langle A_2^* v_1, v_1\rangle_{V^*,V} + \frac{1}{2}\|v_1\|_H^2 \\
&\quad + \frac{1}{2\eta}\langle B_2^*(\eta v_1 + A_1\theta_1), \eta v_1 + A_1\theta_1\rangle_{V^*,V} + C_2 h\sum_{n=0}^{\ell-1}\|v_{n+1}\|_V \|z_{n+1}\|_H.
\end{aligned}
$$

Thus from (A2) and (A3) we have

$$
\begin{aligned}
&\frac{c_L}{2}\|z_\ell\|_H^2 + h\sum_{n=1}^{\ell-1}\|B_1^{1/2} z_{n+1}\|_H^2 + \frac{\omega_{2,1}}{2}\|v_\ell\|_V^2 + \frac{\omega_{2,1}}{2}h^2\sum_{n=1}^{\ell-1}\|z_{n+1}\|_V^2 \\
&+ \frac{1}{2\eta}\langle B_2^*(\eta v_\ell + A_1\theta_\ell), \eta v_\ell + A_1\theta_\ell\rangle_{V^*,V} \\
&\le \frac{1}{2}\|L^{1/2} z_1\|_H^2 + \frac{1}{2}\langle A_2^* v_1, v_1\rangle_{V^*,V} + \frac{1}{2}\|v_1\|_H^2 \\
&\quad + \frac{1}{2\eta}\langle B_2^*(\eta v_1 + A_1\theta_1), \eta v_1 + A_1\theta_1\rangle_{V^*,V} + C_2 h\sum_{n=0}^{\ell-1}\|v_{n+1}\|_V\|z_{n+1}\|_H
\end{aligned}
\tag{4.21}
$$

for all $h \in (0, h_2)$ and $\ell = 2, \dots, N$. Therefore we infer from (4.21), the boundedness of L and A_2^*, and Lemma 4.2 that there exists a constant $C_3 = C_3(T) > 0$ such that

$$
\begin{aligned}
&\frac{c_L}{2}\|z_m\|_H^2 + h\sum_{n=0}^{m-1}\|B_1^{1/2} z_{n+1}\|_H^2 + \frac{\omega_{2,1}}{2}\|v_m\|_V^2 + \frac{\omega_{2,1}}{2}h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_V^2 \\
&\le C_3 + C_2 h\sum_{n=0}^{m-1}\|v_{n+1}\|_V\|z_{n+1}\|_H
\end{aligned}
\tag{4.22}
$$

for all $h \in (0, h_2)$ and $m = 1, \dots, N$. Moreover, we see from (4.22) and the Young inequality that

$$
\begin{aligned}
&\frac{1}{2}(c_L - C_2 h)\|z_m\|_H^2 + h\sum_{n=0}^{m-1}\|B_1^{1/2} z_{n+1}\|_H^2 + \frac{1}{2}(\omega_{2,1} - C_2 h)\|v_m\|_V^2 \\
&+ \frac{\omega_{2,1}}{2}h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_V^2 \\
&\le C_3 + \frac{C_2}{2}h\sum_{j=0}^{m-1}\|v_j\|_V^2 + \frac{C_2}{2}h\sum_{j=0}^{m-1}\|z_j\|_H^2
\end{aligned}
\tag{4.23}
$$

for all $h \in (0, h_2)$ and $m = 1, \dots, N$. Hence there exist constants $h_3 \in (0, h_2)$ and $C_4 = C_4(T) > 0$ such that

$$\begin{aligned} &\|z_m\|_H^2 + h \sum_{n=0}^{m-1} \|B_1^{1/2} z_{n+1}\|_H^2 + \|v_m\|_V^2 + h^2 \sum_{n=0}^{m-1} \|z_{n+1}\|_V^2 \\ &\le C_4 + C_4 h \sum_{j=0}^{m-1} \|v_j\|_V^2 + C_4 h \sum_{j=0}^{m-1} \|z_j\|_H^2 \end{aligned}$$

for all $h \in (0, h_3)$ and $m = 1, \dots, N$. Therefore, owing to the discrete Gronwall lemma (see e.g., [8, Prop. 2.2.1]), there exists a constant $C_5 = C_5(T) > 0$ satisfying

$$\|z_m\|_H^2 + h \sum_{n=0}^{m-1} \|B_1^{1/2} z_{n+1}\|_H^2 + \|v_m\|_V^2 + h^2 \sum_{n=0}^{m-1} \|z_{n+1}\|_V^2 \le C_5$$

for all $h \in (0, h_3)$ and $m = 1, \dots, N$. □

Lemma 4.4. *Let h_2 be as in Lemma 4.1. Then there exists a constant $C = C(T) > 0$ such that*

$$\|\Phi \overline{\varphi}_h\|_{L^\infty(0,T;H)} \le C$$

for all $h \in (0, h_2)$.

The above lemma follows from (A6) and Lemma 4.1.

Lemma 4.5. *Let h_3 be as in Lemma 4.3. Then there exist constants $h_4 \in (0, h_3)$ and $C = C(T) > 0$ such that*

$$\Big\|\frac{d\widehat{\theta}_h}{dt}\Big\|_{L^2(0,T;H)}^2 + h\Big\|\frac{d\widehat{\theta}_h}{dt}\Big\|_{L^2(0,T;V)}^2 + \|\overline{\theta}_h\|_{L^\infty(0,T;V)}^2 \le C$$

for all $h \in (0, h_4)$.

