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GLOBAL WELL-POSEDNESS FOR KLEIN-GORDON-HARTREE AND FRACTIONAL HARTREE EQUATIONS ON MODULATION SPACES

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Abstract. We study the Cauchy problems for the Klein-Gordon (HNLKG), wave (HNLW), and Schrödinger (HNLS) equations with cubic convolution (of Hartree type) nonlinearity. Some global well-posedness and scattering are obtained for the (HNLKG) and (HNLS) with small Cauchy data in some modulation spaces. Global well-posedness for fractional Schrödinger (fNLSH) equation with Hartree type nonlinearity is obtained with Cauchy data in some modulation spaces. Local well-posedness for (HNLW), (fHNLS) and (HNLKG) with rough data in modulation spaces is shown. As a consequence, we get local and global well-posedness and scattering in larger than usual L^p-Sobolev spaces.

1. Introduction and statement of results

1.1. Klein-Gordon-Hartree and wave-Hartree equations. We study the Cauchy problem for the Klein-Gordon and wave equations with Hartree type nonliearity

$$u_{tt} + (I - \Delta)u = (V * |u|^2)u, u(0) = u_0, u_t(0) = u_1 \tag{1.1}$$

and

$$u_{tt} - \Delta u = (V * |u|^2),\ u(0) = u_0, u_t(0) = u_1, \tag{1.2}$$

where $u(t,x)$ is a complex valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, $i = \sqrt{-1}$, $u_t = \frac{\partial}{\partial t}, u_{tt} = \frac{\partial^2}{\partial^2 t}$, I is the identity operator, Δ is the Laplace operator, u_0 and u_1 are complex valued functions of $x \in \mathbb{R}^d$, $*$ denotes the convolution in $\mathbb{R}^d$, and V is of the type

$$V(x) = \frac{\lambda}{|x|^\gamma}, \quad \lambda \in \mathbb{R},\ x \in \mathbb{R}^d,\ 0 < \gamma < d. \tag{1.3}$$

The stationary equation $-\Delta u + (V * |u|^2)u = \sigma u$ is obtained by looking for separated solutions of (1.1) and (1.2), where $u = e^{i\lambda t}u(x)(\sigma = \lambda^2 - 1$ and $\sigma = \lambda^2)$. In the case $V(x) = |x|^{-1}$, the stationary equations were proposed by Hartree as a model for the helium atom. Thus the homogeneous kernel of the form (1.3) is known as Hartree potential. A class of a “nonlocal” nonlinearity that we call “Hartree type” occurs in the modeling of quantum semiconductor devices.

Menzala-Strauss [21] studied the well-posedness and asymptotic behavior of equations (1.1) and (1.2). Mochizuki [26] and Hidano [16] studied scattering theory

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in the energy space (see also [10, 28]). Recently Miao-Zhang [23, 25] and Miao-Zhang-Zheng [24] studied global well-posedness and scattering theory for equations (1.1) and (1.2) below energy space. We remark that all previous authors have studied equations (1.1) and (1.2) on L^2-based Sobolev spaces. Mainly because generally Klein-Gordon $G(t)=e^{it(I-\Delta)^{1/2}}$ and wave $W(t)=e^{it(-\Delta)^{1/2}}$ semigroups fails to be bounded on $L^p(\mathbb{R}^d)$ if $p\neq 2$. Hence we cannot expect to solve equations (1.1) and (1.2) in $L^p(\mathbb{R}^d)(p\neq 2)$-spaces. The question arises if it is possible to remove L^2 constraint and consider equations (1.1) and (1.2) in function spaces which are not L^2 based.

This question has inspired to study equations (1.1) and (1.2) in other function spaces (e.g., modulation spaces $M^{p,q}(\mathbb{R}^d)$, see Definition 2.1 below) arising in harmonic analysis. Pioneering steps in this direction were taken by Wang-Lifeng-Boling [31], Wang-Hudzik [29] and Bényi-Gröchenig-Okoudjou-Rogers [1]. In fact, in [29] it is proved that Klein-Gordon equation with power type nonlinarity is globally well-posed with small Cauchy data in $M^{2,1}(\mathbb{R}^d)$. In [1, 31] it is proved that the Fourier multiplier operator with multiplier $e^{it|\xi|^\alpha}$ ($\alpha\in[0,2]$) is bounded on $M^{p,q}(\mathbb{R}^d)$ ($1\leq p,q\leq\infty$). (The cases $\alpha=1$ and $\alpha=2$ occurs in the time evolution of the free wave and Schrödinger equations respectively.) Many authors [2, 6, 9, 12, 17, 27, 32] have studied Klein-Gordon and wave equations with power type nonliterary in modulation spaces. However, there is not much progress concerning well-posedness and scattering theory for the equations (1.1) and (1.2) in modulation spaces.

Taking these considerations into account, we are inspired to study equations (1.1) and (1.2) with Cauchy data in modulation spaces. To sate results, we set up notation. Set $2\sigma(p)=(d+2)(\frac12-\frac1p)$ $(2<p<\infty, d\in\mathbb{N})$, $1/p+1/p'=1$. We call pair (p,r) is Klein-Gordon admissible if there exists another exponent β such that

$$\begin{gathered}\frac{1}{\beta}+\frac{2}{r}=1,\\ \frac13\leq\frac1\beta\leq\frac{d}{d+2}\wedge d(\frac12-\frac1p),\\ \frac14\leq p<\frac12-\frac{1}{3d}.\end{gathered}\tag{1.4}$$

We remark that if pair (p,r) is Klein-Gordon admissible, then $3\leq r<\infty$ and $rd(\frac12-\frac1p)>1$.

Theorem 1.1 (Global well-posedness). *Let $2<p<3$, $\frac1p+\frac{\gamma}{d}-1=\frac{1}{2p'}$, $s\in\mathbb{R}$, and pair (p,r) is Klein-Gordon admissible. Assume that*

$$(u_0,u_1)\in M^{p',1}_{s+2\sigma(p)}(\mathbb{R}^d)\times M^{p',1}_{s+2\sigma(p)-1}(\mathbb{R}^d)$$

and there exists a small $\delta>0$ such that $\|u_0\|_{M^{p',1}_{s+2\sigma(p)}}+\|u_1\|_{M^{p',1}_{s+2\sigma(p)-1}}\leq\delta$. Then (1.1) has a unique global solution

$$u\in C(\mathbb{R},M^{p,1}_s(\mathbb{R}^d))\cap C^1(\mathbb{R},M^{p,1}_{s-1}(\mathbb{R}^d))\cap L^r(\mathbb{R},M^{p,1}_s(\mathbb{R}^d)).$$

One also has the bound $\|u\|_{L^r(\mathbb{R},M^{p,1}_s(\mathbb{R}^d))}\lesssim\|u_0\|_{M^{p',1}_{s+2\sigma(p)}}+\|u_1\|_{M^{p',1}_{s+2\sigma(p)-1}}$.

Noticing $L^p_s(\mathbb{R}^d)\subset M^{p,1}(\mathbb{R}^d)$ for $s>d$ and taking $s=-2\sigma(p)$ (see Theorem 2.4 below), Theorem 1.1 reveals that we can control initial Cauchy data beyond

L^p_s-Sobolev spaces. To prove Theorem 1.1 we use some algebraic properties (see Proposition 2.5 below) and the integrability of time decay terms for Klein-Gordon semigroup:

$$\|G(t)f\|_{M^{p,q}_s} \lesssim (1+|t|)^{-d\theta(1/2-1/p)}\|f\|_{M^{p',q}_{s+\theta 2\sigma(p)}},$$

where $s\in\mathbb{R}, 2\le p\le\infty, 1\le q<\infty, \theta\in[0,1]$ (see Proposition 2.9 below). We remark that there is no singularity at $t=0$ and but preserve the same decay as in the below $L^p-L^{p'}$ estimate of $G(t)$. This is a special characteristic of modulation spaces. Recall standard $L^p-L^{p'}$ estimate of $G(t)$;

$$\|G(t)\|_{L^p_{2\sigma(p)}} \le C|t|^{-d(1/2-1/p)}\|f\|_{L^{p'}}, \quad 2\le p<\infty$$

and since $|t|^{-d(1/2-1/p)}$ is not integrable, we do not know whether we can use the similar argument under L^p, Besov, or Sobolev spaces.

Theorem 1.1 reveals that we have $L^r_t(\mathbb{R}, M^{p,1}_s)$ bound for the solution of (1.1) if the initial data is small enough. This implies we obtain scattering. Specifically, we have the following result.

Corollary 1.2 (Scattering)**.** *Let $u_0\in M^{p,1}_s(\mathbb{R}^d), u_1\in M^{p,1}_{s-1}(\mathbb{R}^d)$, and let u is the global solution to (1.1) such that $\|u\|_{L^r_t(\mathbb{R},M^{p,1}_s)}\le M$ for some constant $M>0$. Then there exist $v_1^\pm, v_2^\pm\in M^{p,1}_s(\mathbb{R}^d)$ such that $v^\pm = G(t)v_1^\pm + G(t)v_2^\pm$ are solutions to the free Klein-Gordon equation $u_{tt}+(I-\Delta)u=0$ and*

$$\|u(t)-v^\pm\|_{M^{p,1}_s}\to 0 \quad \text{as } t\to\pm\infty.$$

It remains open question to obtain the global well-posedness for equations (1.1) and (1.2) and for the large data in modulation spaces. However, we can obtain local existence with persistency of solutions. Specifically, we have the following theorem.

Theorem 1.3 (Local wellposedness)**.** *Let V is given by (1.3) and $X=M^{p,q}(\mathbb{R}^d)$ $(1\le p\le 2, 1\le q<\frac{2d}{d+\gamma})$ or $M^{p,1}_s(\mathbb{R}^d)$ $(1<p<\infty, s\in\mathbb{R}, \frac{1}{p}+\frac{\gamma}{d}-1=\frac{1}{p+\epsilon}, \epsilon>0)$. Assume that $u_0,u_1\in X$. Then*

1. *there exists $T^*=T^*(\|u_0\|_X,\|u_1\|_X)$ such that (1.1) has a unique solution $u\in C([0,T^*),X)$. Moreover, if $T^*<\infty$, then $\limsup_{t\to T^*}\|u(\cdot,t)\|_X=\infty$.*
2. *there exists $T^*=T^*(\|u_0\|_X,\|u_1\|_X)$ such that (1.2) has a unique solution $u\in C([0,T^*),X)$. Moreover, if $T^*<\infty$, then $\limsup_{t\to T^*}\|u(\cdot,t)\|_X=\infty$.*

Up to now we cannot know if equations (1.1) and (1.2) are locally well posed in $L^p(\mathbb{R}^d)$, but by Theorem 1.3, in $M^{p,1}(\mathbb{R}^d)\subset L^p(\mathbb{R}^d)$ (see Lemma 2.3 (2) below). $M^{p,1}_{s_1}(\mathbb{R}^d)$ $(p\ge 2$, some $s_1\in\mathbb{R})$ contains a class of data which are out of control of $H^s(\mathbb{R}^d)$. Notice that taking $s_1=-d/2$, it follows that $H^s(\mathbb{R}^d)=L^2_s(\mathbb{R}^d)\subsetneq M^{2,1}_{s_1}(\mathbb{R}^d)\subsetneq M^{p,1}_{s_1}(\mathbb{R}^d)$ for any $s>0$ (see Theorem 2.4), Theorem 1.3 reveals that we can get local well-posedness for (1.1) and (1.2) below energy spaces and in any dimension.

Remark 1.4. The analogue of Theorem 1.3 holds for the generalized equations (1.1) and (1.2), that is, Klein-Gordon and wave equations with nonlinearity $(V*|u|^{2k})u$ $(k\in\mathbb{N})$ when $X=M^{p,1}_s(\mathbb{R}^d)$.

