*Electronic Journal of Differential Equations*, Vol. 2021 (2021), No. 102, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# $p$-LAPLACIAN EQUATION WITH FINITELY MANY CRITICAL NONLINEARITIES

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ABSTRACT. This article concerns the ground state solution of the $p$-Laplacian equation with finitely many critical nonlinearities. By using the refined Sobolev inequality with Morrey norm and variational methods, we establish the existence of nonnegative ground state solution.

## 1. INTRODUCTION

We consider the $p$-Laplacian equation

$$-\Delta_p u-\zeta\frac{|u|^{p-2}u}{|x|^p}=\sum_{i=1}^{k}\big(I_{\alpha_i}*|u|^{p^*_{\alpha_i}}\big)|u|^{p^*_{\alpha_i}-2}u+|u|^{p^*-2}u,\quad x\in\mathbb{R}^N,\tag{1.1}$$

where $N\geqslant 3$, $p\in(1,N)$, $\zeta\in(0,\Lambda)$, $\Lambda=(\frac{N-p}{p})^p$, $\Delta_p:=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian, $p^*_{\alpha_i}=\frac{p}{2}(\frac{N+\alpha_i}{N-p})$ are the Hardy-Littlewood-Sobolev critical upper exponents, and the parameters $\alpha_i$ satisfy the following assumption:

(H1) $0<\alpha_1<\dots<\alpha_k<N$, for $k\in\mathbb{N}$, $2\leqslant k<\infty$.

Problem (1.1) is related to the nonlinear Choquard equation

$$-\Delta u+V(x)u=(I_\alpha*|u|^q)\,|u|^{q-2}u,\ \ x\in\mathbb{R}^N,\tag{1.2}$$

where $\frac{N+\alpha}{N}\leqslant q\leqslant\frac{N+\alpha}{N-2}$ and $\alpha\in(0,N)$. For $q=2$ and $\alpha=2$, problem (1.2) goes back to the description of the quantum theory of a polaron at rest by Pekar in 1954 [10] and the modeling of an electron trapped in its own hole in 1976 in the work of Choquard, as a certain approximation to Hartree-Fock theory of one-component plasma [11]. The existence and qualitative properties of solutions of Choquard equations or other related equations have been widely studied in the previous decades, see [2, 3, 4, 5, 7, 8, 12, 15].

Recall that the Sobolev space $D^{1,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the semi-norm

$$\|u\|^p_{D^{1,p}(\mathbb{R}^N)}=\int_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x.$$

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2010 *Mathematics Subject Classification.* 35A15, 35J20.
*Key words and phrases.* p-Laplacian equation; critical exponent; ground state solution; mountain pass theorem.

Submitted June 22, 2021. Published December 24, 2021.

It is well known that $\Lambda = \left(\frac{N-p}{p}\right)^p$ is the best constant in the Hardy inequality

$$\Lambda \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \mathrm{d}x \leqslant \|u\|^p_{D^{1,p}(\mathbb{R}^N)} \quad \text{for all } u \in D^{1,p}(\mathbb{R}^N).$$

By the Hardy inequality for $\zeta \in [0, \Lambda)$, we derive that $\|u\|^p_\zeta = \|u\|^p_{D^{1,p}(\mathbb{R}^N)} - \zeta \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \mathrm{d}x$ is an equivalent norm in $D^{1,p}(\mathbb{R}^N)$, and

$$\left(1 - \frac{\zeta}{\Lambda}\right) \|u\|^p_{D^{1,p}(\mathbb{R}^N)} \leqslant \|u\|^p_\zeta \leqslant \|u\|^p_{D^{1,p}(\mathbb{R}^N)}.$$

For $\alpha \in [0, N)$, $\zeta \in [0, \Lambda)$ and $p \in (1, N)$, we define the best constant:

$$S_{\zeta,\alpha} := \inf_{u \in D^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^p_{D^{1,p}(\mathbb{R}^N)} - \zeta \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \mathrm{d}x}{\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p^*_\alpha} |u(y)|^{p^*_\alpha}}{|x-y|^{N-\alpha}} \mathrm{d}x\mathrm{d}y\right)^{\frac{p}{2 \cdot p^*_\alpha}}}. \tag{1.3}$$

