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CONTINUITY OF ATTRACTORS FOR $\mathcal{C}^1$ PERTURBATIONS OF A SMOOTH DOMAIN

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Abstract. We consider a family of semilinear parabolic problems with nonlinear boundary conditions

$$u_t(x,t) = \Delta u(x,t) - au(x,t) + f(u(x,t)), \quad x \in \Omega_\epsilon,\ t>0\,,$$
$$\frac{\partial u}{\partial N}(x,t) = g(u(x,t)), \quad x \in \partial\Omega_\epsilon,\ t>0\,,$$

where $\Omega_0 \subset \mathbb{R}^n$ is a smooth (at least $\mathcal{C}^2$) domain, $\Omega_\epsilon = h_\epsilon(\Omega_0)$ and h_ϵ is a family of diffeomorphisms converging to the identity in the $\mathcal{C}^1$-norm. Assuming suitable regularity and dissipative conditions for the nonlinearites, we show that the problem is well posed for $\epsilon > 0$ sufficiently small in a suitable scale of fractional spaces, the associated semigroup has a global attractor $\mathcal{A}_\epsilon$ and the family $\{\mathcal{A}_\epsilon\}$ is continuous at $\epsilon = 0$.

1. Introduction

Let $\Omega = \Omega_0 \subset \mathbb{R}^n$ be a $\mathcal{C}^2$ domain, a a positive number, $f, g : \mathbb{R} \to \mathbb{R}$ real functions, and consider the family of semilinear parabolic problems with nonlinear Neumann boundary conditions,

$$\begin{gathered} u_t(x,t) = \Delta u(x,t) - au(x,t) + f(u(x,t)), \quad x \in \Omega_\epsilon,\ t>0\,, \\ \frac{\partial u}{\partial N}(x,t) = g(u(x,t)), \quad x \in \partial\Omega_\epsilon,\ t>0\,, \end{gathered} \tag{1.1}$$

where $\Omega_\epsilon = \Omega_{h_\epsilon} = h_\epsilon(\Omega_0)$ and $h_\epsilon : \Omega_0 \to \mathbb{R}^n$ is a family of $\mathcal{C}^m (m \geq 2)$ maps satisfying suitable conditions to be specified later.

One of the central questions concerning this problem is the existence and properties of *global attractors* since, as it is well known, they determine the dynamics of the entire system (see, for example [8] or [19]). The continuity with respect to parameters present in the equation is also of interest, since it can be seen as a desirable property of "robustness" in the model. In many cases, however, the form of the equation is fixed, so the 'parameter'of interest is the domain where the problem is posed.

The existence of a global compact attractor for the problem (1.1) has been proved in [6, 13], under stronger smoothness hypotheses on the domains and growth and dissipative conditions on the nonlinearities f and g.

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The problem of existence and continuity of global attractors for semilinear parabolic problems, with respect to change of domains has also been considered in [3], for the problem with homogeneous boundary conditions

$$u_t = \Delta u + f(x,u) \quad \text{in } \Omega_\epsilon$$
$$\frac{\partial u}{\partial N} = 0 \quad \text{on } \partial\Omega_\epsilon\,,$$

where Ω_ϵ, $0 \le \epsilon \le \epsilon_0$ are bounded domains with Lipschitz boundary in $\mathbb{R}^N$, $N \ge 2$. The authors proved that, if the perturbations are such that the convergence of the eigenvalues and eigenfunctions of the linear part of the problem can be shown, than the upper semicontinuity of attractors follow. With the additional assumption that the equilibria are all hyperbolic, the lower semicontinuity is also obtained.

The behavior of the equilibria of (1.1) was studied in [1, 2]. In these papers, the authors consider a family of smooth domains $\Omega_\epsilon \subset \mathbb{R}^N$, $N \ge 2$ and $0 \le \epsilon \le \epsilon_0$ whose boundary oscillates rapidly when the parameter $\epsilon \to 0$ and prove that the equilibria, as well as the spectra of the linearized problem around them, converge to the solution of a "limit problem".

In [16] the authors prove the continuity of the attractors of (1.1) with respect to C^2-perturbations of a smooth domain of $\mathbb{R}^n$. These results do not extend immediately to the case considered here, due to the lack of smoothness of the domains considered and the fact that the perturbations do not converge to the inclusion in the C^2-norm.

In this work, we follow the general approach of [16], which basically consists in "pull-backing" the perturbed problems to the fixed domain Ω and then considering the family of abstract semilinear problems thus generated. We present a brief overview of this approach in the next section for convenience. Our aim here is then to prove well-posedness, establish the existence of a global attractor $\mathcal{A}_\epsilon$, for sufficiently small $\epsilon \ge 0$ and prove that the family of attractors is continuous at $\epsilon = 0$.

These results were obtained in our previous paper [5] for the family of perturbations of the unit square in $\mathbb{R}^2$ given by

$$h_\epsilon(x_1,x_2) = (x_1, x_2 + x_2\epsilon\sin(x_1/\epsilon^\alpha)) \tag{1.2}$$

with $0 < \alpha < 1$ and $\epsilon > 0$ sufficiently small, (see Figure 1).

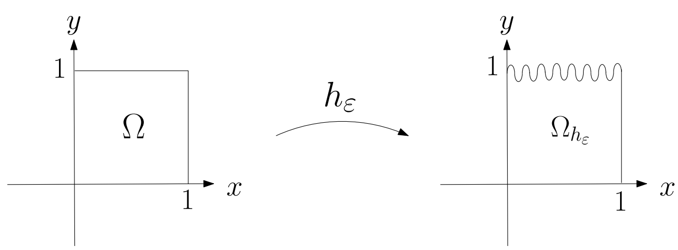

FIGURE 1. Perturbed region

In this article, we generalize these results in two directions: we consider the problem in arbitrary spatial dimensions and, also, instead of a specific family of perturbations, we consider a general family $h_\epsilon : \Omega_0 \to \mathbb{R}^n$ of $\mathcal{C}^m, (m \ge 2)$ maps satisfying the following abstract hypotheses:

(H1) $\|h_\epsilon - i_{\Omega_0}\|_{\mathcal{C}^1(\Omega)} \to 0$ as $\epsilon \to 0$, where i_{Ω_0} is the inclusion map of Ω_0 into $\mathbb{R}^n$.
(H2) The Jacobian determinant Jh_ϵ of h_ϵ is differentiable, and $\|\nabla Jh_\epsilon\|_\infty = \sup\{\|\nabla Jh_\epsilon(x)\|,\ x \in \Omega\} \to 0$ as $\epsilon \to 0$.

In section 4 we show that the family h_ϵ considered in [5] satisfies the conditions (H1) and (H2). Since the domain Ω is not of class $\mathcal{C}^1$, the results obtained here do not immediately apply. However, since the perturbations occur only in a smooth portion of the boundary, they could easily be adapted to this case. We also give more general examples of families satisfying our properties.

This article is organized as follows: in section 2 we show how the problem can be reduced to a family of problems in the initial domain and collect some results needed later. In section 3 we give some rather general examples of families satisfying our basic assumptions. In section 4 we show that the perturbed linear operators are sectorial operators in suitable spaces and study properties of the linear semigroup generated by them. In section 5 we show that the problem (1.1) can be reformulated as an abstract problem in a scale of Banach spaces which are shown to be locally well-posed in section 6, under suitable growth assumptions on f and g. In section 7, assuming a dissipative condition for the problem, we use comparison results to prove that the solutions are globally defined and the family of associated semigroups are uniformly bounded. In section 8 we prove the existence of global attractors. In section 9, we show that these attractors behave upper semicontinuously. Finally, in section 10, with some additional properties on the nonlinearities and on the set of equilibria, we show that they are also lower semicontinuous at $\epsilon = 0$.

2. Reduction to a fixed domain

One of the difficulties encountered in problems of perturbation of the domain is that the function spaces change with the change of the region. One way to overcome this difficulty is to effect a “change of variables” in order to bring the problem back to a fixed region. This approach was developed by D. Henry in [9] and is the one we adopt here. We describe it briefly here, for convenience of the reader. For a different approach, see [1, 2, 3].

Given an open bounded C^m region $\Omega \subset \mathbb{R}^n$, $m \geq 1$, denote by $\mathrm{Diff}^m(\Omega), m \geq 0$, the set of $\mathcal{C}^m$ embeddings (i.e. diffeomorphisms from Ω to its image).

We define a topology in $\mathrm{Diff}^m(\Omega)$, by declaring that Ω is in a ϵ neighborhood of Ω_0, if $\Omega = h(\Omega_0)$, with $\|h - i_{\Omega_0}\|_{\mathcal{C}^m(\Omega_0)} < \epsilon$. It has been shown in [12] that this topology is metrizable and we denote by $\mathcal{M}_m(\Omega)$ or simply $\mathcal{M}_m$ this (separable) metric space. We say that a function F defined in the space $\mathcal{M}_m$ with values in a Banach space is C^m or analytic if $h \mapsto F(h(\Omega))$ is C^m or analytic as a map of Banach spaces (h near i_Ω in $C^m(\Omega, \mathbb{R}^n)$). In this sense, we may express problems of perturbation of the boundary of a boundary value problem as problems of differential calculus in Banach spaces.

If $h : \Omega \mapsto \mathbb{R}^n$ is a C^k, $k \leq m$ embedding, we may consider the ‘pull-back’ of h

$$h^* : C^k(h(\Omega)) \to C^k(\Omega) \quad (0 \leq k \leq m)$$

defined by $h^*(\varphi) = \varphi \circ h$, which is an isomorphism with inverse ${h^{-1}}^*$. Other function spaces can be used instead of C^k, and we will actually be interested mainly in Sobolev spaces and fractional power spaces.

Now, if $F_{h(\Omega)} : C^m(h(\Omega)) \to C^0(h(\Omega))$ is a (generally nonlinear) differential operator in $\Omega_h = h(\Omega)$ we may consider the operator $h^* F_{h(\Omega)} {h^*}^{-1}$, which is a differential operator in the fixed region Ω.

Let now $h_\epsilon : \Omega_0 \to \mathbb{R}^n$ be a family of maps satisfying the conditions (H1) and (H2) and $\Omega_\epsilon = h_\epsilon(\Omega)$ the corresponding family of "perturbed domains".

Lemma 2.1. *If $\epsilon > 0$ is sufficiently small, the map h_ϵ belongs to $\mathit{Diff}^m(\Omega) =$ diffeomorphisms from Ω to its image.*

The proof of the above lemma is straightforward; we omit it.

Lemma 2.2. *If $0 < s \le m$ and $\epsilon > 0$ is small enough, the map $h_\epsilon^* : H^s(\Omega_\epsilon) \to H^s(\Omega)$ given by $u \mapsto u \circ h_\epsilon$ is an isomorphism, with inverse ${h_\epsilon^*}^{-1} = (h_\epsilon^{-1})^*$.*

For a proof of the above lemma see [5]. Using Lemma 2.1 we may bring the problem (1.1) back to the fixed region Ω_0. For this purpose, observe that $v(\cdot, t)$ is a solution of (1.1) in the perturbed region $\Omega_\epsilon = h_\epsilon(\Omega)$, if and only if $u(\cdot, t) = h_\epsilon^* v(\cdot, t)$ satisfies

$$\begin{gathered} u_t(x,t) = h_\epsilon^* \Delta_{\Omega_\epsilon} {h_\epsilon^*}^{-1} u(x,t) - au(x,t) + f(u(x,t)), \quad x \in \Omega,\ t > 0, \\ h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} {h_\epsilon^*}^{-1} u(x,t) = g(u(x,t)), \quad x \in \partial\Omega,\ > 0\,, \end{gathered} \tag{2.1}$$

where $h_\epsilon^* \Delta_{\Omega_\epsilon} {h_\epsilon^*}^{-1}$ and $h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} {h_\epsilon^*}^{-1}$ are defined by

$$\begin{gathered} h_\epsilon^* \Delta_{\Omega_\epsilon} {h_\epsilon^*}^{-1} u(x) = \Delta_{\Omega_\epsilon} (u \circ h_\epsilon^{-1})(h_\epsilon(x)), \\ h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} {h_\epsilon^*}^{-1} u(x) = \frac{\partial}{\partial N_{\Omega_\epsilon}} (u \circ h_\epsilon^{-1})(h_\epsilon(x)) \end{gathered}$$

(in appropriate spaces). In particular, if $\mathcal{A}_\epsilon$ is the global attractor of (1.1) in $H^s(\Omega_\epsilon)$, then $\tilde{\mathcal{A}}_\epsilon = \{v \circ h_\epsilon : v \in \mathcal{A}_\epsilon\}$ is the global attractor of (2.1) in $H^s(\Omega)$ and conversely. In this way we can consider the problem of continuity of the attractors as $\epsilon \to 0$ in a fixed phase space.

For later use, we compute an expression for the differential operator $h_\epsilon^* \Delta_{\Omega_\epsilon} {h_\epsilon^*}^{-1}$ in the fixed region Ω, in terms of h_ϵ. Writing

$$h_\epsilon(x) = h_\epsilon(x_1, x_2, \dots, x_n) = ((h_\epsilon)_1(x), (h_\epsilon)_2(x), \dots, (h_\epsilon)_n(x)) = (y_1, y_2, \dots, y_n) = y,$$

for $i = 1, 2, \dots, n$, we obtain

$$\begin{aligned} \Big(h_\epsilon^* \frac{\partial}{\partial y_i} {h_\epsilon^*}^{-1}(u)\Big)(x) &= \frac{\partial}{\partial y_i} (u \circ h_\epsilon^{-1})(h_\epsilon(x)) \\ &= \sum_{j=1}^n \Big[\Big(\frac{\partial h_\epsilon}{\partial x_j}\Big)^{-1}\Big]_{j,i}(x) \frac{\partial u}{\partial x_j}(x) \\ &= \sum_{j=1}^n b_{ij}^\epsilon(x) \frac{\partial u}{\partial x_j}(x)\,, \end{aligned} \tag{2.2}$$

where $b_{ij}^\epsilon(x)$ is the i,j-entry of the inverse transpose of the Jacobian matrix of h_ϵ. From now on, we omit the ϵ from the notation for simplicity. Therefore,

$$\begin{aligned}
&h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1}(u)(x) \\
&= \sum_{i=1}^n \Big(h_\epsilon^* \frac{\partial^2}{\partial y_i^2} h_\epsilon^{*-1}(u) \Big)(x) \\
&= \sum_{i=1}^n \Big(\sum_{k=1}^n b_{i\,k} \frac{\partial}{\partial x_k} \Big(\sum_{j=1}^n b_{ij} \frac{\partial u}{\partial x_j} \Big) \Big)(x) \\
&= \sum_{k=1}^n \frac{\partial}{\partial x_k} \Big(\sum_{j=1}^n \sum_{i=1}^n b_{ij} b_{ik} \frac{\partial u}{\partial x_j} \Big)(x) - \sum_{j=1}^n \Big(\sum_{i,k=1}^n \frac{\partial}{\partial x_k}(b_{ik}) b_{ij} \Big) \frac{\partial u}{\partial x_j}(x) \\
&= \sum_{k=1}^n \frac{\partial}{\partial x_k} \Big(\sum_{j=1}^n C_{kj} \frac{\partial u}{\partial x_j} \Big)(x) - \sum_{j=1}^n A_j \frac{\partial u}{\partial x_j}(x),
\end{aligned} \tag{2.3}$$

where $C_{kj} = \sum_{i=1}^n b_{ij} b_{ik}$ and $A_j = \sum_{i,k=1}^n \frac{\partial}{\partial x_k}(b_{ik}) b_{ij}$.