1.2. **Fractional Hartree equation.** We study fractional Schrödinger equation with cubic convolution nonlinearity

$$i\partial_t u-(-\Delta)^{\alpha/2}u=(V*|u|^2)u, u(x,0)=u_0(x) \tag{1.5}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}, u_0 : \mathbb{R}^d \to \mathbb{C}$, V is defined by (1.3), and $\alpha > 0$. The fractional Laplacian is defined as

$$\mathcal{F}[(-\Delta)^{\alpha/2} u](\xi) = |\xi|^\alpha \mathcal{F}u(\xi)$$

where $\mathcal{F}$ denotes the Fourier transform. Equation (1.5) is known as the fractional Hartree equation. Equation (1.5) describes the dynamics of Bose-Einstein condensate, in which all particles are in the same state $u(t,x)$. There is an extensive study of (1.5) with Cauchy data in Sobolev spaces, e.g., [22, 11, 7] and the references therein.

Recently, for $0 < \gamma < \min\{\alpha, d/2\}$, Bhimani [4] proved global well-posedness for (1.5) in $M^{p,q}(\mathbb{R}^d)$ $(1 \le p \le 2, 1 \le q < 2d/(d+\gamma))$ when $\alpha = 2, d \ge 1$, and with radial Cauchy data when $d \ge 2, \frac{2d}{2d-1} < \alpha < 2$ (cf. [3, 19]). Manna [20] proved small data global well-posedness for (1.5) with the potential $V \in M^{1,\infty}(\mathbb{R}^d)$. On the other hand, many authors [31, 29, 2, 15, 6] have studied nonlinear Schrödinger equation in modulation spaces. In this paper, using time integrablity of time decay factors of time decay estimate (see Proposition 2.8), we obtain global well-posedness and scattering for small Cauchy data in modulation spaces. To state result, we set up notations. We call pair (p,r) Schrödinger admissible if there exists another exponent β such that

$$\begin{gathered} \frac{1}{\beta} + \frac{2}{r} = 1, \\ \frac{1}{3} \le \frac{1}{\beta} \le 1 \wedge d(\frac{1}{2} - \frac{1}{p}), \\ \frac{1}{4} \le p < \frac{1}{2} - \frac{1}{3d}, \end{gathered} \tag{1.6}$$

and

$$(p,r) \neq \big(\frac{2d}{d-2}, \infty\big).$$

Notice that if pair (p,r) is Schrödinger admissible, then $3 \le r \le \infty$ and $rd(\frac{1}{2} - \frac{1}{p}) > 1$. We are now ready to state following theorem.

Theorem 1.5 (Global well-posedness)**.** *Let $2 < p < 3$, $\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{2p'}$, $s \in \mathbb{R}$, $\alpha = 2$, and (p,r) be a Schrödinger admissible pair. Assume that $u_0 \in M_s^{p',1}(\mathbb{R}^d)$ and there exists a small $\delta > 0$ such that $\|u_0\|_{M_s^{p',1}} \le \delta$. Then (1.5) has a unique global solution*

$$u \in C(\mathbb{R}, M_s^{p,1}(\mathbb{R}^d)) \cap L^r(\mathbb{R}, M_s^{p,1}(\mathbb{R}^d)).$$

One also has the bound $\|u\|_{L^r(\mathbb{R}, M_s^{p,1}(\mathbb{R}^d))} \lesssim \|u_0\|_{M_s^{p',1}}$.

In [4, Theorem 1.1] global well-posedness for (1.5) studied with the range of $\gamma < \min\{d/2, 2\}$. Notice that Theorem 1.5 covers range of $\gamma > d/2$ as $\frac{\gamma}{d} = 1 + \frac{p-3}{2p}$ and $\gamma/d > 1/2 \Leftrightarrow p > 3/2$.

Corollary 1.6 (Scattering)**.** *Let $u_0 \in M_s^{p,1}(\mathbb{R}^d)$ and let u is the global solution to (1.5) with initial $u(0) = u_0$ such that $\|u\|_{L_t^r(\mathbb{R}, M_s^{p,1})} \le M$ for some constant $M > 0$ and $r < \infty$. Then there exist solutions $e^{it\Delta}u_\pm$ to the free Schrödinger equation $iu_t + \Delta u = 0$ such that*

$$\|u(t) - e^{it\Delta}u_\pm\|_{M_s^{p,1}} \to 0 \quad \text{as } t \to \pm\infty.$$

Remark 1.7. Taking Proposition 2.8 into account, the method of proof of Theorem 1.5 may further be applied to equation (1.5) with $\alpha > 2$ to obtain the global well-posedness for the small data in modulation spaces.

Theorem 1.8 (Global well-posedness)**.** *Let* $V \in M^{\infty,1}(\mathbb{R}^d)$ *and* $\frac{1}{2} < \alpha \leq 2$*. Assume that* $u_0 \in M^{p,q}(\mathbb{R}^d)(1 \leq p, q \leq 2)$*. Then there exists a unique global solution of* (1.5) *such that* $u \in C(\mathbb{R}, M^{p,q}(\mathbb{R}^d))$*.*

In [19, Theorem 1.2] it is proved that (1.5) with potential $V \in M^{\infty,1}(\mathbb{R}^d)$ and $\alpha = 2$ is globally well-posed in $M^{p,q}(\mathbb{R}^d)(1 \leq q \leq p \leq 2)$. Notice that Theorem 1.8 generalize this result for (1.5) with $\frac{1}{2} < \alpha < 2$.

Up to now we cannot know (1.5) is locally well-posed in $L^p(\mathbb{R}^d)$ but, by Theorem 1.9, in $M^{p,1}(\mathbb{R}^d)$. Local well-posedness for (1.5) are studied by many authors in Sobolev spaces. Modulation spaces enjoy lower derivative regularity (see Proposition 2.4 below) and we can solve (1.5) with the lower regularity assumption for the Cauchy data.

Theorem 1.9 (Local well-posedness)**.** *Let* V *is given by* (1.3)*,* $1/2 < \alpha \leq 2$ *and* $u_0 \in M_s^{p,1}(\mathbb{R}^d)$ $(1 < p < \infty, s \in \mathbb{R}, \frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{p+\epsilon}, \epsilon > 0)$*. Then there exists* $T^* = T^*(\|u_0\|_{M_s^{p,1}})$ *such that* (1.5) *has a unique solution* $u \in C([0, T^*), M_s^{p,1}(\mathbb{R}^d))$*. Moreover, if* $T^* < \infty$*, then* $\limsup_{t \to T^*} \|u(\cdot, t)\|_{M_s^{p,1}} = \infty$*.*

Remark 1.10.

(1) The analogue of Theorem 1.9 holds for the generalized equation (1.5) and (1.2), that is, fractional Schrödinger equation with nonlinearity $(V * |u|^{2k})u$ $(k \in \mathbb{N})$ when $X = M_s^{p,1}(\mathbb{R}^d)$.

(2) We have obtain local well-posedness for generalized equations (1.1), (1.2) and (1.5) with potential $V \in \mathcal{F}L^q(\mathbb{R}^d)$ $(1 < q < \infty)$ or $M^{\infty,1}(\mathbb{R}^d)$ or $V \in M^{1,\infty}(\mathbb{R}^d)$. See Theorems 6.1 and Remark 6.2 below.

The remainder of this paper is organized as follows. In Section 2, we introduce notations and preliminaries which will be used in the sequel. In Section 3, we prove some Strichartz type estimates and boundedness of Hartree nonlinearity in modulation spaces. In Section 4, we prove Theorems 1.1, 1.3 and Corollary 1.2. In Section 5, we prove Theorems 1.5, 1.8 and 1.9, and Corollary 1.6. In Section 6, we give sketch proof of Remark 1.10 (2).

2. Preliminaries

2.1. **Notation.** The notation $A \lesssim B$ means $A \leq cB$ for a some constant $c > 0$, whereas $A \asymp B$ means $c^{-1}A \leq B \leq cA$ for some $c \geq 1$ and $a \wedge b = \min\{a, b\}$. The symbol $A_1 \hookrightarrow A_2$ denotes the continuous embedding of the topological linear space A_1 into A_2. The $L^p(\mathbb{R}^d)$ norm is denoted by

$$\|f\|_{L^p} = \Big(\int_{\mathbb{R}^d} |f(x)|^p dx \Big)^{1/p} \quad (1 \leq p < \infty),$$

the $L^\infty(\mathbb{R}^d)$ norm is $\|f\|_{L^\infty} = \text{ess.sup}_{x \in \mathbb{R}^d} |f(x)|$. For $1 \leq p \leq \infty$, p' denotes the Hölder conjugate of p, that is, $1/p + 1/p' = 1$. We use $L_t^r(I, X)$ to denote the space time norm

$$\|u\|_{L_t^r(I,X)} = \Big(\int_I \|u\|_X^r dt \Big)^{1/r},$$

where $I \subset \mathbb{R}$ is an interval and X is a Banach space. The Schwartz space is denoted by $\mathcal{S}(\mathbb{R}^d)$ (with it's usual topology), and the space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$. For $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbb{R}^d$, we put $x \cdot y = \sum_{i=1}^d x_i y_i$. Let $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ be the Fourier transform defined by

$$\mathcal{F}f(w) = \widehat{f}(w) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i t \cdot w} dt, \quad w \in \mathbb{R}^d.$$

Then $\mathcal{F}$ is a bijection and the inverse Fourier transform is given by

$$\mathcal{F}^{-1} f(x) = f^{\vee}(x) = \int_{\mathbb{R}^d} f(w)\, e^{2\pi i x \cdot w} dw, \quad x \in \mathbb{R}^d,$$

and this Fourier transform can be uniquely extended to $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. The Fourier-Lebesgue spaces $\mathcal{F}L^p(\mathbb{R}^d)$ is defined by

$$\mathcal{F}L^p(\mathbb{R}^d) = \big\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\mathcal{F}L^p} := \|\hat{f}\|_{L^p} < \infty \big\}.$$

The standard Sobolev spaces $W^{s,p}(\mathbb{R}^d)$ $(1 < p < \infty, s \geq 0)$ have a different character according to whether s is integer or not. Namely, for s integer, they consist of L^p-functions with derivatives in L^p up to order s, hence coincide with the L^p_s-Sobolev spaces (also known as Bessel potential spaces), defined for $s \in \mathbb{R}$ by

$$L^p_s(\mathbb{R}^d) = \big\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{L^p_s} := \|\mathcal{F}^{-1}[\langle \cdot \rangle^s \mathcal{F}(f)]\|_{L^p} < \infty \big\},$$

where $\langle \xi \rangle^s = (1 + |\xi|^2)^{s/2}$ $(\xi \in \mathbb{R}^d)$. Note that $L^p_{s_1}(\mathbb{R}^d) \hookrightarrow L^p_{s_2}(\mathbb{R}^d)$ if $s_2 \leq s_1$.

2.2. **Modulation spaces.** Feichtinger [13] introduced a class of Banach spaces, the so called modulation spaces, which allow a measurement of space variable and Fourier transform variable of a function or distribution on $\mathbb{R}^d$ simultaneously using the short-time Fourier transform(STFT). The STFT of a function f with respect to a window function $g \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$V_g f(x, w) = \int_{\mathbb{R}^d} f(t) \overline{g(t - x)} e^{-2\pi i w \cdot t} dt, \quad (x, w) \in \mathbb{R}^{2d}$$

whenever the integral exists. For $x, y \in \mathbb{R}^d$ the translation operator T_x and the modulation operator M_y are defined by $T_x f(t) = f(t - x)$ and $M_y f(t) = e^{2\pi i y \cdot t} f(t)$. In terms of these operators the STFT may be expressed as

$$V_g f(x, y) = \langle f, M_y T_x g \rangle$$

where $\langle f, g \rangle$ denotes the inner product for L^2 functions, or the action of the tempered distribution f on the Schwartz class function g. Thus $V : (f, g) \to V_g(f)$ extends to a bilinear form on $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ and $V_g(f)$ defines a uniformly continuous function on $\mathbb{R}^d \times \mathbb{R}^d$ whenever $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$.