**Lemma 1.1** ([6]). *Let $t, r > 1$ and $0 < \alpha < N$ with $\frac{1}{t} + \frac{1}{r} + \frac{N-\alpha}{N} = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(N, \alpha, t, r) > 0$, independent of $f, g$ such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)||h(y)|}{|x-y|^{N-\alpha}} \mathrm{d}x\mathrm{d}y \leqslant C(N, \alpha, t, r) \|f\|_t \|h\|_r .$$

*If $t = r = \frac{2N}{N+\alpha}$, then*

$$C(N, \alpha, t, r) = C(N, \alpha) = \pi^{\frac{N-\alpha}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(N + \frac{\alpha}{2})} \left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right)^{\alpha/N}.$$

We introduce the energy functional associated with problem (1.1) by

$$I_\zeta(u) = \frac{1}{p} \|u\|^p_\zeta - \sum_{i=1}^k \frac{1}{2 \cdot p^*_{\alpha_i}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{p^*_{\alpha_i}} |u(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y - \frac{1}{p^*} \int_{\mathbb{R}^N} |\bar{u}_n|^{p^*} \mathrm{d}x.$$

We define the Nehari manifold

$$\mathcal{N}_\zeta = \{u \in D^{1,p}(\mathbb{R}^N) : \langle I'_\zeta(u), u \rangle = 0, \ u \neq 0\},$$

and

$$\bar{c}_\zeta = \inf_{u \in \mathcal{N}_\zeta} I_\zeta(u), \quad \bar{\bar{c}}_\zeta = \inf_{u \in D^{1,p}(\mathbb{R}^N)} \max_{t \geqslant 0} I_\zeta(tu), \quad c_\zeta = \inf_{\Upsilon \in \Gamma} \max_{t \in [0,1]} I_\zeta(\Upsilon(t)),$$

where

$$\Gamma = \{\Upsilon \in C([0,1], D^{1,p}(\mathbb{R}^N)) : \Upsilon(0) = 0, I_\zeta(\Upsilon(1)) < 0\}.$$

Because of lack of compactness of the Sobolev embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ and that the functional $I_\zeta$ is invariant under the weighted dilation, it is hard to show that the Palais-Smale sequence of $I_\zeta$ has a convergent subsequence.

Recently, Su et al. [14] studied the existence of ground state solution for (1.1) with the additional condition

(H2) $\frac{S_{0,0}}{C(N,\alpha_i)^{\frac{p}{2 \cdot p^*_{\alpha_i}}}} \geqslant (k+1)^{-\frac{p}{N+\alpha_k}}$ for $i = 1, \dots, k$, where $C(N, \alpha_i)$ and $S_{0,0}$ are defined in Lemma 2.1 and (1.3) respectively.

Applying this condition, they showed that

$$\bar{c}_0 > \bar{c}_\zeta. \tag{1.4}$$

As an application of inequality (1.4), they proved that the dilated subsequence of Palais-Smale sequence weak converges to nonzero function. And then they established the existence of ground state solution to (1.1). Hence, condition (H2) plays a key role in [14].

It is natural to ask

Can we find a nontrivial solution to (1.1) without assuming (H2)?

To the best of our knowledge, there is no affirmative answer in the literature. An answer to this question is given in the main result of this article:

**Theorem 1.2.** *Assume that $N \geqslant 3$, $p \in (1, N)$, $\zeta \in (0, \Lambda)$ and* (H1) *holds. Then equation* (1.1) *has a nonnegative ground state solution.*

This article is organized as follows: In Section 2, we study the ground state solution of limit equation. In Section 3, we prove Theorem 1.2.

## 2. Ground state solution of the limit equation

To study (1.1), we consider the problem

$$-\Delta_p u = \sum_{i=1}^{k} \big(I_{\alpha_i} * |u|^{p^*_{\alpha_i}}\big)|u|^{p^*_{\alpha_i}-2}u + |u|^{p^*-2}u, \quad x \in \mathbb{R}^N. \tag{2.1}$$

We introduce the energy functional associated with problem (2.1) by

$$I_0(u) = \frac{1}{p}\|u\|^p_{D^{1,p}(\mathbb{R}^N)} - \sum_{i=1}^{k} \frac{1}{2 \cdot p^*_{\alpha_i}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p^*_{\alpha_i}}|u(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*}\mathrm{d}x.$$

