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EXISTENCE RESULTS FOR NONLINEAR SCHRÖDINGER EQUATIONS INVOLVING THE FRACTIONAL (p,q)-LAPLACIAN AND CRITICAL NONLINEARITIES

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Abstract. In this article, we consider the existence of ground state positive solutions for nonlinear Schrödinger equations of the fractional (p,q)-Laplacian with Rabinowitz potentials defined in $\mathbb{R}^n$,

$$(-\Delta)_p^{s_1}u+(-\Delta)_q^{s_2}u+V(\varepsilon x)(|u|^{p-2}u+|u|^{q-2}u)=\lambda f(u)+\sigma|u|^{q^*_{s_2}-2}u.$$

We prove existence by confining different ranges of the parameter λ under the subcritical or critical nonlinearities caused by $\sigma=0$ or 1, respectively. In particular, a delicate calculation for the critical growth is provided so as to avoid the failure of a global Palais-Smale condition for the energy functional.

1. Introduction

Let $0<s_1<s_2<1$, $1<p<q<n/s_2$, $\sigma\in\{0,1\}$, $\varepsilon>0$ be small number and $q^*_{s_2}=\frac{nq}{n-s_2q}$ be the fractional critical exponent for s_2 and q. We are to consider the existence results for the following Schrödinger equations of a fractional (p,q)-Laplacian in $\mathbb{R}^n$:

$$(-\Delta)_p^{s_1}u+(-\Delta)_q^{s_2}u+V(\varepsilon x)(|u|^{p-2}u+|u|^{q-2}u)=\lambda f(u)+\sigma|u|^{q^*_{s_2}-2}u, \tag{1.1}$$

where $\lambda>0$ is a parameter specified later for the parameter $\sigma=0$ or 1, $V:\mathbb{R}^n\to\mathbb{R}$ is a continuous function satisfying global Rabinowitz condition, and $f:\mathbb{R}\to\mathbb{R}$ is a continuous function with subcritical growth. Here, the fractional t-Laplace operator $(-\Delta)_t^s$, for $s\in\{s_1,s_2\}$ and $t\in\{p,q\}$, is defined as

$$(-\Delta)_t^s u(x)=2\lim_{\epsilon\to0}\int_{\mathbb{R}^n\setminus B_\epsilon(x)}\frac{|u(x)-u(y)|^{t-2}(u(x)-u(y))}{|x-y|^{n+st}}dy.$$

Problems of type (1.1) are well known as double-phase equations, appearing in the case of two different materials, where the fractional operator $(-\Delta)_t^s$ with $s\in\{s_1,s_2\}$ and $t\in\{p,q\}$ described the geometry of a composite of two materials. Recently, a considerable attention has been devoted to the work on nonlocal problems driven by fractional operators, particularly on fractional p-Laplacian due to both its interesting theoretical structure and concrete applications such as finance,

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obstacle problems, phase transitions, optimization, anomalous diffusion, conservation laws, image processing and many others. For more details see [23, 22, 12, 31] and the references therein.

In the special case $s_1 = s_2 = 1$ and $V \equiv 1$, Problem (1.1) reduces to the following well-known (p, q)-Laplacian equation in $\mathbb{R}^n$,

$$-\Delta_p u - \Delta_q u + |u|^{p-2}u + |u|^{q-2}u = f(x, u), \tag{1.2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. As explained in [16], the study of Equation (1.2) can be extended to a more general reaction-diffusion system

$$u_t = \operatorname{div}(A(u)\nabla u) + r(x, u) \quad \text{with } A(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}. \tag{1.3}$$

This has a great number of applications in plasma physics, solid state physics, biophysics and chemical reactions. In such applications, u usually corresponds to the concentration term, $\operatorname{div}(A(u)\nabla u)$ represents the diffusion with diffusion coefficient $A(u)$, and the reaction term $r(x, u)$ relates to source and loss processes. In chemical and biological applications, the reaction term $r(x, u)$ has a polynomial form with respect to the concentration u with variable coefficients, see [16]. The existence results for several classes of equations of (p, q)-Laplacian type defined in bounded domains or in the whole of $\mathbb{R}^n$ can be found in [4, 19, 30] and the references therein.

It is a well-known fact that as a special model of the differential operator (1.3) it is the following so-called double-phase one

$$\mathcal{L}u := \operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u),$$

which is related to the energy functional

$$u \mapsto \int_\Omega (|\nabla u|^p + a(x)|\nabla u|^q)dx \tag{1.4}$$

with $1 < p < q$ and $a \in L^\infty(\Omega)$ with $a(x) \geq 0$ for almost all $x \in \Omega$. Roughly speaking, the integral functional (1.4) is characterized by the fact that the energy density changes its growth properties and ellipticity according to the point in the domain. To be more precise, the modulating potential $a(x)$ dictates the geometry of a composite material made of two different components, with distinct power hardening exponents p and q. Following Marcellini's terminology in [26], the integrand $H(x, \xi) = |\xi|^p + a(x)|\xi|^q$ for all $(x, \xi) \in \Omega \times \mathbb{R}^n$ has different growth near the origin and at infinity (unbalanced growth), that is,

$$|\xi|^p \leq H(x, \xi) \leq |\xi|^q + 1 \quad \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^n.$$

The study of the double-phase functionals of the form (1.4) has generated considerable interests after the initial work by Zhikov in [40] to describe models of strongly anisotropic materials, see for instance [7, 8, 17, 18] and the references therein. While $s_1 = s_2$ and $\sigma = 0$, the problem (1.1) boils down to the following fractional (p, q)-Laplacian equations

$$(-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) \quad \text{in } \mathbb{R}^n, \tag{1.5}$$

which has been extensively considered by several authors in recent years. For $p = q \neq 2$, there has been a source of inspiration around its existence and multiplicity results in the last decade due to two phenomena: the nonlocal character of the operator and its nonlinearity; see for instance [33, 14, 24, 38] and the references therein. Indeed, we would like to stress that standard arguments used to investigate the linear case $p = q = 2$ seem to be inapplicable to the nonlinear case on account of the lack of Hilbertian structure of $W^{s,p}(\mathbb{R}^n)$ for $p \neq 2$. For the reader's convenience, we refer to [15, 21, 35, 39], where existence and multiplicity results were obtained in the case $p = q = 2$. However, for $p \neq q$ in the nonlocal framework only few recent papers deal with problems like (1.5). For example, Filippis and Palatucci [20] proved a Hölder regularity result for nonlocal double-phase equations. And Lin and Zheng [25] established the multiplicity and asymptotic behavior of solutions to fractional (p, q)-Kirchhoff type problems with critical Sobolev-Hardy exponent. We refer the readers to some recent papers [5, 6, 13, 27] for other interesting double-phase problems in both local and nonlocal cases.

While $s_1 = s_2$, $\sigma = 1$ and $p = q \neq 2$, this becomes the following fractional nonlocal Schrödinger equation involving a critical Sobolev exponent

$$(-\Delta)_p^s u + V(\varepsilon x)|u|^{p-2}u = f(u) + |u|^{p_s^*-2}u \quad \text{in } \mathbb{R}^n.$$

We refer to [3] for this class of p-fractional Schrödinger equation, where the authors presented the existence and multiplicity results of it. For $p \neq q$, Ambrosio in [2] established an existence result for the fractional (p, q)-Laplacian equation with critical growth. The multiplicity results for fractional (p, q)-Laplacian equations involving critical nonlinearities in bounded domains has been obtained in [9]. Motivated by all these works, we are to consider doubly fractional Schrödinger equations of (p, q)-Laplacian with critical growth. More precisely, we are interested in the existence result of positive solutions to Problem (1.1).

It is an important observation that the lack of compactness due to the critical exponent greatly increases the methodological difficulties. The main contribution of our work is to deal with the possibility of loss of compactness due to the critical nonlinearity of the double-phase Schrödinger equations in the fractional setting suitably. We then apply this to obtain sufficient existence conditions for equations like (1.1) in all $\mathbb{R}^n$, which generalizes the results of Ambrosio's paper [4] for the local case.

Before stating main result, let us introduce main assumptions imposed on the potential V and the nonlinearity f. Throughout this paper, we assume that $V : \mathbb{R}^n \to \mathbb{R}$ is a continuous function satisfying the following condition by Rabinowitz as in [34]:

$$0 < V_0 = \inf_{x \in \mathbb{R}^N} V(x) < \liminf_{|x| \to \infty} V(x) = V_\infty \in (0, \infty], \tag{1.6}$$

and we shall consider it in the two cases of $V_\infty < \infty$ and $V_\infty = \infty$ in the following. For the nonlinearity $f : \mathbb{R} \to \mathbb{R}$, it is assumed that:

(A1) $f \in C^0(\mathbb{R}, \mathbb{R})$ and $f(t) = 0$ for all $t \leq 0$;

(A2) $\lim_{|t| \to 0} |f(t)|/|t|^{p-1} = 0$;

(A3) there exists $r \in (q, q_{s_2}^*)$ with $q_{s_2}^* = nq/(n - s_2 q)$, such that

$$\lim_{|t| \to \infty} \frac{|f(t)|}{|t|^{r-1}} = 0;$$

(A4) there exists $\theta \in (q, q_{s_2}^*)$ such that $0 < \theta F(t) = \theta \int_0^t f(\tau)d\tau \leq tf(t)$ for all $t > 0$;

(A5) the map $t \mapsto f(t)/t^{q-1}$ is increasing in $(0, \infty)$.

Regarding the existence result, we prove that there exists at least one non-negative non-trivial solution to Problem (1.1) in the subcritical for all $\lambda > 0$ and for small ε. For the critical case, we prove the existence of at least one non-negative non-trivial solution to Problem (1.1) provided small ε and λ large enough. To be more precise, our first result of the paper is the following.

Theorem 1.1. *Let $\sigma = 0$. Assume that* (1.6) *and* (A1)–(A5) *hold. Then there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, Problem* (1.1) *with subcritical growth admits at least one positive ground state solution for any $\lambda > 0$.*

The proof of this theorem relies heavily on existence of the corresponding autonomous problem motivated from [3]. As usual, the presence of the fractional (p, q)-Laplacian operator leads to more intriguing analysis. Indeed, the arguments used in the study of ((1.5) seem not to be trivially adaptable to handling two totally different components. Therefore, some appropriate technical lemmas (see Lemma 2.4 and Lemma 2.5) and much more delicate estimates will be needed.