Definition 2.1 (modulation spaces)**.** Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and $0 \neq g \in \mathcal{S}(\mathbb{R}^d)$. The weighted modulation space $M^{p,q}_s(\mathbb{R}^d)$ is defined to be the space of all tempered distributions f for which the following norm is finite:

$$\|f\|_{M^{p,q}_s} = \Big(\int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |V_g f(x, y)|^p dx \Big)^{q/p} (1 + |y|^2)^{sq/2}\, dy \Big)^{1/q},$$

for $1 \leq p, q < \infty$. If p or q is infinite, $\|f\|_{M^{p,q}_s}$ is defined by replacing the corresponding integral by the essential supremum.

For $s = 0$, we write $M^{p,q}_0(\mathbb{R}^d) = M^{p,q}(\mathbb{R}^d)$.

Remark 2.2. The definition of the modulation space given above, is independent of the choice of the particular window function. See [14, Proposition 11.3.2(c)].

Applying the frequency-uniform localization techniques, one can get an equivalent definition of modulation spaces [29] as follows. Let Q_k be the unit cube with the center at k, so $\{Q_k\}_{k\in\mathbb{Z}^d}$ constitutes a decomposition of $\mathbb{R}^d$, that is, $\mathbb{R}^d = \cup_{k\in\mathbb{Z}^d} Q_k$. Let $\rho \in \mathcal{S}(\mathbb{R}^d)$, $\rho : \mathbb{R}^d \to [0,1]$ be a smooth function satisfying $\rho(\xi) = 1$ if $|\xi|_\infty \le \frac{1}{2}$ and $\rho(\xi) = 0$ if $|\xi|_\infty \ge 1$. Let ρ_k be a translation of ρ, that is,

$$\rho_k(\xi) = \rho(\xi - k), \quad k \in \mathbb{Z}^d.$$

Denote

$$\sigma_k(\xi) = \frac{\rho_k(\xi)}{\sum_{l\in\mathbb{Z}^d} \rho_l(\xi)}, \quad k \in \mathbb{Z}^d.$$

Then $\{\sigma_k(\xi)\}_{k\in\mathbb{Z}^d}$ satisfies the following

$$\begin{gathered} |\sigma_k(\xi)| \ge c, \quad \forall z \in Q_k, \\ \operatorname{supp} \sigma_k \subset \{\xi : |\xi - k|_\infty \le 1\}, \\ \sum_{k\in\mathbb{Z}^d} \sigma_k(\xi) \equiv 1, \quad \forall \xi \in \mathbb{R}^d, \\ |D^\alpha \sigma_k(\xi)| \le C_{|\alpha|}, \quad \forall \xi \in \mathbb{R}^d, \alpha \in (\mathbb{N} \cup \{0\})^d. \end{gathered}$$

The frequency-uniform decomposition operators can be exactly defined by

$$\square_k = \mathcal{F}^{-1} \sigma_k \mathcal{F}.$$

For $1 \le p, q \le \infty$, $s \in \mathbb{R}$, it is known [13] that

$$\|f\|_{M_s^{p,q}} \asymp \Big(\sum_{k\in\mathbb{Z}^d} \|\square_k(f)\|_{L^p}^q (1 + |k|)^{sq} \Big)^{1/q},$$

with natural modifications for $p, q = \infty$. We notice almost orthogonality relation for the frequency-uniform decomposition operators

$$\square_k = \sum_{\|\ell\|_\infty \le 1} \square_{k+\ell} \square_k, \quad k, \ell \in \mathbb{Z}^d,$$

where $\|\ell\|_\infty = \max\{|\ell_i| : \ell_i \in \mathbb{Z}, i = 1, \dots, d\}$.

Lemma 2.3 ([30, 14, 27]). *Let $p, q, p_i, q_i \in [1, \infty]$ $(i = 1, 2)$, $s, s_1, s_2 \in \mathbb{R}$. Then*

(1) $M_{s_1}^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M_{s_2}^{p_2,q_2}(\mathbb{R}^d)$ *whenever* $p_1 \le p_2$ *and* $q_1 \le q_2$ *and* $s_2 \le s_1$.
(2) $M^{p,q_1}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p,q_2}(\mathbb{R}^d)$ *holds for* $q_1 \le min\{p, p'\}$ *and* $q_2 \ge max\{p, p'\}$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$.
(3) $M^{\min\{p',2\},p}(\mathbb{R}^d) \hookrightarrow \mathcal{F}L^p(\mathbb{R}^d) \hookrightarrow M^{\max\{p',2\},p}(\mathbb{R}^d)$, $\frac{1}{p} + \frac{1}{p'} = 1$.
(4) $\mathcal{S}(\mathbb{R}^d)$ *is dense in* $M^{p,q}(\mathbb{R}^d)$ *if* p *and* $q < \infty$.
(5) $M^{p,p}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p,p'}(\mathbb{R}^d)$ *for* $1 \le p \le 2$ *and* $M^{p,p'}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow M^{p,p}(\mathbb{R}^d)$ *for* $2 \le p \le \infty$.
(6) *The Fourier transform* $\mathcal{F} : M_s^{p,p}(\mathbb{R}^d) \to M_s^{p,p}(\mathbb{R}^d)$ *is an isomorphism.*
(7) *The space* $M_s^{p,q}(\mathbb{R}^d)$ *is a Banach space.*
(8) *The space* $M_s^{p,q}(\mathbb{R}^d)$ *is invariant under complex conjugation.*

Theorem 2.4 ([18, 27]). *Let $1 \le p, q \le \infty$, $s_1, s_2 \in \mathbb{R}$, and*

$$\tau(p, q) = \max\big\{0, d(\frac{1}{q} - \frac{1}{p}), d(\frac{1}{q} + \frac{1}{p} - 1)\big\}.$$

Then $L^p_{s_1}(\mathbb{R}^d) \subset M^{p,q}_{s_2}(\mathbb{R}^d)$ *if and only if one of the following conditions is satisfied:*

(i) $q \geq p > 1$, $s_1 \geq s_2 + \tau(p,q)$;
(ii) $p > q$, $s_1 > s_2 + \tau(p,q)$;
(iii) $p = 1, q = \infty$, $s_1 \geq s_2 + \tau(1,\infty)$;
(iv) $p = 1$, $q \neq \infty$, $s_1 > s_2 + \tau(1,q)$.

Proposition 2.5 (Algebra property [2])**.** *Let* $m \in \mathbb{N}, s \geq 0$. *Assume that* $\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p_0}$, $\sum_{i=1}^m \frac{1}{q_i} = m - 1 + \frac{1}{q_0}$ *with* $0 < p_i \leq \infty, 1 \leq q_i \leq \infty$ *for* $1 \leq i \leq m$. *Then we have*

$$\|\prod_{i=1}^m u_i\|_{M^{p_0,q_0}_s} \lesssim \prod_{i=1}^m \|u_i\|_{M^{p_i,q_i}_s}.$$

Proposition 2.6 (isomorphism [13])**.** *Let* $0 < p, q \leq \infty, s, \sigma \in \mathbb{R}$. *Then* $J_\sigma : (I - \Delta)^{\sigma/2} : M^{p,q}_s(\mathbb{R}^d) \to M^{p,q}_{s-\sigma}(\mathbb{R}^d)$ *is an isomorphic mapping. (We denote* $J_1 = J$.*)*

Lemma 2.7. *Let* $s \in \mathbb{R}, 1 \leq p, q < \infty$, *and* Ω *be a compact subset of* $\mathbb{R}^d$. *Then* $\mathcal{S}^\Omega = \{f : f \in \mathcal{S}(\mathbb{R}^d)$ *and supp* $\hat{f} \subset \Omega\}$ *is dense in* $M^{p,q}_s(\mathbb{R}^d)$.

For $f \in \mathcal{S}(\mathbb{R}^d)$, we define the fractional Schrödinger propagator $e^{it(-\Delta)^{\alpha/2}}$ for $t, \alpha \in \mathbb{R}$ as follows:

$$U(t)f(x) = e^{it(-\Delta)^{\alpha/2}} f(x) = \int_{\mathbb{R}^d} e^{i\pi t|\xi|^\alpha} \hat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi.$$

When $\alpha = 2$, we write $U(t) = S(t) = e^{-it\Delta}$ (corresponding to usual Schrödinger equation). The next proposition shows that the uniform boundedness and truncated decay estimates of the Schrödinger propagator $e^{it(-\Delta)^{\alpha/2}}$ on modulation spaces.

Proposition 2.8 ([8, 29])**.**

(1) *Let* $1/2 < \alpha \leq 2$, $1 \leq p, q \leq \infty$. *Then* $\|U(t)f\|_{M^{p,q}} \leq (1+|t|)^{d|\frac{1}{p}-\frac{1}{2}|}\|f\|_{M^{p,q}}$.
(2) *Let* $\alpha \geq 2$ *and* $2 \leq p, q \leq \infty$. *Then* $\|U(t)f\|_{M^{p,q}} \leq (1+|t|)^{-\frac{2d}{\alpha}(\frac{1}{2}-\frac{1}{p})}\|f\|_{M^{p',q}}$.

Now we consider the truncated decay estimate and uniform bounded estimates for the Klein-Gordon semigroup $G(t)$.

Proposition 2.9 (See [29, Proposition 4.2])**.** *Let* $G(t) = e^{it(I-\Delta)^{1/2}}$ $(t \in \mathbb{R})$.

(1) *Let* $s \in \mathbb{R}, 2 \leq p \leq \infty$, $1 \leq q < \infty, \theta \in [0,1]$, *and* $2\sigma(p) = (d+2)(\frac{1}{2} - \frac{1}{p})$. *Then we have*

$$\|G(t)f\|_{M^{p,q}_s} \lesssim (1+|t|)^{-d\theta(1/2-1/p)}\|f\|_{M^{p',q}_{s+\theta 2\sigma(p)}}.$$

(2) *Let* $s \in \mathbb{R}$ *and* $1 \leq p, q \leq \infty$. *Then we have*

$$\|G(t)f\|_{M^{p,q}_s} \leq C(1+|t|)^{d|1/2-1/p|}\|f\|_{M^{p,q}_s}.$$

Proposition 2.10 (Uniform boundedness of wave propagator [2])**.** *For* $\sigma^1(\xi) = \sin(2\pi t|\xi|)/2\pi|\xi|$, $\sigma^2(\xi) = \cos(2\pi t|\xi|)$, *and* $f \in \mathcal{S}(\mathbb{R}^d)$, *we define* $H_{\sigma^i}f(x) = (\sigma^i \hat{f})^\vee(x)$ $(x \in \mathbb{R}^d, i = 1,2)$. *Let* $s \in \mathbb{R}$ *and* $1 \leq p, q \leq \infty$. *Then we have*

$$\|H_{\sigma^i}f\|_{M^{p,q}_s} \leq c_d(1+t^2)^{d/4}\|f\|_{M^{p,q}_s}.$$

Proposition 2.11 (Bernstein multiplier theorem [30])**.** *Let $L \in \mathbb{Z}$, $L > d/2$, $\partial_{x_i}^{\alpha}\rho \in L^2$, $i = 1, 2, \dots, d$, $0 \le \alpha \le L$. Then ρ is a multiplier on L^p $(1 \le p \le \infty)$. Moreover there exists a constant C such that*

$$\|\rho\|_{M_p} \le C\|\rho\|_{L^2}^{1-d/2L}\Big(\sum_{i=1}^{d}\|\partial_{x_i}^{L}\rho\|_{L^2}\Big)^{d/2L}.$$

Proposition 2.12 ([30])**.** *Let $\Omega \subset \mathbb{R}^d$ be a compact subset and let $1 \le p \le \infty$, $s_p = d(\frac{1}{p\wedge 1} - \frac{1}{2})$. If $s > s_p$, then there exists a $C > 0$ such that $\|\mathcal{F}^{-1}\phi\mathcal{F}\phi\|_{L^p} \le C\|\phi\|_{H^s}\|f\|_{L^p}$ holds for all $f \in L^p\Omega$ and $\phi \in H^s(\mathbb{R}^d) = L_s^2(\mathbb{R}^d)$.*

3. Nonlinear estimates in $M_s^{p,q}(\mathbb{R}^d)$

In this section we prove estimates for Hartree nonlinearity (Corollary 3.3 and Lemmas 3.4 and 3.5) and Strichartz type estimates (Proposition 3.6). We shall apply these to prove main theorems in the following sections.