We also need to compute the boundary condition $h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{*-1} u = 0$ in the fixed region Ω in terms of h_ϵ. Let $N_{h_\epsilon(\Omega)}$ denote the outward unit normal to the boundary of $h_\epsilon(\Omega) := \Omega_\epsilon$, and $b_{ij}^\epsilon(x)$ the i,j-entry of the inverse transpose of the Jacobian matrix of h_ϵ. From (2.2), we obtain

$$\begin{aligned}
\Big(h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{*-1} u \Big)(x) &= \sum_{i=1}^n \Big(h_\epsilon^* \frac{\partial}{\partial y_i} h_\epsilon^{*-1} u \Big)(x) \big(N_{\Omega_\epsilon} \big)_i (h_\epsilon(x)) \\
&= \sum_{i=1}^n \frac{\partial}{\partial y_i} (u \circ h_\epsilon^{-1})(h_\epsilon(x)) \, (N_{\Omega_\epsilon})_i \, (h_\epsilon(x)) \\
&= \sum_{i,j=1}^n b_{ij}(x) \frac{\partial u}{\partial x_j}(x) \, (N_{\Omega_\epsilon})_i \, (h_\epsilon(x)).
\end{aligned} \tag{2.4}$$

Since

$$N_{\Omega_\epsilon}(h_\epsilon(x)) = h_\epsilon^* N_{\Omega_\epsilon}(x) = \frac{[h_\epsilon^{-1}]_x^T N_\Omega(x)}{\| [h_\epsilon^{-1}]_x^T N_\Omega(x) \|}$$

(see [9]), we obtain

$$(N_{\Omega_\epsilon}(h_\epsilon(x)))_i = \frac{1}{\| [h_\epsilon^{-1}]_x^T N_\Omega(x) \|} \sum_{k=1}^n b_{ik} (N_\Omega)_k(x).$$

Thus, from (2.4),

$$\begin{aligned}
&\Big(h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{*-1} u \Big)(x) \\
&= \frac{1}{\| [h_\epsilon^{-1}]_x^T N_\Omega(x) \|} \sum_{k=1}^n \Big(\sum_{i,j=1}^n b_{ik} b_{ij}(x) \frac{\partial u}{\partial x_j}(x) \Big) (N_\Omega)_k(x) \\
&= \frac{1}{\| [h_\epsilon^{-1}]_x^T N_\Omega(x) \|} \sum_{k=1}^n \Big(\sum_{j=1}^n C_{kj} \frac{\partial u}{\partial x_j}(x) \Big) (N_\Omega)_k(x)
\end{aligned} \tag{2.5}$$

Thus, the boundary condition $\big(h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{*-1} u\big)(x) = 0$ becomes

$$\sum_{j,k=1}^{n} \left(N_\Omega(x)\right)_k \left(C_{kj} D_j u\right) = 0 \quad \text{on } \partial\Omega\,.$$

Therefore, the boundary condition is exactly the "oblique normal derivative" with respect to the divergence part of the operator $h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1}$.

3. Basic assumptions and examples on domain perturbations

We assume that the unperturbed domain Ω_0 is of class $\mathcal{C}^2$, and consider rather general examples of families $h_\epsilon : \Omega_0 \to \mathbb{R}^n$ of $\mathcal{C}^2$ maps satisfying the hypotheses (H1) and (H2) stated in the introduction.

Example 3.1. The family h_ϵ of perturbations of the unit square in $\mathbb{R}^2$ considered in [5], given by

$$h_\epsilon(x_1, x_2) = (x_1, x_2 + x_2 \epsilon \sin(x_1/\epsilon^\alpha)) \tag{3.1}$$

with $0 < \alpha < 1$ and $\epsilon > 0$ sufficiently small, (see figure (1)) satisfies the conditions (H1) and (H2). We observe that the unperturbed region is not of class $\mathcal{C}^2$ and, therefore, it does not strictly satisfies our hypothesis. However, since the perturbation occurs only at a smooth portion of the boundary and the elliptic problem in this case is well posed, (see [7]), the problem can actually be included in the framework considered here, with only minor modifications.

In fact, hypothesis (H1) was shown in [5, Lemma 2.1]. A simple computation gives $\nabla J h_\epsilon = (\epsilon^{(1-\alpha)} \cos(x_1/\epsilon^\alpha), 0)$, from which (H2) follows easily.

From (H1), it follows that the boundary Jacobian $\mu_\epsilon = J_{\partial\Omega} h_\epsilon\big|_{\partial\Omega} \to 1$ uniformly as $\epsilon \to 0$. It can be checked by explicitly computation, as done in [5]:

$$\mu_\epsilon = \begin{cases} \frac{\sqrt{1+\epsilon^{2-2\alpha} cos^2(x_1/\epsilon^\alpha)}}{1+\epsilon \sin(x_1/\epsilon^\alpha)} & \text{for } x \in I_1 := \{(x_1, 1) : 0 \le x_1 \le 1\}, \\ \frac{1}{1+\epsilon \sin(x_1/\epsilon^\alpha)} & \text{for } x \in I_3 := \{(x_1, 0) : 0 \le x_1 \le 1\}, \\ 1 & \text{for } x \in I_2 := \{(1, x_2) : 0 \le x_2 \le 1\} \\ & \text{or } x \in I_4 := \{(0, x_2) : 0 \le x_2 \le 1\}. \end{cases}$$

Much more general families satisfying the conditions (H1) and (H2) are given in the examples below.

Example 3.2. Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^2$ domain, and $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$ a smooth (say $\mathcal{C}^1$) vector field defined in an open set containing Ω and $x(t, x_0)$ the solution of

$$\begin{aligned} \frac{dx}{dt} &= X(x) \\ x(0) &= x_0. \end{aligned}$$

Then, the map

$$x : (t, \xi) \mapsto x(t, \xi) : (-r, r) \times \partial\Omega \to V \subset \mathbb{R}^n$$

is a diffeomorphism for some $r > 0$ and some open neighborhood V of $\partial\Omega$. Let W be a (smaller) open neighborhood of $\partial\Omega$, that is, with $\overline{W} \subset V$ and define $h_\epsilon : W \to \mathbb{R}^n$ by $h_\epsilon(x(t, \xi)) = (x(t + \eta(t) \cdot \theta_\epsilon(\xi), \xi))$, where $\theta_\epsilon : \partial\Omega \to \mathbb{R}$ is a $\mathcal{C}^1$ function, with $\|\theta_\epsilon\|_{\mathcal{C}^1(\partial\Omega)} \to 0$ as $\epsilon \to 0$, $\eta : [-r, r] \to [0, 1]$ is a $\mathcal{C}^2$ function, with $\eta(0) = 1$ and $\eta(t) = 0$ if $|t| \ge r/2$. Observe that h_ϵ is well defined and $\{h_\epsilon, 0 \le \epsilon \le \epsilon_0\}$ is a family of $\mathcal{C}^1$ maps for ϵ_0 sufficiently small, with $\|h_\epsilon - i_{B_r(\partial\Omega)}\|_{\mathcal{C}^1(W)} \to 0$ as $\epsilon \to 0$.

We may extend h_ϵ to a diffeomorphism of $\mathbb{R}^n$, satisfying (H1), which we still write simply as h_ϵ by defining it as the identity outside W.

If $\phi : U \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$ is a local coordinate system for $\partial\Omega$ in a neighborhood of $x_0 \in \partial\Omega$, then the map $\Psi(t, y) = x(t, \phi(y)) : (-r, r)\times U \to \mathbb{R}^n$ is a $\mathcal{C}^1$ coordinate system around the point $x_0 \in \mathbb{R}^n$ and $\Psi^{-1}h_\epsilon\Psi(t, y) = (t+\eta(t)\theta_\epsilon(\phi(y)), y)$. By an easy computation, we find that the Jacobian of $\Psi^{-1}h_\epsilon\Psi$ is given by $J(\Psi^{-1}h_\epsilon\Psi(t, y)) = 1+\eta'(t)\theta_\epsilon(\phi(y))$ and, therefore $Jh_\epsilon(x) = [1 + \eta'(t(x))\theta_\epsilon(\phi(\pi(x)))]\cdot J\Psi\big(\Psi^{-1}(h_\epsilon(x))\big)\cdot J\Psi^{-1}(x)$ for $x \in W$. Since $\|h_\epsilon - Id_{R^n}\|_{\mathcal{C}^1} \to 0$, the condition (H2) follows.

We can also compute $J_{\partial\Omega}h_{\epsilon|\partial\Omega}$, the Jacobian of h_ϵ restricted to $\partial\Omega$. We drop the subscript $\partial\Omega$ to simplify the notation. Note that the coordinate system Ψ above takes $\{0\}\times U$ into a neighborhood of $x_0 \in \partial\Omega$, and $\Psi^{-1}h_{\epsilon|\partial\Omega}\Psi(0, y) = (\theta_\epsilon(\phi(y)), y)$.

A straightforward computation gives $J(\Psi^{-1}h_{\epsilon|\partial\Omega}\Psi(0, y)) = \sqrt{1 + \|\nabla\theta_\epsilon(\phi(y))\|^2}$ and, therefore

$$Jh_\epsilon\big|_{\partial\Omega}(\phi(y)) = \big[\sqrt{1 + \|\nabla\theta_\epsilon(\phi(y))\|^2}\big] J\Psi\big(\Psi^{-1}(h_\epsilon(\phi(y)))\big) \cdot J\Psi^{-1}(\Psi(0, y))$$

for $y \in U$, where Ψ_0 and Ψ_ϵ denote the restriction of Ψ to $\{(0, y)\,|y \in U\}$ and $\{(\theta_\epsilon(\phi(y)), y)\,|y \in U\}$, respectively. Since $\|h_\epsilon - Id_{R^n}\|_{\mathcal{C}^1}$ and $\|\theta_\epsilon(\xi)\|_{\mathcal{C}^1(\partial\Omega)} \to 0$, it follows that $Jh_{\epsilon|\partial\Omega}(\phi(y)) \to 1$ as $\epsilon \to 0$, uniformly in $\partial\Omega$.

Example 3.3. We can choose the vector field X in the previous example as an extension of $N : \partial\Omega \to \mathbb{R}^n$ the unit outward normal to $\partial\Omega$, $t(x) = \pm\operatorname{dist}(x, \partial\Omega)$, ("+" outside, "-" inside), $\phi(x) =$ the point of $\partial\Omega$ nearest x and $B_r(\partial\Omega) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial\Omega) < r\}$.

Then, the map $\rho : (t, \xi) \mapsto \xi+tN(\xi) : (-r, r)\times\partial\Omega \to B_r(\partial\Omega)$ is a diffeomorphism, for some $r > 0$, with inverse $x \mapsto (t(x), \pi(x))$ (see [9]).

Define $h_\epsilon : B_r(\partial\Omega) \to \mathbb{R}^n$ by $h_\epsilon(\rho(t, \xi)) = \xi + tN(\xi) + \eta(t)\theta_\epsilon(\xi)N(\xi) = \rho(t, \xi) + \eta(t)\theta_\epsilon(\xi)N(\xi)$, where $\theta_\epsilon : \partial\Omega \to \mathbb{R}$ is a $\mathcal{C}^1$ function, with $\|\theta_\epsilon\|_{\mathcal{C}^1(\partial\Omega)} \to 0$ as $\epsilon \to 0$, $\eta : [-r, r] \to [0, 1]$ is a $\mathcal{C}^2$ function, with $\eta(0) = 1$ and $\eta(t) = 0$ if $|t| \geq \frac{r}{2}$. Then, $\{h_\epsilon,\ 0 \leq \epsilon \leq \epsilon_0\}$ is a family of $\mathcal{C}^1$ maps for ϵ_0 sufficiently small, with $\|h_\epsilon - i_{B_r(\partial\Omega)}\|_{\mathcal{C}^1} \to 0$ as $\epsilon \to 0$. We may extend h_ϵ to a diffeomorphism of $\mathbb{R}^n$, satisfying (H1), which we still write simply as h_ϵ by defining it as the identity outside $B_r(\partial\Omega)$.

If $\phi : U \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$ is a local coordinate system for $\partial\Omega$ in a neighborhood of $x_0 \in \partial\Omega$, then the map $\Psi(t, y) = \phi(y) + tN(\phi(y)) = \rho(t, \phi(y)) : (-r, r) \times U \to \mathbb{R}^n$ is a $\mathcal{C}^1$ coordinate system around the point $x_0 \in \mathbb{R}^n$ and $\Psi^{-1}h_\epsilon\Psi(t, y) = (t + \eta(t)\theta_\epsilon(\phi(y)), y)$. The condition (H2) can now be checked as in the previous example.

Remark 3.4. We may choose the function θ_ϵ with "oscillatory behavior", so the example above essentially includes the case considered in [5], since the perturbation there is nonzero only in a smooth portion of the boundary.

4. Linear semigroup

In this section we consider the linear semigroups generated by the family of differential operators $-h_\epsilon^*\Delta_{\Omega_\epsilon}h_\epsilon^{*-1} + aI$, appearing in (2.1).

4.1. Strong form in L^p spaces.

Consider the operator in $L^p(\Omega)$, $p \geq 2$, given by

$$A_\epsilon := \big(- h_\epsilon^*\Delta_{\Omega_\epsilon}h_\epsilon^{*-1} + aI\big) \tag{4.1}$$

with domain

$$D(A_\epsilon) = \big\{u \in W^{2,p}(\Omega) : h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{*^{-1}} u = 0, \text{ on } \partial\Omega\big\}. \tag{4.2}$$

We will denote simply by A the unperturbed operator $\big(-\Delta_\Omega + aI\big)$.

Theorem 4.1. *If $\epsilon > 0$ is sufficiently small and $h_\epsilon \in \mathrm{Diff}^1(\Omega)$, then the operator $A_\epsilon = \big(-h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} + aI\big)$ defined by (4.1) and (4.2) is sectorial.*

Proof. Consider the operator $-\Delta_{\Omega_\epsilon}$ defined in $L^p(h_\epsilon(\Omega))$, with domain

$$D(-\Delta_{\Omega_\epsilon}) = \big\{u \in W^{2,p}(\Omega_\epsilon) : \frac{\partial}{\partial N_{\Omega_\epsilon}} u = 0 \text{ on } \partial\Omega_\epsilon\big\},$$

where $\Omega_\epsilon = h_\epsilon(\Omega)$. It is well known that $-\Delta_{\Omega_\epsilon}$ is sectorial, with the spectra contained in the interval $(0, \infty) \subset \mathbb{R}$.

If $\lambda \in \mathbb{C}$ and $f \in L^2(\Omega)$, we have

$$\begin{aligned}
&(h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} + \lambda I)u(x) = f(x) \\
&\Leftrightarrow (\Delta_{\Omega_\epsilon} + \lambda I) u \circ h_\epsilon^{-1}(h_\epsilon(x)) = f \circ h_\epsilon^{-1}(h_\epsilon(x)) \\
&\Leftrightarrow (\Delta_{\Omega_\epsilon} + \lambda I) v(y) = g(y).
\end{aligned}$$

Since $u \mapsto h_\epsilon^* u := u \circ h_\epsilon$ is an isomorphism from $L^2(\Omega_\epsilon)$ to $L^2(\Omega)$ with inverse $(h_\epsilon^{-1})^*$, it follows that the first equation is uniquely solvable in $L^2(\Omega)$ if, and only if, the last equation is uniquely solvable in $L^2(\Omega_\epsilon)$.

Suppose λ belongs to $\rho(-\Delta_{\Omega_\epsilon})$, the resolvent set of $-\Delta_{\Omega_\epsilon}$. Then

$$\begin{aligned}
\|u\|_{L^p(\Omega)}^p &= \int_\Omega |u(x)|^p \, dx \\
&= \int_\Omega |v \circ h_\epsilon(x)|^p \, dx \\
&= \int_{\Omega_\epsilon} |v(y)|^p |J h_\epsilon^{-1}(y)| dy \\
&\le \|J h_\epsilon^{-1}\|_\infty \|v\|_{L^p(\Omega_\epsilon)}^p \\
&\le \|J h_\epsilon^{-1}\|_\infty \| \left(\Delta_{\Omega_\epsilon} + \lambda I\right)^{-1} \|_{\mathcal{L}(L^p(\Omega_\epsilon))} \|g\|_{L^p(\Omega_\epsilon)}^p.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|g\|_{L^p(\Omega_\epsilon)}^p &= \int_{\Omega_\epsilon} |g(x)|^p \, dy \\
&= \int_{\Omega_\epsilon} |f \circ h_\epsilon^{-1}(y)|^p \, dy \\
&= \int_\Omega |f(x)|^p |J h_\epsilon(x)| dx \\
&\le \|J h_\epsilon\|_\infty \|f\|_{L^p(\Omega)}^p.
\end{aligned}$$

It follows that

$$\|u\|_{L^p(\Omega)}^p \le \|J h_\epsilon\|_\infty \|J h_\epsilon^{-1}\|_\infty \|(\Delta_{\Omega_\epsilon} + \lambda I)^{-1}\|_{\mathcal{L}(L^p(\Omega_\epsilon))} \|f\|_{L^p(\Omega)}^p.$$

Therefore, $\lambda \in \rho(-h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1})$ and

$$\|(h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} + \lambda I)^{-1}\|_{\mathcal{L}(L^p(\Omega))} \le \|J h_\epsilon\|_\infty \|J h_\epsilon^{-1}\|_\infty \|(\Delta_{\Omega_\epsilon} + \lambda I)^{-1}\|_{\mathcal{L}(L^p(\Omega_\epsilon))}. \tag{4.3}$$

It can be proved similarly that $\lambda \in \rho(-h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1}) \Rightarrow \lambda \in \rho(-\Delta_{\Omega_\epsilon})$.