Further, we consider the existence result of Problem (1.1) for the critical case with $\sigma = 1$.

Theorem 1.2. *Let $\sigma = 1$. Assume that* (1.6) *and* (A1)–(A5) *hold. Then there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, Problem* (1.1) *with critical growth admits at least one positive ground state solution for any $\lambda \geq \lambda_*$, where λ_* is a positive constant.*

The idea for proving this theorem is also based on suitable variational techniques. Owing to the combination of two nonhomogeneous fractional involved operators with different scaling properties, it is a rather delicate situation, and more estimates will be needed to achieve our result. Particularly, we would like to point out that the calculations performed for the setting $\sigma = 1$ to recover compactness are much more complicated with respect to the case $\sigma = 0$ due to the presence of critical exponent. More precisely, the main difficulty in the critical case is that the energy functional fails to satisfy the Palais-Smale condition globally. This is different from the calculations performed as in [29] and the optimal asymptotic behavior of p-minimizers established as in [10]. Instead, we provide some technical results which allow us to avoid unnecessary calculations and prove the Palais-Smale condition for the critical case (see Lemma 4.4 and Lemma 4.5). To the best of our knowledge, these are new contributions to show the existence results of Problem (1.1) for double-phase nonlocal Schrödinger equations involved in both subcritical and critical growth.

The remainder of this paper is organized as follows. In Section 2, we give some related notations, recall basic facts about the involved fractional Sobolev spaces and provide various useful lemmas. We devoted Section 3 to the proof of existence result for the subcritical case when $\sigma = 0$. In Section 4, we deal with the critical case when $\sigma = 1$, and finally give the proof of Theorem 1.2.

2. Preliminaries

This section is devoted to some well-known facts about the fractional Sobolev spaces and some technical lemmas we will use later. Throughout this paper, $C(n, \nu, L, \cdots)$ stands for a universal constant depending only on prescribed quantities and possibly varying from line to line. However, the ones we need to emphasize will be denoted with special symbols, such as C_1, C_2, C_*, C_ξ.

For $p \in [1, \infty]$ and $A \subset \mathbb{R}^n$, we denote by $|u|_{L^p(A)}$ the $L^p(A)$-norm of a function $u : \mathbb{R}^n \to \mathbb{R}$ belonging to $L^p(A)$, and by $|u|_p$ its $L^p(\mathbb{R}^n)$-norm. We define $D^{s,p}(\mathbb{R}^n)$

as the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the following semi-norm

$$[u]_{s,p}^p = \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy$$

for $s \in (0,1)$ and $p \in (1,\infty)$. The fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ is defined as the set of all functions $u \in L^p(\mathbb{R}^n)$ such that $[u]_{s,p} < \infty$, equipped with the norm

$$\|u\|_{s,p}^p = [u]_{s,p}^p + |u|_p^p.$$

Furthermore, for $u, v \in W^{s,p}(\mathbb{R}^n)$, we put

$$\langle u, v\rangle_{s,p} = \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}(v(x)-v(y))\,dx\,dy.$$

In the following, we denote by $B_R(x)$ the ball of radius R with the center at x. While $x = 0$, we briefly write $B_R = B_R(0)$. First let us begin with the following Sobolev embedding relation.

Lemma 2.1 ([31])**.** *Let $s \in (0,1)$ and $p \in [1,\infty)$ such that $n > sp$. Then there exists a constant $S_* > 0$ such that, for any $u \in D^{s,p}(\mathbb{R}^n)$ it holds*

$$|u|_{p_s^*}^p \le S_*^{-1}[u]_{s,p}^p.$$

Moreover, $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p, p_s^*]$ and compactly embedded in $L^q_{\rm loc}(\mathbb{R}^n)$ for any $q \in [1, p_s^*)$.

The following compactness result of Lions-type is recalled which will be used in the main proof later.

Lemma 2.2 ([3])**.** *Let $n > sp$ and $r \in [p, p_s^*)$. If $\{u_n\}$ is a bounded sequence in $W^{s,p}(\mathbb{R}^n)$ with*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^n} \int_{B_R(y)} |u_n|^r dx = 0,$$

where $R > 0$, then $u_n \to 0$ in $L^t(\mathbb{R}^n)$ for all $t \in (p, p_s^*)$.

Lemma 2.3 ([3])**.** *Let $u \in W^{s_1,p}(\mathbb{R}^n) \cap W^{s_2,q}(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \phi \le 1$, $\phi = 1$ in B_1 and $\phi = 0$ in $\mathbb{R}^n \backslash B_2$. Set $\phi_r(x) = \phi(\frac{x}{r})$. Then*

$$\lim_{r\to\infty} \|u\phi_r - u\|_1 = 0 \quad \text{and} \quad \lim_{r\to\infty} \|u\phi_r - u\|_2 = 0\,.$$

For any $\varepsilon > 0$, we define the space related to the potential $V(\varepsilon x)$,

$$X_\varepsilon = \big\{u \in W^{s_1,p}(\mathbb{R}^n) \cap W^{s_2,q}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(\varepsilon x)(|u|^p + |u|^q)dx < \infty\big\}$$

equipped with the norm

$$\|u\|_\varepsilon = \|u\|_1 + \|u\|_2,$$

where

$$\|u\|_i^t = [u]_{s_i,t}^t + \int_{\mathbb{R}^n} V(\varepsilon x)|u|^t dx \quad \text{for all } t > 1 \text{ and } i = \{1,2\}.$$

Thanks to the assumption (1.6) and Lemma 2.1, it is easy to check that it holds the following result.

Lemma 2.4. *The space X_ε is continuously embedded into $W^{s_1,p}(\mathbb{R}^n) \cap W^{s_2,q}(\mathbb{R}^n)$. Moreover, X_ε is continuously embedded in $L^t(\mathbb{R}^n)$ for any $t \in \big[p, q_{s_2}^*\big]$, and compactly embedded in $L^t_{\rm loc}\big(\mathbb{R}^n\big)$ for any $t \in \big[1, q_{s_2}^*\big)$.*

Proof. For any $u \in X_\varepsilon$, by the assumption (1.6) and the definition of $\|\cdot\|_i$ for $i = \{1,2\}$ we have

$$\min\{1, V_0\}\|u\|^p_{s_1,p} \le \|u\|^p_1,$$
$$\min\{1, V_0\}\|u\|^p_{s_2,q} \le \|u\|^p_2.$$

Therefore, it follows that the embedding $X_\varepsilon \hookrightarrow W^{s_1,p}(\mathbb{R}^n) \cap W^{s_2,q}(\mathbb{R}^n)$ is continuous. By Lemma 2.1, it is evident that the embedding $X_\varepsilon \hookrightarrow L^t(\mathbb{R}^n)$ is continuous for any $t \in [p, q^*_{s_2}]$ and the embedding $X_\varepsilon \hookrightarrow L^t_{\rm loc}(\mathbb{R}^n)$ is obviously compact for any $t \in [1, q^*_{s_2})$. □

In addition, by considering V being coercive, we obtain the following compactness lemma.

Lemma 2.5. *Let $V_\infty = \infty$. Then X_ε is compactly embedded in $L^t(\mathbb{R}^n)$ for any $t \in [p, q^*_{s_2})$.*

Proof. For $t = p$, let $\{u_n\}$ be a sequence such that $u_n \rightharpoonup 0$ in X_ε. By Lemma 2.4 we know that $X_\varepsilon \subset L^p(\mathbb{R}^n)$. Then $u_n \rightharpoonup 0$ in $W^{s_1,p}(\mathbb{R}^n) \cap W^{s_2,q}(\mathbb{R}^n)$ and $u_n \to 0$ in $L^p(B_R)$. Therefore, for any $\epsilon > 0$ there exists $n_0 > 0$ such that

$$\int_{B_R} |u|^p dx \le \epsilon \quad \text{for any } n \ge n_0. \tag{2.1}$$

Since $V(x)$ is coercive, there exists $R = R_\epsilon > 0$ such that

$$\frac{1}{V(\varepsilon x)} < \epsilon \quad \text{for any } |x| > R. \tag{2.2}$$

Let us set

$$T := \sup_{n\in\mathbb{N}} \|u_n\|_\varepsilon < \infty. \tag{2.3}$$

Hence, for any $n \ge n_0$, by using (2.1), (2.2) and (2.3) we conclude that

$$\begin{aligned}
\int_{\mathbb{R}^n} |u|^p dx &= \int_{B_R} |u|^p dx + \int_{\mathbb{R}^n\setminus B_R} |u|^p dx \\
&\le \epsilon + \epsilon \int_{\mathbb{R}^n\setminus B_R} V(\varepsilon x)|u|^p dx \le \epsilon(1 + T^p).
\end{aligned}$$

Then we have $u_n \to 0$ in $L^p(\mathbb{R}^n)$. As for $p < t < q^*_{s_2}$, by using the interpolation inequality and Lemma 2.1 we see that

$$|u|_t \le C[u]^\alpha_{s_2,q}|u|^{1-\alpha}_p,$$

where $\frac{1}{t} = \frac{\alpha}{p} + \frac{1-\alpha}{q^*_{s_2}}$, which yields the required result. □

Finally, we recall the following splitting lemma which will be very useful in our main proof, which is proved by Ambrosio and Repovš (cf. [4]) following the arguments developed by Brezis and Lieb [11].

Lemma 2.6. *Let $\{u_n\}$ be a sequence such that $u_n \rightharpoonup u$ in X_ε, and $v_n = u_n - u$. Then we have*

(i) $[v_n]^p_{s_1,p} + [v_n]^q_{s_2,q} = \big([u_n]^p_{s_1,p} + [u_n]^q_{s_2,q}\big) - \big([u]^p_{s_1,p} + [u]^q_{s_2,q}\big) + o_n(1)$;

(ii) $\int_{\mathbb{R}^n} V(\varepsilon x)(|v_n|^p + |v_n|^q)dx = \int_{\mathbb{R}^n} V(\varepsilon x)\Big((|u_n|^p + |u_n|^q)dx - (|u|^p + |u|^q)\Big)dx + o_n(1)$;

(iii) $\int_{\mathbb{R}^n} (F(v_n) - F(u_n) + F(u))dx = o_n(1)$;

(iv) $\sup_{\|w\|_\varepsilon \le 1} \int_{\mathbb{R}^n} |(f(v_n) - f(u_n) + f(u))w| dx = o_n(1)$.