We define fractional integral operator $T_\gamma(0 < \gamma < d)$ as follows

$$T_\gamma f(x) = V_\gamma * f(x) = \pm\int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^\gamma}dy, \quad (f \in \mathcal{S}(\mathbb{R}^d),\ V_\gamma(x) = \pm|x|^{-\gamma}).$$

It is known T_γ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ for some specific p, q and γ.

Proposition 3.1 (Hardy-Littlewood-Sobolev inequality)**.** *Assume that $0 < \gamma < d$ and $1 < p < q < \infty$ with $\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{q}$. Then we have $\|T_\gamma f\|_{L^q} \le C_{d,\gamma,p}\|f\|_{L^p}$.*

We prove an analogue of Hardy-Littlewood-Sobolev inequality in case of modulation spaces.

Proposition 3.2. *Assume that $0 < \gamma < d, 1 < p_1 < p_2 < \infty$ with*

$$\frac{1}{p_1} + \frac{\gamma}{d} - 1 = \frac{1}{p_2}$$

and $1 \le q \le \infty$, $s \ge 0$. Then the map T_γ is bounded from $M_s^{p_1,q}(\mathbb{R}^d)$ to $M_s^{p_2,q}(\mathbb{R}^d)$:

$$\|T_\gamma f\|_{M_s^{p_2,q}} \lesssim \|f\|_{M_s^{p_1,q}}.$$

Proof. We may rewrite the STFT as $V_g(x,w) = e^{-2\pi i x\cdot w}(f * M_w g^*)(x)$ where $g^*(y) = \overline{g(-y)}$. Using Hardy-Littlewood-Sobolev inequality, we obtain

$$\begin{aligned}\|T_\gamma f\|_{M_s^{p_2,q}} &= \|\,\|V_\gamma * (f * M_w g^*)\|_{L^{p_2}}\langle w\rangle^s\|_{L_w^q}\\ &\lesssim \|\,\|f * M_w g^*)\|_{L^{p_1}}\langle w\rangle^s\|_{L_w^q}\\ &\lesssim \|f\|_{M_s^{p_1,q}}.\end{aligned}$$

This completes the proof. □

Corollary 3.3. *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{p+\epsilon}$ for some $\epsilon > 0$. Then*

$$\|(V_\gamma * |f|^{2k})f\|_{M_s^{p,1}} \lesssim \|f\|_{M_s^{p,1}}^{2k+1} \quad (k \in \mathbb{N}).$$

Proof. By Proposition 2.5 and Lemma 2.3(1), we have

$$\|(V_\gamma * |f|^{2k})f\|_{M_s^{p,1}} \lesssim \|T_\gamma|f|^{2k}\|_{M_s^{\infty,1}}\|f\|_{M_s^{p,1}} \lesssim \|T_\gamma|f|^{2k}\|_{M_s^{p+\epsilon,1}}\|f\|_{M_s^{p,1}},$$

for some $\epsilon > 0$. By Propositions 3.2 and 2.5, we have

$$\|T_\gamma|f|^{2k}\|_{M_s^{p+\epsilon,1}} \lesssim \||f|^{2k}\|_{M_s^{p,1}} \lesssim \|f\|_{M_s^{p,1}}^{2k}.$$

This completes the proof. □

Lemma 3.4. *Let* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{p+\epsilon}$ *for some* $\epsilon > 0$. *Then we have*

$$\|(V_\gamma * |f|^2)f - (V_\gamma * |g|^2)g\|_{M_s^{p,1}} \lesssim (\|f\|^2_{M_s^{p,1}} + \|f\|_{M_s^{p,1}}\|g\|_{M_s^{p,1}} + \|g\|^2_{M_s^{p,1}})\|f-g\|_{M_s^{p,1}}.$$

Proof. Using the ideas of proof as in Corollary 3.3, we obtain

$$\|(V_\gamma * |f|^2)(f-g)\|_{M_s^{p,1}} \lesssim \|f\|^2_{M_s^{p,1}}\|f-g\|_{M_s^{p,1}},$$

and

$$\begin{aligned}\|(V_\gamma * (|f|^2 - |g|^2))g\|_{M_s^{p,1}} &\lesssim \||f|^2 - |g|^2\|_{M_s^{p,1}}\|g\|_{M_s^{p,1}} \\ &\lesssim \Big(\|f\|_{M_s^{p,1}}\|g\|_{M_s^{p,1}} + \|g\|^2_{M_s^{p,1}}\Big)\|f-g\|_{M_s^{p,1}}.\end{aligned}$$

This together with the following identity

$$(V_\gamma * |f|^2)f - (V_\gamma * |g|^2)g = (V_\gamma * |f|^2)(f-g) + (V_\gamma * (|f|^2 - |g|^2))g,$$

gives the desired inequality. □

Lemma 3.5. *Let* $2 < p < 2p'$ *and* $\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{2p'}$. *Then we have*

$$\|(V_\gamma * |f|^2)f - (V_\gamma * |g|^2)g\|_{M_s^{p',1}} \lesssim (\|f\|^2_{M_s^{p,1}} + \|f\|_{M_s^{p,1}}\|g\|_{M_s^{p,q}} + \|g\|^2_{M_s^{p,1}})\|f-g\|_{M_s^{p,1}}.$$

Proof. By Proposition 2.5, we have

$$\begin{aligned}\|(V_\gamma * |f|^2)(f-g)\|_{M_s^{p',1}} &\lesssim \|V_\gamma * |f|^2\|_{M_s^{2p',1}}\|f-g\|_{M_s^{2p',1}} \\ &\lesssim \||f|^2\|_{M_s^{p,1}}\|f-g\|_{M_s^{p,1}}\end{aligned}$$

and

$$\begin{aligned}\|(V_\gamma * (|f|^2 - |g|^2))g\|_{M_s^{p',1}} &\lesssim \|V_\gamma * (|f|^2 - |g|^2)\|_{M_s^{2p',1}}\|g\|_{M_s^{2p',1}} \\ &\lesssim \||f|^2 - |g|^2\|_{M_s^{p,1}}\|g\|_{M_s^{p,1}} \\ &\lesssim (\|f\|_{M_s^{p,1}}\|g\|_{M_s^{p,1}} + \|g\|^2_{M_s^{p,1}})\|f-g\|_{M_s^{p,1}}.\end{aligned}$$

□

Recall that equation (1.1) have the following equivalent form

$$u(t) = K'(t)u_0 + K(t)u_1 - \mathcal{B}f(u),$$

where we denote $\omega = (I - \Delta)$,

$$K(t) = \frac{\sin t\omega^{1/2}}{\omega^{1/2}}, \quad K'(t) = \cos t\omega^{1/2}, \quad \mathcal{B} = \int_0^t K(t-\tau)\cdot d\tau.$$

We prove following Strichartz type estimates in modulation spaces.

Proposition 3.6. *Let* $F(u) = (V_\gamma * |u|^2)u, p \in (2,3)$, $\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{2p'}$ *and pair* (p, r) *is Klein-Gordon admissible. Then we have*

$$\Big\|\int_0^t K(t-\tau)F(u(\tau))d\tau\Big\|_{L_t^r(\mathbb{R}, M_s^{p,1})} \lesssim \|F(u)\|_{L_t^{r/3}(\mathbb{R}, M_s^{p',1})} \lesssim \|u\|^3_{L_t^r(\mathbb{R}, M_s^{p,1})}.$$

Proof. Since $G(t) = e^{it\omega^{1/2}}$, we have $K(t)\omega^{1/2} = (G(t) - G(-t))/2i$. By the general Minkowski inequality, Propositions 2.9 and 2.6, we have

$$\Big\|\int_0^t K(t-\tau)F(u(\tau))d\tau\Big\|_{L_t^r(\mathbb{R}, M_s^{p,1})}$$

$$
\begin{aligned}
&\lesssim \Big\| \int_0^t \|K(t-\tau)F(u(\tau))\|_{M_s^{p,1}} d\tau \Big\|_{L_t^r(\mathbb{R})} \\
&\lesssim \Big\| \int_0^t (1+|t-\tau|)^{-d\theta(1/2-1/p)} \|F(u)\|_{M^{p',1}_{s+\theta 2\sigma(p)-1}} d\tau \Big\|_{L_t^r(\mathbb{R})} \\
&\lesssim \Big\| \int_{\mathbb{R}} (1+|t-\tau|)^{-d\theta(1/2-1/p)} h(\tau) d\tau \Big\|_{L_t^r(\mathbb{R})} \\
&\lesssim \|g * h\|_{L_t^r},
\end{aligned}
$$

where $h(\tau) = \|F(u)\|_{M^{p',1}_{s+\theta 2\sigma(p)-1}}$, $g(t) = (1+|t|)^{-d\theta(1/2-1/p)}$ and $\theta \in [0,1]$. We divide Klein-Gordon admissible pairs (see (1.4)) into two cases.

Case I: $\frac{1}{\beta} = \frac{d}{d+2} \wedge d(\frac{1}{2} - \frac{1}{p})$. In this case $\frac{1}{\beta} < 1$ and there exists $\theta \in (0,1]$ such that

$$
\frac{1}{\beta} = \theta d(\frac{1}{2} - \frac{1}{p}) = \frac{d}{d+2} \wedge d(\frac{1}{2} - \frac{1}{p}).
$$

With this θ, we have $\theta 2\sigma(p) - 1 \leq 0$. Since pair (p,r) is Klein-Gordon admissible, we have

$$
\frac{1}{r} = \frac{3}{r} - \frac{1 - d\theta(1/2-1/p)}{1}
$$

and $r/3 > 1$. With this θ, by Hardy-Littlewood-Sobolev inequality in dimension one, we have

$$
\begin{aligned}
\Big\| \int_0^t K(t-\tau)F(u(\tau)) d\tau \Big\|_{L_t^r(\mathbb{R}, M_s^{p,1})} &\lesssim \|g * h\|_{L_t^r(\mathbb{R})} \\
&\lesssim \big\| \|F(u)\|_{M_s^{p',1}} \big\|_{L^{r/3}} \\
&= \|F(u)\|_{L^{r/3}(\mathbb{R}, M_s^{p',1})}.
\end{aligned}
$$

Case II: $\frac{1}{\beta} < \frac{d}{d+2} \wedge d(\frac{1}{2} - \frac{1}{p})$. In this case there exists $\theta \in [0,1]$ such that

$$
\frac{1}{\beta} < \theta d(\frac{1}{2} - \frac{1}{p}) \leq \frac{d}{d+2} \wedge d(\frac{1}{2} - \frac{1}{p}).
$$

With this θ, we have $\beta\theta d(\frac{1}{2} - \frac{1}{p}) > 1$, and $\theta 2\sigma(p) - 1 \leq 0$. By Young and Hölder inequalities, we have

$$
\begin{aligned}
\Big\| \int_0^t K(t-\tau)F(u(\tau)) d\tau \Big\|_{L_t^r(\mathbb{R}, M_s^{p,1})} &\lesssim \|g * h\|_{L_t^r} \\
&\lesssim \|g\|_{L^\beta} \big\| \|F(u)\|_{M_s^{p',1}} \big\|_{L^{r/3}} \\
&\lesssim \|F(u)\|_{L^{r/3}(\mathbb{R}, M_s^{p',1})}.
\end{aligned}
$$