Finally, if $B_\epsilon = -\Delta_{\Omega_\epsilon} + aI$ is sectorial with $\|(\lambda - B_\epsilon)^{-1}\| \leq \frac{M}{|\lambda - a'|}$ for all λ in the sector $S_{a',\phi_0} = \{\lambda : \phi_0 \leq |\arg(\lambda - a')| \leq \pi, \lambda \neq a'\}$, for some $a' \in \mathbb{R}$ and $0 \leq \phi_0 < \pi/2$, it follows from (4.3) that $A_\epsilon = a - h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1}$ satisfies $\|(\lambda - A)^{-1}\| \leq \frac{M'}{|\lambda - a'|}$ for all λ in the sectoriality of A_ϵ follows from the sectoriality of B_ϵ. □

Remark 4.2. From Theorem 4.1 and results in [10], it follows that A_ϵ generates a linear analytic semigroup in $L^p(\Omega)$, for each $\epsilon \geq 0$.

4.2. **Weak form in L^p spaces.** One would like to prove that the operators A_ϵ defined by (4.1) and (4.2) become close to the operator A as $\epsilon \to 0$ in a certain sense. This is possible when the perturbation diffeomorphisms h_ϵ converge to the identity in the $\mathcal{C}^2$-norm (see, for example [14, 16]). To obtain similar results here, we need to consider the problem in weaker topologies, that is, we need to extend those operators. To this end, we now want to consider the operator $A_\epsilon = \big(-h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} + aI\big)$ as an operator $\widetilde{A}_\epsilon$ in $(W^{1,q}(\Omega))'$ with $D(\widetilde{A}_\epsilon) = W^{1,p}(\Omega)$, where q is the conjugate exponent of p, that is $\frac{1}{p} + \frac{1}{q} = 1$.

If $u \in D(A_\epsilon) = \{u \in W^{2,p}(\Omega) : h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{*-1} u = 0\}$, $\psi \in W^{1,q}(\Omega)$, and $v = u \circ h_\epsilon^{-1}$, integrating by parts, we obtain

$$
\begin{aligned}
&\langle A_\epsilon u, \psi\rangle_{-1,1} \\
&= -\int_\Omega (h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} u)(x)\psi(x)\,dx + a\int_\Omega u(x)\psi(x)\,dx \\
&= -\int_\Omega \Delta_{\Omega_\epsilon}(u \circ h_\epsilon^{-1})(h_\epsilon(x))\psi(x)\,dx + a\int_\Omega u(x)\psi(x)\,dx \\
&= -\int_{\Omega_\epsilon} \Delta_{\Omega_\epsilon} v(y)\psi(h_\epsilon^{-1}(y))\frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|}dy \\
&\quad + a\int_{\Omega_\epsilon} u(h_\epsilon^{-1}(y))\psi(h_\epsilon^{-1}(y))\frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|}dy \\
&= -\int_{\partial\Omega_\epsilon} \frac{\partial v}{\partial N_{\Omega_\epsilon}}(y)\psi(h_\epsilon^{-1}(y))\frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|}\,d\sigma(y) \\
&\quad + \int_{\Omega_\epsilon} \nabla_{\Omega_\epsilon} v(y) \cdot \nabla_{\Omega_\epsilon}\big[\psi(h_\epsilon^{-1}(y))\frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|}\big]\,dy \\
&\quad + a\int_{\Omega_\epsilon} u(h_\epsilon^{-1}(y))\psi(h_\epsilon^{-1}(y))\frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|}\,dy \\
&= \int_{\Omega_\epsilon} \nabla_{\Omega_\epsilon} v(y) \cdot \nabla_{\Omega_\epsilon}\big[\psi(h_\epsilon^{-1}(y))\frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|}\big]\,dy \\
&\quad + a\int_{\Omega_\epsilon} u(h_\epsilon^{-1}(y))\psi(h_\epsilon^{-1}(y))\frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|}\,dy \\
&= \int_\Omega \nabla_{\Omega_\epsilon} v(h_\epsilon(x)) \cdot \nabla_{\Omega_\epsilon}\big[\psi \circ h_\epsilon^{-1}\frac{1}{|Jh_\epsilon \circ h_\epsilon^{-1}|}(h_\epsilon(x))\big]\|Jh_\epsilon(x)|\,dx + a\int_\Omega u(x)\psi(x)dx \\
&= \int_\Omega (h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} u)(x) \cdot \big[h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1}\frac{\psi}{Jh_\epsilon}\big](x)|Jh_\epsilon(x)|\,dx \ + a\int_\Omega u(x)\psi(x)\,dx \\
&= \int_\Omega (h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} u)(x) \cdot h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1}\psi(x)\,dx + a\int_\Omega u(x)\psi(x)\,dx
\end{aligned}
$$

$$+\int_\Omega (h_\epsilon^*\nabla_{\Omega_\epsilon} h_\epsilon^{*-1}u)(x)(h_\epsilon^*\nabla_{\Omega_\epsilon} h_\epsilon^{*-1}Jh_\epsilon)(x)\frac{1}{Jh_\epsilon}\psi(x)\,dx. \tag{4.4}$$

Since (4.4) is well defined for $u \in W^{1,p}(\Omega)$, we define an extension $\widetilde{A}_\epsilon$ of A_ϵ, with domain $W^{1,p}(\Omega)$ and values in $(W^{1,q}(\Omega))'$, by

$$\begin{aligned}&\langle \widetilde{A}_\epsilon u\,,\psi\rangle_{-1,1}\\ &:= \int_\Omega (h_\epsilon^*\nabla_{\Omega_\epsilon} h_\epsilon^{*-1}u)(x)\cdot h_\epsilon^*\nabla_{\Omega_\epsilon} h_\epsilon^{*-1}\psi(x)\,dx + a\int_\Omega u(x)\psi(x)\,dx\\ &\quad+\int_\Omega (h_\epsilon^*\nabla_{\Omega_\epsilon} h_\epsilon^{*-1}u)(x)\cdot(h_\epsilon^*\nabla_{\Omega_\epsilon} h_\epsilon^{*-1}Jh_\epsilon)(x)\,\cdot\frac{1}{Jh_\epsilon}\psi(x)\,dx,\end{aligned} \tag{4.5}$$

for any $\Psi \in (W^{1,q}(\Omega))$.

Remark 4.3. If u is regular enough, then $\widetilde{A}u = Au$ implies that u must satisfy the boundary condition $h_\epsilon^*\frac{\partial}{\partial N_{\Omega_\epsilon}}h_\epsilon^{*^{-1}}u = 0$, on $\partial\Omega$ but, since this is not well defined in $(W^{1,q}(\Omega))$, the domain of $\widetilde{A}$ does not incorporate this boundary condition.

For simplicity, we still denote this extension by A_ϵ, whenever there is no danger of confusion. Also, from now on, we drop the absolute value in $|Jh_\epsilon(x)|$, since the Jacobian of h_ϵ is positive for sufficiently small ϵ.

Next we now prove the following basic inequality.

Theorem 4.4. *$D\big(A_\epsilon\big) \supset D\big(A\big)$ for any $\epsilon \ge 0$ and there exists a positive function $\tau(\epsilon)$ such that*

$$\big|\big|\big(A_\epsilon - A\big)u\,\big|\big|_{W^{1,q}(\Omega)'} \le \tau(\epsilon)\big|\big|A\,u\,\big|\big|_{W^{1,q}(\Omega)'}\,,$$

for all $u \in D\big(A\big)$, with $\lim_{\epsilon\to 0^+}\tau(\epsilon) = 0$.

Proof. The assertion about the domain is immediate. The inequality is equivalent to

$$|\langle(A_\epsilon - A)u,\psi\rangle_{-1,1}| \le \tau(\epsilon)\|Au\|_{(W^{1,q}(\Omega))'}\|\psi\|_{W^{1,q}(\Omega)}\,,$$

for all $u \in W^{1,p}(\Omega)$, $\psi \in W^{1,q}(\Omega)$, with $\lim_{\epsilon\to 0^+}\tau(\epsilon) = 0$. For $\epsilon > 0$, We have

$$\begin{aligned}&|\langle\big(A_\epsilon - A)u,\psi\rangle_{-1,1}|\\ &\le \Big|\int_\Omega \big(h_\epsilon^*\nabla_{\Omega_\epsilon} h_\epsilon^{*-1}u\big)(x)\cdot\big[\big(\,h_\epsilon^*\nabla_{\Omega_\epsilon} h_\epsilon^{*-1}\psi\big)(x) - \big(\nabla_\Omega\psi\big)(x)\big]\,dx\Big|\\ &\quad+\Big|\int_\Omega \big(h_\epsilon^*\nabla_{\Omega_\epsilon} h_\epsilon^{*-1}u - \nabla_\Omega u\big)(x)\cdot\big(\nabla_\Omega\psi\big)(x)\,dx\Big|\\ &\quad+\Big|\int_\Omega (h_\epsilon^*\nabla_{\Omega_\epsilon} h_\epsilon^{*-1}u)(x)\cdot(h_\epsilon^*\nabla_{\Omega_\epsilon} h_\epsilon^{*-1}Jh_\epsilon)(x)\cdot\frac{1}{Jh_\epsilon}\psi(x)\,dx\Big|.\end{aligned} \tag{4.6}$$

Now, writing $|v|_p = (\sum_{i=1}^n |v_i|^p)^{1/p}$, $1 \le p < \infty$, $|v|_\infty = \sup(|v_i|, i = 1,2,\dots,n)$ for the p-norm of the vector $v = (v_1, v_2,\dots,v_n) \in \mathbb{R}^n$, we observe that

$$\begin{aligned}|h_\epsilon^*\nabla_{\Omega_\epsilon} h_\epsilon^{*-1}u\,(x)|_p &= \Big(\sum_i \big|h_\epsilon^*\frac{\partial}{\partial y_i}h_\epsilon^{*-1}u\,(x)\big|^p\Big)^{1/p}\\ &= \Big(\sum_i \Big(\sum_j \big|b_{i,j}^\epsilon(x)\frac{\partial u}{\partial x_j}(x)\big|\Big)^p\Big)^{1/p}\\ &\le \Big[\sum_i \Big(\sum_j |b_{i,j}^\epsilon|^q(x)\Big)^{p/q}\Big(\sum_j \Big(\big|\frac{\partial u}{\partial x_j}\big|\Big)^p(x)\Big)\Big]^{1/p}\end{aligned}$$

$$
\begin{aligned}
&\le \Big[\sum_i \Big(\sum_j |b^\epsilon_{i,j}|^q(x)\Big)^{p-1}\Big]^{1/p} |\nabla u(x)|_p \\
&\le \|b^\epsilon\|_\infty \Big[\sum_i n^{p-1}\Big]^{1/p} |\nabla u(x)|_p \\
&\le n\|b^\epsilon\|_\infty |\nabla u(x)|_p \\
&\le B(\epsilon)|\nabla u(x)|_p\,,
\end{aligned}
$$

$$
\begin{aligned}
&|h^*_\epsilon \nabla_{\Omega_\epsilon} h^{*-1}_\epsilon u\,(x) - \nabla_\Omega u(x)|_p \\
&= \Big(\sum_i \big|h^*_\epsilon \frac{\partial}{\partial y_i} h^{*-1}_\epsilon u\,(x) - \frac{\partial u}{\partial x_i}(x)\big|^p\Big)^{1/p} \\
&= \Big(\sum_i \Big(\sum_j \Big|(b^\epsilon_{i,j}(x) - \delta_{i,j})\frac{\partial u}{\partial x_j}(x)\Big|\Big)^p\Big)^{1/p} \\
&\le \Big[\sum_i \Big(\sum_j |b^\epsilon_{i,j} - \delta_{i,j}|^q(x)\Big)^{p/q} \Big(\sum_j \Big(|\frac{\partial u}{\partial x_j}|\Big)^p(x)\Big)\Big]^{1/p} \\
&\le \Big[\sum_i \Big(\sum_j |b^\epsilon_{i,j} - \delta_{i,j}|^q(x)\Big)^{p-1}\Big]^{1/p} |\nabla u(x)|_p \\
&\le \|b^\epsilon - \delta\|_\infty \Big[\sum_i n^{p-1}\Big]^{1/p} |\nabla u(x)|_p \\
&\le n\|b^\epsilon - \delta\|_\infty |\nabla u(x)|_p \\
&\le \eta(\epsilon)|\nabla u(x)|_p\,,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{Jh_\epsilon(x)}|h^*_\epsilon \nabla_{\Omega_\epsilon} h^{*-1}_\epsilon Jh_\epsilon(x)|_\infty
&= \frac{1}{Jh_\epsilon(x)} \sup_i \big\{\big|h^*_\epsilon \frac{\partial}{\partial y_i} h^{*-1}_\epsilon Jh_\epsilon\,(x)\big|\big\} \\
&= \frac{1}{Jh_\epsilon(x)} \sup_i \big\{\sum_j \big|b^\epsilon_{i,j}(x)\frac{\partial Jh_\epsilon}{\partial x_j}(x)\big|\big\} \\
&= \frac{1}{Jh_\epsilon(x)} \|b^\epsilon\|_\infty \sum_j \big|\frac{\partial Jh_\epsilon}{\partial x_j}(x)\big| \\
&\le \frac{1}{Jh_\epsilon(x)} \|b^\epsilon\|_\infty |\nabla Jh_\epsilon(x)|_1 \le n\|b^\epsilon\|_\infty |\nabla Jh_\epsilon(x)|_\infty \\
&\le \frac{1}{Jh_\epsilon(x)} B(\epsilon)|\nabla Jh_\epsilon(x)|_\infty \le \frac{1}{Jh_\epsilon(x)} B(\epsilon)\|\nabla Jh_\epsilon\|_\infty \\
&\le \mu(\epsilon)\,,
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{Jh_\epsilon(x)}\big|h^*_\epsilon \nabla_{\Omega_\epsilon} h^{*-1}_\epsilon Jh_\epsilon(x)\psi(x)\big|_q
&= \frac{1}{Jh_\epsilon(x)} \Big(\sum_i \big|h^*_\epsilon \frac{\partial}{\partial y_i} h^{*-1}_\epsilon Jh_\epsilon\,(x)\cdot\psi(x)\big|^q\Big)^{1/q} \\
&\le \frac{1}{Jh_\epsilon(x)} \big|h^*_\epsilon \nabla h^{*-1}_\epsilon Jh_\epsilon\,(x)\big|_\infty \Big(\sum_i \big|\psi(x)\big|^q\Big)^{1/q} \\
&\le n\mu(\epsilon)\psi(x)\,,
\end{aligned}
$$

where $\|b^\epsilon\|_\infty := \sup\{|b^\epsilon_{i,j}(x)|,\ 1 \le i,j \le n,\ x \in \Omega\}$, $\|b^\epsilon - \delta\|_\infty := \sup\{|b^\epsilon_{i,j} - \delta_{i,j}|(x)|,\ 1 \le i,j \le n,\ x \in \Omega\}$, $B(\epsilon) \to n$ and $\eta(\epsilon)$, $\mu(\epsilon) \to 0$ as $\epsilon \to 0$, by hypotheses (H1) and (H2).

In a similar way, we obtain

$$|h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1}\psi(x)|_p \le B(\epsilon)|\nabla\psi(x)|_p,$$
$$|h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1}\psi(x) - \nabla\psi(x)|_p \le \eta(\epsilon)|\nabla\psi(x)|_p.$$

It follows that

$$\begin{aligned}
&|\langle (A_\epsilon - A)u, \psi\rangle_{-1,1}| \\
&\le \int_\Omega \big|\big(h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} u\big)(x)\big|_p \big|\big(h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1}\psi\big)(x) - \big(\nabla_\Omega \psi\big)(x)\big|_q \, dx \\
&\quad + \int_\Omega |\big(h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} u - \nabla_\Omega u\big)(x)|_p \big|\big(\nabla_\Omega \psi\big)(x)\big|_q \, dx \\
&\quad + \int_\Omega \big|(h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} u)(x)\big|_p \Big|\frac{1}{Jh_\epsilon(x)}(h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1})(x)\psi(x)\Big|_q \, dx \\
&\le B(\epsilon)\Big[\int_\Omega |\nabla u(x)|_p^p \, dx\Big]^{1/p} \eta(\epsilon)\Big[\int_\Omega |\nabla\psi(x)|_q^q \, dx\Big]^{1/q} \\
&\quad + \eta(\epsilon)\Big[\int_\Omega |\nabla u(x)|_p^p \, dx\Big]^{1/p}\Big[\int_\Omega |\nabla\psi(x)|_q^q \, dx\Big]^{1/q} \\
&\quad + B(\epsilon)n\cdot\mu(\epsilon)\Big[\int_\Omega |\nabla u(x)|_p^p \, dx\Big]^{1/p}\Big[\int_\Omega |\psi(x)|_q^q \, dx\Big]^{1/q} \\
&\le ((1 + B(\epsilon))\eta(\epsilon) + n\beta(\epsilon))\mu(\epsilon))\, \|u\|_{W^{1,p}(\Omega)}\|\psi\|_{W^{1,q}(\Omega)} \\
&\le K(\epsilon)\|u\|_{W^{1,p}(\Omega)}\|\psi\|_{W^{1,q}(\Omega)}
\end{aligned}$$

with $\lim_{\epsilon\to 0^+} K(\epsilon) = 0$ (independently of u). We conclude that

$$\|\,(A_\epsilon - A)\,u\|_{W^{1,q}(\Omega)'} \le K(\epsilon)\|u\|_{W^{1,p}(\Omega)} \le \tau(\epsilon)\|Au\|_{W^{1,q}(\Omega)'} \tag{4.7}$$

with $\lim_{\epsilon\to 0^+} \tau(\epsilon) = 0$, (and $\tau(\epsilon)$ does not depend on u). □

4.3. **Existence and continuity of the linear semigroup.** Using well known facts about the "unperturbed operator" A and Theorem 4.4, one can now establish existence and continuity of the linear semigroup, based on the following results.

