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AN ASYMPTOTIC MONOTONICITY FORMULA FOR MINIMIZERS OF ELLIPTIC SYSTEMS OF ALLEN-CAHN TYPE AND THE LIOUVILLE PROPERTY

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Abstract. We prove an asymptotic monotonicity formula for bounded, globally minimizing solutions (in the sense of Morse) to a class of semilinear elliptic systems of the form $\Delta u = W_u(u)$, $x \in \mathbb{R}^n$, $n \geq 2$, with $W : \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, nonnegative and vanishing at exactly one point (at least in the closure of the image of the considered solution u). As an application, we can prove a Liouville type theorem under various assumptions.

1. Introduction and statement of main results

We consider the semilinear elliptic equation

$$\Delta u = W_u(u) \quad \text{in } \mathbb{R}^n,\ n \geq 2, \tag{1.1}$$

where $W : \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, is sufficiently smooth and *nonnegative* (we use the notation $W_u = \nabla_u W$). This system has variational structure, as solutions (in a smooth, bounded domain $\Omega \subset \mathbb{R}^n$) are critical points of the energy

$$E(v;\Omega) = \int_\Omega \Big\{\frac{1}{2}|\nabla v|^2 + W(v)\Big\}\,dx \tag{1.2}$$

(subject to their own boundary conditions), where $|\nabla v|^2 = \sum_{i=1}^n |v_{x_i}|^2$. A solution $u \in C^2(\mathbb{R}^n;\mathbb{R}^m)$ is called *globally minimizing* (in the sense of Morse) if

$$E(u;\Omega) \leq E(u+\varphi;\Omega) \tag{1.3}$$

for every smooth, bounded domain $\Omega \subset \mathbb{R}^n$ and for every $\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ (see [7, 30] and the references therein).

If $m \geq 2$, there are two main categories of such potentials W:

- Those that vanish only on a discrete set of points (usually finite); in this case (1.1) is known as the vectorial Allen-Cahn equation and models multiphase transitions (see [7, 9, 16, 27] and some of the references that will follow).
- Those that vanish on a continuum of points, as in the Ginzburg-Landau system (see [14]) or the elliptic system modeling phase-separation in [13].

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This article is motivated from the first class. In this setting, an effective way to construct entire, nontrivial solutions to (1.1) is to assume that W is symmetric with respect to a finite reflection group and to look for equivariant solutions (one first minimizes $E(\cdot;B_R)$ in this class, under suitable boundary conditions on ∂B_R, and then lets $R\to\infty$). Under proper assumptions, this roughly amounts to studying bounded, globally minimizing solutions to (1.1) such that the closure of their image contains exactly one global minimum of W. In the scalar case, that is $m=1$, this approach has been utilized, among others, in [17] and [22]. On the other hand, recent progress has been made in the vector case in [4, 7, 11, 12, 29, 40]. In our opinion, the main obstruction in the vector case is the lack of the maximum principle. This short discussion motivates our main result which is the following.

Theorem 1.1. *Assume that $W\in C^1(\mathbb{R}^m;\mathbb{R})$, $m\geq 1$, and that there exists $a\in\mathbb{R}^m$ such that*

$$W>0 \text{ in } \mathbb{R}^m\setminus\{a\} \quad \text{and} \quad W(a)=0. \tag{1.4}$$

If $u\in C^2(\mathbb{R}^n;\mathbb{R}^m)$, $n\geq 2$, is a bounded, globally minimizing solution to the elliptic system (1.1)*, then*

$$\lim_{R\to\infty}\Big(\frac{1}{R^{n-1}}\int_{B_R}\big\{\frac{1}{2}|\nabla u|^2+W(u)\big\}\,dx\Big)=0, \tag{1.5}$$

where B_R stands for the n-dimensional ball of radius R and center at the origin.

The above result may be interpreted as an *asymptotic monotonicity formula* (see (2.19) below). We emphasize that there is no assumption for the behavior of W near a. Our proof of Theorem 1.1 is based on an adaptation to this setting of the famous "bad discs" construction of [14] from the study of vortices in the Ginzburg-Landau model.