Proof. We multiply the first equation in (1.2) by $\theta_{n+1} - \theta_n$ and by $h\theta_{n+1}$, respectively, and use the Young inequality to obtain that

$$\begin{aligned} &h\Big\|\frac{\theta_{n+1} - \theta_n}{h}\Big\|_H^2 + \langle A_1^* \theta_{n+1}, \theta_{n+1} - \theta_n \rangle_{V^*,V} + (\theta_{n+1} - \theta_n, \theta_{n+1})_H \\ &\quad + h(A_1 \theta_{n+1}, \theta_{n+1})_H \\ &= -\eta h \Big(v_{n+1}, \frac{\theta_{n+1} - \theta_n}{h} \Big)_H - \eta h (v_{n+1}, \theta_{n+1})_H \\ &\le \eta^2 h \|v_{n+1}\|_H^2 + \frac{1}{2} h \Big\|\frac{\theta_{n+1} - \theta_n}{h}\Big\|_H^2 + \frac{1}{2} h \|\theta_{n+1}\|_H^2. \end{aligned} \tag{4.24}$$

Here it holds that

$$\begin{aligned} &\langle A_1^* \theta_{n+1}, \theta_{n+1} - \theta_n \rangle_{V^*,V} + (\theta_{n+1} - \theta_n, \theta_{n+1})_H \\ &= \frac{1}{2} \langle A_1^* \theta_{n+1}, \theta_{n+1} \rangle_{V^*,V} - \frac{1}{2} \langle A_1^* \theta_n, \theta_n \rangle_{V^*,V} \\ &\quad + \frac{1}{2} \langle A_1^* (\theta_{n+1} - \theta_n), \theta_{n+1} - \theta_n \rangle_{V^*,V} \\ &\quad + \frac{1}{2} \|\theta_{n+1}\|_H^2 - \frac{1}{2} \|\theta_n\|_H^2 + \frac{1}{2} \|\theta_{n+1} - \theta_n\|_H^2. \end{aligned} \tag{4.25}$$

From (4.24), (4.25) and the continuity of the embedding $V \hookrightarrow H$, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} &\frac{1}{2}h\|\frac{\theta_{n+1}-\theta_n}{h}\|_H^2 + \frac{1}{2}\langle A_1^*\theta_{n+1}, \theta_{n+1}\rangle_{V^*,V} - \frac{1}{2}\langle A_1^*\theta_n, \theta_n\rangle_{V^*,V} \\ &+ \frac{1}{2}\langle A_1^*(\theta_{n+1}-\theta_n), \theta_{n+1}-\theta_n\rangle_{V^*,V} + \frac{1}{2}\|\theta_{n+1}\|_H^2 - \frac{1}{2}\|\theta_n\|_H^2 \\ &+ \frac{1}{2}\|\theta_{n+1}-\theta_n\|_H^2 \\ &\le \eta^2 h\|v_{n+1}\|_H^2 + C_1 h\|\theta_{n+1}\|_V^2 \end{aligned} \tag{4.26}$$

for all $h \in (0, h_3)$. Therefore we can prove Lemma 4.5 by summing (4.26) over $n = 0, \dots, m-1$ with $1 \le m \le N$, the condition (A3), Lemma 4.1 and the discrete Gronwall lemma (see e.g., [8, Prop. 2.2.1]). □

Lemma 4.6. *Let h_4 be as in Lemma 4.5. Then there exists a constant $C = C(T) > 0$ such that*

$$\|\frac{d\widehat{\theta}_h}{dt}\|_{L^2(0,T;V)}^2 + \|A_1\overline{\theta}_h\|_{L^\infty(0,T;H)}^2 \le C$$

for all $h \in (0, h_4)$.

Proof. It follows from the first equation in (1.2) that

$$\begin{aligned} &h\Big\langle A_1^*\frac{\theta_{n+1}-\theta_n}{h}, \frac{\theta_{n+1}-\theta_n}{h}\Big\rangle_{V^*,V} + h\|\frac{\theta_{n+1}-\theta_n}{h}\|_H^2 \\ &+ \frac{1}{2}\|A_1\theta_{n+1}\|_H^2 - \frac{1}{2}\|A_1\theta_n\|_H^2 + \frac{1}{2}\|A_1(\theta_{n+1}-\theta_n)\|_H^2 \\ &= -\eta h\Big\langle A_1^*\frac{\theta_{n+1}-\theta_n}{h}, v_{n+1}\Big\rangle_{V^*,V} + h\|\frac{\theta_{n+1}-\theta_n}{h}\|_H^2 \end{aligned}$$

and then we can prove this lemma by (A3), the boundedness of the operator $A_1^* : V \to V^*$, the Young inequality, Lemma 4.3, summing over $n = 0, \dots, m-1$ with $1 \le m \le N$ and Lemma 4.5. □

Lemma 4.7. *Let h_4 be as in Lemma 4.5. Then there exists a constant $C = C(T) > 0$ such that*

$$\|B_2\overline{\theta}_h\|_{L^\infty(0,T;H)}^2 + \|B_1\overline{v}_h\|_{L^2(0,T;H)}^2 + \|A_2\overline{\varphi}_h\|_{L^2(0,T;H)}^2 \le C$$

for all $h \in (0, h_4)$.