Lemma 4.5. *Suppose A is a sectorial operator with $\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda - a|}$ for all λ in the sector $S_{a,\phi_0} = \{\lambda : \phi_0 \le |\arg(\lambda - a)| \le \pi, \lambda \ne a\}$, for some $a \in \mathbb{R}$ and $0 \le \phi_0 < \pi/2$. Suppose also that B is a linear operator with $D(B) \supset D(A)$ and $\|Bx - Ax\| \le \varepsilon\|Ax\| + K\|x\|$, for any $x \in D(A)$, where K and ε are positive constants with $\varepsilon \le \frac{1}{4(1+LM)}$, $K \le \frac{\sqrt{5}}{20M}\frac{\sqrt{2}L-1}{L^2-1}$, for some $L > 1$.*

Then B is also sectorial. More precisely, if $b = \frac{L^2}{L^2-1}a - \frac{\sqrt{2}L}{L^2-1}|a|$, $\phi = \max\{\phi_0, \frac{\pi}{4}\}$ and $M' = 2M\sqrt{5}$, then

$$\|(\lambda - B)^{-1}\| \le \frac{M'}{|\lambda - b|},$$

in the sector $S_{b,\phi} = \{\lambda \mid \phi \le |arg(\lambda - b)| \le \pi, \lambda \ne b\}$.

For a proof of the above lemma, see [16, p. 346].

Remark 4.6. Observe that b can be made arbitrarily close to a by taking L sufficiently large. In particular, if $a > 0$ then $b > 0$.

Theorem 4.7. *Suppose that A is as in Lemma 4.5, Λ a topological space and $\{A_\gamma\}_{\gamma\in\Lambda}$ is a family of operators in X with $A_{\gamma_0}=A$ satisfying the following conditions:*

1. *$D(A_\gamma)\supset D(A)$, for all $\gamma\in\Lambda$;*
2. *$\|A_\gamma x-Ax\|\le\epsilon(\gamma)\|Ax\|+K(\gamma)\|x\|$ for any $x\in D(A)$, where $K(\gamma)$ and $\epsilon(\gamma)$ are positive functions with $\lim_{\gamma\to\gamma_0}\epsilon(\gamma)=0$ and $\lim_{\gamma\to\gamma_0}K(\gamma)=0$.*

Then, there exists a neighborhood V of γ_0 such that A_γ is sectorial if $\gamma\in V$ and the family of (linear) semigroups e^{-tA_γ} satisfies

$$\begin{gathered}\|e^{-tA_\gamma}-e^{-tA}\|\le C(\gamma)e^{-bt}\\ \|A\left(e^{-tA_\gamma}-e^{-tA}\right)\|\le C(\gamma)\frac{1}{t}e^{-bt}\\ \|A^\alpha\left(e^{-tA_\gamma}-e^{-tA}\right)\|\le C(\gamma)\frac{1}{t^\alpha}e^{-bt},\quad 0<\alpha<1\end{gathered}\tag{4.8}$$

for $t>0$, where b is as in Lemma 4.5, and $C(\gamma)\to 0$ as $\gamma\to\gamma_0$.

For a proof of the above lemma, see [16, p. 349].

Theorem 4.8. *The operators A_ϵ given by (4.5) in the space $X=(W^{1,q}(\Omega))'$, with domain $W^{1,p}(\Omega)$, $1<p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, are sectorial operators with sectors and constant in the sectorial inequality independent of ϵ, for ϵ_0 sufficiently small. The family of analytic linear semigroups e^{-tA_ϵ} generated by A_ϵ in the "base space" X, satisfies (4.8).*

The first assertion of the above theorem follows from Theorem 4.5, and the second from Theorem 4.7.

5. The abstract problem in a scale of Banach spaces

Our goal in this section is to pose the problem (1.1) in a convenient abstract setting. We proved in Theorem 4.1 that, if ϵ is small, the operator A_ϵ in $L^p(\Omega)$ defined by (4.1) with domain given in (4.2) is sectorial and, in Theorem 4.8 that the same is true for its extension $\widetilde{A}_\epsilon$ to $(W^{1,q}(\Omega))'(\Omega)$.

It is then well-known that the domains X^α_ϵ (resp. $\widetilde{X}^\alpha_\epsilon$), $\alpha\ge 0$ of the fractional powers of A_ϵ (resp. $\widetilde{A}_\epsilon$) are Banach spaces, $X^0_\epsilon=L^p(\Omega)$, (resp. $\widetilde{X}^0_\epsilon=(W^{1,q}(\Omega))'(\Omega)$), $X^1_\epsilon=D(A_\epsilon)=W^{2,p}(\Omega)$, (resp. $\widetilde{X}^1_\epsilon=D(\widetilde{A}_\epsilon)=W^{1,p}(\Omega)$), X^α_ϵ, ($\widetilde{X}^\alpha_\epsilon$) is compactly embedded in X^β_ϵ, ($\widetilde{X}^\beta_\epsilon$) when $0\le\alpha<\beta<1$, and $X^\alpha_\epsilon=W^{2\alpha,p}$, when 2α is an integer number.

Since $X^{1/2}_\epsilon=\widetilde{X}^1_\epsilon$, it follows easily that $X^{\alpha-\frac12}_\epsilon=\widetilde{X}^\alpha_\epsilon$, for $\frac12\le\alpha\le1$ and, by an abuse of notation, we will still write $X^{\alpha-\frac12}_\epsilon$ instead of $\widetilde{X}^\alpha_\epsilon$, for $0\le\alpha\le\frac12$ so we may denote by $\{X^\alpha_\epsilon,\ -\frac12\le\alpha\le1\}=\{X^\alpha_\epsilon,\ 0\le\alpha\le1\}\cup\{\widetilde{X}^\alpha_\epsilon,\ 0\le\alpha\le1\}$, the whole family of fractional power spaces. We will denote simply by X^α the fractional power spaces associated to the unperturbed operator A.

For any $-1/2\le\beta\le0$, we may now define an operator in these spaces as the restriction of $\widetilde{A}_\epsilon$. We then have the following result.

Theorem 5.1. *For any $-1/2\le\beta\le0$ and ϵ sufficiently small, the operator $(A_\epsilon)_\beta$ in X^β_ϵ, obtained by restricting $\widetilde{A}_\epsilon$, with domain $X^{\beta+1}_\epsilon$ is a sectorial operator.*

Proof. Writing $\beta = -\frac{1}{2} + \delta$, for some $0 \leq \delta \leq 1/2$, we have $(A_\epsilon)_\beta = \widetilde{A}_\epsilon^{-\delta} \widetilde{A}_\epsilon \widetilde{A}_\epsilon^{\delta}$. Since $\widetilde{A}_\epsilon^{\delta}$ is an isometry from X_ϵ^β to $X_\epsilon^{-\frac{1}{2}} = (W^{1,q}(\Omega))'$, the result follows easily. □

We can now pose the problem (2.1) as an abstract problem in the scale of Banach spaces $\{X_\epsilon^\beta, -1/2 \leq \beta \leq 0\}$.

$$\begin{gathered} u_t = -(A_\epsilon)_\beta u + (H_\epsilon)_\beta u, \quad t > t_0\,; \\ u(t_0) = u_0 \in X_\epsilon^\eta\,, \end{gathered} \tag{5.1}$$

where

$$(H_\epsilon)_\beta = H(\cdot, \epsilon) := (F_\epsilon)_\beta + (G_\epsilon)_\beta : X_\epsilon^\eta \to X_\epsilon^\beta, \quad \epsilon > 0, \quad 0 \leq \eta \leq \beta + 1, \tag{5.2}$$

(i) $(F_\epsilon)_\beta = F(\cdot, \epsilon) : X_\epsilon^\eta \to X_\epsilon^\beta$ is given by

$$\langle F(u, \epsilon), \Phi \rangle_{\beta, -\beta} = \int_\Omega f(u) \Phi \, dx, \quad \text{for any } \Phi \in (X_\epsilon^\beta)', \tag{5.3}$$

(ii) $(G_\epsilon)_\beta = G(\cdot, \epsilon) : X_\epsilon^\eta \to X_\epsilon^\beta$ is given by

$$\langle G(u, \epsilon), \Phi \rangle_{\beta, -\beta} = \int_{\partial\Omega} g(\gamma(u)) \gamma(\Phi) \Big| \frac{J_{\partial\Omega} h_\epsilon}{J h_\epsilon} \Big| \, d\sigma(x), \quad \text{for any } \Phi \in (X_\epsilon^\beta)', \tag{5.4}$$

where γ is the trace map and $J_{\partial\Omega} h_\epsilon$ is the determinant of the Jacobian matrix of the diffeomorphism $h_\epsilon : \partial\Omega \longrightarrow \partial h_\epsilon(\Omega)$.

We will choose β, small enough in order that $X_\epsilon^{\beta+1}$ does not incorporate the boundary conditions, that is, the closure of the subset defined by smooth functions with Neumann boundary condition is the whole space. It is not difficult to show, integrating by parts, that a regular enough solution of (5.1), must satisfy (2.1) (see [6, 13]).

6. Local well-posedness

To prove local well-posedness for the abstract problem, without assuming growth conditions in the nonlinearities, we want to have two somewhat conflicting requirements for our phase space X_ϵ^η: we need it to be continuously embedded in L^∞ and we do not want it to incorporate the boundary conditions. To this end, we need to choose η and p big enough so that the inclusion holds and, on the other hand, we need η small enough so that the normal derivative does not have a well defined trace. To achieve both requirements we will henceforth assume that

p and η are such that the inclusion $X_\epsilon^\eta \hookrightarrow L^\infty(\Omega_\epsilon)$ holds, for some $\mu \geq 0$ and $\eta < 1/2$. (6.1)

It is easy to check that (6.1) holds, for instance, if $p = 2n$, and $1/4 < \eta < 1/2$. Also, the last inequality is automatically satisfied if we choose our base space $X_\epsilon^\beta = X_\epsilon^{-1/2} = (W^{1,q}(\Omega))'$, where q and p are conjugate exponents, since we must have $\eta - \beta < 1$.

Lemma 6.1. *Suppose that p and η are such that* (6.1) *holds and f is locally Lipschitz. Then, the operator $(F_\epsilon)_\eta : X_\epsilon^\eta \to X_\epsilon^{-1/2}$ given by* (5.3) *is well defined and Lipschitz in bounded sets.*

Proof. Suppose $u \in X_\epsilon^\eta$. From (6.1), it follows that $u \in L^\infty(\Omega)$ and, therefore, if L_f is the Lipschitz constant of f on the interval $[-\|u\|_\infty, \|u\|_\infty]$, it follows that $|f(u(x) - f(0)| \le L_f|u(x)|$, for any $x \in \Omega$. If $\Phi \in (X_\epsilon^{-1/2})' = W^{1,q}(\Omega)$, then

$$\begin{aligned}
\big|\langle (F_\epsilon)_\eta(u), \Phi\rangle_{\beta,-\beta}\big| &\le \int_\Omega |f(u)||\Phi|\,dx \\
&\le L_f \int_\Omega |u||\Phi|\,dx + \int_\Omega |f(0)||\Phi|\,dx \\
&\le L_f\|u\|_{L^p(\Omega)} \cdot \|\Phi\|_{L^q(\Omega)} + \|f(0)\|_{L^p(\Omega)} \cdot \|\Phi\|_{L^q(\Omega)}
\end{aligned}$$

Since $W^{1,q}(\Omega) \subset L^q(\Omega)$ and $X_\epsilon^\eta \subset L^p(\Omega)$ with stronger norms, we have

$$\big|\langle (F_\epsilon)_\eta(u), \Phi\rangle_{\beta,-\beta}\big| \le L_f\|u\|_{L^p(\Omega)}\|\Phi\|_{W^{1,q}(\Omega)} + \|f(0)\|_{L^p(\Omega)}\|\Phi\|_{W^{1,q}(\Omega)},$$

so $(F_\epsilon)_\eta$ is well defined and

$$\begin{aligned}
\|(F_\epsilon)_\eta(u)\|_{(W^{1,q}(\Omega))'} &\le L_f\|u\,\|_{L^p(\Omega)} + \|f(0)\|_{L^p(\Omega)} &(6.2)\\
&\le L_f\|u\,\|_{X_\epsilon^\eta} + \|f(0)\|_{L^p(\Omega)} &(6.3)
\end{aligned}$$

where L_f is the Lipschitz constant of f on the interval $[-\|u\|_\infty, \|u\|_\infty]$

Alternatively, if $M_f = M_f(u) := \sup\{|f(x)|\ x \in [-\|u\|_\infty, \|u\|_\infty]\}$, it follows that

$$\begin{aligned}
\big|\langle (F_\epsilon)_\eta(u), \Phi\rangle_{\beta,-\beta}\big| &\le \int_\Omega |f(u)||\Phi|\,dx \\
&\le M_f|\Omega|^{1/p}\|\Phi\|_{L^q(\Omega)} \\
&\le M_f|\Omega|^{1/p}\|\Phi\|_{W^{1,q}(\Omega)}.
\end{aligned}$$

Thus

$$\|(F_\epsilon)_\eta(u)\|_{(W^{1,q}(\Omega))'} \le M_f|\Omega|^{1/p}. \tag{6.4}$$

Suppose now that u_1, u_2 belong to a bounded set $B \in X_\epsilon^\eta$. From (6.1) it follows now that u_1, u_2 belong to a ball of radius $R = \sup_{u\in B}\|u\|_\infty$ in $L^\infty(\Omega)$ and, therefore, if L is the Lipschitz constant of f in the interval $[-R, R]$, we have $|f(u_1(x)) - f(u_2(x)))| \le L|u_1(x) - u_2(x)|$, for any $x \in \Omega$. If $\Phi \in (X_\epsilon^{-1/2})' = W^{1,q}(\Omega)$, we obtain

$$\begin{aligned}
\big|\langle (F_\epsilon)_\eta(u_1) - (F_\epsilon)_\eta(u_2), \Phi\rangle_{\beta,-\beta}\big| &= \big|\int_\Omega [f(u_1) - f(u_2)]\,\Phi\,dx\big| \\
&\le \int_\Omega L|u_1 - u_2||\Phi|\,dx \\
&\le L_f\|u_1 - u_2\|_{L^p(\Omega)} \cdot \|\Phi\|_{L^q(\Omega)} \\
&\le L_f\|u_1 - u_2\|_{X_\epsilon^\eta} \cdot \|\Phi\|_{W^{1,q}(\Omega)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|(F_\epsilon)_\eta(u_1) - (F_\epsilon)_\eta(u_2)\|_{(W^{1,q}(\Omega))'} &\le L_f\|u_1 - u_2\|_{L^p(\Omega)} &(6.5)\\
&\le L_f\|u_1 - u_2\|_{X_\epsilon^\eta}. &(6.6)
\end{aligned}$$

This conclude the proof. □

Lemma 6.2. *Suppose that p and η are such that* (6.1) *holds and g is locally Lipschitz. Then, if ϵ_0 is sufficiently small, the operator $(G_\epsilon)_\eta = G : X_\epsilon^\eta \to (W^{1,q}(\Omega))'$ given by* (5.4) *is well defined, for $0 \le \epsilon < \epsilon_0$ and bounded in bounded sets.*

Proof. Suppose $u \in X^\eta_\epsilon$. From (6.1) it follows that $u \in L^\infty(\Omega)$ and, therefore, if L_g is the Lipschitz constant of g in the interval $[-\|u\|_\infty, \|u\|_\infty]$, it follows that $|g(\gamma(u)(x) - g(0)| \le L_g|\gamma(u)(x)|$, for any $x \in \partial\Omega$.