3. Subcritical case while $\sigma = 0$

3.1. **Functional setting in the subcritical case.** In this section we consider the problem

$$
\begin{gathered}
(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u + V(\varepsilon x)\big(|u|^{p-2}u + |u|^{q-2}u\big) = \lambda f(u) \quad \text{in } \mathbb{R}^n, \\
u \in W^{s_1,p}(\mathbb{R}^n) \cap W^{s_2,q}(\mathbb{R}^n), \quad u > 0 \quad \text{in } \mathbb{R}^n.
\end{gathered} \tag{3.1}
$$

Let us define the energy functional associated with (3.1),

$$
I_\varepsilon(u) = \frac{1}{p}\|u\|_1^p + \frac{1}{q}\|u\|_2^q - \lambda \int_{\mathbb{R}^n} F(u) dx,
$$

which is well-defined for all $u : \mathbb{R}^n \to \mathbb{R}$ belonging to the fractional space X_ε. Therefore, from the assumptions on f it is easy to check that $I_\varepsilon \in C^1(X_\varepsilon, \mathbb{R})$ and its differential is given by

$$
\begin{aligned}
&\langle I_\varepsilon'(u), v\rangle \\
&= \langle u, v\rangle_{s_1,p} + \langle u, v\rangle_{s_2,q} + \int_{\mathbb{R}^n} V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u)v \, dx - \lambda \int_{\mathbb{R}^n} f(u) v \, dx
\end{aligned}
$$

for any $u, v \in X_\varepsilon$.

Now we check that I_ε possesses a mountain pass geometry (cf. [1]).

Lemma 3.1. *The functional has a mountain pass geometry shown as follows:*

(i) *there exist $\alpha, \rho > 0$ such that $I_\varepsilon(u) \ge \alpha$ with $\|u\|_\varepsilon = \rho$;*
(ii) *there exists $e \in X_\varepsilon$ with $\|e\|_\varepsilon > \rho$ such that $I_\varepsilon(e) < 0$.*

Proof. (i) By growth assumptions (A2) and (A3) on f, we readily see that for any $\xi > 0$ there exists a constant $C_\xi > 0$ such that

$$
|f(t)| \le \xi |t|^{p-1} + C_\xi |t|^{r-1} \quad \forall t \in \mathbb{R}, \tag{3.2}
$$

$$
|F(t)| \le \frac{\xi}{p}|t|^p + \frac{C_\xi}{r}|t|^r \quad \forall t \in \mathbb{R}. \tag{3.3}
$$

Therefore, using $V_0 \le V(\varepsilon x)$ and taking $\xi \in (0, \frac{V_0}{\lambda})$, we have

$$
\begin{aligned}
I_\varepsilon(u) &\ge \frac{1}{p}\|u\|_1^p + \frac{1}{q}\|u\|_2^q - \lambda\frac{\xi}{p}|t|_p^p - \lambda\frac{C_\xi}{r}|t|_r^r \\
&\ge C_1\|u\|_1^p + \frac{1}{q}\|u\|_2^q - \lambda\frac{C_\xi}{r}|t|_r^r.
\end{aligned}
$$

By choosing $\|u\|_\varepsilon = \rho \in (0,1)$ and using $1 < p < q$, we have $\|u\|_1^p < 1$ which leads to $\|u\|_1^p \ge \|u\|_1^q$. This fact combined with an elementary convex inequality $a^t + b^t \ge 2^{-t+1}(a+b)^t$, $\forall a, b \ge 0$ and $t > 1$, and Lemma 2.4 yield

$$
I_\varepsilon(u) \ge C_2\|u\|_\varepsilon^q - \lambda\frac{C_\xi}{r}|t|_r^r \ge C_2\|u\|_\varepsilon^q - C_3\|u\|_\varepsilon^r.
$$

Therefore, we can find $\alpha > 0$ such that $I_\varepsilon(u) \ge \alpha$ for $\|u\|_\varepsilon = \rho$ due to $r > q$.

(ii) By assumption (A4) we can infer that for some $C_1, C_2 > 0$ it holds

$$
F(t) \ge C_1 t^\theta - C_2 \quad \text{for any } t > 0.
$$

By taking $u \in C_c^\infty(\mathbb{R}^n)$ such that $u \geq 0$ and $u \neq 0$, then we know that

$$I_\varepsilon(tu) \leq \frac{t^p}{p}\|u\|_1^p + \frac{t^q}{q}\|u\|_2^q - \lambda t^\theta C_1 \int_{\mathbb{R}^n} u^\theta dx + C_3 \to -\infty \quad \text{as } t \to \infty,$$

where we have used $\theta > q > p$. This completes the proof. $\square$

Therefore, invoking a variant of the mountain pass theorem without Palais-Smale condition (cf. [37]), we can see that there exists a sequence $\{u_n\} \subset X_\varepsilon$ such that

$$I_\varepsilon(u_n) \to c_\varepsilon \text{ and } I'_\varepsilon(u_n) \to 0,$$

where

$$c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\varepsilon(\gamma(t)) \quad \text{with} \quad \Gamma = \{\gamma \in C^0([0,1], X_\varepsilon) : \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0\}.$$

Further, as in [37] we can use the equivalent characterization of c_ε, which is more appropriate to our aim, given by

$$c_\varepsilon = \inf_{u \in X_\varepsilon \backslash \{0\}} \max_{t \geq 0} I_\varepsilon(\gamma(t)).$$

We also introduce the Nehari manifold associated with I_ε, which is defined by

$$N_\varepsilon = \{u \in X_\varepsilon \backslash \{0\} : \langle I'_\varepsilon(u), u\rangle = 0\}.$$

Next, we prove that any Palais-Smale sequence of I_ε is bounded.

Lemma 3.2. *Let $\{u_n\}$ be a Palais-Smale sequence of I_ε at level c. Then $\{u_n\}$ is bounded in X_ε.*

Proof. Let $\{u_n\} \subset X_\varepsilon$ be a Palais-Smale sequence at the level c, that is

$$I_\varepsilon(u_n) = c + o_n(1) \quad \text{and} \quad I'_\varepsilon(u_n) = o_n(1).$$

Using (A4) and considering $\theta > q > p$ we can deduce that

$$\begin{aligned} c(1 + \|u_n\|_\varepsilon) &\geq I_\varepsilon(u_n) - \frac{1}{\theta}\langle I'_\varepsilon(u_n), u_n\rangle \\ &= \big(\frac{1}{p} - \frac{1}{\theta}\big)\|u_n\|_1^p + \big(\frac{1}{q} - \frac{1}{\theta}\big)\|u_n\|_2^q + \lambda \int_{\mathbb{R}^n} \big(\frac{1}{\theta} f(u_n)u_n - F(u_n)\big)dx \\ &\geq \big(\frac{1}{q} - \frac{1}{\theta}\big)(\|u_n\|_1^p + \|u_n\|_2^q). \end{aligned}$$

Next we prove it by contradiction. Assume that $\|u_n\|_\varepsilon = \|u\|_1 + \|u\|_2 \to \infty$, we distinguish it in the following three cases:

Case 1. If $\|u_n\|_1 \to \infty$ or $\|u_n\|_2 \to \infty$, then for n sufficiently large we have $\|u_n\|_2^{q-p} \geq 1$, that is$\|u_n\|_2^q \geq \|u_n\|_2^p$. Thus

$$c(1 + \|u_n\|_\varepsilon) \geq \big(\frac{1}{q} - \frac{1}{\theta}\big)(\|u_n\|_1^p + \|u_n\|_2^p) \geq C(\|u_n\|_1 + \|u_n\|_2)^p = C\|u_n\|_\varepsilon^p,$$

which gives a contradiction because $p > 1$.

Case 2. If $\|u_n\|_1 \to \infty$ and $\|u_n\|_2$ is bounded, then we have

$$c(1 + \|u_n\|_1 + \|u_n\|_2) = c(1 + \|u_n\|_\varepsilon) \geq \big(\frac{1}{q} - \frac{1}{\theta}\big)\|u_n\|_1^p,$$

which implies that

$$c\Big(\frac{1}{\|u_n\|_1^p} + \frac{1}{\|u_n\|_1^{p-1}} + \frac{\|u_n\|_2}{\|u_n\|_1^p}\Big) \geq \big(\frac{1}{q} - \frac{1}{\theta}\big).$$

Passing to the limit as $n \to \infty$, we obtain $0 < \frac{1}{q} - \frac{1}{\theta} \leq 0$, which leads to a contradiction.