Under even more general assumptions on W, it is well known that every bounded and globally minimizing solution to (1.1) satisfies

$$\limsup_{R\to\infty}\Big(\frac{1}{R^{n-1}}\int_{B_R}\big\{\frac{1}{2}|\nabla u|^2+W(u)\big\}\,dx\Big)<\infty,$$

(see for example [7, Ch. 5], [19]). The above relation can be proven by comparing the energy of u in B_R to that of a suitable test function which agrees with u on ∂B_R and is equal to some zero of W in B_{R-1}. This simple idea, which can actually be traced back to the theory of minimal surfaces (see [20]), will also play an important role in our analysis.

As an application of Theorem 1.1, we can prove the following Liouville type theorem.

Theorem 1.2. *Assume that W and u are as in Theorem 1.1. Then*

$$u\equiv a,$$

provided that one of the following additional conditions is satisfied:

(a) *$m=1$ and $W\in C^{1,1}_{\rm loc}(\mathbb{R};\mathbb{R})$; or $m\geq 1$ and Modica's gradient bound holds, that is*

$$\frac{1}{2}|\nabla u|^2\leq W(u) \quad \text{in } \mathbb{R}^n. \tag{1.6}$$

(b) *$n=2$ and there exists a small $r_0>0$ such that the functions*

$$r\mapsto W(a+r\nu), \quad \nu\in\mathbb{S}^{m-1} \quad \text{are nondecreasing for } r\in(0,r_0]; \tag{1.7}$$

or $n=2$ and $m=1$.

The above Liouville property was originally proven, for all $n \geq 2$, by different techniques in [30] (see also the earlier paper [29]), under the conditions that $W \in C^2(\mathbb{R}^m;\mathbb{R})$ and u satisfy the assumptions of Theorem 1.1, and that the functions in (1.7) have a strictly positive second order derivative in $(0, r_0)$. In particular, the approach of the latter references is based on a quantitative refinement of the replacement lemmas in [4] and [28], combined with a rather involved iterative procedure. If W additionally satisfies the stronger assumption that a is a non-degenerate minimum, this theorem was recently reproven in [6] by extending to this setting the density estimates of [19]. In the aforementioned references, the Liouville type theorem was proven by an application of a basic pointwise estimate. However, it is not difficult to convince oneself that going in the opposite direction is also possible, i.e., the pointwise estimate follows from the Liouville property (see also [41] for this viewpoint). We note that the pointwise estimate is the one that is directly applicable in relation to the discussion preceding Theorem 1.1. This pointwise estimate roughly says that if W (as in Theorem 1.1) is such that the Liouville type theorem holds, then a globally minimizing solution, defined in a sufficiently large ball (with the appropriate modifications in the definition) and bounded independently of the size of the ball, has to be close to a in the ball of half the radius (with the same center).

In light of the recent density estimates of [23], we expect that the assertions of Theorems 1.1 and 1.2 should also remain valid under the complementary set of assumptions that $W \in C^1$ satisfies

$$c|u-a|^p \leq W(u) \leq C|u-a|^p, \quad u \in \mathbb{R}^m,\ m \geq 1,$$

for some constants $c, C > 0$, where

$$p \in \begin{cases} (2,\infty), & n=2,\\ \big(2,\frac{2n}{n-2}\big), & n \geq 3. \end{cases}$$

In the scalar case, under the assumptions of the first part of Case (a) above, this Liouville property can also be proven by using radial barriers as in [41]. On the other hand, in the ODE case (i.e. $n = 1$, $m \geq 1$) the Liouville property is valid solely under the assumptions of Theorem 1.1 on W and u (see [8]).

In our opinion, three are the main advantages of our approach. Firstly, we can treat in a unified and coordinate way the various situations in Theorem 1.2. Secondly, we find that our approach is considerably simpler than those in the aforementioned references. Lastly, to the best of our knowledge, it provides the strongest available result when $n = 2$ for any $m \geq 1$, *even for the extensively studied scalar case*. This may seem too restrictive at first, but keep in mind that the dimensions $n = 2, 3$ are the ones with physical interest. In fact, the majority of papers on the subject deal exclusively with these dimensions (see [1, 2, 15, 16, 31, 38] for $n = 2$, and [34] for $n = 3$). If $n = 2$, we believe that our results strongly indicate that the convexity of W near its global minima, that was assumed in some of the aforementioned papers that deal with the existence of equivariant solutions to (1.1) (for instance in [12]), can be relaxed to the monotonicity condition that is described in (1.7). In this regard, we emphasize that systems of the form (1.1) where the potentials have degenerate minima arise naturally in various physical models (see for example [10]).