Proof. By (A5) and Lemmas 4.5 and 4.6, there exists a constant $C_1 = C_1(T) > 0$ such that

$$\|B_2\overline{\theta}_h\|_{L^\infty(0,T;H)}^2 \le C_1 \tag{4.27}$$

for all $h \in (0, h_4)$. The second equation in (1.2) yields

$$\begin{aligned} &h\|B_1 v_{n+1}\|_H^2 = h(B_1 v_{n+1}, B_1 v_{n+1})_H \\ &= -h(Lz_{n+1}, B_1 v_{n+1})_H - h(A_2\varphi_{n+1}, B_1 v_{n+1})_H - h(\Phi\varphi_{n+1}, B_1 v_{n+1})_H \\ &\quad - h(\mathcal{L}\varphi_{n+1}, B_1 v_{n+1})_H + h(B_2\theta_{n+1}, B_1 v_{n+1})_H \end{aligned}$$

and then by Young's inequality, the boundedness of the operator $L : H \to H$, (A11) and Lemma 4.1, there exists a constant $C_2 = C_2(T) > 0$ satisfying

$$\begin{aligned} h\|B_1 v_{n+1}\|_H^2 &\le C_2 h\|z_{n+1}\|_H^2 - h(A_2\varphi_{n+1}, B_1 v_{n+1})_H + C_2 h\|\Phi\varphi_{n+1}\|_H^2 \\ &\quad + C_2 h\|B_2\theta_{n+1}\|_H^2 + C_2 h \end{aligned} \tag{4.28}$$

for all $h \in (0, h_4)$. From (A4) we have

$$\begin{aligned} -h(A_2\varphi_{n+1}, B_1 v_{n+1})_H &= -(A_2\varphi_{n+1}, B_1\varphi_{n+1} - B_1\varphi_n)_H \\ &= -\frac{1}{2}(A_2\varphi_{n+1}, B_1\varphi_{n+1})_H + \frac{1}{2}(A_2\varphi_n, B_1\varphi_n)_H \\ &\quad - \frac{1}{2}(A_2(\varphi_{n+1} - \varphi_n), B_1(\varphi_{n+1} - \varphi_n))_H. \end{aligned} \tag{4.29}$$

Thus summing (4.28) over $n = 0, \dots, m-1$ with $1 \le m \le N$ and using (4.27), (4.29), Lemmas 4.3 and 4.4 imply the existence of a constant $C_3 = C_3(T) > 0$ such that

$$\|B_1\overline{v}_h\|^2_{L^2(0,T;H)} \le C_3 \tag{4.30}$$

for all $h \in (0, h_4)$. Moreover, from the second equation in (1.7), (4.27), (4.30), Lemmas 4.3 and 4.4, (A11) and Lemma 4.1 there exists a constant $C_4 = C_4(T) > 0$ satisfying

$$\|A_2\overline{\varphi}_h\|^2_{L^2(0,T;H)} \le C_4$$

for all $h \in (0, h_4)$. □

Lemma 4.8. *Let h_4 be as in Lemma 4.5. Then there exists a constant $C = C(T) > 0$ such that*

$$\begin{aligned} &\|\widehat{\varphi}_h\|_{W^{1,\infty}(0,T;V)} + \|\widehat{v}_h\|_{W^{1,\infty}(0,T;H)} + \|\widehat{v}_h\|_{L^\infty(0,T;V)} \\ &+ \|\widehat{\theta}_h\|_{H^1(0,T;V)} + \|\widehat{\theta}_h\|_{L^\infty(0,T;V)} \le C \end{aligned}$$

for all $h \in (0, h_4)$.

The above lemma follows from (1.8)-(1.10) and Lemmas 4.1, 4.3, 4.5 and 4.6.

Proof of Theroem 1.4 (existence part). By Lemmas 4.1, 4.3-4.8, and (1.11)-(1.13), there exist functions

$$\begin{gathered} \theta \in H^1(0,T;V) \cap L^\infty(0,T;V) \cap L^\infty(0,T;D(A_1)), \\ \varphi \in L^\infty(0,T;V) \cap L^2(0,T;D(A_2)), \\ \xi \in L^\infty(0,T;H) \end{gathered}$$

such that

$$\frac{d\varphi}{dt} \in L^\infty(0,T;V) \cap L^2(0,T;D(B_1)), \ \frac{d^2\varphi}{dt^2} \in L^\infty(0,T;H)$$

and

$$\widehat{\varphi}_h \to \varphi \quad \text{weakly}^* \text{ in } W^{1,\infty}(0,T;V), \tag{4.31}$$

$$\overline{v}_h \to \frac{d\varphi}{dt} \quad \text{weakly}^* \text{ in } L^\infty(0,T;V), \tag{4.32}$$

$$\widehat{v}_h \to \frac{d\varphi}{dt} \quad \text{weakly}^* \text{ in } W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V), \tag{4.33}$$