Case 3. If $\|u_n\|_2 \to \infty$ and $\|u_n\|_1$ is bounded, we can do it by a similar way as Case 2. Putting the three cases together completes the proof. $\square$

3.2. **Autonomous subcritical problem.** We devote this subsection to the following autonomous problem associated with Problem (3.1) for any $\mu > 0$:

$$\begin{gathered}(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u + \mu\big(|u|^{p-2}u + |u|^{q-2}u\big) = \lambda f(u) \quad \text{in } \mathbb{R}^n, \\ u \in W^{s_1,p}(\mathbb{R}^n) \cap W^{s_2,q}(\mathbb{R}^n), \quad u > 0 \quad \text{in } \mathbb{R}^n.\end{gathered} \tag{3.4}$$

The corresponding energy functional is

$$I_\mu(u) = \frac{1}{p}\|u\|_{\mu,1}^p + \frac{1}{q}\|u\|_{\mu,2}^q - \lambda \int_{\mathbb{R}^n} F(u)dx,$$

which is well-defined on the space $Y_\mu = W^{s_1,p}(\mathbb{R}^n) \cap W^{s_2,q}(\mathbb{R}^n)$ equipped with the norm

$$\|u\|_\mu = \|u\|_{\mu,1} + \|u\|_{\mu,2},$$

where

$$\|u\|_{\mu,i}^t = [u]_{s_i,t}^t + \mu|u|_t^t \quad \text{for all } t > 1 \text{ and } i = \{1, 2\}.$$

It is easy to check that $I_\mu \in C^1(Y_\mu, \mathbb{R})$ and its differential is given by

$$\langle I_\mu'(u), v\rangle = \langle u, v\rangle_{s_1,p} + \langle u, v\rangle_{s_2,q} + \mu \int_{\mathbb{R}^n} (|u|^{p-2}u + |u|^{q-2}u)v\, dx - \lambda \int_{\mathbb{R}^n} f(u)v\, dx$$

for any $u, v \in Y_\mu$. Let us define the Nehari manifold associated with I_μ as follows:

$$N_\mu = \{u \in Y_\mu\backslash\{0\} : \langle I_\mu'(u), u\rangle = 0\}.$$

It follows from (A4) that

$$\begin{aligned} I_\mu(u) &= I_\mu(u) - \frac{1}{q}\langle I_\mu'(u), u\rangle \\ &= \big(\frac{1}{p} - \frac{1}{q}\big)\|u\|_{\mu,1}^p - \lambda \int_{\mathbb{R}^n} \Big(F(u) - \frac{1}{q}f(u)v\Big)dx \\ &\geq \big(\frac{1}{p} - \frac{1}{q}\big)\|u\|_{\mu,1}^p \quad \text{for all } u \in N_\mu. \end{aligned} \tag{3.5}$$

We easily check that I_μ has a mountain pass geometry, and we denote by c_μ its mountain pass level. Moreover, by the standard arguments from [37] and (3.5) we can show that

$$0 < c_\mu = \inf_{u \in N_\mu} I_\mu(u) = \inf_{u \in Y_\mu\backslash\{0\}} \max_{t \geq 0} I_\mu(tu).$$

With these facts in hand, we arrive to the following lemma.

Lemma 3.3. *Let $t \in \big[p, q_{s_2}^*\big)$, and $\{u_n\} \subset N_\mu$ be a minimizing sequence for I_μ. Then, $\{u_n\}$ is bounded in Y_μ; moreover, there exist a sequence $\{y_n\} \subset \mathbb{R}^n$ and constants $R, \beta > 0$ with*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^t dx \geq \beta > 0.$$

Proof. Arguing as in the proof of Lemma 3.2, we immediately see that $\{u_n\}$ is bounded in Y_μ. To prove the latter conclusion of this Lemma, by contradiction we assume that for any $R > 0$ it holds

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^n} \int_{B_R(y)} |u_n|^t dx = 0.$$

Then, it follows from Lemma 2.2 that

$$u_n \to 0 \quad \text{in } L^t(\mathbb{R}^n) \quad \text{for all } t \in \big(p, q^*_{s_2}\big). \tag{3.6}$$

We fix $\xi \in (0, \frac{\mu}{\lambda})$, and take into account $\{u_n\} \in N_\mu$ and (3.2) to deduce that

$$\begin{aligned}
0 &= \langle I'_\mu(u_n), u_n\rangle \\
&\geq \|u_n\|^p_{\mu,1} + \|u_n\|^q_{\mu,2} - \lambda\xi|t|^p_p - \lambda C_\xi |t|^r_r \\
&\geq C_1\|u_n\|^p_{\mu,1} + \|u_n\|^q_{\mu,2} - \lambda C_\xi |t|^r_r,
\end{aligned}$$

which combined with (3.6) implies that $\|u_n\|_\mu \to 0$ as $n \to \infty$. This leads to a contradiction because of $I_\mu(u_n) \to c_\mu > 0$, which completes the proof. □

Next we prove a useful compactness result for the autonomous problem (3.4).

Theorem 3.4. *Under assumptions* (A1)–(A5), *Problem* (3.4) *admits a positive ground state solution.*

Proof. We see from a variant of the mountain pass theorem without Palais-Smale condition (cf. [37]) that there exists a Palais-Smale sequence $\{u_n\} \subset Y_\mu$ for I_μ at the level c_μ. Then, using Lemma 3.3 we know that $\{u_n\}$ is bounded in Y_μ, and we may assume that

$$\begin{gathered}
u_n \rightharpoonup u \quad \text{in } Y_\mu, \\
u_n \to u \quad \text{in } L^t_{\rm loc}(\mathbb{R}^n) \quad \text{for all } t \in \big[1, q^*_{s_2}\big).
\end{gathered}$$

Now show that the weak limit u is a critical point of I_μ. To this end, we consider the sequence

$$h_n(x,y) = \frac{|u_n(x) - u_n(x)|^{p-2}(u_n(x) - u_n(x))}{|x-y|^{\frac{n+s_1p}{p'}}},$$

and let

$$h(x,y) = \frac{|u(x) - u(x)|^{p-2}(u(x) - u(x))}{|x-y|^{\frac{n+s_1p}{p'}}}$$

with $p' = \frac{p}{p-1}$. We easily check that $\{h_n\}$ is a bounded sequence in the reflexive Banach space $L^{p'}\big(\mathbb{R}^{2n}\big)$ with $h_n \to h$ a.e. in $\mathbb{R}^{2n}$. Then there exists a subsequence, still denoted by $\{h_n\}$, such that $h_n \rightharpoonup h$ in $L^{p'}\big(\mathbb{R}^{2n}\big)$, that is to say,

$$\int_{\mathbb{R}^{2n}} h_n(x,y) g(x,y)\, dx\, dy \to \int_{\mathbb{R}^{2n}} h(x,y) g(x,y)\, dx\, dy$$

for all $g \in L^p\big(\mathbb{R}^{2n}\big)$. For any $v \in C^\infty_c(\mathbb{R}^n)$, by taking

$$g(x,y) = \frac{v(x) - v(x)}{|x-y|^{\frac{n+s_1p}{p}}} \in L^p\big(\mathbb{R}^{2n}\big),$$

we can see that

$$\int_{\mathbb{R}^{2n}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{n+s_1p}} (v(x) - v(y))\, dx\, dy$$

$$\to \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+s_1p}}(v(x)-v(y))\,dx\,dy.$$

Similarly, we also prove that

$$\begin{aligned}&\int_{\mathbb{R}^{2n}} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))}{|x-y|^{n+s_2q}}(v(x)-v(y))\,dx\,dy\\ &\to \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))}{|x-y|^{n+s_2q}}(v(x)-v(y))\,dx\,dy.\end{aligned}$$

Note that

$$\begin{gathered}\int_{\mathbb{R}^n} |u_n|^{p-2}u_n v\,dx \to \int_{\mathbb{R}^n} |u|^{p-2}uv\,dx,\\ \int_{\mathbb{R}^n} |u_n|^{q-2}u_n v\,dx \to \int_{\mathbb{R}^n} |u|^{q-2}uv\,dx,\\ \int_{\mathbb{R}^n} f(u_n)v\,dx \to \int_{\mathbb{R}^n} f(u)v\,dx\end{gathered}$$

and the fact that $\langle I'_\mu(u_n), v\rangle = o_n(1)$, we can deduce that $\langle I'_\mu(u), v\rangle = 0$ for all $v \in C_c^\infty(\mathbb{R}^n)$. By the density of $C_c^\infty(\mathbb{R}^n)$ in Y_μ, we obtain that u is a critical point of I_μ, which implies that $\langle I'_\mu(u), u\rangle = 0$.

Next, we prove that $I_\mu(u) = c_\mu$, which is divided into two cases:

Case 1. For $u \neq 0$, it suffices only to show that

$$\|u_n\|_{\mu,1}^p \to \|u\|_{\mu,1}^p, \tag{3.7}$$

and then similarly, we can see that $\|u_n\|_{\mu,2}^q \to \|u\|_{\mu,2}^q$. Lemma 2.6 leads to that $u_n \to u$ in Y_μ. This together with the fact that $I_\mu(u_n) \to c_\mu$ deduces the desired result.

To prove (3.7), we observe that Fatou's lemma yields

$$\|u\|_{\mu,1}^p \le \liminf_{n\to\infty} \|u_n\|_{\mu,1}^p.$$

By contradiction, let us assume that

$$\|u\|_{\mu,1}^p < \limsup_{n\to\infty} \|u_n\|_{\mu,1}^p. \tag{3.8}$$

We notice that

$$\begin{aligned}c_\mu + o_n(1) &= I_\mu(u_n) - \frac{1}{q}\langle I'_\mu(u_n), u_n\rangle\\ &= \big(\frac{1}{p}-\frac{1}{q}\big)\|u_n\|_{\mu,1}^p + \lambda\int_{\mathbb{R}^n}\Big(\frac{1}{q}f(u_n)u_n - F(u_n)\Big)dx.\end{aligned} \tag{3.9}$$

Recalling that

$$\limsup_{n\to\infty}(a_n+b_n) \ge \limsup_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

and $p<q$, using (3.8), (3.9), Fatou's lemma again and the fact that $\langle I'_\mu(u), u\rangle = 0$, it yields that

$$\begin{aligned}c_\mu &\ge \big(\frac{1}{p}-\frac{1}{q}\big)\limsup_{n\to\infty}\|u_n\|_{\mu,1}^p + \lambda\liminf_{n\to\infty}\int_{\mathbb{R}^n}\Big(\frac{1}{q}f(u_n)u_n - F(u_n)\Big)dx\\ &> \big(\frac{1}{p}-\frac{1}{q}\big)\|u\|_{\mu,1}^p + \lambda\int_{\mathbb{R}^n}\Big(\frac{1}{q}f(u)u - F(u)\Big)dx\\ &= I_\mu(u) - \frac{1}{q}\langle I'_\mu(u), u\rangle = I_\mu(u) \ge c_\mu,\end{aligned}$$

which gives a contradiction.

Case 2. For $u = 0$, we argue it as in the proof of Lemma 3.3, we can find a sequence $\{y_n\} \subset \mathbb{R}^n$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^q dx \geq \beta > 0.$$

Set $v_n(x) = u_n(x + y_n)$. From the invariance by translations of $\mathbb{R}^n$, it is clear to show that $\{v_n\} \subset Y_\mu$ is also a bounded Palais–Smale sequence for I_μ at the level c_μ, and $v_n \rightharpoonup v \neq 0$ in Y_μ. Hence, we can proceed as before to check that $\{v_n\}$ converges strongly in Y_μ.