The proof of Theorem 1.2 is based on combining Theorem 1.1 with a variety of results that are available in the literature. Next, we will provide the proofs of our main results.

2. Proofs of the main results

Proof of Theorem 1.1. Throughout this proof, we denote the energy density of u by

$$e(x)=\frac{1}{2}|\nabla u(x)|^2+W\big(u(x)\big),\quad x\in\mathbb{R}^n. \tag{2.1}$$

Firstly, we note that standard elliptic regularity theory and Sobolev imbeddings [25, 33], in combination with the fact that u is bounded and $W\in C^1$, yield

$$\|u\|_{C^{1,\alpha}(\mathbb{R}^n;\mathbb{R}^m)}\le C_1, \tag{2.2}$$

for some constants $\alpha\in(0,1)$ and $C_1>0$ (in fact, it holds for any $\alpha\in(0,1)$ provided that $C_1=C_1(\alpha)>0$).

Since u is a globally minimizing solution, by comparing its energy to that of a suitable test function which agrees with u on ∂B_R and is identically equal to a in B_{R-1}, thanks to (2.2), we find that

$$\int_{B_R} e(x)\,dx\le C_2R^{n-1},\quad R\ge 1, \tag{2.3}$$

for some $C_2>0$ (see [7, Ch. 5], [19]).

Therefore, by (2.3), the coarea formula (see for instance [25, Ap. C]), the non-negativity of W, and the mean value theorem, there exists

$$S_R\in(R,2R) \tag{2.4}$$

such that

$$\int_{\partial B_{S_R}} e(x)\,dS(x)\le C_3R^{n-2},\quad R\ge 1, \tag{2.5}$$

for some $C_3>0$.

Let $\epsilon>0$ be any small number. We will show in the sequel that the subset of ∂B_{S_R} where $e(x)$ is above ϵ is contained in at most $\mathcal{O}(R^{n-2})$ many geodesic balls of radius 1 as $R\to\infty$ (the so-called "bad balls", see [14]). More precisely, we will establish that there exist $N_{\epsilon,R}\ge 0$ points $\{x_{R,1},\cdots,x_{R,N_{\epsilon,R}}\}$ on ∂B_{S_R} such that

$$N_{\epsilon,R}\le M_\epsilon R^{n-2},\quad R\gg 1\quad (\text{with } M_\epsilon>0 \text{ independent of } R), \tag{2.6}$$

and

$$e(x)\le\epsilon\quad \text{if } x\in\partial B_{S_R}\setminus\cup_{i=1}^{N_{\epsilon,R}}U_R(x_{R,i},1), \tag{2.7}$$

for $R\gg 1$, where $U_R(p,r)\subset\partial B_{S_R}$ stands for the geodesic ball with center at p and radius r (for convenience, we have suppressed the explicit dependence of $x_{R,i}$ on ϵ). We will prove the above properties by adapting some arguments from [14].

First, we will show the following *clearing-out property*, which is in the spirit of [14, Thm. III.3] and is actually valid for any function u that satisfies (2.2). For any $\epsilon\in(0,1)$, there exists a $\mu_\epsilon<\epsilon$ such that if

$$\int_{U_R(y,2)} e(x)dS(x)<\mu_\epsilon\quad \text{for some } y\in\partial B_{S_R},$$

then $e(x) \leq \epsilon$ for $x \in U_R(y,1)$, with $R \geq 1$. We will show this property by arguing by contradiction. So, let us suppose that, no matter how small μ is, we have

$$e(z) \geq \epsilon \quad \text{for some } z \in U_R(y,1). \tag{2.8}$$

From (2.2), using again that $W \in C^1$, there exists a $C_4 > 0$ such that

$$\|e\|_{C^{0,\alpha}(\mathbb{R}^n;\mathbb{R})} \leq C_4.$$

It then follows that

$$e(x) \geq \epsilon - C_4 d^\alpha, \quad x \in U_R(z,d),$$

for all $d < \min\{1, (\frac{\epsilon}{2C_4})^{1/\alpha}\}$ (see also [43, Lem. 2.3]). Since $e \geq 0$, we find that

$$\begin{aligned}\int_{U_R(y,2)} e(x)dS(x) &\geq \int_{U_R(z,d)} e(x)dS(x)\\ &\geq (\epsilon - C_4 d^\alpha)|U_R(z,d)|\\ &\geq \frac{\epsilon}{2}|U_R(z,d)| = \frac{\epsilon}{2}|\mathbb{S}^{n-1}|d^{n-1}.\end{aligned}$$