We define

$$c_0 = \inf_{\Upsilon\in\Gamma}\max_{t\in[0,1]} I_0(\Upsilon(t)),$$

where

$$\Gamma = \{\Upsilon \in C([0,1], D^{1,p}(\mathbb{R}^N)) : \Upsilon(0) = 0, I_0(\Upsilon(1)) < 0\}.$$

**Lemma 2.1.** *Assuming the conditions of Theorem 1.2. The following conclusions hold:*

(i) *there exists $\{v_n\} \subset D^{1,p}(\mathbb{R}^N)$ such that*
$$I_0(v_n) \to c_0, \quad \|I_0'(v_n)\|_{D^{-1,p}(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty$$
*and $\{v_n\}$ is uniformly bounded in $D^{1,p}(\mathbb{R}^N)$, and $\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^{p^*}\mathrm{d}x > 0$;*

(ii) *for each $u \in D^{1,p}(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}^0$;*

(iii) $c_0 = \bar{c}_0 = \bar{\bar{c}}_0 = \inf_{u\in\mathcal{N}^0} I_0(u) > 0$.

*Proof.* (i) Clearly, $I_0$ satisfies the mountain pass geometry. Then there exists a $(PS)_c$ sequence $\{v_n\} \subset D^{1,p}(\mathbb{R}^N)$ of $I_0$ at level $c_0 > 0$. It is not hard to prove that $\{v_n\}$ is uniformly bounded in $D^{1,p}(\mathbb{R}^N)$.

We now show that $\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^{p^*}\mathrm{d}x > 0$. Suppose on the contrary that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^{p^*}\mathrm{d}x = 0. \tag{2.2}$$

It follows from (2.2) and Lemma 1.1 that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(x)|^{p^*_{\alpha_i}}|v_n(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y=0,\quad \text{for } i=1,\dots,k. \tag{2.3}$$

By using (2.3) and the definition of $(PS)_{c_0}$ sequence, we obtain

$$c_0+o(1)=\frac{1}{p}\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)},$$
$$o(1)=\|u_n\|^p_{D^{1,p}(\mathbb{R}^N)}.$$

These equalities yield $c_0=0$ which contradicts $c_0>0$.

(ii) For $u\in D^{1,p}(\mathbb{R}^N)\setminus\{0\}$ and $t\in(0,\infty)$, we set

$$\begin{aligned}
g_1(t)&=I_0(tu)\\
&=\frac{t^p}{p}\|u\|^p_{D^{1,p}(\mathbb{R}^N)}-\sum_{i=1}^k\frac{t^{2\cdot p^*_{\alpha_i}}}{2\cdot p^*_{\alpha_i}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p^*_{\alpha_i}}|u(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y\\
&\quad-\frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N}|u|^{p^*}\mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
g_1'(t)&=t^{p-1}\|u\|^p_{D^{1,p}(\mathbb{R}^N)}-\sum_{i=1}^k t^{2\cdot p^*_{\alpha_i}-1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p^*_{\alpha_i}}|u(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y\\
&\quad-t^{p^*-1}\int_{\mathbb{R}^N}|u|^{p^*}\mathrm{d}x.
\end{aligned}$$

We know that $g_1'(\cdot)=0$ if and only if

$$\|u\|^p_{D^{1,p}(\mathbb{R}^N)}=\sum_{i=1}^k t^{2\cdot p^*_{\alpha_i}-p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p^*_{\alpha_i}}|u(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y+t^{p^*-p}\int_{\mathbb{R}^N}|u|^{p^*}\mathrm{d}x.$$

We set

$$g_2(t)=\sum_{i=1}^k t^{2\cdot p^*_{\alpha_i}-p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p^*_{\alpha_i}}|u(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y+t^{p^*-p}\int_{\mathbb{R}^N}|u|^{p^*}\mathrm{d}x.$$

Then we obtain that $\lim_{t\to 0}g_2(t)\to 0$, $\lim_{t\to\infty}g_2(t)\to\infty$, and $g_2(\cdot)$ is strictly increasing on $(0,\infty)$. Then there exists a unique $0<t_u<\infty$ such that

$$g_2(t)\begin{cases}>\|u\|^p_{D^{1,p}(\mathbb{R}^N)}, & t_u<t<\infty,\\ =\|u\|^p_{D^{1,p}(\mathbb{R}^N)}, & t=t_u,\\ <\|u\|^p_{D^{1,p}(\mathbb{R}^N)}, & 0<t<t_u.\end{cases}$$

This shows that $t_u u\in\mathcal{N}^0$. Moreover, we have

$$g_1'(t)\begin{cases}<0, & t_u<t<\infty,\\ =0, & t=t_u,\\ >0, & 0<t<t_u.\end{cases}$$

This implies that $g_1(\cdot)$ admits a unique critical point $t_u$ on $(0,\infty)$ such that $g_1(\cdot)$ takes the maximum at $t_u$.