By Propositions 2.5 and 3.2 and Lemma 2.3 (1), we have

$$
\begin{aligned}
\|F(u)\|_{L^{r/3}(\mathbb{R}, M_s^{p',1})} &\lesssim \Big(\int (\|T_\gamma |u|^2\|_{M_s^{2p',1}} \|u\|_{M_s^{2p',1}})^{r/3} dt \Big)^{3/r} \\
&\lesssim \Big(\int (\||u|^2\|_{M_s^{p,1}} \|u\|_{M_s^{p,1}})^{r/3} dt \Big)^{3/r} \\
&\lesssim \Big(\int \|u\|^r_{M_s^{p,1}} dt \Big)^{3/r} \\
&\lesssim \|u\|^3_{L_t^r(\mathbb{R}, M_s^{p,1})}.
\end{aligned}
$$

This completes the proof. □

Lemma 3.7. *Let* $F(u) = (V_\gamma * |u|^2)u, p \in (2,3)$, $\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{2p'}$ *and pair* (p,r) *is Klein-Gordon admissible. Then*

$$\begin{aligned}&\|\int_0^t K(t-\tau)[F(u(\tau)) - F(v(\tau))]d\tau\|_{L_t^r(\mathbb{R},M_s^{p,1})}\\&\lesssim (\|u\|^2_{L_t^r(\mathbb{R},M_s^{p,1})} + \|u\|_{L_t^r(\mathbb{R},M_s^{p,1})}\|v\|_{L_t^r(\mathbb{R},M_s^{p,1})} + \|v\|^2_{L_t^r(\mathbb{R},M_s^{p,1})})\|u-v\|^3_{L_t^r(\mathbb{R},M_s^{p,1})}.\end{aligned}$$

Proof. By Proposition 3.6, we have

$$\|\int_0^t K(t-\tau)[F(u(\tau)) - F(v(\tau))]d\tau\|_{L_t^r(\mathbb{R},M_s^{p,1})} \lesssim \|F(u) - F(v))\|_{L_t^{r/3}(\mathbb{R},M_s^{p',1})}.$$

By Proposition 2.5, Lemma 2.3(1) and Hölder inequality, we obtain

$$\|(V_\gamma * |u|^2)(u-v)\|_{L^{r/3}(\mathbb{R},M_s^{p',1})} \lesssim \|u\|^2_{L^r(\mathbb{R},M_s^{p,1})}\|u-v\|_{L^r(\mathbb{R},M_s^{p,1})}$$

and

$$\begin{aligned}&\|(V_\gamma * (|u|^2 - |v|^2))v\|_{L^{r/3}(\mathbb{R},M_s^{p',1})}\\&\lesssim \Big(\|u\|_{L^r(\mathbb{R},M_s^{p,1})}\|v\|_{L^r(\mathbb{R},M_s^{p,1})} + \|v\|^2_{L^r(\mathbb{R},M_s^{p,1})}\Big)\|u-v\|_{L^r(\mathbb{R},M_s^{p,1})}.\end{aligned}$$

□

Lemma 3.8 ([4]). *Let* V *be given by* (1.3), $1 \le p \le 2, 1 \le q < \frac{2d}{d+\gamma}$. *Then for any* $f, g \in M^{p,q}(\mathbb{R}^d)$, *we have*

(1) $\|(V * |f|^2)f\|_{M^{p,q}} \lesssim \|f\|^3_{M^{p,q}}$.
(2)

$$\|(V*|f|^2)f - (K*|g|^2)g\|_{M^{p,q}} \lesssim (\|f\|^2_{M^{p,q}} + \|f\|_{M^{p,q}}\|g\|_{M^{p,q}} + \|g\|^2_{M^{p,q}})\|f-g\|_{M^{p,q}}.$$

4. Proofs of theorems 1.1 and 1.3

Proof of Theorem 1.1. Recall that equation (1.1) have the equivalent form

$$u(t) = K'(t)u_0 + K(t)u_1 - \int_0^t K(t-\tau)F(u(\tau))d\tau =: \mathcal{J}(u)$$

where

$$K(t) = \frac{\sin t(I-\Delta)^{1/2}}{(I-\Delta)^{1/2}}, \quad K'(t) = \cos t(I-\Delta)^{1/2}, \quad F(u) = (V_\gamma * |u|^2)u.$$

Denote $X = L^r(\mathbb{R}, M_s^{p,1}(\mathbb{R}^d))$. For $\delta > 0$, put $B_\delta = \{u \in X : \|u\|_X \le \delta\}$ which is the closed ball of radius δ, and centered at the origin in X. Since $rd(\frac{1}{2} - \frac{1}{p}) > 1$, we have $(1+|t|)^{-d(\frac{1}{2}-\frac{1}{p})} \in L^r(\mathbb{R})$. Now by Proposition 2.9, we have

$$\|K(t)u_0\|_X \lesssim \|(1+|t|)^{-d(\frac{1}{2}-\frac{1}{p})}\|u_0\|_{M^{p',1}_{s+2\sigma(p)}}\|_{L^r} \lesssim \|u_0\|_{M^{p',1}_{s+2\sigma(p)}}.$$

By Propositions 2.9 and 2.6, we have

$$\|K'(t)u_1\|_X \lesssim \|(1+|t|)^{-d(\frac{1}{2}-\frac{1}{p})}\|u_1\|_{M^{p',1}_{s+2\sigma(p)-1}}\|_{L^r} \lesssim \|u_1\|_{M^{p',1}_{s+2\sigma(p)-1}}.$$

By Proposition 3.6, we have

$$\|\int_0^t K(t-\tau))F(u(\tau))d\tau\|_X \lesssim \|u\|^3_X.$$

Thus we have

$$\|\mathcal{J}(u)\|_X \lesssim \|u_0\|_{M^{p',1}_{s+2\sigma(p)}} + \|u_1\|_{M^{p',1}_{s+2\sigma(p)-1}} + \|u\|_X^3.$$

By Lemma 3.7, for any $u, v \in B_\delta$, we have

$$\|\mathcal{J}u - \mathcal{J}v\|_X \lesssim (\|u\|_X^2 + \|u\|_X\|v\|_X + \|v\|_X^2)\|u - v\|_X.$$

If we assume that $\delta > 0$ is sufficiently small, then $\mathcal{J} : X \to X$ is a strict contraction. Therefor $\mathcal{J}$ has a unique fixed point and we have $u \in L^r(\mathbb{R}, M^{p,1}_s(\mathbb{R}^d))$. We shall now verify this $u \in C(\mathbb{R}, M^{p,1}_s(\mathbb{R}^d)) \cap C^1(\mathbb{R}, M^{p,1}_{s-1}(\mathbb{R}^d))$ and $\|u\|_{L^r(\mathbb{R}, M^{p,1}_s(\mathbb{R}^d))} \lesssim \|u_0\|_{M^{p',1}_{s+2\sigma(p)}} + \|u_1\|_{M^{p',1}_{s+2\sigma(p)-1}}$. To prove $u \in C(\mathbb{R}, M^{p,1}_s(\mathbb{R}^d))$. It is equivalent to prove that

$$\|u(t_n, \cdot) - u(t, \cdot)\|_{M^{p,1}_s} \to 0 \tag{4.1}$$

as $t_n \to t$ for arbitrary fixed $t > 0$. We note that

$$\begin{aligned}
\|u(t_n, \cdot) - u(t, \cdot)\|_{M^{p,1}_s} &\le \|K'(t_n)u_0 - K'(t)u_0\|_{M^{p,1}_s} + \|K(t_n)u_1 - K(t)u_1\|_{M^{p,1}_s} \\
&\quad + \Big\| \int_0^{t_n} K(t_n - \tau)F(u(\tau)) - \int_0^t K(t - \tau)F(u(\tau))\Big\|_{M^{p,1}_s} \\
&= I + II + III.
\end{aligned}$$

Recall that $u_0, J^{-1}u_1 \in M^{p,1}_s(\mathbb{R}^d)$ (see Proposition 2.6). For I and II, by density Lemma 2.7, Proposition 2.9, triangle inequality, and since $G(t) = e^{it\omega^{1/2}}$ ($\omega = I - \Delta$), we only need to prove that $G(t)v \in C(\mathbb{R}, M^{p,1}_s(\mathbb{R}^d))$ for $v \in \mathcal{S}^\Omega$. By Hausdroff-Young inequality, we have

$$\begin{aligned}
\|\Box_k(G(t_n)v - G(t)v)\|_{L^p} &\lesssim \|\sigma_k(e^{it_n(1+|\xi|^2)^{1/2}} - e^{it(1+|\xi|^2)^{1/2}})\hat{v}(\xi)\|_{L^{p'}} \\
&\lesssim \|(e^{it_n(1+|\xi|^2)^{1/2}} - e^{it(1+|\xi|^2)^{1/2}})\hat{v}(\xi)\|_{L^{p'}} \to 0
\end{aligned}$$

as $t_n \to t$, by Lebesgue dominated convergence theorem. Since $\hat{v} \in \mathcal{S}^\Omega$, there exists only finite number of k such that $\Box_k(G(t_n)v - G(t)v) \neq 0$, so we have $\|G(t_n)v - G(t)v\|_{M^{p,1}_s} \to 0$ as $t_n \to t$. It follows that I and II tends to 0 as $t_n \to t$. For III, we note that

$$\begin{aligned}
III &\lesssim \Big\| \int_0^{t_n} K(t_n - \tau)F(u(\tau))d\tau - \int_0^{t_n} K(t - \tau)F(u(\tau))d\tau \Big\|_{M^{p,1}_s} \\
&\quad + \Big\| \int_0^{t_n} K(t - \tau)F(u(\tau))d\tau - \int_0^{t} K(t - \tau)F(u(\tau))d\tau \Big\|_{M^{p,1}_s} \\
&\lesssim \int_0^{t_n} \|(K(t_n - \tau) - K(t - \tau))F(u(\tau))\|_{M^{p,1}_s} d\tau \\
&\quad + \int_{t_n}^{t} \|K(t - \tau)F(u(\tau))\|_{M^{p,1}_s} d\tau \\
&= \tilde{I} + \tilde{I}I.
\end{aligned}$$

For $\|(K(t_n - \tau) - K(t - \tau))F(u(\tau)\|_{M^{p,1}_s} \lesssim \|F(u(\tau))\|_{M^{p',1}_s} \lesssim \|u\|^3_{M^{p,1}_s} \in L^r(\mathbb{R})$. Since $3 \le r$, we have $L^r[0, t] \subset L^1[0, t]$ and so $\|u\|^3_{M^{p,1}_s} \in L^1[0, t]$, hence

$$\|(K(t_n - \tau) - K(t - \tau))F(u(\tau)\|_{M^{p,1}_s} \in L^1[0, t].$$

Since $\|(K(t_n-\tau)-K(t-\tau))F(u(\tau)\|_{M_s^{p,1}} \to 0$ as $t_n \to t$, therefore we have $\tilde{I} \to 0$ as $t_n \to t$. Secondly as in the proof of Proposition 3.6, we obtain

$$\begin{aligned}
\tilde{I}I &\lesssim \int_{t_n}^{t} (1+|t-\tau|)^{-d(1/2-1/p)} \|F(u(\tau))\|_{M_s^{p',1}} d\tau \\
&\lesssim \int_{t_n}^{t} \|F(u(\tau))\|_{M_s^{p',1}} d\tau \\
&\lesssim \int_{t_n}^{t} \|u\|^3_{M_s^{p,1}} d\tau \to 0
\end{aligned}$$

as $t_n \to t$ as $\|u\|^3_{M_s^{p,1}} \in L^1([0,t])$. It follows that (4.1) holds.