Finally, we prove that the ground state obtained above is positive. In fact, thanks to $\langle I'_\mu(u), u^-\rangle = 0$, assumption (A1) and the inequality

$$|x-y|^{t-2}(x-y)(x^- - y^-) \geq |x^- - y^-|^t \quad \text{for all } t \geq 1,$$

for $u^- = \min\{u, 0\}$ we have that

$$\|u^-\|^p_{\mu,1} + \|u^-\|^q_{\mu,2} \leq 0,$$

which implies that $u^- = 0$, which meads that $u \geq 0$ in $\mathbb{R}^n$. By employing a variant of the maximum principle (cf. [32]) we conclude that $u > 0$ in $\mathbb{R}^n$. This completes the proof. □

3.3. **Proof of Theorem 1.1.** In this subsection, we concentrate on the existence of the solution to Problem (3.1) and then give the proof of Theorem 1.1 provided that ε is sufficiently small. We shall start with the following inevitable lemma.

Lemma 3.5. *Let $t \in \big[p, q^*_{s_2}\big)$, and $\{u_n\} \subset N_\varepsilon$ be a sequence such that $I_\varepsilon(u_n) \to c_\varepsilon$ and $u_n \rightharpoonup 0$ in X_ε. Then, one of the following alternatives occurs:*

(a) *$u_n \to 0$ in X_ε;*

(b) *there exist a sequence $\{y_n\} \subset \mathbb{R}^n$ and constants $R, \beta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^t dx \geq \beta > 0.$$

Proof. It is natural that (b) fails while (a) happens. Conversely, we assume that (b) does not hold. Then for any $R > 0$ it holds

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^n} \int_{B_R(y)} |u_n|^t dx = 0.$$

By using Lemma 3.2 and Lemma 2.2, it follows that

$$u_n \to 0 \quad \text{in } L^t(\mathbb{R}^n) \text{ for all } t \in \big(p, q^*_{s_2}\big).$$

We argue it as in the proof of Lemma 3.3, and we conclude that $\|u_n\|_\varepsilon \to 0$ as $n \to \infty$. Thus we complete the proof. □

Before establishing a compactness result for I_ε, it is necessary to prove the following auxiliary lemma.

Lemma 3.6. *For $V_\infty < \infty$, let $\{v_n\} \subset N_\varepsilon$ be a sequence such that $I_\varepsilon(v_n) \to c$ and $v_n \rightharpoonup 0$ in X_ε. If $v_n \not\to 0$ in X_ε, then we have $c \geq c_\infty$, where c_∞ is the infimum of I_{V_∞} over N_{V_∞}.*

Proof. It follows from Lemma 3.2 that $\{v_n\}$ is bounded in X_ε. Let $\{t_n\} \subset (0,\infty)$ be such that $\{t_n v_n\} \subset N_{V_\infty}$. Hence, it suffices to prove that

$$\limsup_{n\to\infty} t_n \le 1.$$

In fact, by contradiction we suppose that there exist $\delta > 0$ such that

$$t_n \ge 1+\delta \quad \text{for all } n \in \mathbb{N}. \tag{3.10}$$

Note that $\{v_n\} \subset X_\varepsilon$ is a bounded Palais-Smale sequence for I_ε, we easily see that $\langle I'_\varepsilon(v_n), v_n\rangle = 0$, which means that

$$[v_n]^p_{s_1,p} + [v_n]^q_{s_2,q} + \int_{\mathbb{R}^n} V(\varepsilon x)(|v_n|^p + |v_n|^q)dx - \lambda \int_{\mathbb{R}^n} f(v_n)v_n dx = 0.$$

This combined with the fact that $t_n v_n \in N_{V_\infty}$ yields

$$t_n^{p-q}[v_n]^p_{s_1,p} + [v_n]^q_{s_2,q} + t_n^{p-q} V_\infty \int_{\mathbb{R}^n} |v_n|^p dx + V_\infty \int_{\mathbb{R}^n} |v_n|^q dx - \lambda \int_{\mathbb{R}^n} \frac{f(t_n v_n)v_n^q}{(t_n v_n)^{q-1}} dx = 0,$$

which implies that

$$\begin{aligned} &\lambda \int_{\mathbb{R}^n} \Big(\frac{f(t_n v_n)}{(t_n v_n)^{q-1}} - \frac{f(v_n)}{v_n^{q-1}}\Big) v_n^q dx \\ &\le \int_{\mathbb{R}^n} (V_\infty - V(\varepsilon x))|v_n|^p dx + \int_{\mathbb{R}^n} (V_\infty - V(\varepsilon x))|v_n|^q dx \end{aligned} \tag{3.11}$$

for any required t_n and $p < q$. By using assumption (1.6), for any $\zeta > 0$, there exists a constant $R > 0$ such that

$$V(\varepsilon x) \ge V_\infty - \zeta \quad \text{for all } |x| \ge R. \tag{3.12}$$

In addition, using the boundedness of $\{v_n\}$ in X_ε together with the fact that $v_n \to 0$ in $L^p(B_R)$, we can deduce that

$$\begin{aligned} &\int_{\mathbb{R}^n} (V_\infty - V(\varepsilon x))|v_n|^p dx \\ &= \int_{B_R} (V_\infty - V(\varepsilon x))|v_n|^p dx + \int_{\mathbb{R}^n\setminus B_R} (V_\infty - V(\varepsilon x))|v_n|^p dx \\ &\le V_\infty \int_{B_R} |v_n|^p dx + \zeta \int_{\mathbb{R}^n\setminus B_R} |v_n|^p dx \\ &\le o_n(1) + \zeta C. \end{aligned} \tag{3.13}$$

Similarly, we also find that

$$\int_{\mathbb{R}^n} (V_\infty - V(\varepsilon x))|v_n|^q dx \le o_n(1) + \zeta C. \tag{3.14}$$

Combining (3.11), (3.13) and (3.14) we have

$$\int_{\mathbb{R}^n} \Big(\frac{f(t_n v_n)}{(t_n v_n)^{q-1}} - \frac{f(v_n)}{v_n^{q-1}}\Big) v_n^q dx \le o_n(1) + \zeta C. \tag{3.15}$$

With the help of Lemma 3.5, we can infer that there exist a sequence $\{y_n\} \subset \mathbb{R}^n$, such that for the constants $R, \beta > 0$ it holds

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |v_n|^t dx \ge \beta > 0 \tag{3.16}$$

with $t \in \big[p, q^*_{s_2}\big)$. By considering $\hat{v}_n = v_n(x + y_n)$, then, up to a subsequence we can assume that $\hat{v}_n \rightharpoonup \hat{v}$ in X_ε. By formula (3.16) there exists $\Omega \subset \mathbb{R}^n$ with positive measure such that $\hat{v} > 0$ in Ω, which combined (3.10) and (3.15) yields the inequality

$$\int_\Omega \Big(\frac{f((1+\delta)\hat{v}_n)}{((1+\delta)\hat{v}_n)^{q-1}} - \frac{f(\hat{v}_n)}{\hat{v}_n^{q-1}}\Big)\hat{v}_n^q dx \le o_n(1) + \zeta C.$$

Therefore, passing to the limit as $n \to \infty$, using Fatou's lemma and (A5), yields

$$0 < \int_\Omega \Big(\frac{f((1+\delta)\hat{v})}{((1+\delta)\hat{v})^{q-1}} - \frac{f(\hat{v})}{\hat{v}^{q-1}}\Big)\hat{v}^q dx \le \zeta C \quad \text{for all } \zeta > 0,$$

which leads to a contradiction. For the remainder we consider the following two cases.

Case 1. For $\limsup_{n\to\infty} t_n = 1$, up to a subsequence, there exists $\{t_n\}$ such that $t_n \to 1$. Considering that $I_\varepsilon(v_n) \to c$, we have

$$\begin{aligned} c + o_n(1) &= I_\varepsilon(v_n) \\ &= I_\varepsilon(v_n) - I_{V_\infty}(t_n v_n) + I_{V_\infty}(t_n v_n) \\ &\ge I_\varepsilon(v_n) - I_{V_\infty}(t_n v_n) + c_\infty. \end{aligned} \tag{3.17}$$

Note that

$$\begin{aligned} &I_\varepsilon(v_n) - I_{V_\infty}(t_n v_n) \\ &= \frac{1-t_n^p}{p}[v_n]^p_{s_1,p} + \frac{1-t_n^q}{q}[v_n]^q_{s_2,q} + \frac{1}{p}\int_{\mathbb{R}^n}(V(\varepsilon x) - t_n^p V_\infty)|v_n|^p dx \\ &\quad + \frac{1}{q}\int_{\mathbb{R}^n}(V(\varepsilon x) - t_n^q V_\infty)|v_n|^q dx + \lambda\int_{\mathbb{R}^n}(F(t_n v_n) - F(v_n))dx. \end{aligned} \tag{3.18}$$

Taking into account that $v_n \to 0$ in $L^p(B_R)$ as $t_n \to 1$, Inequality (3.12) and assumption (1.6) imply that

$$V(\varepsilon x) - t_n^p V_\infty = (V(\varepsilon x) - V_\infty) + (1 - t_n^p)V_\infty \ge -\zeta + (1-t_n^p)V_\infty \quad \text{for all } |x| \ge R.$$

Therefore, we conclude that

$$\begin{aligned} &\int_{\mathbb{R}^n}(V(\varepsilon x) - t_n^p V_\infty)|v_n|^p dx \\ &= \int_{B_R}(V(\varepsilon x) - t_n^p V_\infty)|v_n|^p dx + \int_{\mathbb{R}^n\setminus B_R}(V(\varepsilon x) - t_n^p V_\infty)|v_n|^p dx \\ &\ge (V_0 - t_n^p V_\infty)\int_{B_R}|v_n|^p dx - \zeta\int_{\mathbb{R}^n\setminus B_R}|v_n|^p dx + (1-t_n^p)V_\infty\int_{\mathbb{R}^n\setminus B_R}|v_n|^p dx \\ &\ge o_n(1) - \zeta C. \end{aligned} \tag{3.19}$$