Hence, we can arrive at a contradiction by choosing

$$\mu_\epsilon = \frac{\epsilon}{2}|\mathbb{S}^{n-1}|\Big(\min\big\{1, \big(\frac{\epsilon}{2C_4}\big)^{1/\alpha}\big\}\Big)^{n-1}. \tag{2.9}$$

We consider a finite family of geodesic balls $\{U_R(x_i,1)\}_{i\in I_R}$, $I_R \subset \mathbb{N}$, such that

$$U_R\big(x_i, \frac{1}{4}\big) \cap U_R\big(x_k, \frac{1}{4}\big) = \emptyset \quad \text{if } i \neq k, \tag{2.10}$$

$$\cup_{i\in I_R} U_R(x_i,1) = \partial B_{S_R}, \tag{2.11}$$

for all $R \geq 1$ (having suppressed the obvious dependence of x_i on R). This is indeed possible by the Vitali covering theorem (see [24, Sec. 1.5] and keep in mind that ∂B_{S_R} becomes a metric space when equipped with the geodesic distance). We say that the ball $U_R(x_i,1)$ is a *good ball* if

$$\int_{U_R(x_i,2)} e(x)dS(x) < \mu_\epsilon,$$

and that $U_R(x_i,1)$ is a *bad ball* if

$$\int_{U_R(x_i,2)} e(x)dS(x) \geq \mu_\epsilon.$$

The collection of bad balls is indexed over

$$J_R = \{i \in I_R \ : \ U_R(x_i,1) \text{ is a bad ball}\}.$$

The main observation is that, by (2.10), there is a universal constant $C_5 > 0$ (independent of both ϵ and R) such that

$$\sum_{i\in I_R} \int_{U_R(x_i,2)} e(x)dS(x) \leq C_5 \int_{\partial B_{S_R}} e(x)dS(x),$$

owing to the fact that each point on ∂B_{S_R} is covered by at most C_5 geodesic balls $U_R(x_i,2)$ (see also [14, Ch. IV]). The latter property plainly follows by observing that all such balls that contain the same point are certainly contained in a geodesic ball of radius 10, and from the basic fact that any $(n-1)$-dimensional ball of radius

10 can contain only a certain number of disjoint balls of radius 1/4 (keep in mind that ∂B_{S_R} is essentially a flat manifold for $R \gg 1$). Using (2.5), we then infer that

$$\operatorname{card} J_R \le \frac{C_5 C_3}{\mu_\epsilon} R^{n-2}, \quad R \gg 1. \tag{2.12}$$

Now, let us consider an $x \in \partial B_{S_R} \setminus \cup_{i \in J_R} U_R(x_i, 1)$. By (2.11), there exists some $k \in I_R \setminus J_R$ such that $x \in U_R(x_k, 1)$ which is a good ball. It follows from the definition of μ_ϵ that

$$e(x) \le \epsilon,$$

thereby completing the proof of (2.6) and (2.7).

In view of (1.4) and (2.7), we have

$$|\nabla u(x)|^2 \le 2\epsilon \text{ and } |u(x) - a| \le \sigma_\epsilon \quad \text{if } x \in \partial B_{S_R} \setminus \cup_{i=1}^{N_{\epsilon,R}} U_R(x_{R,i}, 1),\ R \gg 1, \tag{2.13}$$

where

$$\sigma_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0, \tag{2.14}$$

(we point out that σ_ϵ depends only on ϵ).

We consider the function $v_R \in W^{1,2}(B_{S_R}; \mathbb{R}^m) \cap L^\infty(B_{S_R}; \mathbb{R}^m)$ which is defined in terms of polar coordinates as

$$v_R(r, \theta) = \begin{cases} u(S_R, \theta) + \big(a - u(S_R, \theta)\big)(S_R - r), & r \in [S_R - 1, S_R],\ \theta \in \mathbb{S}^{n-1}, \\ a, & r \in [0, S_R - 1],\ \theta \in \mathbb{S}^{n-1}, \end{cases}$$