(iii) Clearly, $I_0$ is bounded from below on $\mathcal{N}^0$, and $\bar{c}_0>0$. Indeed, it follows form Lemma 3.1 (ii) that $\bar{c}_0=\bar{\bar{c}}_0$. Notice that for any $u\in D^{1,p}(\mathbb{R}^N)\setminus\{0\}$, there exists $\tilde{t} > 0$ large, such that $I_0(\tilde{t}u) < 0$. We define a path $\gamma : [0, 1] \to D^{1,p}(\mathbb{R}^N)$ by $\gamma(t) = t\tilde{t}u$. Clearly, $\gamma \in \Gamma$ and consequently, $c_0 \leqslant \bar{\bar{c}}_0$. On the other hand, for every path $\gamma \in \Gamma$, we let $g_3(t) := \langle I_0'(\gamma(t)), \gamma(t)\rangle$. Then $g_3(0) = 0$ and $g_3(t) > 0$ for $t > 0$ small. We have

$$\begin{aligned}
&I_0(\gamma(1)) - \frac{1}{p}\langle I_0'(\gamma(1)), \gamma(1)\rangle \\
&\geqslant \sum_{i=1}^{k}\Big(\frac{1}{p} - \frac{1}{2\cdot p^*_{\alpha_i}}\Big)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{p^*_{\alpha_i}}|u(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y - \Big(\frac{1}{p} - \frac{1}{p^*}\Big)\int_{\mathbb{R}^N}|u|^{p^*}\mathrm{d}x \geqslant 0
\end{aligned}$$

which implies

$$\langle I_0'(\gamma(1)), \gamma(1)\rangle \leqslant pI_0(\gamma(1)) = pI_0(\tilde{t}u) < 0.$$

Then there exists $\tilde{\tilde{t}} \in (0, 1)$ such that $g_3(\tilde{\tilde{t}}) = 0$, i.e., $\gamma(\tilde{\tilde{t}}) \in \mathcal{N}^0$. So $c_0 \geqslant \bar{c}_0$. $\square$

We recall that a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Morrey space $\|u\|_{\mathcal{L}^{q,\varpi}(\mathbb{R}^N)}$ with $q \in [1,\infty)$ and $\varpi \in (0, N]$, if and only if

$$\|u\|^q_{\mathcal{L}^{q,\varpi}(\mathbb{R}^N)} = \sup_{R>0,x\in\mathbb{R}^N} R^{\varpi-N}\int_{B(x,R)}|u(y)|^q\mathrm{d}y < \infty.$$

**Lemma 2.2** ([9, Theorem 2]). *For $p \in (1, N)$, there exists $C > 0$ such that for $\iota$ and $\vartheta$ satisfying $\frac{p}{p^*} \leqslant \iota < 1$, $1 \leqslant \vartheta < p^*$, we have*

$$\Big(\int_{\mathbb{R}^N}|u|^{p^*}\mathrm{d}x\Big)^{1/p^*} \leqslant C\|u\|^{\iota}_{D^{1,p}(\mathbb{R}^N)}\|u\|^{1-\iota}_{\mathcal{L}^{\vartheta,\frac{\vartheta(N-p)}{p}}(\mathbb{R}^N)},$$

*for all $u \in D^{1,p}(\mathbb{R}^N)$.*

*Proof of ground state solution for* (2.1). We divided our proof into two steps.