Similarly, we can prove that

$$\int_{\mathbb{R}^n}(V(\varepsilon x) - t_n^q V_\infty)|v_n|^q dx \ge o_n(1) - \zeta C. \tag{3.20}$$

Using the boundedness of $\{v_n\}$ in X_ε, we obtain

$$\frac{1-t_n^p}{p}[v_n]^p_{s_1,p} = o_n(1) \quad \text{and} \quad \frac{1-t_n^q}{q}[v_n]^q_{s_2,q} = o_n(1). \tag{3.21}$$

Combining (3.18), (3.19), (3.20) and (3.21) we have

$$I_\varepsilon(v_n)-I_{V_\infty}(t_nv_n)\geq \lambda\int_{\mathbb{R}^n}(F(t_nv_n)-F(v_n))dx+o_n(1)-\zeta C. \tag{3.22}$$

Then it suffices to show that

$$\int_{\mathbb{R}^n}(F(t_nv_n)-F(v_n))dx=o_n(1). \tag{3.23}$$

Employing Lemma 2.6, (3.3) and the boundedness of $\{v_n\}$ in X_ε, we can infer that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\big(F(t_nv_n)-F(v_n)\big)\,dx\\
&=\int_{\mathbb{R}^n}F((t_n-1)v_n)dx+o_n(1)\\
&\leq \frac{\xi}{p}|t_n-1|^p\int_{\mathbb{R}^n}|v_n|^pdx+\frac{C_\xi}{r}|t_n-1|^r\int_{\mathbb{R}^n}|v_n|^rdx+o_n(1)\leq o_n(1).
\end{aligned}$$

Thus, putting (3.17), (3.22) and (3.23) together we obtain

$$c+o_n(1)\geq o_n(1)-\zeta C+c_\infty,$$

and then passing to the limit as $\zeta\to 0$ yields $c\geq c_\infty$.

Case 2. For $\limsup_{n\to\infty}t_n=t_0<1$, there exists a subsequence, still denoted by $\{t_n\}$ such that $t_n\to t_0$ and $t_n<1$ for any $n\in\mathbb{N}$. Considering that $I_\varepsilon'(v_n)\to 0$, we have

$$\begin{aligned}
c+o_n(1)&=I_\varepsilon(v_n)-\frac{1}{q}\langle I_\varepsilon'(v_n),v_n\rangle\\
&=\big(\frac{1}{p}-\frac{1}{q}\big)\|v_n\|_1^p+\lambda\int_{\mathbb{R}^n}\Big(\frac{1}{q}f(v_n)v_n-F(v_n)\Big)dx.
\end{aligned} \tag{3.24}$$

Moreover, recalling the fact that $t_nv_n\in N_{V_\infty}$, Inequality (3.13) and assumption (A5) yields

$$\begin{aligned}
c_\infty&\leq I_{V_\infty}(t_nv_n)\\
&=I_{V_\infty}(t_nv_n)-\frac{1}{q}\langle I_{V_\infty}'(t_nv_n),t_nv_n\rangle\\
&=\big(\frac{1}{p}-\frac{1}{q}\big)\|t_nv_n\|_{V_\infty,1}^p+\lambda\int_{\mathbb{R}^n}\Big(\frac{1}{q}f(t_nv_n)t_nv_n-F(t_nv_n)\Big)dx\\
&\leq\big(\frac{1}{p}-\frac{1}{q}\big)(\|v_n\|_1^p+o_n(1)+\zeta C)+\lambda\int_{\mathbb{R}^n}\Big(\frac{1}{q}f(v_n)v_n-F(v_n)\Big)dx\\
&=c+o_n(1)+\zeta C,
\end{aligned}$$

where we used (3.24) in the last inequality. Therefore, by passing to the limit as $\zeta\to 0$ and $n\to\infty$ it implies that $c\geq c_\infty$, which completes the proof. □

We are now in a position to prove the compactness result as follows.

Lemma 3.7. *Let $\{u_n\}\subset N_\varepsilon$ be such that $I_\varepsilon(u_n)\to c_\varepsilon$, where $c_\varepsilon<c_\infty$ for $V_\infty<\infty$, and any $c_\varepsilon\in\mathbb{R}$ for $V_\infty=\infty$. Then $\{u_n\}$ has a convergent subsequence in X_ε.*

Proof. We argue as before, we immediately see that $\{u_n\}$ is bounded in X_ε, which means that we can take a sequence $\{u_n\}$ such that

$$\begin{gathered}
u_n\rightharpoonup u\quad\text{in } X_\varepsilon,\\
u_n\to u\quad\text{in } L^t_{\rm loc}(\mathbb{R}^n)\text{ for all } t\in\big[1,q_{s_2}^*\big).
\end{gathered}$$

Similar to the proof of Theorem 3.4, it is easy to check that $I'_\varepsilon(u) = 0$. Let $v_n = u_n - u$ and $I_\varepsilon(v_n) \to c$. Considering that $I_\varepsilon(u_n) \to c_\varepsilon$ and Lemma 2.6, we have

$$\begin{aligned} c + o_n(1) &= I_\varepsilon(v_n) \\ &= I_\varepsilon(u_n) - I_\varepsilon(u) + o_n(1) \\ &= c_\varepsilon - I_\varepsilon(u) + o_n(1). \end{aligned} \tag{3.25}$$

Arguing it as in the proof of [3, Proposition 3.1], we can see that $I'_\varepsilon(u) = 0$. Note that

$$I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{q}\langle I'_\varepsilon(u), u\rangle \geq 0. \tag{3.26}$$

By using (3.25) and (3.26), and considering $V_\infty < \infty$, we can see that

$$c \leq c_\varepsilon < c_\infty,$$

which together with Lemma 3.6 yields $v_n \to 0$ in X_ε, which means that $u_n \to u$ in X_ε.

On the other hand, by considering $V_\infty = \infty$, and using Lemma 2.5 we can infer that $v_n \to 0$ in $L^t(\mathbb{R}^n)$ for any $t \in \big[p, q^*_{s_2}\big)$. This combined with assumptions (A2) and (A3) yields

$$\int_{\mathbb{R}^n} f(v_n)v_n dx = o_n(1).$$

In addition, in accordance with $\langle I'_\varepsilon(v_n), v_n\rangle = o_n(1)$ we deduce that

$$\|v_n\|_1^p + \|v_n\|_2^q = o_n(1),$$

which leads to $\|u_n - u\|_\varepsilon = o_n(1)$ as $n \to \infty$. This completes the proof. $\square$

Proof of Theorem 1.1. According to Lemma 3.1, we know that there exists a sequence $\{u_n\} \subset X_\varepsilon$ such that $I_\varepsilon(u_n) \to c_\varepsilon$ and $I'_\varepsilon(u_n) \to 0$, where

$$c_\varepsilon = \inf_{u \in X_\varepsilon \setminus \{0\}} \max_{t \geq 0} I_\varepsilon(tu).$$

By standard arguments, we obtain that $\{u_n\}$ is bounded in X_ε, which yields that there exists a subsequence $\{u_n\}$ such that $u_n \rightharpoonup u$ in X_ε, where u is denoted by its weak limit. Moreover, $I'_\varepsilon(u) = 0$. With the help of Lemma 3.7, it is clear to check that $u_n \to u$ in X_ε while $V_\infty = \infty$. Then applying the mountain pass theorem yields the existence result. It is rather clear to check that $c_\varepsilon \leq I_\varepsilon(u)$. On the other hand, it follows from Fatou's lemma that

$$I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{q}\langle I'_\varepsilon(u), u\rangle \leq \liminf_{n\to\infty} \Big(I_\varepsilon(u_n) - \frac{1}{q}\langle I'_\varepsilon(u_n), u_n\rangle \Big) = c_\varepsilon,$$

which implies that the solution obtained above is a ground state solution, that is, $c_\varepsilon = I_\varepsilon(u)$.

To complete the proof, it suffices to show that $c_\varepsilon < c_\infty$ for small ε while $V_\infty < \infty$. Without loss of generality, let us suppose that

$$V(0) = V_0 = \inf_{x \in \mathbb{R}^n} V(x)$$

and $\mu \in (V_0, V_\infty)$. It is clear that $c_0 < c_\mu < c_\infty$, where c_0 is the infimum of I_{V_0} over N_{V_0}. By Theorem 3.4, there exists $w \in W^{s_1,p}(\mathbb{R}^n) \cap W^{s_2,q}(\mathbb{R}^n)$ as a positive ground state of the autonomous problem (3.4). Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function such that $0 \le \phi \le 1$, $\phi = 1$ in B_1, and $\phi = 0$ in $\mathbb{R}^n \backslash B_2$. We set $\phi_r(x) = \phi(\frac{x}{r})$ and consider the function $w_r(x) = \phi_r(x)w(x)$. Using Lemma 2.3 we see that

$$\lim_{r\to\infty} \|w_r - w\|_\varepsilon = 0. \tag{3.27}$$

Take $t_r > 0$ such that

$$I_\mu(t_r w_r) = \max_{t\ge 0} I_\mu(tw_r),$$

which leads to that $t_r w_r \in N_\mu$. Now we prove that there exists r large enough such that $I_\mu(t_r w_r) < c_\infty$. By contradiction we assume $I_\mu(t_r w_r) \ge c_\infty$ for any $r > 0$. By using the assumption (A5), (3.27), $t_r w_r \in N_\mu$ and $w \in N_\mu$, we can infer that $t_r \to 1$. Therefore, we conclude that

$$c_\infty \le \liminf_{r\to\infty} I_\mu(t_r w_r) = I_\mu(w_r) = c_\mu < c_\infty,$$

which leads to a contradiction. In addition, by assumption (1.6) we obtain that for some $\varepsilon_0 > 0$,

$$V(\varepsilon x) \le \mu \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

Hence, we deduce that

$$c_\varepsilon \le \max_{t\ge 0} I_\varepsilon(tw_r) \le \max_{t\ge 0} I_\mu(tw_r) = I_\mu(t_r w_r) < c_\infty$$

for all $\varepsilon \in (0, \varepsilon_0)$, which completes the proof. $\square$