(having slightly abused notation, keep in mind that $x = r\theta$). We note that v_R belongs in $W^{1,2}$ because it is the composition of a smooth function with a Lipschitz continuous one (see [35, pg. 54] and keep in mind that we only use the polar coordinates away from the origin). Clearly, we have

$$v_R = u \quad \text{on } \partial B_{S_R}. \tag{2.15}$$

Let

$$\mathcal{A}_R = B_{S_R} \setminus B_{(S_R - 1)} \quad \text{and} \quad \mathcal{C}_R = \cup_{i=1}^{N_{\epsilon,R}} \left(\bar{B}_{10}(x_{R,i}) \cap \bar{\mathcal{A}}_R \right),$$

where $B_{10}(x_{R,i})$ stands for the n-dimensional ball of radius 10 and center at $x_{R,i}$. If $x = r\theta \in \mathcal{A}_R \setminus \mathcal{C}_R$, via (2.13), we obtain

$$|v_R(x) - a| \le 2|u(S_R, \theta) - a| \le 2\sigma_\epsilon. \tag{2.16}$$

Moreover for such x, using (2.4) and (2.13), we find that

$$\begin{aligned} |\nabla_{\mathbb{R}^n} v_R|^2 &= |u(S_R, \theta) - a|^2 + \frac{1}{r^2} |(1 + r - S_R) \nabla_{\mathbb{S}^{n-1}} u(S_R, \theta)|^2 \\ &\le \sigma_\epsilon^2 + \frac{2}{S_R^2} |\nabla_{\mathbb{S}^{n-1}} u(S_R, \theta)|^2 \\ &\le \sigma_\epsilon^2 + 2|\nabla_{\mathbb{R}^n} u(S_R \theta)|^2 \\ &\le \sigma_\epsilon^2 + 4\epsilon, \end{aligned} \tag{2.17}$$

where we made repeated use of the identity

$$|\nabla_{\mathbb{R}^n} v|^2 = |\partial_r v|^2 + \frac{1}{R^2} |\nabla_{\mathbb{S}^{n-1}} v|^2 \quad \text{on } \partial B_R,\ R > 0;$$

see [44, Ch. 8]. It follows that

$$\int_{B_{S_R}} \{ \frac{1}{2} |\nabla v_R|^2 + W(v_R) \} \, dx$$

$$
\begin{aligned}
&= \int_{\mathcal{A}_R} \{\frac{1}{2}|\nabla v_R|^2 + W(v_R)\}\,dx \\
&\le C_6 N_{\epsilon,R} + \int_{\mathcal{A}_R\setminus\mathcal{C}_R} \{\frac{1}{2}|\nabla v_R|^2 + W(v_R)\}\,dx \quad \text{(using (2.2) and part of (2.17))} \\
&\le C_6 N_{\epsilon,R} + \Big(\frac{\sigma_\epsilon^2}{2} + 2\epsilon + C_7\sigma_\epsilon\Big)|\mathcal{A}_R\setminus\mathcal{C}_R| \quad \text{(using (2.16), (2.17))} \\
&\le C_6 N_{\epsilon,R} + C_8(\sigma_\epsilon+\epsilon)S_R^{n-1},
\end{aligned}
$$

where $C_6, C_7, C_8 > 0$ are independent of both small ϵ and large R.

Since u is a globally minimizing solution, thanks to (2.15), we obtain

$$
\begin{aligned}
\int_{B_{S_R}} e(x)\,dx &\le C_6 N_{\epsilon,R} + C_8(\sigma_\epsilon+\epsilon)S_R^{n-1} \\
&\le C_6 M_\epsilon R^{n-2} + 2^{n-1}C_8(\sigma_\epsilon+\epsilon)R^{n-1}
\end{aligned} \tag{2.18}
$$

for $R \gg 1$, were we used (2.4) and (2.6). Since $\epsilon > 0$ is arbitrary, in light of (2.14), we infer that (1.5) holds, as desired. □

Proof of Theorem 1.2. **Case (a)** If u satisfies (1.6), since $W \ge 0$, it is known that the following strong monotonicity formula holds

$$
\frac{d}{dR}\Big(\frac{1}{R^{n-1}}\int_{B_R}\{\frac{1}{2}|\nabla u|^2 + W(u)\}\,dx\Big) \ge 0, \quad R > 0, \tag{2.19}
$$

(see [18, 37] for $m = 1$, and [3] for arbitrary $m \ge 1$). Let us point out in passing that u being a globally minimizing solution is not used for this. Hence, for any positive $r < R$, we have

$$
\frac{1}{r^{n-1}}\int_{B_r}\{\frac{1}{2}|\nabla u|^2 + W(u)\}\,dx \le \frac{1}{R^{n-1}}\int_{B_R}\{\frac{1}{2}|\nabla u|^2 + W(u)\}\,dx.
$$

By Theorem 1.1, letting $R \to \infty$ in the above relation yields $u \equiv a$ as desired.