We now prove that $u_t(t)$ exists and is continuous in $M_s^{p,1}$ sense. For $u_0, J^{-1}u_1 \in M_s^{p,1}(\mathbb{R}^d)$ (see Proposition 2.6), and since $G(t)=e^{it\omega^{1/2}}$ ($\omega = I-\Delta$), we should only deal with the derivative of $G(t)\psi(x)$ for $\psi \in M_s^{p,1}(\mathbb{R}^d)$ and $\int_0^t K(t-\tau)F(u(\tau))d\tau$. By Lemma 2.7, for every $\epsilon > 0$, there exists $v \in \mathcal{S}^{\Omega} \cap M_s^{p,1}(\mathbb{R}^d)$ such that $\|\psi - v\|_{M_s^{p,1}} < \epsilon$. For the derivative of $G(t)\psi(x)$ at $t=t_3$ for $\psi \in M_s^{p,1}(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\|\frac{G(t)\psi - G(t_3)\psi}{t-t_3} - i\omega^{1/2}G(t_3)\psi\|_{M_{s-1}^{p,1}} \\
&= \|\frac{G(t)\psi - G(t_3)\psi}{(t-t_3)\omega^{1/2}} - iG(t_3)\psi\|_{M_s^{p,1}} \\
&\le \|\frac{G(t)(\psi-v) - G(t_3)(\psi-v)}{(t-t_3)\omega^{1/2}}\|_{M_s^{p,1}} + \|\frac{G(t)(v)-G(t_3)(v)}{(t-t_3)\omega^{1/2}} - iG(t_3)v\|_{M_s^{p,1}} \\
&\quad + \|iG(t_3)(\psi-v)\|_{M_s^{p,1}} \\
&= IV + V + VI.
\end{aligned}$$

For V, by the Hausdroff-Young inequality and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\|\Box_k(\frac{G(t)(v)-G(t_3)(v)}{(t-t_3)\omega^{1/2}} - iG(t_3)v)\|_{L^p} &\lesssim \|\sigma_k(\frac{e^{it\langle\xi\rangle} - e^{it_3\langle\xi\rangle}}{(t-t_3)\langle\xi\rangle} - ie^{it_3\langle\xi\rangle})\hat{v}\|_{L^{p'}} \\
&\to 0 \quad \text{as } t \to t_3.
\end{aligned}$$

As $v \in \mathcal{S}^{\Omega} \cap M_s^{p,1}(\mathbb{R}^d)$, so there is only the finite number of k such that

$$\Big(\frac{G(t)(v)-G(t_3)(v)}{(t-t_3)\omega^{1/2}} - iG(t_3)v\Big) \neq 0.$$

Thus we get $V \to 0$ as $t \to t_3$, that is, $(G(t)v(x))_t = i\omega^{1/2}G(t)v(x)$ in $M_{s-1}^{p,1}(\mathbb{R}^d)$ for $v \in \mathcal{S}^{\Omega} \cap M_s^{p,1}(\mathbb{R}^d)$. For IV, by the Bernstein multiplier theorem, we have

$$\|\Box_l(\frac{G(t)(\psi-v) - G(t_3)(\psi-v)}{(t-t_3)\omega^{1/2}})\|_{L^p} \lesssim \|\psi - v\|_{L^p}.$$

Using the almost orthogonality of modulation space, we have $IV \lesssim \|\psi - v\|_{M_s^{p,1}} < \epsilon$. For VI, by Proposition 2.9 (2), we have $VI = \|iG(t_3)(\psi-v)\|_{M_s^{p,1}} \lesssim \|\psi-v\|_{M_s^{p,1}} < \epsilon$. Accordingly, for $\psi \in M_s^{p,1}(\mathbb{R}^d)$,

$$(G(t)\psi)_t = i\omega^{1/2}G(t)\psi \text{ in } M_s^{p,1}(\mathbb{R}^d). \tag{4.2}$$

For the nonlinear part,

$$
\begin{aligned}
&\Big\|\frac{\int_0^t K(t-\tau)F(u(\tau))d\tau-\int_0^{t_3} K(t_3-\tau)F(u(\tau))d\tau}{t-t_3}-\int_0^{t_3} K'(t_3-\tau)F(u)d\tau\Big\|_{M^{p,1}_{s-1}}\\
&\le \Big\|\frac{\int_0^{t_3}(K(t-\tau)-K(t_3-\tau)F(u(\tau))d\tau}{t-t_3}-\int_0^{t_3} K'(t_3-\tau)F(u)d\tau\Big\|_{M^{p,1}_{s-1}}\\
&\quad+\|\frac{\int_{t_3}^t K(t-\tau)F(u(\tau))d\tau}{t-t_3}\|_{M^{p,1}_{s-1}}\\
&\lesssim \int_0^{t_3}\|(\frac{(K(t-\tau)-K(t_3-\tau)}{t-t_3}-K'(t_3-\tau))F(u)\|_{M^{p,1}_{s-1}}d\tau\\
&\quad+\max_{\tau\in[t_3,t]}\|K(t-\tau)F(u(\tau))\|_{M^{p,1}_{s-1}}.
\end{aligned}
$$

If $\omega(t,x)\in C(I,M^{p,1}_s(\mathbb{R}^d))$, then we have $K(t)\omega(t,x)\in C(I,M^{p,1}_{s-1}(\mathbb{R}^d))$. In fact taking taking advantage of (4.2) and the Lebesgue dominated convergence theorem, we obtain

$$
\begin{aligned}
&\|K(t)\omega(t,x)-K(t_3)\omega(t_3,x)\|_{M^{p,1}_{s-1}}\\
&\le \|(K(t)-K(t_3))\omega(t_3,x)\|_{M^{p,1}_{s-1}}+\|K(t)(\omega(t,x)-\omega(t_3,x))\|_{M^{p,1}_{s-1}}\\
&\to 0\quad \text{as } t\to t_3.
\end{aligned}
$$

Recall that $F(u)\in C(\mathbb{R},M^{p,1}_s(\mathbb{R}^d))$ and apply (4.2) and the Lebesgue dominated convergence theorem, we can get

$$
\Big(\int_0^t K(t-\tau)F(u(\tau))d\tau\Big)'_t\Big|_{t=t_3}=\int_{t=0}^{t_3} K'(t_3-\tau)F(u(\tau))d\tau \text{ in } M^{p,1}_{s-1}(\mathbb{R}^d).
$$

Consequently,

$$
u_t(t)=-J^2K(t)u_0+K'(t)u_1-\int_0^t K'(t-\tau)F(u(\tau))d\tau\quad \text{in } M^{p,1}_s(\mathbb{R}^d).
$$

Next, the proof of time continuity of u_t is similar to u. It only needs to take care of the difference of smoothness and the action of the Bessel potential. Finally, we obtain $u\in C(\mathbb{R},M^{p,1}_s(\mathbb{R}^d))\cap C^1(\mathbb{R},M^{p,1}_{s-1}(\mathbb{R}^d))$. □

Proof of Corollary 1.2. Let

$$
\begin{aligned}
2v_1(t)&=u_0+\frac{u_1}{i\omega^{1/2}}-\int_0^t\frac{G(-\tau)F(u(\tau))}{i\omega^{1/2}}d\tau,\\
2v_2(t)&=u_0-\frac{u_1}{i\omega^{1/2}}+\int_0^t\frac{G(-\tau)F(u(\tau))}{i\omega^{1/2}}d\tau.
\end{aligned}
$$

For $0<s<t$, we have

$$
v_1(t)-v_1(s)=-\int_s^t\frac{G(-\tau)F(u(\tau))}{i\omega^{1/2}}d\tau.
$$

Since the pair (p,r) is Klein-Gordon admissible, there exists $\tilde\beta$ such that

$$
\frac{1}{\tilde\beta}+\frac{3}{r}=1,\quad \tilde\beta d(\frac12-\frac1p)>1.
$$

By Proposition 3.6 and Hölder's inequality, we have

$$\begin{aligned}\|v_1(t)-v_1(s)\|_{M_s^{p,1}} &\lesssim \int_s^t (1+|\tau|)^{-d(\frac{1}{2}-\frac{1}{p})}\|F(u(\tau))\|_{M_s^{p',1}}d\tau\\ &\lesssim \int_s^t (1+|\tau|)^{-d(\frac{1}{2}-\frac{1}{p})}\|u\|^3_{M_s^{p,1}}d\tau\\ &\lesssim \|(1+|\tau|)^{-d(\frac{1}{2}-\frac{1}{p})}\|_{L^{\tilde\beta}}\|\|u\|^3_{M_s^{p,1}}\|_{L^{r/3}([s,t],M_s^{p,1})}\\ &\lesssim \|u\|^3_{L^r([s,t],M_s^{p,1})}.\end{aligned}$$

Since $\|u\|_{L^r([s,t],M_s^{p,1})}\leq M$, we have

$$\|v_1(t)-v_1(s)\|_{M_s^{p,1}}\lesssim \|u\|^3_{L^r([s,t],M_s^{p,1})}\to 0 \quad \text{as } t,s\to\infty.$$

This implies that $v_1(t)$ is Cauchy in $M_s^{p,1}(\mathbb{R}^d)$ as $t\to\infty$. Denote v_1^+ to be the limit:

$$2v_1^+=\lim_{t\to+\infty}2v_1(t)=u_0+\frac{u_1}{i\omega^{1/2}}-\int_0^t\frac{G(-\tau)F(u(\tau))}{i\omega^{1/2}}d\tau$$

and

$$2v_1^-=\lim_{t\to+\infty}2v_1(t)=u_0-\frac{u_1}{i\omega^{1/2}}+\int_0^t\frac{G(-\tau)F(u(\tau))}{i\omega^{1/2}}d\tau.$$

Similarly, we obtain

$$v_2^+(t)=\lim_{t\to\infty}v_2(t)\quad\text{and}\quad v_2^-(t)=\lim_{t\to\infty}v_2(t).$$

Recall that $v^\pm=G(t)v_1^\pm+G(t)v_2^\pm$, we note that

$$\begin{aligned}\|u(t)-v^+\|_{M_s^{p,1}} &= \Big\|\int_t^\infty K(t-\tau)F(u(\tau))d\tau\Big\|_{M_s^{p,1}}\\ &\lesssim \|(1+|\tau|)^{-d(\frac{1}{2}-\frac{1}{p})}\|_{L^{\tilde\beta}}\|\,\|u\|^3_{M_s^{p,1}}\|_{L^{r/3}([t,\infty],M_s^{p,1})}\\ &\lesssim \|u\|^3_{L^3([t,\infty],M_s^{p,1})}\to 0\quad\text{as } t\to\infty.\end{aligned}$$

So is v^- respectively. In fact, in our proof we also have $v_1^+\in M_s^{p,1}(\mathbb{R}^d)$. □

Proof of Theorem 1.3. Equation (1.2) can be written in the equivalent form

$$u(\cdot,t)=\tilde K(t)u_0+K(t)u_1-\int_0^t K(t-\tau)[(V_\gamma*|u|^2)(\tau)u(\tau)]d\tau=:\mathcal{J}(u) \tag{4.3}$$

where

$$K(t)=\frac{\sin(t\sqrt{-\triangle})}{\sqrt{-\triangle}},\quad \tilde K(t)=\cos(t\sqrt{-\triangle}).$$

By using Proposition 2.10 for the first two inequalities below, and Propositions 3.2 and 3.8 for the last inequality, we can write

$$\begin{gathered}\|\tilde K(t)u_0\|_X\leq C_T\|u_0\|_X,\\ \|K(t)u_1\|_X\leq C_T\|u_1\|_X,\\ \Big\|\int_0^t K(t-\tau)[(V_\gamma*|u|^2)(\tau)u(\tau)]d\tau\Big\|_X\leq TC_T\|u\|_X^3,\end{gathered}\tag{4.4}$$

where C_T is some constant times $(1+T^2)^{d/4}$, as before. Thus the standard contraction mapping argument can be applied to $\mathcal{J}$ to complete the proof. This completes the proof of Theorem 1.3 (2). Taking Propositions 2.9, 3.8 and Corollary 3.3 and

Lemma 3.4 into account, the standard contraction mapping argument give the proof of Theorem 1.3 (1). □

5. Proofs of theorems 1.5, 1.8 and 1.9

To prove Theorem 1.8 first we shall prove following Strichartz type estimates for Schrödinger admissible pairs.