$$\overline{z}_h \to \frac{d^2\varphi}{dt^2} \quad \text{weakly}^* \text{ in } L^\infty(0,T;H), \tag{4.34}$$

$$L\overline{z}_h \to L\frac{d^2\varphi}{dt^2} \quad \text{weakly}^* \text{ in } L^\infty(0,T;H), \tag{4.35}$$

$$\widehat{\theta}_h \to \theta \quad \text{weakly}^* \text{ in } H^1(0,T;V) \cap L^\infty(0,T;V), \tag{4.36}$$

$$\overline{\varphi}_h \to \varphi \quad \text{weakly}^* \text{ in } L^\infty(0,T;V), \tag{4.37}$$

$$\overline{\theta}_h \to \theta \quad \text{weakly}^* \text{ in } L^\infty(0,T;V), \tag{4.38}$$
$$A_1\overline{\theta}_h \to A_1\theta \quad \text{weakly}^* \text{ in } L^\infty(0,T;H), \tag{4.39}$$
$$B_1\overline{v}_h \to B_1\frac{d\varphi}{dt} \quad \text{weakly in } L^2(0,T;H), \tag{4.40}$$
$$A_2\overline{\varphi}_h \to A_2\varphi \quad \text{weakly in } L^2(0,T;H), \tag{4.41}$$
$$\Phi\overline{\varphi}_h \to \xi \quad \text{weakly}^* \text{ in } L^\infty(0,T;H), \tag{4.42}$$
$$B_2\overline{\theta}_h \to B_2\theta \quad \text{weakly}^* \text{ in } L^\infty(0,T;H) \tag{4.43}$$

as $h = h_j \to +0$. From Lemma 4.8, the compactness of the embedding $V \hookrightarrow H$ and the convergence (4.31) we infer that

$$\widehat{\varphi}_h \to \varphi \quad \text{strongly in } C([0,T];H) \tag{4.44}$$

as $h = h_j \to +0$ (see e.g., [11, Section 8, Corollary 4]). From (1.11) and Lemma 4.3 we have

$$\overline{\varphi}_h \to \varphi \quad \text{strongly in } L^\infty(0,T;H) \tag{4.45}$$

as $h = h_j \to +0$. Hence the convergences (4.42) and (4.45) yield

$$\int_0^T (\Phi\overline{\varphi}_h(t), \overline{\varphi}_h(t))_H\, dt \to \int_0^T (\xi(t), \varphi(t))_H\, dt$$

as $h = h_j \to +0$ and then

$$\xi = \Phi\varphi \quad \text{in } H \text{ a.e. on } (0,T) \tag{4.46}$$

(see e.g., [1, Lemma 1.3, p. 42]). On the other hand, from Lemma 4.8, the compactness of the embedding $V \hookrightarrow H$ and (4.36) it follows that

$$\widehat{\theta}_h \to \theta \quad \text{strongly in } C([0,T];H) \tag{4.47}$$

as $h = h_j \to +0$. Similarly, we derive from (4.33) that

$$\widehat{v}_h \to \frac{d\varphi}{dt} \quad \text{strongly in } C([0,T];H) \tag{4.48}$$

as $h = h_j \to +0$. Therefore, combining (4.31), (4.35), (4.36), (4.39)-(4.48) and (A11), we can verify that there exists a solution of (1.1). □

5. Uniqueness for (1.1)

Proof of Theorem 1.4 (uniqueness part). We let (θ, φ), $(\overline{\theta}, \overline{\varphi})$ be two solutions of (1.1) and put $\widetilde{\theta} := \theta - \overline{\theta}$, $\widetilde{\varphi} := \varphi - \overline{\varphi}$. Then by (1.15), Young's inequality, (A6), (A11), Lemma 4.1, the continuity of the embedding $V \hookrightarrow H$ and (A2), there exists a constant $C_1 = C_1(T) > 0$ satisfying

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|L^{1/2}\frac{d\widetilde{\varphi}}{dt}(t)\|_H^2 + \Big(B_1\frac{d\widetilde{\varphi}}{dt}(t), \frac{d\widetilde{\varphi}}{dt}(t)\Big)_H + \frac{1}{2}\frac{d}{dt}\big\|A_2^{1/2}\widetilde{\varphi}(t)\big\|_H^2 \\
&= \Big(B_2\widetilde{\theta}(t), \frac{d\widetilde{\varphi}}{dt}(t)\Big)_H - \Big(\Phi\varphi(t) - \Phi\overline{\varphi}(t), \frac{d\widetilde{\varphi}}{dt}(t)\Big)_H \\
&\quad - \Big(\mathcal{L}\varphi(t) - \mathcal{L}\overline{\varphi}(t), \frac{d\widetilde{\varphi}}{dt}(t)\Big)_H \\
&\le \Big(B_2\widetilde{\theta}(t), \frac{d\widetilde{\varphi}}{dt}(t)\Big)_H + \frac{C_\Phi^2}{2}(1 + \|\varphi(t)\|_V^p + \|\overline{\varphi}(t)\|_V^q)^2\|\widetilde{\varphi}(t)\|_V^2 \\
&\quad + \frac{C_{\mathcal{L}}^2}{2}\|\widetilde{\varphi}(t)\|_H^2 + \|\frac{d\widetilde{\varphi}}{dt}(t)\|_H^2 \\
&\le \Big(B_2\widetilde{\theta}(t), \frac{d\widetilde{\varphi}}{dt}(t)\Big)_H + C_1\|\widetilde{\varphi}(t)\|_V^2 + \frac{1}{c_L}\|L^{1/2}\frac{d\widetilde{\varphi}}{dt}(t)\|_H^2
\end{aligned} \tag{5.1}
$$