To complete the proof in this case, we note that the gradient estimate (1.6) was shown in [26] to hold for any bounded, entire solution when $m = 1$ and $W \in C^{1,1}_{\rm loc}(\mathbb{R};\mathbb{R})$ is nonnegative (see [18, 36] for earlier proofs which required higher regularity on W).

Case (b) Here we partly follow [42]. Since $n = 2$, by working as in (2.5), and using the assertion of Theorem 1.1, we arrive at

$$
\int_{\partial B_{S_R}} W\left(u(x)\right) dS(x) \to 0, \quad \text{for some } S_R \in (R, 2R), \text{ as } R \to \infty.
$$

By using just the C^1-bound in (2.2), and working as we did in order to exclude (2.8), we deduce that

$$
\max_{|x|=S_R} |u(x) - a| \to 0 \quad \text{as } R \to \infty. \tag{2.20}
$$

Under the assumptions of the first part of Case (b), a recent variational maximum principle from [5], as extended in [42] (to allow for non-strict monotonicity in (1.7)), implies that

$$
\max_{|x|\le S_R} |u(x) - a| \le \max_{|x|=S_R} |u(x) - a|.
$$

In light of (2.20), by letting $R \to \infty$ in the above relation, we can conclude that the assertion of the theorem holds in the first scenario of (b).

We will establish the validity of the Liouville property in the second scenario in (b) by borrowing some ideas from [45], while adopting a slightly more explanatory viewpoint. To this end, we will argue by contradiction. Without loss of generality, we may assume that there exists a sequence $R_j \to \infty$ and a $\delta > 0$ such that

$$u(x_j) = \max_{|x| \le S_{R_j}} u(x) \ge a + \delta, \quad j \ge 1,$$

for some $x_j \in B_{S_{R_j}}$. In particular, there exists a $d \in (0, \delta)$ such that

$$W(a+d) < W\left(u(x_j)\right), \quad j \ge 1.$$

By (2.20), we may further assume that

$$\max_{|x| = S_{R_j}} u(x) \le a + \frac{d}{2}, \quad j \ge 1. \tag{2.21}$$

Let $u_j \in [a+d, u(x_j))$ be such that

$$W(u_j) = \min_{u \in [a+d, u(x_j)]} W(u). \tag{2.22}$$

We consider the competitor function

$$V_j(x) = \min\{u(x), u_j\}, \quad x \in B_{S_{R_j}},$$

which belongs in $W^{1,2}\big(B_{S_{R_j}}; \mathbb{R}^m\big) \cap L^\infty\big(B_{S_{R_j}}; \mathbb{R}^m\big)$ (see for instance [21]) and, thanks to (2.21), agrees with u on $\partial B_{S_{R_j}}$. To conclude, we will show that

$$E\big(V_j; B_{S_{R_j}}\big) < E\big(u; B_{S_{R_j}}\big),$$

which contradicts the energy minimality character of u. To this aim, we set

$$\mathcal{D}_j = \{x \in B_{S_{R_j}} : u(x) > u_j\}.$$

We observe that $\mathcal{D}_j$ is nonempty (since it contains x_j) and strictly contained in $B_{S_{R_j}}$ (from (2.21)). Then, to arrive at a contradiction we plainly note that

$$E\big(V_j; B_{S_{R_j}} \setminus \mathcal{D}_j\big) = E\big(u; B_{S_{R_j}} \setminus \mathcal{D}_j\big) \text{ and } E\big(V_j; \mathcal{D}_j\big) = E\big(u_j; \mathcal{D}_j\big) < E\big(u; \mathcal{D}_j\big),$$

since (2.22) holds and there exists a connected component $\mathcal{E}_j$ of $\mathcal{D}_j$, say the one containing x_j, where u is nonconstant (note that $u = u_j$ on $\partial \mathcal{D}_j$). □

We refer to [32] for a class of systems (1.1) of Allen-Cahn type whose solutions satisfy Modica's gradient bound (1.6). To the best of our knowledge, there are no counterexamples to Modica's gradient bound for systems of Allen-Cahn type in the case of minimizing solutions. In this regard, we refer the interested reader to [39].

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