4. Critical case while $\sigma = 1$

4.1. **Functional setting in the critical case.** In this section we focus on the critical case while $\sigma = 1$

$$\begin{gathered} (-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = \lambda f(u) + |u|^{q_{s_2}^*-2}u \quad \text{in } \mathbb{R}^n, \\ u \in W^{s_1,p}(\mathbb{R}^n) \cap W^{s_2,q}(\mathbb{R}^n), \ u > 0 \quad \text{in } \mathbb{R}^n. \end{gathered} \tag{4.1}$$

It is clear that the energy functional associated with (4.1) is

$$J_\varepsilon(u) = \frac{1}{p}\|u\|_1^p + \frac{1}{q}\|u\|_2^q - \lambda \int_{\mathbb{R}^n} F(u)dx - \frac{1}{q_{s_2}^*}|u|_{q_{s_2}^*}^{q_{s_2}^*}$$

and its differential is given by

$$\begin{aligned} \langle J_\varepsilon'(u), v\rangle = \langle u, v\rangle_{s_1,p} + \langle u, v\rangle_{s_2,q} + \int_{\mathbb{R}^n} V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u)v\,dx \\ - \lambda \int_{\mathbb{R}^n} f(u)v\,dx - \int_{\mathbb{R}^n} |u|^{q_{s_2}^*-2}uv\,dx \end{aligned}$$

for any $u, v \in X_\varepsilon$. We also introduce the Nehari manifold associated with J_ε,

$$M_\varepsilon = \{u \in X_\varepsilon \backslash \{0\} : \langle J_\varepsilon'(u), u\rangle = 0\}.$$

It is easy to check that J_ε possesses a mountain pass geometry shown as follows (cf. [1]). For simplicity, we here omit the proof because of its similarity to Lemma 3.1.

Lemma 4.1. *The functional J_ε satisfies the following conditions:*

(i) *there exists $\alpha, \rho > 0$ such that $J_\varepsilon(u) \ge \alpha$ with $\|u\|_\varepsilon = \rho$;*
(ii) *there exists $e \in X_\varepsilon$ with $\|e\|_\varepsilon > \rho$ such that $J_\varepsilon(e) < 0$.*

In view of Lemma 4.1 we define the mountain pass level

$$d_\varepsilon = \inf_{\xi\in\Lambda}\max_{t\in[0,1]} J_\varepsilon(\xi(t)) = \inf_{u\in X_\varepsilon\setminus\{0\}}\max_{t\geq 0} J_\varepsilon(\xi(t)),$$

where

$$\Lambda = \{\xi \in C^0([0,1], X_\varepsilon) : \xi(0) = 0, J_\varepsilon(\xi(1)) < 0\}.$$

Next we prove that any Palais-Smale sequence of J_ε is bounded.

Lemma 4.2. *If $\{u_n\}$ is a Palais-Smale sequence of J_ε at level c, then $\{u_n\}$ is bounded in X_ε.*

Proof. Using assumption (A4) and $q^*_{s_2} > \theta > q > p$, we conclude that

$$\begin{aligned} c(1+\|u_n\|_\varepsilon) &\geq J_\varepsilon(u_n) - \frac{1}{\theta}\langle J'_\varepsilon(u_n), u_n\rangle \\ &= \big(\frac{1}{p}-\frac{1}{\theta}\big)\|u_n\|_1^p + \big(\frac{1}{q}-\frac{1}{\theta}\big)\|u_n\|_2^q \\ &\quad + \lambda\int_{\mathbb{R}^n}\Big(\frac{1}{\theta}f(u_n)u_n - F(u)\Big)dx + \Big(\frac{1}{\theta}-\frac{1}{q^*_{s_2}}\Big)|u|_{q^*_{s_2}}^{q^*_{s_2}} \\ &\geq \big(\frac{1}{q}-\frac{1}{\theta}\big)(\|u_n\|_1^p + \|u_n\|_2^q). \end{aligned}$$

Then we argue it as in the proof of Lemma 3.2, and we easily get the desired result. □

4.2. **Autonomous critical problem.** First, let us consider the autonomous problem associated with Problem (4.1) as follows:

$$\begin{gathered} (-\Delta)^{s_1}_p u + (-\Delta)^{s_2}_q u + \mu\big(|u|^{p-2}u + |u|^{q-2}u\big) = \lambda f(u) + |u|^{q^*_{s_2}-2}u \quad \text{in } \mathbb{R}^n, \\ \mu > 0, \quad u \in W^{s_1,p}(\mathbb{R}^n)\cap W^{s_2,q}(\mathbb{R}^n), \quad u > 0 \quad \text{in } \mathbb{R}^n. \end{gathered} \tag{4.2}$$

Therefore, the corresponding energy functional is defined as

$$J_\mu(u) = \frac{1}{p}\|u\|^p_{\mu,1} + \frac{1}{q}\|u\|^q_{\mu,2} - \lambda\int_{\mathbb{R}^n} F(u)dx - \frac{1}{q^*_{s_2}}|u|^{q^*_{s_2}}_{q^*_{s_2}},$$

and its differential is given by

$$\begin{aligned} \langle J'_\mu(u), v\rangle = \langle u,v\rangle_{s_1,p} + \langle u,v\rangle_{s_2,q} + \mu\int_{\mathbb{R}^n}(|u|^{p-2}u + |u|^{q-2}u)v\,dx \\ - \lambda\int_{\mathbb{R}^n} f(u)v\,dx - \int_{\mathbb{R}^n}|u|^{q^*_{s_2}-2}uv\,dx \end{aligned}$$

for any $u, v \in Y_\mu$. Moreover, the Nehari manifold associated with J_μ is

$$M_\mu = \{u \in Y_\mu\setminus\{0\} : \langle J'_\mu(u), u\rangle = 0\}.$$

Arguing it as before, it is standard to check that J_μ has a mountain pass geometry, and we denote by d_μ its mountain pass level.

Remark 4.3. As in [36] we have the following variational characterization of the infimum of J_μ over M_μ:

$$0 < d_\mu = \inf_{u\in M_\mu} J_\mu(u) = \inf_{\xi\in\Lambda}\max_{t\in[0,1]} J_\varepsilon(\xi(t)) = \inf_{u\in Y_\mu\setminus\{0\}}\max_{t\geq 0} J_\mu(tu).$$

To prove the existence of a nontrivial solution to Problem (4.2), we firstly need to prove the following fundamental result.

Lemma 4.4. *There exists $\lambda_* > 0$ such that $d_\mu \in \big(0, \frac{s_2}{n} S_*^{\frac{n}{s_2 q}}\big)$ for all $\lambda \geq \lambda_*$.*

Proof. Let $v \in C_c^\infty(\mathbb{R}^n)$ be a non-zero function such that $v \geq 0$ in $\mathbb{R}^n$. Then there exists $t_\lambda > 0$ such that $J_\mu(t_\lambda v) = \max_{t \geq 0} J_\mu(tv)$, which yields $\langle J'_\mu(t_\lambda v), t_\lambda v\rangle = 0$. Therefore,

$$t_\lambda^p \|v\|_{\mu,1}^p + t_\lambda^q \|v\|_{\mu,2}^q = \lambda \int_{\mathbb{R}^n} f(t_\lambda v) t_\lambda v \, dx + t_\lambda^{q_{s_2}^*} |v|_{q_{s_2}^*}^{q_{s_2}^*}. \tag{4.3}$$

Using assumption (A4) we find that

$$t_\lambda^p \|v\|_{\mu,1}^p + t_\lambda^q \|v\|_{\mu,2}^q \geq t_\lambda^{q_{s_2}^*} |v|_{q_{s_2}^*}^{q_{s_2}^*},$$

which combined with $p < q < q_{s_2}^*$ yields that t_λ is bounded. Then there exists a subsequence $\{t_{\lambda_n}\}$ such that $t_{\lambda_n} \to t_0 \geq 0$ as $\lambda_n \to \infty$. Now we prove $t_0 = 0$ by contradiction. Let us suppose that $t_0 > 0$ such that

$$\begin{gathered} t_{\lambda_n}^p \|v\|_{\mu,1}^p + t_{\lambda_n}^q \|v\|_{\mu,2}^q \to T \in (0, \infty), \\ \lambda_n \int_{\mathbb{R}^n} f(t_{\lambda_n} v) t_{\lambda_n} v \, dx + t_{\lambda_n}^{q_{s_2}^*} |v|_{q_{s_2}^*}^{q_{s_2}^*} \to \infty, \end{gathered}$$

which certainly goes against (4.3). Thus, $t_{\lambda_n} \to 0$ as $\lambda_n \to \infty$.