for a.a. $t \in (0, T)$. From Young's inequality, (A2) and the continuity of the embedding $V \hookrightarrow H$, there exists a constant $C_2 > 0$ such that

$$
\frac{1}{2}\frac{d}{dt}\|\widetilde{\varphi}(t)\|_H^2 = \Big(\frac{d\widetilde{\varphi}}{dt}(t), \widetilde{\varphi}(t)\Big)_H \le \frac{1}{2c_L}\|L^{1/2}\frac{d\widetilde{\varphi}}{dt}(t)\|_H^2 + C_2\|\widetilde{\varphi}(t)\|_V^2 \tag{5.2}
$$

for a.a. $t \in (0, T)$. We see from (A3) that

$$
\begin{aligned}
\frac{1}{2}\big\|A_2^{1/2}\widetilde{\varphi}(t)\big\|_H^2 + \frac{1}{2}\|\widetilde{\varphi}\|_H^2 &= \frac{1}{2}\langle A_2^*\widetilde{\varphi}(t), \widetilde{\varphi}(t)\rangle_{V^*,V} + \frac{1}{2}\|\widetilde{\varphi}\|_H^2 \\
&\ge \frac{\omega_{2,1}}{2}\|\widetilde{\varphi}(t)\|_V^2.
\end{aligned} \tag{5.3}
$$

Moreover, the identity (1.14) yields that

$$
\begin{aligned}
\Big(B_2\widetilde{\theta}(t), \frac{d\widetilde{\varphi}}{dt}(t)\Big)_H &= \frac{1}{\eta}\Big(B_2\widetilde{\theta}(t), -\frac{d\widetilde{\theta}}{dt}(t) - A_1\widetilde{\theta}(t)\Big)_H \\
&= -\frac{1}{2\eta}\frac{d}{dt}\big\|B_2^{1/2}\widetilde{\theta}(t)\big\|_H^2 - \frac{1}{\eta}(B_2\widetilde{\theta}(t), A_1\widetilde{\theta}(t))_H.
\end{aligned} \tag{5.4}
$$

From (5.1)-(5.4) and (A4), there exists a constant $C_3 = C_3(T) > 0$ such that

$$
\begin{aligned}
&\frac{1}{2}\|L^{1/2}\frac{d\widetilde{\varphi}}{dt}(t)\|_H^2 + \frac{\omega_{2,1}}{2}\|\widetilde{\varphi}(t)\|_V^2 + \frac{1}{2\eta}\|B_2^{1/2}\widetilde{\theta}(t)\|_H^2 \\
&\le C_3\int_0^t \|L^{1/2}\frac{d\widetilde{\varphi}}{dt}(s)\|_H^2\,ds + C_3\int_0^t \|\widetilde{\varphi}(s)\|_V^2\,ds
\end{aligned}
$$

for a.a. $t \in (0, T)$, whence we obtain that $\frac{d\widetilde{\varphi}}{dt} = \widetilde{\varphi} = 0$ by the Gronwall lemma and (A2). Then (1.14) leads to

$$
\frac{1}{2}\frac{d}{dt}\|\widetilde{\theta}(t)\|_H^2 + (A_1\widetilde{\theta}(t), \widetilde{\theta}(t))_H = 0. \tag{5.5}
$$

Thus $\widetilde{\theta} = 0$. $\square$

6. Error estimates

Proof of Theorem 1.5. Let h_4 be as in Lemma 4.5. Then, putting $z := \frac{dv}{dt}$, we derive from the identity $\frac{d\widehat{v}_h}{dt} = \overline{z}_h$, the second equation in (1.7) and (1.15) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|L^{1/2}(\widehat{v}_h(t) - v(t))\|_H^2 \\
&= (L(\overline{z}_h(t) - z(t)), \widehat{v}_h(t) - \overline{v}_h(t))_H + (L(\overline{z}_h(t) - z(t)), \overline{v}_h(t) - v(t))_H \\
&= (L(\overline{z}_h(t) - z(t)), \widehat{v}_h(t) - \overline{v}_h(t))_H - (B_1(\overline{v}_h(t) - v(t)), \overline{v}_h(t) - v(t))_H \\
&\quad - (A_2(\overline{\varphi}_h(t) - \varphi(t)), \overline{v}_h(t) - v(t))_H - (\Phi\overline{\varphi}_h(t) - \Phi\varphi(t), \overline{v}_h(t) - v(t))_H \\
&\quad - (\mathcal{L}\overline{\varphi}_h(t) - \mathcal{L}\varphi(t), \overline{v}_h(t) - v(t))_H + (B_2(\overline{\theta}_h(t) - \theta(t)), \overline{v}_h(t) - v(t))_H.
\end{aligned} \tag{6.1}$$