If $u \in X^\eta_\epsilon$ and $\Phi \in (X_\epsilon^{-1/2})' = W^{1,q}(\Omega)$, we have

$$
\begin{aligned}
&\big|\langle G(u,\epsilon), \Phi\rangle_{\beta,-\beta}\big| \\
&\le \int_{\partial\Omega} |g(\gamma(u))||\gamma(\Phi)|\big|\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}\big|\, d\sigma(x) \\
&\le \|\mu\|_\infty \int_{\partial\Omega} L_g|\gamma(u)||\gamma(\Phi)| + |g(0)||\gamma(\Phi)| d\sigma(x) \\
&\le \|\mu\|_\infty \left(L_g\|\gamma(u)\|_{L^p(\partial\Omega)}\|\gamma(\Phi)\|_{L^q(\partial\Omega)} + \|g(0)\|_{L^p(\partial\Omega)}\|\gamma(\Phi)\|_{L^q(\partial\Omega)}\right)
\end{aligned}
$$

where $\mu(x,\epsilon) = |\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}|$, and $\|\mu\|_\infty = \sup\{|\mu(x,\epsilon)| : x \in \partial\Omega, 0 \le \epsilon \le \epsilon_0\}$ is finite by hypothesis (H1).

By the imbedding and trace theorems,

$$
\|\gamma(\Phi)\|_{L^q(\partial\Omega)} \le K_1\|\Phi\|_{W^{1,q}(\Omega)}, \quad \|\gamma(u)\|_{L^p(\partial\Omega)} \le K_2\|u\|_{X^\eta_\epsilon},
$$

for some constants K_1, K_2. Thus

$$
\begin{aligned}
&\big|\langle G(u,\epsilon), \Phi\rangle_{\beta,-\beta}\big| \\
&\le \|\mu\|_\infty\Big(L_gK_1\|\gamma(u)\|_{L^p(\partial\Omega)}\|\Phi\|_{W^{1,q}(\Omega)} + K_1\|g(0)\|_{L^p(\partial\Omega)}\cdot\|\Phi\|_{W^{1,q}(\Omega)}\Big)
\end{aligned}
$$

proving that $(G_\epsilon)_\beta$ is well defined and

$$
\begin{aligned}
\|G(u,\epsilon)\|_{(W^{1,q}(\Omega))'} &\le \|\mu\|_\infty \left(L_gK_1\|\gamma(u)\|_{L^p(\partial\Omega)} + K_1\|g(0)\|_{L^p(\partial\Omega)}\right) &(6.7)\\
&\le \|\mu\|_\infty \left(L_gK_1K_2\|u\|_{X^\eta_\epsilon} + K_1\|g(0)\|_{L^p(\partial\Omega)}\right). &(6.8)
\end{aligned}
$$

Alternatively, if $M_g = M_g(u) := \sup\{|g(x)| : x \in [-\|u\|_\infty, \|u\|_\infty]\}$, it follows that

$$
\begin{aligned}
\big|\langle G(u,\epsilon), \Phi\rangle_{\beta,-\beta}\big| &\le \int_{\partial\Omega} |g(\gamma(u))||\gamma(\Phi)|\,\big|\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}\big|\, d\sigma(x) \\
&\le \|\mu\|_\infty M_g \int_{\partial\Omega} |\gamma(\Phi)| d\sigma(x) \\
&\le \|\mu\|_\infty M_g|\partial\Omega|^{1/p}\|\gamma(\Phi)\|_{L^q(\partial\Omega)} \\
&\le \|\mu\|_\infty M_g|\partial\Omega|^{1/p}K_1\|\Phi\|_{W^{1,q}(\Omega)}.
\end{aligned}
$$

Thus

$$
\|G(u,\epsilon)\|_{(W^{1,q}(\Omega))'} \le \|\mu\|_\infty M_g|\partial\Omega|^{1/p}K_1. \tag{6.9}
$$

□

Lemma 6.3. *Suppose the hypotheses of Lemma 6.2 hold. Then the operator* $G(u,\epsilon){=}G(u) : X^\eta_\epsilon \times [0,\epsilon_0] \to (W^{1,q}(\Omega))'$ *given by* (5.4) *is uniformly continuous in* ϵ*, for* u *in bounded sets of* X^η_ϵ *and locally Lipschitz continuous in* u*, uniformly in* ϵ*.*

Proof. We first show that $(G_\epsilon)_\beta$ is locally Lipschitz continuous in $u \in X^\eta_\epsilon$. Suppose that u_1, u_2 belong to a bounded set $B \in X^\eta_\epsilon$. From (6.1), the Trace Theorem and the hypotheses, it follows now that $\gamma(u_1), \gamma(u_2)$ belong to a ball of some radius R in $L^\infty(\partial\Omega)$ and, therefore, if L_g is the Lipschitz constant of g in the interval $[-R, R]$, we have $|g(\gamma(u_1)(x)) - g(\gamma(u_2)(x)))| \le L_g|\gamma(u_1)(x) - \gamma(u_2)(x)|$, for any $x \in \partial\Omega$. If $\Phi \in (W^{1,q}(\Omega))'$ and $\epsilon \in [0, \epsilon_0]$, we obtain

$$\begin{aligned}
\big|\langle G(u_1,\epsilon) - G(u_2,\epsilon), \Phi\rangle_{\beta,-\beta}\big| &\le \int_{\partial\Omega} |g(\gamma(u_1)) - g(\gamma(u_2))|\, |\gamma(\Phi)|\, \Big|\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}\Big|\, d\sigma(x)\\
&\le \int_{\partial\Omega} L_g|\gamma(u_1) - \gamma(u_2)|\, |\gamma(\Phi)|\, \Big|\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}\Big|\, d\sigma(x)\\
&\le L_g\|\mu\|_\infty \int_{\partial\Omega} |\gamma(u_1) - \gamma(u_2)|\, |\gamma(\Phi)|\, d\sigma(x)\\
&\le L_g\|\mu\|_\infty \|\gamma(u_1) - \gamma(u_2)\|_{L^p(\partial\Omega)} \|\gamma(\Phi)\|_{L^q(\partial\Omega)}\\
&\le L_g\|\mu\|_\infty K_1K_2\|u_1 - u_2\|_{X^\eta_\epsilon} \|\Phi\|_{W^{1,q}(\Omega)},
\end{aligned}$$

where K_1, K_2 are the norms of the trace mappings. Therefore,

$$\begin{aligned}
\|G(u_1,\epsilon) - G(u_2,\epsilon)\|_{(W^{1,q}(\Omega))'} &\le L_g\|\mu\|_\infty K_1\|\gamma(u_1) - \gamma(u_2)\|_{L^p(\partial\Omega)} &(6.10)\\
&\le L_g\|\mu\|_\infty K_1K_2\|u_1 - u_2\|_{X^\eta_\epsilon} &(6.11)
\end{aligned}$$

so $(G_\epsilon)_\beta$ is locally Lipschitz in u.

Now, if $u \in X^\eta_\epsilon$, $\Phi \in (W^{1,q}(\Omega))'$ and $\epsilon_1, \epsilon_2 \in [0, \epsilon_0]$, we have

$$\begin{aligned}
&\big|\langle G(u,\epsilon_1) - G(u,\epsilon_2), \Phi\rangle_{\beta,-\beta}\big|\\
&\le \int_{\partial\Omega} |\gamma(g(u))|\, |\gamma(\Phi)|\, \Big|\Big|\frac{J_{\partial\Omega}h_{\epsilon_1}}{Jh_{\epsilon_1}}\Big| - \Big|\frac{J_{\partial\Omega}h_{\epsilon_2}}{Jh_{\epsilon_2}}\Big|\Big|\, d\sigma(x)\\
&\le \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_\infty \int_{\partial\Omega} |g(\gamma(u))||\gamma(\Phi)|\, d\sigma(x)\\
&\le \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_\infty \int_{\partial\Omega} (L_g|\gamma(u)| + |g(0)|)\, |\gamma(\Phi)| d\sigma(x)\\
&\le \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_\infty \big(L_g\|\gamma(u)\|_{L^p(\partial\Omega)} \cdot \|\gamma(\Phi)\|_{L^q(\partial\Omega)} + \|g(0)\|_{L^p(\partial\Omega)}\|\gamma(\Phi)\|_{L^q(\partial\Omega)}\big)\\
&\le \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_\infty \big(L_gK_1K_2\|u\|_\eta\|\Phi\|_{W^{1,q}(\Omega)} + K_1\|g(0)\|_{L^p(\partial\Omega)}\|\Phi\|_{W^{1,q}(\Omega)}\big)
\end{aligned}$$

where $\|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_\infty = \sup\big\{\big|\frac{J_{\partial\Omega}h_{\epsilon_1}}{Jh_{\epsilon_1}}\big| - \big|\frac{J_{\partial\Omega}h_{\epsilon_2}}{Jh_{\epsilon_2}}\big| : x \in \partial\Omega\big\} \to 0$ as $|\epsilon_1 - \epsilon_2| \to 0$, by hypothesis (H1) and K_1, K_2 are trace constants given by the Trace Theorem. It follows that

$$\begin{aligned}
&\|G(u,\epsilon_1) - G(u,\epsilon_2)\|_{(W^{1,q}(\Omega))'}\\
&\le \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_\infty \big(L_gK_1K_2\|u\|_\eta + K_1\|g(0)\|_{L^p(\partial\Omega)}\big).
\end{aligned} \tag{6.12}$$

Alternatively, if $M_g = M_g(u) := \sup\{|g(x)| : x \in [-\|u\|_\infty, \|u\|_\infty]\}$,

$$\begin{aligned}
&\big|\langle G(u,\epsilon_1) - G(u,\epsilon_2), \Phi\rangle_{\beta,-\beta}\big|\\
&\le \int_{\partial\Omega} |\gamma(g(u))||\gamma(\Phi)|\Big|\Big|\frac{J_{\partial\Omega}h_{\epsilon_1}}{Jh_{\epsilon_1}}\Big| - \Big|\frac{J_{\partial\Omega}h_{\epsilon_2}}{Jh_{\epsilon_2}}\Big|\Big|\, d\sigma(x)\\
&\le \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_\infty \int_{\partial\Omega} |g(\gamma(u))||\gamma(\Phi)| d\sigma(x)\\
&\le \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_\infty M_g \int_{\partial\Omega} |\gamma(\Phi)|\, d\sigma(x)\\
&\le \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_\infty M_g |\partial\Omega|^{1/p} \|\gamma(\Phi)\|_{L^p(\partial\Omega)}\\
&\le \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_\infty M_g |\partial\Omega|^{1/p} K_1\|\Phi\|_{W^{1,q}(\Omega)}.
\end{aligned}$$

It follows that

$$\|G(u,\epsilon_1)-G(u,\epsilon_2)\|_{(W^{1,q}(\Omega))'} \le \|\mu_{\epsilon_1}-\mu_{\epsilon_2}\|_\infty M_g|\partial\Omega|^{1/p}K_1. \tag{6.13}$$

□

Corollary 6.4. *Suppose the hypotheses of Lemmas 6.1 and 6.2 hold. Then the map* $(H(u,\epsilon))_\eta := (F_\epsilon(u))_\eta + (G(u,\epsilon))_\eta : X_\epsilon^\eta \times [0,\epsilon_0] \to (W^{1,q}(\Omega))'$ *is well defined, bounded in bounded sets uniformly in* ϵ*, uniformly continuous in* ϵ *for* u *in bounded sets of* X_ϵ^η *and locally Lipschitz continuous in* u *uniformly in* ϵ*.*

Proof. From (6.2), (6.3), (6.7) and (6.8), we obtain

$$\begin{aligned}
&\|(H_\epsilon)_\eta(u)\|_{(W^{1,q}(\Omega))'}\\
&\le L_f\|u\|_{L^p(\Omega)} + L_gK_1\|\gamma(u)\|_{L^p(\partial\Omega)} + \|f(0)\|_{L^p(\Omega)} + K_1\|g(0)\|_{L^p(\partial\Omega)} \quad (6.14)\\
&\le (L_f+L_gK_1K_2)\,\|u\|_{X_\epsilon^\eta} + \|f(0)\|_{L^p(\Omega)} + K_1\|g(0)\|_{L^p(\partial\Omega)}, \quad (6.15)
\end{aligned}$$

where L_f and L_g are Lipschitz constants of f and g in the interval $[-\|u\|_\infty,\|u\|_\infty]$, respectively.

Alternatively, if $M_f = M_f(u) := \sup\{|f(x)| : x \in [-\|u\|_\infty,\|u\|_\infty]\}$, $M_g = M_g(u) := \sup\{|g(x)| : x\in[-\|u\|_\infty,\|u\|_\infty]\}$, from (6.4) and (6.13) we obtain

$$\|(H_\epsilon)_\eta(u)\|_{(W^{1,q}(\Omega))'} \le M_f|\Omega|^{1/p} + \|\mu\|_\infty M_g|\partial\Omega|^{1/p}K_1. \tag{6.16}$$

From (6.5), (6.6), (6.10) and (6.11), we have

$$\begin{aligned}
&\|H(u_1,\epsilon)-H(u_2,\epsilon)\|_{(W^{1,q}(\Omega))'}\\
&\le L_g\|\mu\|_\infty K_1\|\gamma(u_1)-\gamma(u_2)\|_{L^p(\partial\Omega)} + L_f\|u_1-u_2\|_{L^p(\Omega)} \quad (6.17)\\
&\le (L_g\|\mu\|_\infty K_1K_2 + L_f)\,\|u_1-u_2\|_{X_\epsilon^\eta}. \quad (6.18)
\end{aligned}$$

From (6.12),

$$\begin{aligned}
&\|H(u,\epsilon_1)-H(u,\epsilon_2)\|_{(W^{1,q}(\Omega))'}\\
&\le \|\mu_{\epsilon_1}-\mu_{\epsilon_2}\|_\infty\left(L_gK_1K_2\|u\|_\eta + K_1\|g(0)\|_{L^p(\partial\Omega)}\right).
\end{aligned} \tag{6.19}$$

Alternatively, from (6.13), we have

$$\|H(u,\epsilon_1)-H(u,\epsilon_2)\|_{(W^{1,q}(\Omega))'} \le \|\mu_{\epsilon_1}-\mu_{\epsilon_2}\|_\infty M_g|\partial\Omega|^{1/p}K_1. \tag{6.20}$$

In the estimates above K_1 and K_2 are the norms of the trace mappings. □

Theorem 6.5. *Suppose the hypotheses of Corollary 6.4 hold. Then, for any* (t_0,u_0) *in* $\mathbb{R}\times X_\epsilon^\eta$*, problem* (5.1) *has a unique solution* $u(t,t_0,u_0,\epsilon)$ *with initial value* $u(t_0)=u_0$*.*

Proof. From Theorem 5.1 it follows that $(A_\epsilon)_\beta$ is a sectorial operator in $(W^{1,q}(\Omega))'$, with domain $X_\epsilon^{1/2} = W^{1,p}(\Omega)$, if ϵ is small enough. The result follows then from Corollary 6.4 and the results in [10, 16]. □

7. Global existence and boundedness of the semigroup

We will use $T_\epsilon(t)u_0$ for the (local) solution of problem (5.1) given by Theorem 6.5, with initial condition u_0 in some fractional power space of A_ϵ. We now want to show that these solutions are globally defined if an additional (dissipative) hypothesis on f and g is assumed. We use the hypotheses

There exist constants c_0 and d_0 such that

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} \le c_0\,, \quad \limsup_{|u|\to\infty} \frac{g(u)}{u} \le d_0 \tag{7.1}$$

and the first eigenvalue $\mu_1(\epsilon)$ of the problem

$$\begin{gathered} -h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} \Delta u + (a - c_0)u = \mu u \quad \text{in } \Omega \\ h_\epsilon^* \frac{\partial u}{\partial N_\Omega} h_\epsilon^{*-1} = d_0\, u \quad \text{on } \partial\Omega \end{gathered} \tag{7.2}$$

is positive for ϵ sufficiently small.

Remark 7.1. Observe that if hypothesis (7.2) hold for $\epsilon = 0$, then this is also true for ϵ small, since the eigenvalues change continuously with ϵ by (4.7).

Remark 7.2. The arguments bellow are a slight modification of the ones in [13], but we include them here for the sake of completeness. Similar arguments were used in [4] in a somewhat different setting.

For using comparison results, we start by defining the concepts of sub- and super-solutions.

Definition 7.3. Suppose Ω is a $\mathcal{C}^{1,\alpha}$, domain for some $\alpha \in (0,1)$, L is a uniformly elliptic second order differential operator in $\overline{\Omega}$, $u_0 \in \mathcal{C}^\alpha(\Omega)$, $T > 0$ and $\bar{u} : \Omega \subset \mathbb{R}^{\to}\mathbb{R}^n$ ($\underline{u}$ respectively) a function which is continuous in $[0,T] \times \bar{\Omega}$, continuously differentiable in t and twice continuously differentiable in x for $(t,x) \in (0,T] \times \Omega$. Then $\overline{u}$ (respectively, $\underline{u}$) is a super-solution (sub-solution) of the problem

$$\begin{gathered} u_t = Lu + f(u), \quad \text{in } (0,T] \times \Omega, \\ \frac{\partial u}{\partial N} = g(u), \quad \text{on } \partial\Omega \\ u(0) = u_0. \end{gathered} \tag{7.3}$$

if it satisfies

$$\begin{gathered} u_t \ge Lu + f(u), \quad \text{in } (0,T] \times \Omega, \\ \frac{\partial u}{\partial N} \ge g(u), \quad \text{on } \partial\Omega \\ u(0) \ge u_0. \end{gathered} \tag{7.4}$$

(and respectively with the $\ge$ sign replaced by the $\le$ sign).