**Step 1.** Note that $\{v_n\}$ is a bounded sequence in $D^{1,p}(\mathbb{R}^N)$. Up to a subsequence, we assume that

$$v_n \rightharpoonup v \text{ in } D^{1,p}(\mathbb{R}^N), \quad v_n \to v \text{ a.e. in } \mathbb{R}^N, \quad v_n \to v \text{ in } L^r_{\rm loc}(\mathbb{R}^N)$$

for all $r \in [p, p^*)$. From Lemmas 2.1 (i) and 2.2, there exists $C > 0$ such that

$$\|v_n\|_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)} \geqslant C > 0.$$

On the other hand, since the sequence is bounded in $D^{1,p}(\mathbb{R}^N)$, and (see [13]),

$$D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N) \hookrightarrow \mathcal{L}^{p,N-p}(\mathbb{R}^N),$$

we have

$$\|v_n\|_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)} \leqslant C.$$

Hence, there exists a positive constant which we denote again by $C$ such that for any $n$ we obtain

$$C \leqslant \|v_n\|_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)} \leqslant C^{-1}.$$

So we may find $\sigma_n > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\frac{1}{\sigma_n^p}\int_{B(x_n,\sigma_n)}|u_n(y)|^p\mathrm{d}y \geqslant \|u_n\|^p_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)} - \frac{C}{2n} \geqslant C_1 > 0.$$

Let $\bar{v}_n(x) = \sigma_n^{\frac{N-p}{p}} v_n(x_n + \sigma_n x)$. We need to verify that

$$I_0(\bar{v}_n) = I_0(u_n) \to c_0, \quad \|I_0'(\bar{v}_n)\|_{D^{-1,p}(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty,$$

For all $\varphi \in D^{1,p}(\mathbb{R}^N)$, we have

$$\begin{aligned} |\langle I_0'(\bar{v}_n), \varphi\rangle| &= |\langle I_0'(v_n), \bar{\varphi}_n\rangle| \\ &\leqslant \|I_0'(v_n)\|_{D^{-1,p}(\mathbb{R}^N)} \|\bar{\varphi}_n\|_{D^{1,p}(\mathbb{R}^N)} \\ &= o(1)\|\bar{\varphi}_n\|_{D^{1,p}(\mathbb{R}^N)}\,, \end{aligned}$$

where $\bar{\varphi}_n = \sigma_n^{-\frac{N-p}{p}} \varphi(\frac{x-x_n}{\sigma_n})$. From $\|\bar{\varphi}_n\|_{D^{1,p}(\mathbb{R}^N)} = \|\varphi\|_{D^{1,p}(\mathbb{R}^N)}$, we obtain

$$\|I_0'(\bar{v}_n)\|_{D^{-1,p}(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.$$

Thus there exists $\bar{v}$ such that

$$\bar{v}_n \rightharpoonup \bar{v} \text{ in } D^{1,p}(\mathbb{R}^N), \quad \bar{v}_n \to \bar{v} \text{a.e. in } \mathbb{R}^N, \quad \bar{v}_n \to \bar{v} \text{ in } L^r_{\rm loc}(\mathbb{R}^N)$$

for all $r \in [p, p^*)$. Then

$$\int_{B(0,1)} |\bar{v}_n(y)|^p \mathrm{d}y = \frac{1}{\sigma_n^p} \int_{B(x_n,\sigma_n)} |v_n(y)|^p \mathrm{d}y \geqslant C_1 > 0.$$

This implies $\bar{v} \not\equiv 0$.

**Step 2.** For any $\varphi \in D^{1,p}(\mathbb{R}^N)$, applying $\langle I_0'(\bar{v}_n), \varphi\rangle \to 0$ and $\bar{v}_n \rightharpoonup \bar{v}$ weakly in $D^{1,p}(\mathbb{R}^N)$, we obtain

$$\langle I_0'(\bar{v}), \varphi\rangle = 0. \tag{2.4}$$

Moreover, by (2.4) and $\bar{v} \not\equiv 0$, we obtain $\bar{u} \in \mathcal{N}^0$. By the Brézis-Lieb Lemma [1], we have

$$\int_{\mathbb{R}^N} |\bar{u}_n|^{p^*} \mathrm{d}x \geqslant \int_{\mathbb{R}^N} |\bar{u}|^{p^*} \mathrm{d}x + o(1), \tag{2.5}$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\bar{u}_n(x)|^{p_\alpha^*}|\bar{u}_n(y)|^{p_\alpha^*}}{|x-y|^{N-\alpha}} \mathrm{d}x\mathrm{d}y \geqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\bar{u}(x)|^{p_\alpha^*}|\bar{u}(y)|^{p_\alpha^*}}{|x-y|^{N-\alpha}} \mathrm{d}x\mathrm{d}y + o(1). \tag{2.6}$$