The boundedness of the operator $L : H \to H$ implies the existence of a constant $C_1 > 0$ such that

$$\begin{aligned}
(L(\overline{z}_h(t) - z(t)), \widehat{v}_h(t) - \overline{v}_h(t))_H &\le \|L(\overline{z}_h(t) - z(t))\|_H \|\widehat{v}_h(t) - \overline{v}_h(t)\|_H \\
&\le C_1 \|\overline{z}_h(t) - z(t)\|_H \|\widehat{v}_h(t) - \overline{v}_h(t)\|_H
\end{aligned} \tag{6.2}$$

for a.a. $t \in (0, T)$ and all $h \in (0, h_4)$. From the identities $\overline{v}_h = \frac{d\widehat{\varphi}_h}{dt}$, $v = \frac{d\varphi}{dt}$ and the boundedness of the operator $A_2^* : V \to V^*$, there exists a constant $C_2 > 0$ such that

$$\begin{aligned}
&- (A_2(\overline{\varphi}_h(t) - \varphi(t)), \overline{v}_h(t) - v(t))_H \\
&= -\langle A_2^*(\overline{\varphi}_h(t) - \widehat{\varphi}_h(t)), \overline{v}_h(t) - v(t)\rangle_{V^*,V} - \frac{1}{2}\frac{d}{dt}\|A_2^{1/2}(\widehat{\varphi}_h(t) - \varphi(t))\|_H^2 \\
&\le C_2 \|\overline{\varphi}_h(t) - \widehat{\varphi}_h(t)\|_V \|\overline{v}_h(t) - v(t)\|_V - \frac{1}{2}\frac{d}{dt}\|A_2^{1/2}(\widehat{\varphi}_h(t) - \varphi(t))\|_H^2
\end{aligned} \tag{6.3}$$

for a.a. $t \in (0, T)$ and all $h \in (0, h_4)$. From (A6), Lemma 4.1, Young's inequality and (A2), there exists a constant $C_3 = C_3(T) > 0$ such that

$$\begin{aligned}
&- (\Phi\overline{\varphi}_h(t) - \Phi\varphi(t), \overline{v}_h(t) - v(t))_H \\
&\le C_\Phi(1 + \|\overline{\varphi}_h(t)\|_V^p + \|\varphi(t)\|_V^q)\|\overline{\varphi}_h(t) - \varphi(t)\|_V \|\overline{v}_h(t) - v(t)\|_H \\
&\le C_3 \|\overline{\varphi}_h(t) - \varphi(t)\|_V \|\overline{v}_h(t) - v(t)\|_H \\
&\le \frac{C_3}{2}\|\overline{\varphi}_h(t) - \varphi(t)\|_V^2 + \frac{C_3}{2}\|\overline{v}_h(t) - v(t)\|_H^2 \\
&\le C_3\|\overline{\varphi}_h(t) - \widehat{\varphi}_h(t)\|_V^2 + C_3\|\widehat{\varphi}_h(t) - \varphi(t)\|_V^2 \\
&\quad + C_3\|\overline{v}_h(t) - \widehat{v}_h(t)\|_H^2 + \frac{C_3}{c_L}\|L^{1/2}(\widehat{v}_h(t) - v(t))\|_H^2
\end{aligned} \tag{6.4}$$

for a.a. $t \in (0, T)$ and all $h \in (0, h_4)$. By (A11), the continuity of the embedding $V \hookrightarrow H$, Young's inequality and (A2), there exists a constant $C_4 > 0$ such that

$$\begin{aligned}
&- (\mathcal{L}\overline{\varphi}_h(t) - \mathcal{L}\varphi(t), \overline{v}_h(t) - v(t))_H \\
&\le C_4 \|\overline{\varphi}_h(t) - \varphi(t)\|_V \|\overline{v}_h(t) - v(t)\|_H \\
&\le \frac{C_4}{2}\|\overline{\varphi}_h(t) - \varphi(t)\|_V^2 + \frac{C_4}{2}\|\overline{v}_h(t) - v(t)\|_H^2 \\
&\le C_4\|\overline{\varphi}_h(t) - \widehat{\varphi}_h(t)\|_V^2 + C_4\|\widehat{\varphi}_h(t) - \varphi(t)\|_V^2 \\
&\quad + C_4\|\overline{v}_h(t) - \widehat{v}_h(t)\|_H^2 + \frac{C_4}{c_L}\|L^{1/2}(\widehat{v}_h(t) - v(t))\|_H^2
\end{aligned} \tag{6.5}$$

for a.a. $t \in (0,T)$ and all $h \in (0,h_4)$. From the first equation in (1.7) and (1.14) it follows that

$$
\begin{aligned}
&(B_2(\overline{\theta}_h(t)-\theta(t)),\overline{v}_h(t)-v(t))_H\\
&= -\frac{1}{\eta}\Big(B_2(\overline{\theta}_h(t)-\theta(t)),\frac{d\widehat{\theta}_h}{dt}(t)-\frac{d\theta}{dt}(t)\Big)_H\\
&\quad -\frac{1}{\eta}(B_2(\overline{\theta}_h(t)-\theta(t)),A_1(\overline{\theta}_h(t)-\theta(t)))_H\\
&= -\frac{1}{\eta}\Big\langle B_2^*(\overline{\theta}_h(t)-\widehat{\theta}_h(t)),\frac{d\widehat{\theta}_h}{dt}(t)-\frac{d\theta}{dt}(t)\Big\rangle_{V^*,V}\\
&\quad -\frac{1}{2\eta}\frac{d}{dt}\|B_2^{1/2}(\widehat{\theta}_h(t)-\theta(t))\|_H^2\\
&\quad -\frac{1}{\eta}(B_2(\overline{\theta}_h(t)-\theta(t)),A_1(\overline{\theta}_h(t)-\theta(t)))_H.
\end{aligned}
\tag{6.6}
$$