Next, we write $h(t) = tv$ for $t \in [0,1]$. Then $h \in \Lambda$, and we obtain

$$0 < d_\mu \leq \max_{t \in [0,1]} J_\mu(h(t)) \leq \max_{t \geq 0} J_\mu(tv) = J_\mu(t_\lambda v) \leq t_\lambda^p \|v\|_{\mu,1}^p + t_\lambda^q \|v\|_{\mu,2}^q.$$

Taking λ sufficiently large we obtain

$$t_\lambda^p \|v\|_{\mu,1}^p + t_\lambda^q \|v\|_{\mu,2}^q < \frac{s_2}{n} S_*^{\frac{n}{s_2 q}},$$

which completes the proof. $\square$

Lemma 4.5. *Let $t \in \big[p, q_{s_2}^*\big)$, and $\{u_n\} \subset M_\mu$ be a minimizing sequence for J_μ. Then, $\{u_n\}$ is bounded in Y_μ, and there exist a sequence $\{y_n\} \subset \mathbb{R}^n$ and constants $R, \beta > 0$ such that*

$$\liminf_{n \to \infty} \int_{B_R(y_n)} |u_n|^t dx \geq \beta > 0.$$

Proof. It is easy to see that $\{u_n\}$ is bounded in Y_μ. We assume by contradiction that for any $R > 0$ it holds

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^n} \int_{B_R(y)} |u_n|^t dx = 0.$$

Then, it follows from Lemma 2.2 that

$$u_n \to 0 \quad \text{in } L^t(\mathbb{R}^n) \text{ for all } t \in \big(p, q_{s_2}^*\big). \tag{4.4}$$

Employing (3.2), (3.3) and (4.4) we can infer that

$$\begin{gathered} 0 \leq \int_{\mathbb{R}^n} f(u_n) u_n dx \leq \xi \int_{\mathbb{R}^n} |u_n|^p + o_n(1), \\ 0 \leq \int_{\mathbb{R}^n} F(u_n) dx \leq C\,\xi \int_{\mathbb{R}^n} |u_n|^p + o_n(1). \end{gathered}$$

Passing to the limit as $\xi \to 0$, we obtain

$$\int_{\mathbb{R}^n} f(u_n) u_n dx = o_n(1) \quad \text{and} \quad \int_{\mathbb{R}^n} F(u_n) dx = o_n(1), \tag{4.5}$$

which combined with $\langle J'_\mu(u_n), u_n\rangle = 0$ yields

$$\|u_n\|_{\mu,1}^p + \|u_n\|_{\mu,2}^q - |u_n|_{q_{s_2}^*}^{q_{s_2}^*} = o_n(1).$$

Moreover, from the boundedness of $\{u_n\}$ in Y_μ we may assume that

$$\|u_n\|_{\mu,1}^p + \|u_n\|_{\mu,2}^q \to L \geq 0 \quad \text{and} \quad |u_n|_{q_{s_2}^*}^{q_{s_2}^*} \to L \geq 0. \tag{4.6}$$

If $L = 0$, then $\|u_n\|_\mu \to 0$ as $n \to \infty$, which is a contradiction because of $J_\mu(u_n) \to d_\mu > 0$. Therefore, in the following we assume that $L > 0$. By taking into account (4.5) and (4.6), we conclude that

$$\begin{aligned} d_\mu &= J_\mu(u_n) + o_n(1) \\ &= \frac{1}{p}\|u_n\|_{\mu,1}^p + \frac{1}{q}\|u_n\|_{\mu,2}^q - \lambda\int_{\mathbb{R}^n} F(u_n)dx - \frac{1}{q_{s_2}^*}|u_n|_{q_{s_2}^*}^{q_{s_2}^*} \\ &\geq \frac{1}{q}L - \frac{1}{q_{s_2}^*}L + o_n(1) \\ &= \frac{s_2}{n}L + o_n(1). \end{aligned} \tag{4.7}$$

On the other hand, by using Lemma 2.1 it follows that

$$|u_n|_{q_{s_2}^*}^q \leq S_*^{-1}([u_n]_{s_2,q}^q + \mu|u_n|_q^q) = S_*^{-1}\|u\|_{\mu,2}^q \leq S_*^{-1}(\|u_n\|_{\mu,1}^p + \|u_n\|_{\mu,2}^q).$$

Passing to the limit as $n \to \infty$, we find that

$$L^{\frac{q}{q_{s_2}^*}} \leq S_*^{-1}L,$$

which together with (4.7) yields $d_\mu \geq \frac{s_2}{n}S_*^{\frac{n}{s_2 q}}$. This yields a contradiction in view of Lemma 4.4. Thus we completes the proof. □

Let us now prove the existence result for the autonomous critical case.

Theorem 4.6. *Under assumptions* (A1)–(A5)*, Problem* (4.2) *admits a positive ground state solution.*

Proof. This proof follows the argument developed as in Theorem 3.4. We here need to replace (3.9) by

$$\begin{aligned} &d_\mu + o_n(1) \\ &= J_\mu(u_n) - \frac{1}{q}\langle J'_\mu(u_n), u_n\rangle \\ &= (\frac{1}{p} - \frac{1}{q})\|u_n\|_{\mu,1}^p + \lambda\int_{\mathbb{R}^n}\Big(\frac{1}{q}f(u_n)u_n - F(u_n)\Big)dx + \Big(\frac{1}{q} - \frac{1}{q_{s_2}^*}\Big)|u|_{q_{s_2}^*}^{q_{s_2}^*}. \end{aligned}$$

Recalling that

$$\begin{aligned} \limsup_{n\to\infty}(a_n + b_n + c_n) &\geq \limsup_{n\to\infty} a_n + \liminf_{n\to\infty}(b_n + c_n) \\ &\geq \limsup_{n\to\infty} a_n + \liminf_{n\to\infty} b_n + \liminf_{n\to\infty} c_n, \end{aligned}$$

which implies that

$$\begin{aligned} d_\mu \geq (\frac{1}{p} - \frac{1}{q})\limsup_{n\to\infty}\|u_n\|_{\mu,1}^p + \lambda\liminf_{n\to\infty}\int_{\mathbb{R}^n}\Big(\frac{1}{q}f(u_n)u_n - F(u_n)\Big)dx \\ + \Big(\frac{1}{q} - \frac{1}{q_{s_2}^*}\Big)\liminf_{n\to\infty}|u|_{q_{s_2}^*}^{q_{s_2}^*}. \end{aligned}$$

We complete the proof by using Lemma 4.5 instead of Lemma 3.3. ☐

4.3. **Proof of Theorem 1.2.** By a similar argument as Lemma 4.5, the critical version of Lemma 3.5 is presented here.

Lemma 4.7. *Assume that $d_\varepsilon < \frac{s_2}{n} S_*^{\frac{n}{s_2 q}}$. Let $t \in \big[p, q^*_{s_2}\big)$, and $\{u_n\} \subset M_\varepsilon$ be a sequence such that $J_\varepsilon(u_n) \to d_\varepsilon$ and $u_n \rightharpoonup 0$ in X_ε. Then, one of the following alternatives occurs:*

(i) *$u_n \to 0$ in X_ε;*
(ii) *there exist a sequence $\{y_n\} \subset \mathbb{R}^n$ and constants $R, \beta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^t dx \geq \beta > 0.$$

For the case of $V_\infty < \infty$, we immediately obtain the following result along the lines of the proof of Lemma 3.6.

Lemma 4.8. *Assume that $V_\infty < \infty$, and let $\{v_n\} \subset M_\varepsilon$ be a sequence such that $J_\varepsilon(v_n) \to d$ and $v_n \rightharpoonup 0$ in X_ε. If $v_n \not\to 0$ in X_ε, then $d \geq d_\infty$, where d_∞ is the infimum of J_{V_∞} over M_{V_∞}.*

Next, we can give the following compactness result in the critical case.

Lemma 4.9. *Let $\{u_n\} \subset M_\varepsilon$ be such that $J_\varepsilon(u_n) \to d_\varepsilon$, where $d_\varepsilon < d_\infty$ for $V_\infty < \infty$, and $d_\varepsilon < \frac{s_2}{n} S_*^{\frac{n}{s_2 q}}$ for $V_\infty = \infty$. Then $\{u_n\}$ has a convergent subsequence in X_ε.*

Proof. We first argue it as in the proof of Lemma 3.7 so that we know that $\{u_n\}$ is bounded in X_ε, which yields that we may assume that $u_n \rightharpoonup u$ in X_ε. It is clear that $J'_\varepsilon(u) = 0$. Let $v_n = u_n - u$ and $J_\varepsilon(v_n) \to d$. By Brezis-Lieb Lemma in [11] and [28, Lemma 3.3] we find that

$$|v_n|^{q^*_{s_2}}_{q^*_{s_2}} = |u_n|^{q^*_{s_2}}_{q^*_{s_2}} - |u|^{q^*_{s_2}}_{q^*_{s_2}} + o_n(1).$$

We replace (3.25) by

$$\begin{aligned} d + o_n(1) &= J_\varepsilon(v_n) \\ &= J_\varepsilon(u_n) - J_\varepsilon(u) + o_n(1) \\ &= d_\varepsilon - J_\varepsilon(u) + o_n(1) \end{aligned}$$

and notice that

$$\begin{aligned} J_\varepsilon(u) &= J_\varepsilon(u) - \frac{1}{q}\langle J'_\varepsilon(u), u\rangle \\ &= \frac{1}{p}\|u\|_1^p + \lambda \int_{\mathbb{R}^n} \Big(\frac{1}{q} f(u)u - F(u)\Big)dx + \Big(\frac{1}{q} - \frac{1}{q^*_{s_2}}\Big)|u|^{q^*_{s_2}}_{q^*_{s_2}} \geq 0, \end{aligned}$$

where we used assumption (A4) in the last inequality. Therefore, for the case of $V_\infty < \infty$ we deduce that

$$d \leq d_\varepsilon < d_\infty,$$

which together with Lemma 4.8 yields $v_n \to 0$ in X_ε, that is to say, $u_n \to u$ in X_ε.

For the case of $V_\infty = \infty$, by using Lemma 2.5 we can infer that $v_n \to 0$ in $L^t(\mathbb{R}^n)$ for any $t \in \big[p, q^*_{s_2}\big)$. This combined with assumptions (A2) and (A3) implies that

$$\int_{\mathbb{R}^n} f(v_n)v_n \, dx = o_n(1),$$

which together with $\langle J_\varepsilon'(v_n), v_n\rangle = 0$ yields

$$\|v_n\|_1^p + \|v_n\|_2^q - |v_n|_{q_{s_2}^*}^{q_{s_2}^*} = o_n(1).$$

Considering the boundedness of $\{v_n\}$, we may assume that

$$\|v_n\|_1^p + \|v_n\|_2^q \to L \geq 0 \quad \text{and} \quad |v_n|_{q_{s_2}^*}^{q_{s_2}^*} \to L \geq 0.$$

If $L > 0$, we follow the lines of the proof of Lemma 4.5 to get $d \geq \frac{s_2}{n} S_*^{\frac{n}{s_2 q}}$. Note that $d \leq d_\varepsilon < \frac{s_2}{n} S_*^{\frac{n}{s_2 q}}$, which yields a contradiction. This leads to that $L = 0$, which completes the proof. □

We are finally ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. The proof follows the same lines as in the proof of Theorem 1.1, once one replace Lemma 3.1, Lemma 3.2, Theorem 3.4 and Lemma 3.7 by Lemma 4.1, Lemma 4.2, Theorem 4.6 and Lemma 4.9, respectively. □

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