Proposition 5.1. *Let* $F(u) = (V_\gamma * |u|^2)u$, $p \in (2,3)$, $\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{2p'}$ *and pair* (p, r) *is Schrödinger admissible. Then we have*

$$\|\int_0^t S(t-\tau)F(u(\tau))d\tau\|_{L_t^r(\mathbb{R}, M_s^{p,1})} \lesssim \|F(u)\|_{L_t^{r/3}(\mathbb{R}, M_s^{p',1})} \lesssim \|u\|^3_{L_t^r(\mathbb{R}, M_s^{p,1})}.$$

Proof. By the general Minkowski inequality, Proposition 2.8, we have

$$\begin{aligned}
&\Big\|\int_0^t S(t-\tau)F(u(\tau))d\tau\Big\|_{L_t^r(\mathbb{R}, M_s^{p,1})}\\
&\lesssim \Big\|\int_0^t \|S(t-\tau)F(u(\tau))\|_{M_s^{p,1}}d\tau\Big\|_{L_t^r(\mathbb{R})}\\
&\lesssim \Big\|\int_0^t (1+|t-\tau|)^{-d(1/2-1/p)}\|F(u)\|_{M_s^{p',1}}d\tau\Big\|_{L_t^r(\mathbb{R})}\\
&\lesssim \Big\|\int_{\mathbb{R}} (1+|t-\tau|)^{-d(1/2-1/p)}h(\tau)d\tau\Big\|_{L_t^r(\mathbb{R})}\\
&\lesssim \|g*h\|_{L_t^r},
\end{aligned}$$

where $h(\tau) = \|F(u)\|_{M_s^{p',1}}, g(t) = (1+|t|)^{-d(1/2-1/p)}$. We divide Schroödinger admissible pairs (see (1.6)) into several cases.
Case I: $\frac{1}{\beta} < d(\frac{1}{2} - \frac{1}{p}) \wedge 1$. In this case we have

$$d\beta(\frac{1}{2} - \frac{1}{p}) > 1.$$

Using Young inequality and Hölder's inequality we have

$$\begin{aligned}
\Big\|\int_0^t S(t-\tau)F(u(\tau))d\tau\Big\|_{L_t^r(\mathbb{R}, M_s^{p,1})} &\lesssim \|g\|_{L^\beta}\|F(u)\|_{L^{r/3}(\mathbb{R}, M_s^{p',1}(\mathbb{R}^d))}\\
&\lesssim \|u\|^3_{L^r(\mathbb{R}, M_s^{p,1}(\mathbb{R}^d))}.
\end{aligned}$$

Case II: $\frac{1}{\beta} = 1 \wedge d(\frac{1}{2} - \frac{1}{p})$, $d(\frac{1}{2} - \frac{1}{p}) > 1$. In this case, we can get $\beta = 1$ and $r = \infty$. Obviously

$$d\beta(\frac{1}{2} - \frac{1}{p}) > 1,$$

and therefor, we have the desired result by the same way as Case I.
Case III: $\frac{1}{\beta} = 1 \wedge d(\frac{1}{2} - \frac{1}{p})$, $d(\frac{1}{2} - \frac{1}{p}) < 1$. In this case we have

$$d\beta(\frac{1}{2} - \frac{1}{p}) = 1.$$

Since pair (p, r) is Schrödinger admissible, we have

$$\frac{1}{r} = \frac{3}{r} - \frac{1 - d(1/2 - 1/p)}{1}$$

and $r/3 > 1$. By Hardy-Littlewood-Sobolev inequality in one dimension, we have

$$\begin{aligned}\Big\| \int_0^t K(t-\tau)F(u(\tau))d\tau \Big\|_{L_t^r(\mathbb{R}, M_s^{p,1})} &\lesssim \|g * h\|_{L_t^r(\mathbb{R})} \\ &\lesssim \big\| \|F(u)\|_{M_s^{p',1}} \big\|_{L^{r/3}} \\ &\lesssim \|F(u)\|_{L^{r/3}(\mathbb{R}, M_s^{p',1})} \\ &\lesssim \|u\|^3_{L^r(\mathbb{R}, M_s^{p,1})}.\end{aligned}$$

Case IV: $\frac{1}{\beta} = 1 \wedge d(\frac{1}{2} - \frac{1}{p})$, $d(\frac{1}{2} - \frac{1}{p}) = 1$. In this case $(p, r) = (\frac{2d}{d-2}, \infty)$ which is not Schrödinger admissible. $\square$

Lemma 5.2. *Let $F(u) = (V_\gamma * |u|^2)u$, $p \in (2,3)$, $\frac{1}{p} + \frac{\gamma}{d} - 1 = \frac{1}{2p'}$ and pair (p, r) is Schrödinger admissible. Then*

$$\begin{aligned}&\Big\| \int_0^t S(t-\tau)[F(u(\tau)) - F(v(\tau))]d\tau \Big\|_{L_t^r(\mathbb{R}, M_s^{p,1})} \\ &\lesssim (\|u\|^2_{L_t^r(\mathbb{R}, M_s^{p,1})} + \|u\|_{L_t^r(\mathbb{R}, M_s^{p,1})}\|v\|_{L_t^r(\mathbb{R}, M_s^{p,1})} + \|v\|^2_{L_t^r(\mathbb{R}, M_s^{p,1})})\|u-v\|^3_{L_t^r(\mathbb{R}, M_s^{p,1})}.\end{aligned}$$

Proof. Using Propositions 2.5 and 5.1, Lemma 2.3 (1) and Hölder inequality, the proof can be produced. We omit the details. $\square$

Proof of Theorem 1.5. For $\alpha = 2$, we may rewrite equation (1.5) in the form

$$u(t) = S(t)u_0 - \int_0^t S(t-\tau)F(u(\tau))d\tau =: \mathcal{J}(u)$$

where $S(t) = e^{-it\Delta}$ and $F(u) = (V_\gamma * |u|^2)u$. Denote $X = L^r(\mathbb{R}, M_s^{p,1}(\mathbb{R}^d))$. For $\delta > 0$, we put $B_\delta = \{u \in X : \|u\|_X \le \delta\}$ which is the closed ball of radius δ, and centered at the origin in X. Since $rd(\frac{1}{2} - \frac{1}{p}) > 1$, we have $(1+|t|)^{-d(\frac{1}{2}-\frac{1}{p})} \in L^r(\mathbb{R})$. Now by Proposition 2.8, we have

$$\|S(t)u_0\|_X \lesssim \|(1+|t|)^{-d(\frac{1}{2}-\frac{1}{p})}\|u_0\|_{M_s^{p',1}}\|_{L^r} \lesssim \|u_0\|_{M_s^{p',1}}.$$

By Proposition 5.1, we have

$$\Big\| \int_0^t S(t-\tau))F(u(\tau))d\tau \Big\|_X \lesssim \|u\|^3_X.$$

Thus

$$\|\mathcal{J}(u)\|_X \lesssim \|u_0\|_{M_s^{p',1}} + \|u\|^3_X.$$

By Lemma 5.2, for any $u, v \in B_\delta$, we have

$$\|\mathcal{J}u - \mathcal{J}v\|_X \lesssim (\|u\|^2_X + \|u\|_X\|v\|_X + \|v\|^2_X)\|u-v\|_X.$$

If we assume that $\delta > 0$ is sufficiently small, then $\mathcal{J} : X \to X$ is a strict contraction. Therefor $\mathcal{J}$ has a unique fixed point and we have $u \in L^r(\mathbb{R}, M_s^{p,1}(\mathbb{R}^d))$ and $\|u\|_{L^r(\mathbb{R}, M_s^{p,1}(\mathbb{R}^d))} \lesssim \|u_0\|_{M_s^{p',1}}$. We want to show that if $f \in M_s^{p,1}(\mathbb{R}^d)$ then $S(t)f \in C(\mathbb{R}, M_s^{p,1}(\mathbb{R}^d))$. Let $t > 0$ and $t_n \to t$. By Lemma 2.7, Proposition 2.8 and the triangle inequality, we have

$$\begin{aligned}\|S(t)f - S(t_n)f\|_{M_s^{p,1}} \le \|S(t)f - S(t)g\|_{M_s^{p,1}} + \|S(t)g - S(t_n)g\|_{M_s^{p,1}} \\ + \|S(t_n)f - S(t_n)g\|_{M_s^{p,1}}.\end{aligned}$$

We only need to treat the case $f \in \mathcal{S}^\Omega$. Using Lemma 2.12 and the Hausdroff-Young inequality, we have

$$\|\Box_k(S(t_n) - S(t))f\|_{L^p} \lesssim \|(S(t_n) - S(t))f\|_{L^p} \lesssim \|(e^{it_n|\xi|^2} - e^{it|\xi|^2})\hat{f}\|_{L^{p'}} \to 0$$

as $t_n \to t$ by the Lebesgue dominated convergence theorem. Since $f \in \mathcal{S}^\Omega$, there exist only finite number of k such that $\Box_k(S(t_n) - S(t))f \neq 0$, and thus

$$\|S(t)f - S(t_n)f\|_{M_s^{p,1}} \to 0 \quad \text{as } t_n \to t.$$

We write

$$\begin{aligned}
I &= \int_0^t S(t-\tau)F(u(\tau))d\tau - \int_0^{t_n} S(t_n-\tau)F(u(\tau))d\tau \\
&= \Big(\int_0^{t_n} S(t-\tau)F(u(\tau))d\tau - \int_0^{t_n} S(t_n-\tau)F(u(\tau))d\tau\Big) \\
&\quad + \Big(\int_0^{t} S(t-\tau)F(u(\tau))d\tau - \int_0^{t_n} S(t-\tau)F(u(\tau))d\tau\Big) \\
&= I_1 + I_2.
\end{aligned}$$

For I_2, we have

$$\begin{aligned}
\|I_2\|_{M_s^{p,1}} &\lesssim \int_{t_n}^t \|S(t-\tau)F(u)(\tau)\|_{M_s^{p,1}} d\tau \\
&\lesssim \int_{t_n}^t (1+|t-\tau|)^{-d(1/2-1/p)} \|F(u)(\tau)\|_{M_s^{p',1}} d\tau \\
&\lesssim \int_{t_n}^t \|u\|^3_{M_s^{p,1}} d\tau \\
&\lesssim |t-t_n|^\beta \|u\|^3_{L^r([0,t],M_s^{p,1})} \to 0.
\end{aligned}$$

For I_1, we have

$$\begin{aligned}
I_1 &\lesssim \int_0^{t_n} \|S(\tau)(S(t_n) - S(t))F(u(\tau))\|_{M_s^{p,1}} d\tau \\
&\lesssim \int_I \|(S(t_n) - S(t))F(u(\tau))\|_{M_s^{p,1}} d\tau.
\end{aligned}$$

We note that $\|(S(t_n) - S(t))F(u(\tau))\|_{M_s^{p,1}} \lesssim \|F(u)(\tau)\|^3_{M_s^{p,1}}$ and recalling $r \geq 3$ and $u \in L^r(\mathbb{R}, M_s^{p,1}(\mathbb{R}^d))$, we have $\|u(\tau)\|^3_{M_s^{p,1}} \in L^1[0,t]$. Since $F(u) \in M_s^{p,1}(\mathbb{R}^d)$, for every $\tau \in [0,t]$ we have $\|(S(t_n) - S(t))F(u(\tau))\|_{M_s^{p,1}} \to 0$. $\square$