From (6.1)-(6.6), the integration over $(0,t)$, where $t \in [0,T]$, the boundedness of the operator $B_2^* : V \to V^*$, (1.11)-(1.13), Lemmas 4.3, 4.6, and the inequalities $0 < h_4 < 1$, there exists a constant $C_5 = C_5(T) > 0$ such that

$$
\begin{aligned}
&\frac{1}{2}\|L^{1/2}(\widehat{v}_h(t)-v(t))\|_H^2+\frac{1}{2}\|A_2^{1/2}(\widehat{\varphi}_h(t)-\varphi(t))\|_H^2\\
&+\int_0^t\|B_1^{1/2}(\overline{v}_h(s)-v(s))\|_H^2\,ds+\frac{1}{2\eta}\|B_2^{1/2}(\widehat{\theta}_h(t)-\theta(t))\|_H^2\\
&+\frac{1}{\eta}\int_0^t(B_2(\overline{\theta}_h(s)-\theta(s)),A_1(\overline{\theta}_h(s)-\theta(s)))_H\,ds\\
&\le C_5h+C_5\int_0^t\|\widehat{\varphi}_h(s)-\varphi(s)\|_V^2\,ds+C_5\int_0^t\|L^{1/2}(\widehat{v}_h(s)-v(s))\|_H^2\,ds
\end{aligned}
\tag{6.7}
$$

for all $t \in [0,T]$ and all $h \in (0,h_4)$. From $\frac{d\widehat{\varphi}_h}{dt} = \overline{v}_h$, $\frac{d\varphi}{dt} = v$, Young's inequality, (A2) and the continuity of the embedding $V \hookrightarrow H$, there exists a constant $C_6 > 0$ such that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widehat{\varphi}_h(t)-\varphi(t)\|_H^2\\
&=(\overline{v}_h(t)-v(t),\widehat{\varphi}_h(t)-\varphi(t))_H\\
&\le\frac{1}{2}\|\overline{v}_h(t)-v(t)\|_H^2+\frac{1}{2}\|\widehat{\varphi}_h(t)-\varphi(t)\|_H^2\\
&\le\|\overline{v}_h(t)-\widehat{v}_h(t)\|_H^2+\frac{1}{c_L}\|L^{1/2}(\widehat{v}_h(t)-v(t))\|_H^2+C_6\|\widehat{\varphi}_h(t)-\varphi(t)\|_V^2
\end{aligned}
\tag{6.8}
$$

for a.a. $t \in (0,T)$ and all $h \in (0,h_4)$. Thus, integrating (6.8) over $(0,t)$, where $t \in [0,T]$, we deduce from (6.7) and (A3) that there exists a constant $C_7 = C_7(T) > 0$

such that

$$
\begin{aligned}
&\frac{1}{2}\|L^{1/2}(\widehat{v}_h(t)-v(t))\|_H^2+\frac{\omega_{2,1}}{2}\|\widehat{\varphi}_h(t)-\varphi(t)\|_V^2 \\
&+\int_0^t\|B_1^{1/2}(\overline{v}_h(s)-v(s))\|_H^2\,ds+\frac{1}{2\eta}\|B_2^{1/2}(\widehat{\theta}_h(t)-\theta(t))\|_H^2 \\
&+\frac{1}{\eta}\int_0^t(B_2(\overline{\theta}_h(s)-\theta(s)),A_1(\overline{\theta}_h(s)-\theta(s)))_H\,ds \\
&\le C_7h+C_7\int_0^t\|\widehat{\varphi}_h(s)-\varphi(s)\|_V^2\,ds+C_7\int_0^t\|L^{1/2}(\widehat{v}_h(s)-v(s))\|_H^2\,ds
\end{aligned}
\tag{6.9}
$$

for all $t\in[0,T]$ and all $h\in(0,h_4)$.