We set

$$K(u) = \sum_{i=1}^k \Big(\frac{1}{p} - \frac{1}{2\cdot p_{\alpha_i}^*}\Big) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{p_{\alpha_i}^*}|u(y)|^{p_{\alpha_i}^*}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y + \Big(\frac{1}{p} - \frac{1}{p^*}\Big) \int_{\mathbb{R}^N} |u|^{p^*} \mathrm{d}x. \tag{2.7}$$

Applying Lemma 2.1, (2.5), (2.6), (2.7) and $\bar{u} \in \mathcal{N}^0$, we obtain

$$\begin{aligned} \bar{c}_0 = c_0 = I_0(\bar{v}_n) - \frac{1}{p}\langle I_0'(\bar{v}_n), \bar{v}_n\rangle &= \lim_{n\to\infty} K(\bar{v}_n) \\ \geqslant K(\bar{v}) = I_0(\bar{v}) - \frac{1}{p}\langle I_0'(\bar{v}), \bar{v}\rangle & \\ = I_0(\bar{v}) \geqslant \bar{c}_0. & \end{aligned}$$

Therefore, the inequalities above have to be equalities. We obtain $I_0(\bar{v}) = c_0$, which means that $\tilde{v}$ is a ground state solution of problem (2.1) at the energy level $c_0$. We know that $|\bar{v}| \in D^{1,p}(\mathbb{R}^N)$ and $|\nabla|\bar{v}|| = |\nabla \bar{v}|$ a.e. in $\mathbb{R}^N$. Hence, we can choose $\bar{v} \geqslant 0$. $\square$

## 3. Proof of Theorem 1.2

As in Lemma 2.1, we have the following results without proof.

**Lemma 3.1.** *Under the conditions of Theorem 1.2, the following conclusions hold:*

(i) *there exists* $\{u_n\} \subset D^{1,p}(\mathbb{R}^N)$ *such that*

$$I_\zeta(u_n) \to c_\zeta, \quad \|I'_\zeta(u_n)\|_{D^{-1,p}(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty,$$

*and* $\{u_n\}$ *is uniformly bounded in* $D^{1,p}(\mathbb{R}^N)$, *and* $\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p^*} \mathrm{d}x > 0$;

(ii) *for each* $u \in D^{1,p}(\mathbb{R}^N) \setminus \{0\}$, *there exists a unique* $t_u > 0$ *such that* $t_u u \in \mathcal{N}^\zeta$;

(iii) $c_\zeta = \bar{c}_\zeta = \bar{\bar{c}}_\zeta = \inf_{u\in\mathcal{N}^\zeta} I_\zeta(u) > 0$.

We now prove inequality (1.4).

**Lemma 3.2.** *Assume that the conditions of Theorem 1.2 hold. Then* $\bar{c}_0 > \bar{c}_\zeta$ *for all* $\zeta \in (0, \Lambda)$.

*Proof.* Since $\bar{v}$ is a nonnegative ground state solution of equation (2.1), so we have $I_0(\bar{v}) = c_0$ and

$$\|\bar{v}\|^p_{D^{1,p}(\mathbb{R}^N)} = \sum_{i=1}^k \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\bar{v}(x)|^{p^*_{\alpha_i}} |\bar{v}(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} |\bar{v}|^{p^*} \mathrm{d}x. \tag{3.1}$$

By Lemma 3.1, there exists $t_{\bar{v}} > 0$ such that $t_{\bar{v}}\bar{v} \in \mathcal{N}^\zeta$. We now claim $t_{\bar{v}} < 1$. It follows from $t_{\bar{v}}\bar{v} \in \mathcal{N}^\zeta$ that

$$\begin{aligned} &\|\bar{v}\|^p_{D^{1,p}(\mathbb{R}^N)} - \zeta \int_{\mathbb{R}^N} \frac{|\bar{v}|^p}{|x|^p} \mathrm{d}x \\ &= \sum_{i=1}^k t_{\bar{v}}^{2\cdot p^*_{\alpha_i} - p} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\bar{v}(x)|^{p^*_{\alpha_i}} |\bar{v}(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y + t_{\bar{v}}^{p^*-p} \int_{\mathbb{R}^N} |\bar{v}|^{p^*} \mathrm{d}x. \end{aligned} \tag{3.2}$$