The following is a basic result for our arguments.

Theorem 7.4 ([15])**.** *If f is locally Lipschitz and $\bar{u}$ and $\underline{u}$ are respectively a super and sub-solution of problem* (7.3)*, satisfying*

$$\underline{u} \le \bar{u}, \ \text{in } \Omega \times (0,T),$$

then there exists a solution u of (7.3) *such that*

$$\underline{u} \le u \le \bar{u}, \quad \text{in } \Omega \times (0,T).$$

Let φ_ϵ be the first positive normalized eigenfunction of (7.2) and let $m_\epsilon = \min_{x\in\bar{\Omega}} \varphi_\epsilon(x)$. We know that $m_\epsilon > 0$. For each $\theta > 0 \in \mathbb{R}$, define

$$\Sigma_\theta^\epsilon = \{u \in X_\epsilon^\eta : |u(x)| \le \theta\varphi_\epsilon(x), \text{ for all } x \in \bar{\Omega}\}.$$

From the dissipative hypothesis (7.1) on f and g, we know that there exists $\xi \in \mathbb{R}$, such that

$$\frac{f(s)}{s} \le c_0 \quad \text{and} \quad \frac{g(s)}{s} \le d_0,$$

for all s with $|s| \ge \xi$. To simplify notation, we take the $\epsilon = 0$, in the proofs below, since the argument is the same for any ϵ such that (7.2) is true (see Remark 7.1).

Lemma 7.5. *In addition to the hypotheses of Theorem 6.5, suppose hat* (7.1) *and* (7.2) *hold. Then, if $\theta m_\epsilon \ge \xi$ and ϵ is small enough, the set Σ_θ^ϵ is a positively invariant set for $T(t)$.*

Proof. Let

$$\Sigma_\theta^1 = \{u \in X^\eta : u(x) \le \theta\varphi(x), \text{ for all } x \in \bar{\Omega}\},$$
$$\Sigma_\theta^2 = \{u \in X^\eta : u(x) \ge -\theta\varphi(x), \text{ for all } x \in \bar{\Omega}\}.$$

Since $\Sigma_\theta = \Sigma_\theta^1 \cap \Sigma_\theta^2$ it is sufficient to show that Σ_θ^1 and Σ_θ^2 are positively invariant.

Let $u_0 \in \Sigma_\theta^1$, and suppose, for contradiction, that there exists $t_0 \in [0, t_{\max}[$ and $x_0 \in \bar{\Omega}$ such that

$$T(t_0)u_0(x_0) > \theta\varphi(x_0).$$

Consider $\bar{v}(t) = e^{-\mu(t-t_0)}\theta\varphi$, where μ is the eigenvalue associated with φ. We have

$$\frac{\partial \bar{v}}{\partial t} = (\Delta\bar{v} - a\bar{v} + c_0\bar{v}) \ge \Delta\bar{v} - a\bar{v} + f(\bar{v})$$
$$\frac{\partial \bar{v}}{\partial N} = d_0\bar{v} \ge g(\bar{v}),$$

for all $t \in]0, t_0]$.

Thus $\bar{v}$ is a super-solution for problem (2.1). It follows from Theorem 7.4 that

$$T(t)u_0 \le \bar{v}(t), \quad \text{in } \bar{\Omega} \text{ for all } t \in [0, t_0[.$$

In particular, $T(t_0)u_0(x_0) \le \theta\varphi(x_0)$ and we reach a contradiction.

To prove that Σ_θ^2 is positively invariant we proceed in a similar way, using now that $\underline{v} = -\bar{v}$ is a sub-solution for the problem (2.1). □

Lemma 7.6. *Suppose the hypotheses of Lemma 7.5 hold. If $\theta m_\epsilon \ge \xi$, and $\eta \le \alpha < \frac{1}{2}$, there exists a constant $R = R(\theta, \eta)$, and $T > 0$ independent of ϵ, such that the orbit of any bounded subset V of $X_\epsilon^\eta \cap \Sigma_\theta^\epsilon$ under $T_\epsilon(t)$ is in the ball of radius R of X_ϵ^α, for $t > T$. In particular, the solutions with initial condition in $X_\epsilon^\eta \cap \Sigma_\theta$ are globally defined.*

Proof. Lemma 7.5 implies that $T_\epsilon(t)u_0 \in \Sigma_\theta^\epsilon$, for all $t \in [0, t_{\max}[$ so

$$\|T_\epsilon(t)u_0\|_\infty \le \theta\|\varphi\|_\infty.$$

Applying the variation of constants formula, we obtain (see [10])

$$\begin{aligned} \|T(t)u_0\|_\alpha &\le Mt^{-(\alpha-\eta)}e^{-\delta t}\|u_0\|_\eta \\ &\quad + M\int_0^t (t-s)^{-(\alpha+\frac{1}{2})}e^{-\delta(t-s)}\|(H_\epsilon)_\eta(T(s)u_0)\|_{X^{-1/2}}\,ds, \end{aligned}$$

where $M, \delta > 0$ are constants depending only on the decay of the linear semigroup $e^{A_\epsilon t}$, and can be chosen independently of ϵ. By (6.16),

$$\|(H_\epsilon)_\eta(T(s)u_0)\|_{X^{-1/2}} \le M_f|\Omega|^{1/p} + \|\mu\|_\infty M_g|\partial\Omega|^{1/p}K_1,$$

where $M_f = M_f(u) := \sup\{|f(x)| : x \in I\}$, $M_g = M_g(u) := \sup\{|g(x)| : x \in I\}$, with $[-\theta\|\varphi_\epsilon\|_\infty, \theta\|\varphi_\epsilon\|_\infty] \subset I$, for all ϵ sufficiently small. Thus, writing $K = M_f|\Omega|^{1/p} + \|\mu\|_\infty M_g|\partial\Omega|^{1/p}K_1$, we obtain

$$\begin{aligned}\|T_\epsilon(t)u_0\|_\alpha &\le Mt^{-(\alpha-\eta)}e^{-\delta t}\|u_0\|_\eta + KM\int_0^t (t-s)^{-(\alpha+\frac{1}{2})}e^{-\delta(t-s)}\,ds\\ &\le Mt^{-(\alpha-\eta)}e^{-\delta t}\|u_0\|_\eta + KM\frac{\Gamma(\frac{1}{2}-\alpha)}{\delta^{\frac{1}{2}-\alpha}},\end{aligned}$$

for all $t \in [0, t_{\max}[$.

Therefore $\|T_\epsilon(t)u_0\|_\alpha$ is bounded by a constant for any $t > 0$. Since X^α is compactly embedded in X^η, if $\alpha > \eta$, it follows that the solution is globally defined. Also, if T is such that $t^{-(\alpha-\eta)}e^{-\delta t}\|u_0\|_\eta \le K\frac{\Gamma(\frac{1}{2}-\alpha)}{\delta^{\frac{1}{2}-\alpha}}$, then $\|T_\epsilon(t)u_0\|_\alpha$ belongs to the ball of X^α of radius $R(\theta) = 2KM\frac{\Gamma(\frac{1}{2}-\alpha)}{\delta^{\frac{1}{2}-\alpha}}$, for $t \ge T$. ☐

8. Existence of global attractors

The first step to show the existence of global attractors will be to obtain a "contraction property" of the sets Σ_θ, similar to the property for rectangles, considered by Smoller [18].

Lemma 8.1. *Suppose that the hypotheses of Lemma 7.5 hold and $\bar\theta \in \mathbb{R}$ satisfy $\bar\theta m_\epsilon > \xi$. Then, for any θ there exists a $\bar t$, which can be chosen independently of ϵ, such that*

$$T_\epsilon(t)\Sigma_\theta^\epsilon \subset \Sigma_{\bar\theta}^\epsilon,$$

for all $t \ge \bar t$.

Proof. Let $u \in \Sigma_\theta$. We can suppose without loss of generality that $\theta \ge \bar\theta$. Let $\bar v = e^{-t\mu_\epsilon}\theta\varphi$, $\underline{v} = -\bar v$. As in Lemma 7.5, we can prove that $\bar v$ and $\underline{v}$ are super- and sub-solutions respectively. Thus, using Theorem 7.4 and the uniqueness of solution, we have

$$\underline{v} \le T_\epsilon(t)u \le \bar v.$$

Therefore $T_\epsilon(t)u$ enters $\Sigma_{\bar\theta}$ after a time depending only on θ, and on the first eigenvalue μ_ϵ of A_ϵ (and not on the particular solution $u \in \Sigma_\theta$). Since μ_ϵ is bigger than a constant μ, for ϵ sufficiently small, and $\Sigma_{\bar\theta}$ is positively invariant, the result follows. ☐

Theorem 8.2. *Suppose that the hypotheses of Lemma 7.5 hold. Then problem* (5.1) *has a global attractor $\mathcal{A}_\epsilon$ in X_ϵ^η. Furthermore $\mathcal{A}_\epsilon \subset \Sigma_\theta^\epsilon$ if $\theta m_\epsilon \ge \xi$.*

Proof. Let V be a bounded subset of X^η, and $\bar\theta \in \mathbb{R}$ be such that $\bar\theta m \ge \xi$. If u is any element of X^η, it follows from the continuity of the embedding $X^\eta \hookrightarrow C^0(\bar\Omega)$ that $u \in \Sigma_\theta$, for some θ and then, applying Lemma 8.1, we conclude that $T(t)u \in \Sigma_{\bar\theta}$, for t big enough. From Lemma 7.6, it follows that V enters and remains in a ball of X^α, with $\alpha > \eta$ of radius $R(\alpha, \bar\theta)$, which does not depend on V. Since this ball is a compact set of X^α, the existence of a global compact attractor $\mathcal{A}$ follows immediately. Furthermore, since $\Sigma_{\bar\theta}$ is positively invariant by Lemma 7.5 it also follows that $\mathcal{A} \subset \Sigma_{\bar\theta}$, as claimed. ☐

Corollary 8.3. *Suppose that the hypotheses of Lemma 7.5 hold. If ϵ_0 is sufficiently small, the attractor $\mathcal{A}_\epsilon$ is uniformly bounded in $L^\infty(\Omega)$, for $0 \le \epsilon \le \epsilon_0$.*

Proof. From (4.4) and results in [11], it follows that the first eigenvalue and eigenfunction of A_ϵ are continuous in $W^{1,p}(\Omega)$ and, therefore, also in $L^\infty(\Omega)$, Thus the sets Σ_θ^ϵ are uniformly bounded in $L^\infty(\Omega)$ and the result follows from Theorem 8.2. □

9. Upper semicontinuity of the family of global attractors

Recall that a family of subsets $\mathcal{A}_\lambda$ of a metric space (X, d) is said to be *upper-semicontinuous* at $\lambda = \lambda_0$ if $\delta(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) \to 0$ as $\lambda \to \lambda_0$, where $\delta(A, B) = \sup_{x\in A} d(x, B) = \sup_{x\in A} \inf_{y\in B} d(x, y)$ and *lower-semicontinuous* if $\delta(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda) \to 0$ as $\lambda \to \lambda_0$.

To prove the upper semicontinuity of the family of attractors A_ϵ, given by Theorem 8.2 in the (fixed) fractional space X^η, $0 < \eta < \frac{1}{2}$, we will need two main ingredients: the uniform boundedness of the family and the continuity of the nonlinear semigroup T_ϵ with respect to ϵ. This is the content of the next two results. In view of the uniform boundedness of the solutions, proved in Corollary 8.3 we may suppose, without loss of generality, the following hypothesis on the nonlinearites.

$$\text{f and g are globally bounded, and globally Lipschtiz with constants L_f and L_g respectively.} \tag{9.1}$$

Lemma 9.1. *Suppose that the hypotheses of Lemma 7.5 and* (9.1) *hold. If ϵ_0 is sufficiently small, the family of attractors $\mathcal{A}_\epsilon$ given by Theorem 8.2 is uniformly bounded in the (fixed) fractional space X^η, $0 < \eta < 1/2$, for $0 \le \epsilon \le \epsilon_0$.*

Proof. Let b be the exponential rate of decay of the linear semigroup generated by A_ϵ, for ϵ small, given by Theorem 4.7. Let $u \in \mathcal{A}_\epsilon$. By the variation of constants formula, Lemma 4.5 and Theorem 4.7, we obtain

$$\begin{aligned}
&\|T_\epsilon(t)(u)\|_\eta\\
&\le \|e^{A_\epsilon(t)}u\|_\eta + \int_0^t \|e^{A_\epsilon(t-s)}H_\epsilon(T_\epsilon(s)u)\|_\eta\,ds\\
&\le \|e^{A(t)}u\|_\eta + \|\Big(e^{A_\epsilon(t)} - e^{A(t)}\Big)u\|_\eta + \int_0^t \|e^{A(t-s)}H_\epsilon(T_\epsilon(s)u)\|_\eta\,ds\\
&\quad + \int_0^t \|\Big(e^{A_\epsilon(t-s)} - e^{A(t-s)}\Big)H_\epsilon(T_\epsilon(s)u)\|_\eta\,ds\\
&\le \big(Ce^{-at} + C(\epsilon)e^{-bt}\big)\frac{1}{t^{\eta+\frac{1}{2}}}\|u\| + \int_0^t Ce^{-a(t-s)}\frac{1}{(t-s)^\eta + \frac{1}{2}}\|H_\epsilon(T_\epsilon(s)u)\|ds\\
&\quad + \int_0^t Ce^{-b(t-s)}\frac{1}{(t-s)^{\eta+\frac{1}{2}}}\|H_\epsilon(T_\epsilon(s)u)\|ds.
\end{aligned}$$

By (6.16),

$$\begin{aligned}
\|(H_\epsilon)_\eta(T(s)u_0)\|_{X^{-1/2}} &\le M_f|\Omega|^{1/p} + \|\mu\|_\infty M_g|\partial\Omega|^{1/p}K_1\\
&\le \|f\|_\infty|\Omega|^{1/p} + \|\mu\|_\infty\|g\|_\infty|\partial\Omega|^{1/p}\,K_1,
\end{aligned}$$

where K_1 is a constant of the trace mapping. Thus

$$\begin{aligned}
\|T_\epsilon(t)(u)\|_\eta &\le C'e^{-bt}\frac{1}{t^{\eta+\frac{1}{2}}}\|u\|_\infty\\
&\quad + C''\Big(\|f\|_\infty|\Omega|^{1/p} + \|\mu\|_\infty\|g\|_\infty|\partial\Omega|^{1/p}\Big)\int_0^t e^{-b(t-s)}\frac{1}{(t-s)^\eta + \frac{1}{2}}\,ds,
\end{aligned}$$

where the constants C' and C'' do not depend on ϵ.

Since the right hand side is uniformly bounded for $u \in \mathcal{A}_\epsilon$ and $t > 0$, and the attractors are invariant, the result follows immediately. $\square$

Lemma 9.2. *Suppose that the hypotheses of Lemma 9.1 hold. Then the map*

$$(u,\epsilon) \in X^\eta \times [0,\epsilon_0] \mapsto T_\epsilon u \in X^\eta$$

is continuous at $\epsilon = 0$, uniformly for u in bounded sets and $0 < t \le T < \infty$.