Next the first equation in (1.7) and (1.14) lead to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widehat{\theta}_h(t)-\theta(t)\|_H^2 \\
&=-\eta(\overline{v}_h(t)-v(t),\widehat{\theta}_h(t)-\theta(t))_H-(A_1(\overline{\theta}_h(t)-\theta(t)),\widehat{\theta}_h(t)-\overline{\theta}_h(t))_H \\
&\quad-\langle A_1^*(\overline{\theta}_h(t)-\theta(t)),\overline{\theta}_h(t)-\theta(t)\rangle_{V^*,V}.
\end{aligned}
\tag{6.10}
$$

Here we use the Young inequality and (A2) to infer that

$$
\begin{aligned}
&-(\overline{v}_h(t)-v(t),\widehat{\theta}_h(t)-\theta(t))_H \\
&\le\frac{1}{2}\|\overline{v}_h(t)-v(t)\|_H^2+\frac{1}{2}\|\widehat{\theta}_h(t)-\theta(t)\|_H^2 \\
&\le\|\overline{v}_h(t)-\widehat{v}_h(t)\|_H^2+\|\widehat{v}_h(t)-v(t)\|_H^2+\frac{1}{2}\|\widehat{\theta}_h(t)-\theta(t)\|_H^2 \\
&\le\|\overline{v}_h(t)-\widehat{v}_h(t)\|_H^2+\frac{1}{c_L}\|L^{1/2}(\widehat{v}_h(t)-v(t))\|_H^2+\frac{1}{2}\|\widehat{\theta}_h(t)-\theta(t)\|_H^2.
\end{aligned}
\tag{6.11}
$$

We have from (A3) that

$$
\begin{aligned}
&-\langle A_1^*(\overline{\theta}_h(t)-\theta(t)),\overline{\theta}_h(t)-\theta(t)\rangle_{V^*,V} \\
&\le-\omega_{1,1}\|\overline{\theta}_h(t)-\theta(t)\|_V^2+\|\overline{\theta}_h(t)-\theta(t)\|_H^2 \\
&\le-\omega_{1,1}\|\overline{\theta}_h(t)-\theta(t)\|_V^2+2\|\overline{\theta}_h(t)-\widehat{\theta}_h(t)\|_H^2+2\|\widehat{\theta}_h(t)-\theta(t)\|_H^2.
\end{aligned}
\tag{6.12}
$$

Hence, owing to (6.10)-(6.12), the integration over $(0,t)$, where $t\in[0,T]$, (1.12), (1.13), Lemmas 4.3 and 4.6, there exists a constant $C_8=C_8(T)>0$ such that

$$
\begin{aligned}
&\frac{1}{2}\|\widehat{\theta}_h(t)-\theta(t)\|_H^2+\omega_{1,1}\int_0^t\|\overline{\theta}_h(s)-\theta(s)\|_V^2\,ds \\
&\le C_8h+C_8\int_0^t\|L^{1/2}(\widehat{v}_h(s)-v(s))\|_H^2\,ds+C_8\int_0^t\|\widehat{\theta}_h(s)-\theta(s)\|_H^2\,ds
\end{aligned}
\tag{6.13}
$$

for all $t\in[0,T]$ and all $h\in(0,h_4)$.

Therefore we can obtain Theorem 1.5 by combining (6.9), (6.13) and by applying the Gronwall lemma. □

Acknowledgments. The author would like to thank the anonymous referees for their careful reading, helpful comments and suggestions. The author is supported by JSPS Research Fellowships for Young Scientists (No. 18J21006).

References

[1] V. Barbu; *Nonlinear Semigroups and Differential Equations in Banach spaces*, Noordhoff International Publishing, Leyden, 1976.

[2] V. Barbu; *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer, New York, 2010.

[3] P. Colli, A. Favini; *Time discretization of nonlinear Cauchy problems applying to mixed hyperbolic-parabolic equations*, Internat. J. Math. Math. Sci., **19** (1996), 481–494.

[4] P. Colli, S. Kurima; *Time discretization of a nonlinear phase-field system in general domains*, Commun. Pure Appl. Anal., **18** (2019), 3161–3179.

[5] M. Grasselli, V. Pata; *Existence of a universal attractor for a parabolic-hyperbolic phase-field system*, Adv. Math. Sci. Appl., **13** (2003), 443-459.

[6] M. Grasselli, V. Pata; *Asymptotic behavior of a parabolic-hyperbolic system*, Comm. Pure Appl. Anal., **3** (2004), 849-881.

[7] M. Grasselli, H. Petzeltová, G. Schimperna; *Convergence to stationary solutions for a parabolic-hyperbolic phase-field system*, Commun. Pure Appl. Anal., **5** (2006), 827–838.

[8] J. W. Jerome; *Approximations of Nonlinear Evolution Systems*, Mathematics in Science and Engineering **164**, Academic Press Inc., Orlando, 1983.

[9] S. Kurima; *Time discretization of an initial value problem for a simultaneous abstract evolution equation applying to parabolic-hyperbolic phase-field systems*, ESAIM Math. Model. Numer. Anal., **54** (2020), 977–1002.

[10] A. Matsubara, T. Yokota; *Applications of the Hille–Yosida theorem to the linearized equations of coupled sound and heat flow*, AIMS Mathematics **1** (2016), 165–177.

[11] J. Simon; *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl., (4) **146** (1987), 65–96.

[12] H. Wu, M. Grasselli, S. Zheng; *Convergence to equilibrium for a parabolic-hyperbolic phase-field system with Neumann boundary conditions*, Math. Models Methods Appl. Sci., **17** (2007), 125–153.

[13] H. Wu, M. Grasselli, S. Zheng; *Convergence to equilibrium for a parabolic-hyperbolic phase-field system with dynamical boundary condition*, J. Math. Anal. Appl., **329** (2007), 948-976.

Shunsuke Kurima
Department of Mathematics, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan
Email address: shunsuke.kurima@gmail.com