Putting (3.1) into (3.2), we have

$$\begin{aligned} &- \zeta \int_{\mathbb{R}^N} \frac{|\bar{v}|^p}{|x|^p} \mathrm{d}x \\ &= \sum_{i=1}^k \big(t_{\bar{v}}^{2\cdot p^*_{\alpha_i} - p} - 1\big) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\bar{v}(x)|^{p^*_{\alpha_i}} |\bar{v}(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y + (t_{\bar{v}}^{p^*-p} - 1) \int_{\mathbb{R}^N} |\bar{v}|^{p^*} \mathrm{d}x, \end{aligned}$$

which implies

$$0 > \sum_{i=1}^k \big(t_{\bar{v}}^{2\cdot p^*_{\alpha_i} - p} - 1\big) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\bar{v}(x)|^{p^*_{\alpha_i}} |\bar{v}(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}} \mathrm{d}x\mathrm{d}y + (t_{\bar{v}}^{p^*-p} - 1) \int_{\mathbb{R}^N} |\bar{v}|^{p^*} \mathrm{d}x.$$

Note that $2p^*_{\alpha_i} > p$ and $p^* > p$. Then $t_{\bar{v}} < 1$. It follows from (3.1) that

$$\begin{aligned}
&I_0(\bar{v}) - I_0(t_{\bar{v}}\bar{v}) \\
&= \frac{1-t_{\bar{v}}^p}{p}\|\bar{v}\|^p_{D^{1,p}(\mathbb{R}^N)} - \frac{1-t_{\bar{v}}^{p^*}}{p^*}\int_{\mathbb{R}^N}|\bar{v}|^{p^*}\mathrm{d}x \\
&\quad - \sum_{i=1}^k \frac{1-t_{\bar{v}}^{2\cdot p^*_{\alpha_i}}}{2\cdot p^*_{\alpha_i}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{v}(x)|^{p^*_{\alpha_i}}|\bar{v}(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y \\
&= \sum_{i=1}^k \Big[\frac{1}{p} - \frac{t_{\bar{v}}^p}{p} - \frac{1}{2\cdot p^*_{\alpha_i}} + \frac{t_{\bar{v}}^{2\cdot p^*_{\alpha_i}}}{2\cdot p^*_{\alpha_i}}\Big]\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\bar{v}(x)|^{p^*_{\alpha_i}}|\bar{v}(y)|^{p^*_{\alpha_i}}}{|x-y|^{N-\alpha_i}}\mathrm{d}x\mathrm{d}y \\
&\quad + \Big[\frac{1}{p} - \frac{t_{\bar{v}}^p}{p} - \frac{1}{p^*} + \frac{t_{\bar{v}}^{p^*}}{p^*}\Big]\int_{\mathbb{R}^N}|\bar{v}|^{p^*}\mathrm{d}x.
\end{aligned} \tag{3.3}$$

We set

$$\begin{gathered}
f_1(t_{\bar{v}}) = \frac{1}{p} - \frac{t_{\bar{v}}^p}{p} - \frac{1}{2\cdot p^*_{\alpha_i}} + \frac{t_{\bar{v}}^{2\cdot p^*_{\alpha_i}}}{2\cdot p^*_{\alpha_i}}, \\
f_2(t_{\bar{v}}) = \frac{1}{p} - \frac{t_{\bar{v}}^p}{p} - \frac{1}{p^*} + \frac{t_{\bar{v}}^{p^*}}{p^*}.
\end{gathered}$$

It is easy to see that

$$\begin{gathered}
f_1(0) > 0, \quad f_1(1) = 0, \quad f_1'(t_{\bar{v}}) = t_{\bar{v}}^{p-1}(t_{\bar{v}}^{2\cdot p^*_{\alpha_i}-p} - 1), \\
f_2(0) > 0, \quad f_2(1) = 0, \quad f_2'(t_{\bar{v}}) = t_{\bar{v}}^{p-1}(t_{\bar{v}}^{p^*-p} - 1).
\end{gathered}$$