Proof. Using the variation of constants formula, (6.16), (6.20) and (6.18), we obtain

$$\begin{aligned}
&\|T_\epsilon(t)(u) - T(t)(u)\|_\eta \\
&\le \|e^{A_\epsilon(t)}u - e^{A(t)}u\|_\eta \\
&\quad + \int_0^t \|\big(e^{A_\epsilon(t-s)} - e^{A(t-s)}\big)H_\epsilon(T_\epsilon(s)u)\|_\eta \, ds \\
&\quad + \int_0^t \|e^{A(t-s)}\left(H_\epsilon(T_\epsilon(s)u) - H(T_\epsilon(s)u)\right)\|_\eta ds \\
&\quad + \int_0^t \|e^{A(t-s)}\left(H(T_\epsilon(s)u) - H(T(s)u)\right)\|_\eta \, ds \\
&\le C(\epsilon)e^{-bt}\frac{1}{t^{\eta+\frac12}}\|u\| + \int_0^t C(\epsilon)e^{-b(t-s)}\frac{1}{(t-s)^{\eta+\frac12}}\|H_\epsilon(T_\epsilon(s)u)\|d\,s \\
&\quad + \int_0^t Ce^{-b(t-s)}\frac{1}{(t-s)^{\eta+\frac12}}\|H_\epsilon(T_\epsilon(s)u) - H(T_\epsilon(s)u)\|ds \\
&\quad + \int_0^t Ce^{-b(t-s)}\frac{1}{(t-s)^{\eta+\frac12}}\|H(T_\epsilon(s)u) - H(T(s)u)\|ds \\
&\le C(\epsilon)e^{-bt}\frac{1}{t^{\eta+\frac12}}\|u\| \\
&\quad + \int_0^t C(\epsilon)e^{-b(t-s)}\frac{1}{(t-s)^{\eta+\frac12}}\Big(\|f\|_\infty|\Omega|^{1/p} + \|\mu\|_\infty\|g\|_\infty|\partial\Omega|^{1/p}K_1\Big)\,d\,s \\
&\quad + \int_0^t Ce^{-b(t-s)}\frac{1}{(t-s)^{\eta+\frac12}}\|\Big(\|\mu_\epsilon - 1\|_\infty M_g|\partial\Omega|^{1/p}K_1\Big)\,ds \\
&\quad + \int_0^t Ce^{-b(t-s)}\frac{1}{(t-s)^{\eta+\frac12}}\|\left(L_g\|\mu\|_\infty K_1K_2 + L_f\right)\|T_\epsilon(s)u - T(s)u\|_{X^\eta_\epsilon}\|ds\,.
\end{aligned}$$

Writing

$$\begin{aligned}
A(\epsilon) := C(\epsilon)\|u\| + t^{\eta+\frac12}&\int_0^t C(\epsilon)e^{bs}\frac{1}{(t-s)^{\eta+\frac12}} \\
&\times\Big(\|f\|_\infty|\Omega|^{1/p} + \|\mu\|_\infty\|g\|_\infty|\partial\Omega|^{1/p}K_1\Big)\,ds \\
&+ t^{\eta+\frac12}\int_0^t Ce^{bs}\frac{1}{(t-s)^{\eta+\frac12}}\Big(\|\mu_\epsilon - 1\|_\infty M_g|\partial\Omega|^{1/p}K_1\Big)\,ds
\end{aligned}$$

and $B := C\big(L_g\|\mu\|_\infty K_1K_2 + L_f\big)$, we obtain

$$e^{bt}\|T_\epsilon(t)(u) - T(t)(u)\|_\eta \le A(\epsilon)t^{-(\eta+\frac12)} + B\int_0^t t^{-(\eta+\frac12)}e^{bs}\|T_\epsilon(s)u - T(s)u\|_{X^\eta_\epsilon}\,ds.$$

From the singular Gronwall's inequality, it follows that

$$\|T_\epsilon(t)(u) - T(t)(u)\|_\eta \le A(\epsilon) M e^{-bt} t^{-(\eta+\frac{1}{2})},$$

for $0 < t \le T$, where the constant M depends on B, η and T, for u in a bounded set of X^η_ϵ. □

Theorem 9.3. *Suppose that the hypotheses of Lemma 9.1 hold. Then the family of attractors $\mathcal{A}_\epsilon$, given by Theorem 8.2 is upper semicontinuous with respect to ϵ at $\epsilon = 0$.*

Proof. From Lemma 9.1 there exists a bounded set $B \subset X^\eta$ such that $\cup_{0\le\epsilon\le\epsilon_0}\mathcal{A}_\epsilon \subset B$. Given $\delta > 0$, there exists $t_\delta > 0$ such that $T(t_\delta)(B) \subset \mathcal{A}_0^{\frac{\delta}{2}}$, where $\mathcal{A}_0^{\frac{\delta}{2}}$ is the $\frac{\delta}{2}$-neighborhood of $\mathcal{A}_0$.

From Lemma 9.2, there exists $\bar\epsilon > 0$ such that $\|T_\epsilon(t_\delta)u - T(t_\delta)u\|_\eta \le \frac{\delta}{2}$, for every $u \in B$ and $0 \le \epsilon \le \bar\epsilon$. It follows that $T_\epsilon(t_\delta)B \subset \mathcal{A}_0^\delta$. In particular, $T_\epsilon(t_\delta)\mathcal{A}_\epsilon \subset \mathcal{A}_0^\delta$. Since $\mathcal{A}_\epsilon$ is invariant under T_ϵ, we conclude that $\mathcal{A}_\epsilon \subset \mathcal{A}_0^\delta$, for $0 \le \epsilon \le \bar\epsilon$, thus proving the claim. □

From the semicontinuity of attractors, we can easily prove the corresponding property for the equilibria.

Corollary 9.4. *Suppose the hypotheses of Theorem 9.3 hold. Then the family of sets of equilibria $\{E_\epsilon \,|\, 0 \le \epsilon \le \epsilon_0\}$, of problem* (5.1) *is upper semicontinuous in X^η.*

Proof. The result is well-known, but we sketch a proof here for completeness. Suppose $u_n \in \mathcal{A}_n$, with $\lim_{n\to\infty} \epsilon_n = 0$. We choose an arbitrary subsequence and still call it (u_n), for simplicity. It is enough to show that, there exists a subsequence (u_{n_k}), which converges to a point $u_0 \in E_0$. Since $(u_n) \to \mathcal{A}_0$, there exists $(v_n) \in \mathcal{A}_0$ with $\|u_n - v_n\|_\eta \to 0$. Since $\mathcal{A}_0$ is compact, there exists a subsequence (v_{n_k}), which converges to a point $u_0 \in \mathcal{A}_0$, so also $(u_{n_k}) \to \mathcal{A}_0$. Now, since the flow $T_\epsilon(t)$ is continuous in ϵ, for any $t > 0$ we have

$$u_{n_k} \to u_0 \Leftrightarrow T_{\epsilon_{n_k}}(t)u_{n_k} \to T_0(t)u_0 \Leftrightarrow u_{n_k} \to T_0(t)u_0.$$

Thus, by uniqueness of the limit, $T_0(t)u_0 = u_0$, for any $t > 0$, so $u_0 \in E_0$. □

10. Lower semicontinuity

For having lower semicontinuity we need to assume the following additional properties for the nonlinearities:

$$f \text{ and } g \text{ belong to } C^1(\mathbb{R},\mathbb{R}) \text{ and have bounded derivatives.} \tag{10.1}$$

Lemma 10.1. *Suppose that η and p are such that* (6.1) *holds and f satisfies* (10.1). *Then the operator $F : X^\eta \times \mathbb{R} \to X^{-1/2}$ given by* (5.3) *is Gateaux differentiable with respect to u, with Gateaux differential $\frac{\partial F}{\partial u}(u,\epsilon)w$ given by*

$$\langle \frac{\partial F}{\partial u}(u,\epsilon)w, \Phi\rangle_{-1/2,1/2} = \int_\Omega f'(u) w \Phi \, dx\,, \tag{10.2}$$

for all $w \in X^\eta$ and $\Phi \in X^{1/2}$.

Proof. Observe first that $F(u,\epsilon)$ is well-defined, since the conditions of Lemma 6.1 are met.

It is clear that $\frac{\partial F}{\partial u}(u,\epsilon)$ is linear. We now show that it is bounded. In fact for all $u,w\in X^\eta$ and $\Phi\in X^{-1/2}=W^{1,q}(\Omega)$ we have

$$\begin{aligned}\big|\langle\frac{\partial F}{\partial u}(u,\epsilon)w,\Phi\rangle_{-1/2,1/2}\big| &\le \int_\Omega |f'(u)||w||\Phi|dx\\ &\le \|f'\|_\infty\int_\Omega |w||\Phi|\,dx\\ &\le \|f'\|_\infty\|w\|_{L^p(\Omega)}\|\Phi\|_{L^q(\Omega)}\,dx\\ &\le \|f'\|_\infty\|w\|_{X^\eta}\|\Phi\|_{X^{1/2}}\,dx\,,\end{aligned}$$

where $\|f'\|_\infty=\sup\{f'(x)|x\in\mathbb{R}\}$. This proves boundedness.

Now, for all $u,w\in X^\eta$ and $\Phi\in X^{1/2}$ we have

$$\begin{aligned}&\big|\frac{1}{t}\langle F(u+tw,\epsilon)-F(u,\epsilon)-t\frac{\partial F}{\partial u}(u,\epsilon)w,\Phi\rangle_{-1/2,1/2}\big|\\ &\le\frac{1}{|t|}\int_\Omega\big|[\,f(u+tw)-f(u)-tf'(u)w\,]\,\Phi\,\big|\,dx\\ &\le\frac{1}{|t|}\Big(\int_\Omega\big|f(u+tw)-f(u)-tf'(u)w\big|^p dx\Big)^{1/p}\|\Phi\|_{X^{1/2}}\\ &\le\underbrace{\Big(\int_\Omega\big|(f'(u+\bar t w)-f'(u))w\big|^p\,dx\Big)^{1/p}}_{(I)}\|\Phi\|_{X^{1/2}},\end{aligned}$$

where $0\le\bar t\le t$. Since f' is bounded and continuous, the integrand of (I) is bounded by an integrable function and goes to 0 as $t\to0$. Thus, the integral (I) goes to 0 as $t\to0$, from Lebesgue's Dominated Convergence Theorem. It follows that

$$\lim_{t\to0}\frac{F(u+tw,\epsilon)-F(u,\epsilon)}{t}=\frac{\partial F}{\partial u}(u,\epsilon)w\quad\text{ in }X^{-1/2},$$

for all $u,w\in X^\eta$; so F is Gateaux differentiable with Gateaux differential given by (10.2). ☐

Now we want to prove that the Gateaux differential of $F(u,\epsilon)$ is continuous in u. Let us denote by $\mathcal{B}(X,Y)$ the space of linear bounded operators from X to Y. We will need the following result, whose simple proof is omitted.

Lemma 10.2. *Suppose X,Y are Banach spaces and $T_n:X\to Y$ is a sequence of linear operators converging strongly to the linear operator $T:X\to Y$. Suppose also that $X_1\subset X$ is a Banach space, the inclusion $i:X_1\hookrightarrow X$ is compact and let $\widetilde{T}_n=T_n\circ i$ and $\widetilde{T}=T\circ i$. Then $\widetilde{T}_n\to\widetilde{T}$ uniformly for x in a bounded subset of X_1 (that is, in the or norm of $\mathcal{B}(X_1,Y)$).*

Lemma 10.3. *Suppose that η and p are such that* (6.1) *holds and f satisfies* (10.1). *Then the Gateaux differential of $F(u,\epsilon)$, with respect to u is continuous in u, that is, the map $u\mapsto\frac{\partial F}{\partial u}(u,\epsilon)\in\mathcal{B}(X^\eta,X^{-1/2})$ is continuous.*

Proof. Let u_n be a sequence converging to u em X^η, and choose $0<\widetilde{\eta}<\eta$, such that the hypotheses still hold. Then, for any $\Phi\in X^{1/2}$ and $w\in X^{\widetilde{\eta}}$ we have

$$\big|\langle\big(\frac{\partial F}{\partial u}(u_n,\epsilon)-\frac{\partial F}{\partial u}(u,\epsilon)\big)w,\Phi\rangle_{-1/2,1/2}\big|$$

$$
\begin{aligned}
&\le \int_\Omega \big|\big(f'(u)-f'(u_n)\big)w\Phi\big|\,dx\\
&\le \Big(\int_\Omega \big|\big(f'(u)-f'(u_n)\big)w\big|^p dx\Big)^{1/p}\Big(\int_\Omega |\Phi|^q dx\Big)^{1/q}\\
&\le \underbrace{\Big(\int_\Omega \big|\big(f'(u)-f'(u_n)\big)w\big|^p dx\Big)}_{(I)}^{1/p}\|\Phi\|_{X^{1/2}}\,.
\end{aligned}
$$

Now, the integrand in (I) is bounded by the integrable function $\|f'\|_\infty^p w^p$ and goes to 0 a.e. as $u_n\to u$ in X^η. Therefore the sequence of operators $\frac{\partial F}{\partial u}(u_n,\epsilon)$ converges strongly in the space $\mathcal{B}(X^{\widetilde{\eta}},X^{-1/2})$ to the operator $\frac{\partial F}{\partial u}(u,\epsilon)$. From Lemma 10.2 the convergence holds in the norm of $\mathcal{B}(X^\eta,X^{-1/2})$, since X^η is compactly embedded in $X^{\widetilde{\eta}}$. ◻

Lemma 10.4. *Suppose that η and p are such that* (6.1) *holds and g satisfies* (10.1)*. Then the operator $G: X^\eta\times\mathbb{R}\to X^{-1/2}$ given by* (5.4) *is Gateaux differentiable with respect to u, with Gateaux differential*

$$
\langle\frac{\partial G}{\partial u}(u,\epsilon)w,\Phi\rangle_{-1/2,1/2}=\int_{\partial\Omega} g'(\gamma(u))\gamma(w)\gamma(\Phi)\big|\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}\big|\,d\sigma(x)\,, \tag{10.3}
$$

for all $w\in X^\eta$ and $\Phi\in X^{1/2}$.

Proof. Observe first that $G(u,\epsilon)$ is well-defined, since the conditions of Lemma 6.2 are met. It is clear that $\frac{\partial G}{\partial u}(u,\epsilon)$ is linear. We now show that it is bounded. In fact, for all $u,w\in X^\eta$ and $\Phi\in X^{1/2}$, we have

$$
\begin{aligned}
\big|\langle\frac{\partial G}{\partial u}(u,\epsilon)w,\Phi\rangle_{-1/2,1/2}\big|
&=\big|\int_{\partial\Omega} g'(\gamma(u))\gamma(w)\gamma(\Phi)\big|\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}\big|\,d\sigma(x)\big|\\
&\le \|\mu\|_\infty\|g'\|_\infty\int_{\partial\Omega}|\gamma(w)|\,|\gamma(\Phi)|\,d\sigma(x)\\
&\le \|\mu\|_\infty\|g'\|_\infty\|\gamma(w)\|_{L^p(\partial\Omega)}\|\gamma(\Phi)\|_{L^q(\partial\Omega)}\\
&\le K_1K_2\|\mu\|_\infty\|g'\|_\infty\|w\|_\eta\|\Phi\|_{X^{1/2}}\,,
\end{aligned}
$$

where $\|g'\|_\infty=\sup\{g'(x):x\in\mathbb{R}\}$, $\|\mu\|_\infty=\sup\{|\mu(x,\epsilon)|:x\in\partial\Omega\}=\sup\{\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}(x):x\in\partial\Omega\}$ and K_1, K_2 are embedding constants. This proves boundedness.

Now, for all $u,w\in X^\eta$ and $\Phi\in X^{1/2}$, we have

$$
\begin{aligned}
&\big|\frac{1}{t}\langle G(u+tw,\epsilon)-G(u,\epsilon)-t\frac{\partial G}{\partial u}(u,\epsilon)w,\Phi\rangle_{-1/2,1/2}\big|\\
&\le\frac{1}{|t|}\int_{\partial\Omega}|[g(\gamma(u+tw))-g(\gamma(u))-tg'(\gamma(u))]\gamma(w)|\,|\gamma(\Phi)|\,\big|\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}\big|\,d\sigma(x)\\
&\le K_1\|\mu\|_\infty\frac{1}{|t|}\Big\{\int_{\partial\Omega}|[g(\gamma(u+tw))-g(\gamma(u))-tg'(\gamma(u))]\gamma(w)|^p\,d\sigma(x)\Big\}^{1/p}\|\Phi\|_{X^{1/2}}\\
&\le K_1\|\mu\|_\infty\underbrace{\Big\{\int_{\partial\Omega}|[\,g'(\gamma(u+\bar{t}w))-g'(\gamma(u))]\gamma(w)|^p\,d\sigma(x)\Big\}}_{(I)}^{1/2}\|\Phi\|_{X^{1/2}}\,,
\end{aligned}
$$

where K_1 is an embedding constant given by Trace Theorem and $0\le\bar{t}\le t$. Since g' is bounded and continuous, the integrand of (I) is bounded by an integrable function and goes to 0 as $t\to 0$. Thus, the integral (I) goes to 0 as $t \to 0$, from Lebesgue's Dominated Convergence Theorem. It follows that $\lim_{t\to 0} \frac{G(u+tw,\epsilon)-G(u,\epsilon)}{t} = \frac{\partial G}{\partial u}(u,\epsilon)w$ in $X^{-1/2}$, for all $u, w \in X^\eta$; so G is Gateaux differentiable with Gateaux differential given by (10.3). $\square$

Lemma 10.5. *Suppose that η and p are such that* (6.1) *holds and g satisfies* (10.1). *Then the Gateaux differential of $G(u,\epsilon)$, with respect to u is continuous in u (that is, the map $u \mapsto \frac{\partial G}{\partial u}(u,\epsilon) \in \mathcal{B}(X^\eta, X^{-1/2})$ is continuous) and uniformly continuous in ϵ for u in bounded sets of X^η and $0 \le \epsilon \le \epsilon_0 < 1$.*

Proof. Let $0 \le \epsilon \le \epsilon_0$, u_n be a sequence converging to u em X^η, and choose $0 < \widetilde{\eta} < \eta$, still satisfying the hypotheses. Then, for any $\Phi \in X^{1/2}$ and $w \in X^{\widetilde{\eta}}$, we have

$$
\begin{aligned}
&\big|\langle\big(\frac{\partial G}{\partial u}(u_n,\epsilon) - \frac{\partial G}{\partial u}(u,\epsilon)\big)w, \Phi\rangle_{-1/2,1/2}\big| \\
&\le \int_{\partial\Omega} |(g'(\gamma(u)) - g'(\gamma(u_n)))\gamma(w)\gamma(\Phi)|\,\big|\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}\big|\,d\sigma(x) \\
&\le \|\mu_\epsilon\|_\infty \Big\{\int_{\partial\Omega} |(g'(\gamma(u)) - g'(\gamma(u_n))\gamma(w)|^p\,d\sigma(x)\Big\}^{1/p} \Big\{\int_{\partial\Omega} |\gamma(\Phi)|^q\,d\sigma(x)\Big\}^{1/q} \\
&\le K_1\|\mu_\epsilon\|_\infty \Big\{\underbrace{\int_{\partial\Omega} |(g'(\gamma(u)) - g'(\gamma(u_n))\gamma(w)|^p\,d\sigma(x)}_{(I)}\Big\}^{1/p} \|\Phi\|_{X^{1/2}}\,,
\end{aligned}
$$

where K_1 is the constant for continuity of the trace map from $X^{1/2}$ into $L^2(\partial\Omega)$, as in Lemma 6.2.