Proof of Corollary 1.6. We only prove the statement for u_+, since the proof for u_- follows similarly. Let us first construct the scattering state $u_+(0)$. For $t > 0$ define $v(t) = e^{-it\Delta}u(t)$. We will show that $v(t)$ converges in $M_s^{p,1}(\mathbb{R}^d)$ as $t \to \infty$, and define u_+ to be the limit. Indeed from Duhamel's formula we have

$$v(t) = u_0 - \int_0^t e^{-i\tau\Delta}F(u(\tau))d\tau \quad (F(u) = (V_\gamma * |u|^2)u). \tag{5.1}$$

Therefore, for $0 < s < t$, we have

$$v(t) - v(s) = -i\int_s^t e^{-i\tau\Delta}F(u(\tau))d\tau.$$

Since the pair (p,r) is a Schrödinger admissible, there exists $\tilde{\beta}$ such that

$$\frac{1}{\tilde{\beta}}+\frac{3}{r}=1, \quad \tilde{\beta}d(\frac{1}{2}-\frac{1}{p})>1.$$

By Proposition 3.6 and Hölder's inequality, we have

$$\begin{aligned}
\|v(t)-v(s)\|_{M_s^{p,1}} &\lesssim \int_s^t (1+|\tau|)^{-d(\frac{1}{2}-\frac{1}{p})}\|F(u(\tau))\|_{M_s^{p',1}}d\tau \\
&\lesssim \int_s^t (1+|\tau|)^{-d(\frac{1}{2}-\frac{1}{p})}\|u\|^3_{M_s^{p,1}}d\tau \\
&\lesssim \|(1+|\tau|)^{-d(\frac{1}{2}-\frac{1}{p})}\|_{L^{\tilde{\beta}}}\| \, \|u\|^3_{M_s^{p,1}}\|_{L^{r/3}([s,t],M_s^{p,1})} \\
&\lesssim \|u\|^3_{L^r([s,t],M_s^{p,1})}.
\end{aligned}$$

Since $\|u\|_{L^r(\mathbb{R},M_s^{p,1})}\leq M$, we have

$$\|v(t)-v(s)\|_{M_s^{p,1}}\lesssim \|u\|^3_{L^r([s,t],M_s^{p,1})}\to 0 \quad \text{as } t,s\to\infty.$$

This implies that $v(t)$ is Cauchy in $M_s^{p,1}(\mathbb{R}^d)$ as $t\to\infty$. We define u_+ to be the limit. In view of (5.1), we see that

$$u_+(0)=u_0-\int_0^\infty e^{-i\tau\Delta}F(u(\tau))d\tau$$

and thus

$$u_+(t)=e^{it\Delta}u_0-\int_0^\infty e^{i(t-\tau)\Delta}F(u(\tau))d\tau.$$

We note that

$$\begin{aligned}
\|u(t)-e^{it\Delta}u_+\|_{M_s^{p,1}} &= \|\int_t^\infty S(t-\tau)F(u(\tau))d\tau\|_{M_s^{p,1}} \\
&\lesssim \|(1+|\tau|)^{-d(\frac{1}{2}-\frac{1}{p})}\|_{L^{\tilde{\beta}}}\| \, \|u\|^3_{M_s^{p,1}}\|_{L^{r/3}([t,\infty],M_s^{p,1})} \\
&\lesssim \|u\|^3_{L^r([t,\infty],M_s^{p,1})}\to 0 \quad \text{as } t\to\infty.
\end{aligned}$$

In fact, in our proof we also have $e^{it\Delta}u_0, e^{it\Delta}u_+\in M_s^{p,1}(\mathbb{R}^d)$. □

To prove Theorem 1.8 first we recall following result.

Lemma 5.3 ([5]). *Let $V\in M^{\infty,1}(\mathbb{R}^d)$, and $1\leq p,q\leq 2$. For $f\in M^{p,q}(\mathbb{R}^d)$, we have*

$$\|(V*|f|^2)f\|_{M^{p,q}}\lesssim \|f\|^3_{M^{p,q}},$$

and

$$\|(V*|f|^2)f-(V*|g|^2)g\|_{M^{p,q}}\lesssim (\|f\|^2_{M^{p,q}}+\|f\|_{M^{p,q}}\|g\|_{M^{p,q}}+\|g\|^2_{M^{p,q}})\|f-g\|_{M^{p,q}}.$$

Proof of Theorem 1.8. Recall (1.5) can be written in the equivalent form

$$u(\cdot,t)=U(t)u_0-i\int_0^t U(t-\tau)[(V*|u|^2)u]\,d\tau=:\mathcal{J}(u).$$

We first prove the local existence on $[0,T)$ for some $T>0$. By Minkowski's inequality for integrals, Proposition 2.8 and Lemma 5.3, we obtain

$$\|\int_0^t U(t-\tau)[(V*|u|^2(\tau))u(\tau)]\,d\tau\|_{M^{p,q}}\leq cT(1+|t|)^{d|\frac{1}{p}-\frac{1}{2}|}\|u(t)\|^3_{M^{p,p}},$$

for some universal constant c. By Proposition 2.8 and the above inequality, we have

$$\|\mathcal{J}u\|_{C([0,T],M^{p,q})} \le C_T(\|u_0\|_{M^{p,q}} + cT\|u\|^3_{M^{p,q}})$$

where $C_T = (1+|T|)^{d|\frac{1}{p}-\frac{1}{2}|}$. For $M>0$, put

$$B_{T,M} = \{u \in C([0,T], M^{p,q}(\mathbb{R}^d)) : \|u\|_{C([0,T],M^{p,q})} \le M\},$$

which is the closed ball of radius M, centered at the origin in $C([0,T], M^{p,q}(\mathbb{R}^d))$. Next, we show that the mapping $\mathcal{J}$ takes $B_{T,M}$ into itself for suitable choice of M and small $T>0$. Indeed, if we let, $M = 2C_T\|u_0\|_{M^{p,p}}$ and $u \in B_{T,M}$, it follows that

$$\|\mathcal{J}u\|_{C([0,T],M^{p,p})} \le \frac{M}{2} + cC_T T M^3.$$

We choose a T such that $cC_T T M^2 \le 1/2$, that is, $T \le \tilde{T}(\|u_0\|_{M^{p,p}})$ and as a consequence we have

$$\|\mathcal{J}u\|_{C([0,T],M^{p,p})} \le \frac{M}{2} + \frac{M}{2} = M,$$

that is, $\mathcal{J}u \in B_{T,M}$. By Lemma 5.3, and the arguments as before, we obtain

$$\|\mathcal{J}u - \mathcal{J}v\|_{C([0,T],M^{p,q})} \le \frac{1}{2}\|u-v\|_{C([0,T],M^{p,q})}.$$

Therefore, using Banach's contraction mapping principle, we conclude that $\mathcal{J}$ has a fixed point in $B_{T,M}$ which is a solution of (1.5).

Indeed, the solution constructed before is global in time: in view of the conservation of L^2 norm, Proposition 2.5 and Lemma 2.3, we have

$$\begin{aligned}
\|u((t)\|_{M^{p,p}} &\lesssim C_T\Big(\|u_0\|_{M^{p,q}} + \int_0^t \|V * |u(\tau)|^2\|_{M^{\infty,1}}\|u(\tau)\|_{M^{p,q}} d\tau\Big) \\
&\lesssim C_T\Big(\|u_0\|_{M^{p,q}} + \int_0^t \|V\|_{M^{\infty,1}}\| \, |u(t)|^2\|_{M^{1,\infty}}\|u(\tau)\|_{M^{p,q}} d\tau\Big) \\
&\lesssim C_T\Big(\|u_0\|_{M^{p,q}} + \int_0^t \| \, |u(t)|^2\|_{L^1}\|u(\tau)\|_{M^{p,q}} d\tau\Big) \\
&\lesssim C_T\Big(\|u_0\|_{M^{p,p}} + \|u_0\|^2_{L^2}\int_0^t \|u(\tau)\|_{M^{p,p}} d\tau\Big)
\end{aligned}$$

and by Gronwall's inequality, we conclude that $\|u(t)\|_{M^{p,q}}$ remains bounded on finite time intervals. This completes the proof. □

Proof of Theorem 1.9. Recall (1.5) can be written in the equivalent form

$$u(\cdot,t) = U(t)u_0 - i\int_0^t U(t-\tau)[(V*|u|^2)u]\,d\tau =: \mathcal{J}(u).$$

By using Proposition 2.8 and Corollary 3.3, we can write

$$\begin{gathered}
\|U(t)u_0\|_{M^{p,1}_s} \le C_T\|u_0\|_{M^{p,1}_s}, \\
\Big\|\int_0^t U(t-\tau)[(V_\gamma * |u|^2)(\tau)u(\tau)]d\tau\Big\|_X \le TC_T\|u\|^3_{M^{p,1}_s},
\end{gathered} \tag{5.2}$$

where C_T is some constant times $(1+T^2)^{d/4}$, as before. Thus the standard contraction mapping argument can be applied to $\mathcal{J}$ to complete the proof. □

6. Local well-posedness with potential $V \in \mathcal{F}L^q$ or $M^{1,\infty}$ or $M^{\infty,1}$

We consider generalized Klein-Gordon equation with Hartree type linearity:

$$u_{tt} + (I-\Delta)u = (V * |u|^{2k})u, u(0) = u_0, u_t(0) = u_1, \quad k \in \mathbb{N}. \tag{6.1}$$

When $k = 1$, equation (6.1) coincides with (1.1).

Theorem 6.1 (Local well-posedness)**.** *Let $i = 0, 1$.*

1. *Let $V \in \mathcal{F}L^q(\mathbb{R}^d)$ $(1 \le q \le \infty)$ and $u_i \in M^{1,1}(\mathbb{R}^d)$. Then there exists $T^* = T^*(\|u_i\|_{M^{1,1}})$ such that (6.1) has a unique solution $u \in C([0,T^*), M^{1,1}(\mathbb{R}^d))$.*
2. *Assume that $V \in \mathcal{F}L^q(\mathbb{R}^d)$ with $1 < q < r \le 2$, and $u_i \in M^{p,\frac{2r}{2r-1}}(\mathbb{R}^d)$. Then there exists $T^* = T^*(\|u_i\|_{M^{p,\frac{2r}{2r-1}}})$ such that (1.1) has a unique solution $u \in C([0,T^*), M^{p,\frac{2r}{2r-1}}(\mathbb{R}^d))$.*
3. *Assume that $V \in M^{\infty,1}(\mathbb{R}^d)$ and $u_i \in M^{p,q}(\mathbb{R}^d)$. Then there exists $T^* = T^*(\|u_i\|_{M^{p,q}})$ such that (1.1) has a unique solution $u \in C([0,T^*), M^{p,q}(\mathbb{R}^d))$.*
4. *Assume that $V \in M^{\infty,1}(\mathbb{R}^d)$ and $u_i \in M^{p,q}(\mathbb{R}^d)$ $(1 \le p, q \le 4,\ 1 \le q \le \frac{2^{2k-2}}{2^{2k-2}-1},\ 1 < k \in \mathbb{N})$. Then there exists $T^* = T^*(\|u_i\|_{M^{p,q}})$ such that (6.1) has a unique solution $u \in C([0,T^*), M^{p,q}(\mathbb{R}^d))$.*
5. *Assume that $V \in M^{1,\infty}(\mathbb{R}^d)$ and $u_i \in M^{p,1}(\mathbb{R}^d)$ $(1 \le p \le \infty)$. Then there exists $T^* = T^*(\|u_i\|_{M^{p,1}})$ such that (1.1) has a unique solution $u \in C([0,T^*), M^{p,q}(\mathbb{R}^d))$.*

Proof. Taking Proposition 2.9 and [5, Lemmas 4.8 and 4.9] and [20, Lemmas 4.2 and 4.3] into account, the standard fixed point argument gives the desired result. We will omit the details. □

Remark 6.2. The analogue of Theorem 6.1 is true for equations (1.2) and (1.5).

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