Now, the integrand in (I) is bounded by the integrable function $\|g'\|_\infty^2|\gamma(w)|^2$ and approaches 0 a.e. as $u_n \to u$ in X^η. Therefore the sequence of operators $\frac{\partial G}{\partial u}(u_n,\epsilon)$ converges strongly in the space $\mathcal{B}(X^{\widetilde{\eta}}, X^{-1/2})$ to the operator $\frac{\partial G}{\partial u}(u,\epsilon)$. From Lemma 10.2 the convergence holds in the norm of $\mathcal{B}(X^\eta, X^{-1/2})$, since X^η is compactly embedded in $X^{\widetilde{\eta}}$ (see [10]).

Finally, if $0 \le \epsilon_1 \le \epsilon_2 < \epsilon_0$, for any $\Phi \in X^{1/2}$ and $w \in X^\eta$, we have

$$
\begin{aligned}
&\big|\langle\big(\frac{\partial G}{\partial u}(u,\epsilon_1) - \frac{\partial G}{\partial u}(u,\epsilon_2)\big)w, \Phi\rangle_{-1/2,1/2}\big| \\
&\le \int_{\partial\Omega} |g'(\gamma(u))\gamma(w)\gamma(\Phi)|\,|\mu_{\epsilon_1} - \mu_{\epsilon_2}|\,d\sigma(x) \\
&\le \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_\infty \Big\{\int_{\partial\Omega} |g'(\gamma(u))\gamma(w)|^p\,d\sigma(x)\Big\}^{1/p} \Big\{\int_{\partial\Omega} |\gamma(\Phi)|^q\,d\sigma(x)\Big\}^{1/q} \\
&\le K_1K_2\|g'\|_\infty\|\|w\|_{X^\eta}\|\Phi\|_{X^{1/2}}\|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_\infty,
\end{aligned}
$$

where K_2 is the constant for the continuity of the trace map from X^η into $L^q(\partial\Omega)$, as before. This proves uniform continuity in ϵ. $\square$

Lemma 10.6. *Suppose that η and p are such that* (6.1) *holds and f and g satisfy* (10.1). *Then the map $(H_\epsilon)_{-\frac{1}{2}} = (F_\epsilon)_{-\frac{1}{2}} + (G_\epsilon)_{-\frac{1}{2}} : X^\eta \times \mathbb{R} \mapsto X^{-1/2}$ given by* (5.2) *is continuously Fréchet differentiable with respect to u and the derivative $\frac{\partial G}{\partial u}$ is uniformly continuous with respect to ϵ, for u in bounded sets of X^η and $0 \le \epsilon \le \epsilon_0 < 1$.*

The above lemma follows from Lemmas 10.3, 10.5 and [17, Proposition 2.8]. We now prove lower semicontinuity for the equilibria.

Theorem 10.7. *If f and g satisfy the conditions of Theorem 7.5 and* (10.1)*, then the equilibria of* (5.1) *with $\epsilon = 0$ are all hyperbolic and $1/4 < \eta < 1/2$, then the family of sets of equilibria $\{E_\epsilon : 0 \leq \epsilon < \epsilon_0\}$ of* (5.1) *is lower semicontinuous in X^η at $\epsilon = 0$.*

Proof. A point $e \in X^\eta$ is an equilibrium of (5.1) if and only if it is a root of the map $Z : W^{1,p}(\Omega) \times \mathbb{R} \to X^{-1/2}$ given by

$$(u, \epsilon) \mapsto (A_\epsilon)_{-1/2}(u) + (H_\epsilon)_{-1/2}(u)\,.$$

By Lemma 10.6 the map $(H_\epsilon)_{-1/2} : X^\eta \to X^{-1/2}$ is continuously Fréchet differentiable with respect to u and by Lemmas 6.3 and 6.1 it is also continuous in ϵ if $\eta = \frac{1}{2} - \delta$, with $\delta > 0$ is sufficiently small. Therefore, the same holds if $\eta = 1/2$.

The map $A_\epsilon = -h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^* + aI$ is a bounded linear operator from $W^{1,p}(\Omega)$ to $X^{-1/2}$. It is also continuous in ϵ since it is analytic as a function of $h_\epsilon \in Diff^1(\Omega)$ and h_ϵ is continuous in ϵ.

Thus, the map Z is continuously differentiable in u and continuous in ϵ. The derivative of $\frac{\partial Z}{\partial u}(e, 0)$ is an isomorphism by hypotheses. Therefore, the Implicit Function Theorem apply, implying that the zeroes of $Z(\cdot, \epsilon)$ are given by a continuous function $e(\epsilon)$. This proves the claim. □

To prove the lower semi continuity of the attractors, we also need the continuity of local unstable manifolds at equilibria.

Theorem 10.8. *Suppose that η and p are such that* (6.1) *holds and f and g satisfy* (10.1)*, u_0 is an equilibrium of* (5.1) *with $\epsilon = 0$, and for each $\epsilon > 0$ sufficiently small, let u_ϵ be the unique equilibrium of* (5.1)*, whose existence is asserted by Corollary 9.4 and Theorem 10.7. Then, for ϵ and δ sufficiently small, there exists a local unstable manifold $W^u_{\rm loc}(u_\epsilon)$ of u_ϵ, and if we denote $W^u_\delta(u_\epsilon) = \{w \in W^u_{\rm loc}(u_\epsilon) : \|w - u_\epsilon\|_\eta < \delta\}$, then*

$$-\frac{1}{2}\Big(W^u_\delta(u_\epsilon), W^u_\delta(u_0)\Big) \quad \textit{and} \quad -\frac{1}{2}\Big(W^u_\delta(u_0), W^u_\delta(u_\epsilon)\Big)$$

approach zero as $\epsilon \to 0$, where $-\frac{1}{2}(O, Q) = \sup_{o \in O} \inf_{q \in Q} \|q - o\|_{X^\eta}$ for O, $Q \subset X^\eta$.

Proof. Let $H_\epsilon(u) = H(u, \epsilon)$ be the map defined by (5.2) and u_ϵ a hyperbolic equilibrium of (5.1). Since $H(u, \epsilon)$ is differentiable by Lemma 10.6, it follows that

$$\begin{aligned} H_\epsilon(u_\epsilon + w, \epsilon) &= H_\epsilon(u_\epsilon, \epsilon) + H_u(u_\epsilon, \epsilon)w + r(w, \epsilon) \\ &= A_\epsilon u_\epsilon + H_u(u_\epsilon, \epsilon)w + r(w, \epsilon), \end{aligned}$$

with $r(w, \epsilon) = o(\|w\|_{X^\eta})$, as $\|w\|_{X^\eta} \to 0$. The claimed was proved in [16], assuming the following properties of H_ϵ:

(a) $\|r(w, 0) - r(w, \epsilon)\|_{X^{-1/2}} \leq C(\epsilon)$, with $C(\epsilon) \to 0$ when $\epsilon \to 0$, uniformly for w in a neighborhood of 0 in X^η.

(b) $\|r(w_1, \epsilon) - r(w_2, \epsilon)\|_{X^{-1/2}} \leq k(\rho)\|w_1 - w_2\|_\eta$, for $\|w_1\|_\eta \leq \rho$, $\|w_2\|_\eta \leq \rho$, with $k(\rho) \to 0$ when $\rho \to 0^+$ and $k(*)$ is non decreasing.

Property (a) follows easily from the fact that both $H(u, \epsilon)$ and $H_u(u, \epsilon)$ are uniformly continuous in ϵ for u in bounded sets of X^η, by Lemmas 6.3, 6.1 and 10.6.

It remains to prove property (b). If $w_1, w_2 \in X^\eta$ and $\epsilon \in [0, \epsilon_0]$, with $0 < \epsilon_0 < 1$ small enough, we have

$$\|r(w_1, \epsilon) - r(w_2, \epsilon)\|_{X^{-1/2}}$$

$$
\begin{aligned}
&= \|H(u_\epsilon + w_1, \epsilon) - H(u_\epsilon, \epsilon) - H_u(u_\epsilon, \epsilon)w_1 \\
&\quad - H(u_\epsilon + w_2, \epsilon) + H_\epsilon(u_\epsilon, \epsilon) + H_u(u_\epsilon, \epsilon)w_2\|_{X^{-1/2}} \\
&\le \|F(u_\epsilon + w_1, \epsilon) - F(u_\epsilon, \epsilon) - F_u(u_\epsilon, \epsilon)w_1 \qquad (10.4)\\
&\quad - F(u_\epsilon + w_2, \epsilon) + F(u_\epsilon, \epsilon) + F_u(u_\epsilon, \epsilon)w_2\|_{X^{-1/2}} \\
&\quad + \|G(u_\epsilon + w_1, \epsilon) - G(u_\epsilon, \epsilon) - G_u(u_\epsilon, \epsilon)w_1 \qquad (10.5)\\
&\quad - G(u_\epsilon + w_2, \epsilon) + G(u_\epsilon, \epsilon) + G_u(u_\epsilon, \epsilon)w_2\|_{X^{-1/2}}\,.
\end{aligned}
$$

We first estimate (10.4). Since f' is bounded by (10.1), we have

$$
\begin{aligned}
&\Big|\langle F(u_\epsilon + w_1, \epsilon) - F(u_\epsilon, \epsilon) - F_u(u_\epsilon, \epsilon)w_1 - F(u_\epsilon + w_2, \epsilon) \\
&+ F(u_\epsilon, \epsilon) + F_u(u_\epsilon, \epsilon)w_2, \Phi\rangle_{-1/2,1/2}\Big| \\
&\le \int_\Omega |[f(u_\epsilon + w_1) - f(u_\epsilon) - f'(u_\epsilon)w_1 - f(u_\epsilon + w_2) + f(u_\epsilon) + f'(u_\epsilon)w_2]\Phi|\,dx \\
&= \int_\Omega |[f'(u_\epsilon + \xi_x) - f'(u_\epsilon)](w_1(x) - w_2(x))\Phi|\,dx \\
&\le K_1\Big\{\int_\Omega |[f'(u_\epsilon + \xi_x) - f'(u_\epsilon)]^p (w_1(x) - w_2(x))^p|\,dx\Big\}^{1/p}\|\Phi\|_{X^{1/2}}\,, \\
&\le K_1K_2\Big\{\int_\Omega |[f'(u_\epsilon + \xi_x) - f'(u_\epsilon)]^p|\,dx\Big\}^{1/p}\|w_1 - w_2\|_{X^\eta}\|\Phi\|_{X^{1/2}}\,,
\end{aligned}
$$

where K_1 is the embedding constant of $X^{1/2}$ into $L^q(\Omega)$, K_2 is the embedding constant of X^η in $L^\infty(\Omega)$ and $w_1(x) \le \xi_x \le w_2(x)$ or $w_2(x) \le \xi_x \le w_1(x)$. Therefore,

$$
\begin{aligned}
&\|F(u_\epsilon + w_1, \epsilon) - F(u_\epsilon, \epsilon) - F_u(u_\epsilon, \epsilon)w_1 - F(u_\epsilon + w_2, \epsilon) \\
&+ F(u_\epsilon, \epsilon) + F_u(u_\epsilon, \epsilon)w_2\|_{X^{-1/2}} \\
&\le K_1K_2\Big\{\int_\Omega [f'(u_\epsilon + \xi_x) - f'(u_\epsilon)]^p dx\Big\}^{1/p}\|w_1 - w_2\|_\eta\,.
\end{aligned}
$$

Now the integrand above is bounded by $2^p\|f'\|_\infty^{2p}$ and approaches 0 a.e. as $\rho \to 0$, since $\|w_1\|_\eta \le \rho$, $\|w_2\|_\eta \le \rho$ and $w_1(x) \le \xi_x \le w_2(x)$. Thus, the integral approaches 0 by Lebesgue's bounded convergence Theorem.

We now estimate (10.5):

$$
\begin{aligned}
&\Big|\langle G(u_\epsilon + w_1, \epsilon) - G(u_\epsilon, \epsilon) - G_u(u_\epsilon, \epsilon)w_1 - G(u_\epsilon + w_2, \epsilon) + G(u_\epsilon, \epsilon) \\
&+ G_u(u_\epsilon, \epsilon)w_2, \Phi\rangle_{-1/2,1/2}\Big| \\
&\le \int_{\partial\Omega} \Big|[g(\gamma(u_\epsilon + w_1)) - g(\gamma(u_\epsilon)) - g'(\gamma(u_\epsilon))w_1 \\
&\quad - g(\gamma(u_\epsilon + w_2)) + g(\gamma(u_\epsilon)) + g'(\gamma(u_\epsilon))w_2]\gamma(\Phi)\gamma\big(\big|\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}\big|\big)\Big|\,d\sigma(x) \\
&= \int_{\partial\Omega} \Big|[g'(\gamma(u_\epsilon + \xi_x)) - g'(\gamma(u_\epsilon))]\gamma(w_1(x) - w_2(x))\gamma(\Phi)\gamma\big(\big|\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}\big|\big)\Big|\,d\sigma(x) \\
&\le K_1\Big\{\int_{\partial\Omega}[\,(g'(\gamma(u_\epsilon + \xi_x)) - g'(\gamma(u_\epsilon)))]^p[\gamma(w_1(x) - w_2(x))]^p
\end{aligned}
$$

$$\times \Big[\gamma\big(\big|\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}\big|\big)\Big]^p d\sigma(x)\Big\}^{1/p}\|\Phi\|_{X^{1/2}}$$

$$\le K_1K_2\mu_\epsilon\Big\{\int_{\partial\Omega}[(g'(\gamma(u_\epsilon+\xi_x))-g'(\gamma(u_\epsilon)))]^p d\sigma(x)\Big\}^{1/p}\|w_1-w_2\|_{X^\eta}\|\Phi\|_{X^{1/2}}\,,$$

where $\mu_\epsilon = |\frac{J_{\partial\Omega}h_\epsilon}{Jh_\epsilon}|$ is bounded, uniformly in ϵ and $w_1(x) \le \xi_x \le w_2(x)$ or $w_2(x) \le \xi_x \le w_1(x)$.

Now the integrand above is bounded by $2^p\|g'\|_\infty^p\|$ and approaches o 0 a.e. as $\rho \to 0$, since $\|w_1\|_\eta \le \rho$, $\|w_2\|_\eta \le \rho$ and $w_1(x) \le \xi_x \le w_2(x)$. Thus, the integral approaches 0 by Lebesgue's dominated convergence Theorem. □

We are now in a position to prove the main result of this section.

Theorem 10.9. *Assume the hypotheses of Theorem 10.7 hold. Then the family of attractors $\{\mathcal{A}_\epsilon : 0 \le \epsilon \le \epsilon_0\}$, of problem (5.1), whose existence is guaranteed by Theorem 8.2 is lower semicontinuous in X^η.*

Proof. The system generated by (5.1) is gradient for any ϵ and its equilibria are all hyperbolic for ϵ in a neighborhood of 0. Also, the equilibria are continuous in ϵ by Theorem 10.7, the linearization is continuous in ϵ as shown during the proof of Theorem 10.7 and the local unstable manifolds of the equilibria are continuous in ϵ, by Theorem 10.8. The result follows then from [16, Theorem 3.10]. □

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