Note that $0 < t_{\bar{v}} < 1$, $2p^*_{\alpha_i} > p$, and $p^* > p$. Then

$$\begin{gathered}
f_1'(t_{\bar{v}}) < 0 \quad \text{for } t_{\bar{v}} \in (0,1), \\
f_2'(t_{\bar{v}}) < 0 \quad \text{for } t_{\bar{v}} \in (0,1).
\end{gathered}$$

Hence,

$$\begin{gathered}
f_1(t_{\bar{v}}) > 0 \quad \text{for } t_{\bar{v}} \in (0,1), \\
f_2(t_{\bar{v}}) > 0 \quad \text{for } t_{\bar{v}} \in (0,1).
\end{gathered}$$

Putting two results above into (3.3), we have

$$I_0(\bar{v}) > I_0(t_{\bar{v}}\bar{v}).$$

On the other hand,

$$I_\zeta(t_{\bar{v}}\bar{v}) = I_0(t_{\bar{v}}\bar{v}) - t_{\bar{v}}^p\zeta\int_{\mathbb{R}^N}\frac{|\bar{v}|^p}{|x|^p}\mathrm{d}x < I_0(t_{\bar{v}}\bar{v}).$$

So in general, we can obtain

$$\bar{c}_\zeta \leqslant I_\zeta(t_{\bar{v}}\bar{v}) < I_0(t_{\bar{v}}\bar{v}) < I_0(\bar{v}) = \bar{c}_0.$$

The proof is complete. □

*Proof of Theorem 1.2.* We divided our proof into four steps.

**Step 1.** Note that $\{u_n\}$ is a bounded sequence in $D^{1,p}(\mathbb{R}^N)$. Up to a subsequence, we assume that

$$u_n \rightharpoonup u \text{ in } D^{1,p}(\mathbb{R}^N), \quad u_n \to u \text{a.e. in } \mathbb{R}^N, \quad u_n \to u \text{ in } L^r_{\mathrm{loc}}(\mathbb{R}^N)$$

for all $r \in [p, p^*)$. By Lemmas 2.2 and 3.1 (i), there exists $C > 0$ such that

$$\|u_n\|_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)} \geqslant C > 0.$$

On the other hand, since the sequence is bounded in $D^{1,p}(\mathbb{R}^N)$, and (see [13])

$$D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N) \hookrightarrow \mathcal{L}^{p,N-p}(\mathbb{R}^N),$$

we have

$$\|u_n\|_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)} \leqslant C,$$

for some $C > 0$ independent of $n$. Hence, there exists a positive constant which we denote again by $C$ such that for any $n$ we obtain

$$C \leqslant \|u_n\|_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)} \leqslant C^{-1}.$$

So we may find $\sigma_n > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\frac{1}{\sigma_n^p}\int_{B(x_n,\sigma_n)} |u_n(y)|^p \mathrm{d}y \geqslant \|u_n\|^p_{\mathcal{L}^{p,N-p}(\mathbb{R}^N)} - \frac{C}{2n} \geqslant C_6 > 0.$$

Let $\bar{u}_n(x) = \sigma_n^{\frac{N-p}{p}} u_n(x_n + \sigma_n x)$. We need to verify that

$$\widetilde{I}_\zeta(\bar{u}_n) = I_\zeta(u_n) \to c_\zeta, \quad \widetilde{I}_\zeta{}'(\bar{u}_n) \to 0 \quad \text{as } n \to \infty,$$

where

$$\begin{aligned}\widetilde{I}_\zeta(\bar{u}_n) &= \frac{1}{p}\|\bar{u}_n\|^p_{D^{1,p}(\mathbb{R}^N)} - \frac{\zeta}{p}\int_{\mathbb{R}^N} \frac{|\bar{u}_n|^p}{|x + \frac{x_n}{\sigma_n}|^p}\mathrm{d}x \\ &\quad - \sum_{i=1}^k \frac{1}{2\cdot p^*_{\alpha_i}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\bar{u}_n(x)|^{p^*_{\alpha_i}}|\bar{u}_n(y)|^{p^*_{\alpha_i}}}{|x-y|^{\alpha_i}}\mathrm{d}x\mathrm{d}y - \frac{1}{p^*}\int_{\mathbb{R}^N} |\bar{u}_n|^{p^*}\mathrm{d}x.\end{aligned}$$

Now, for $\varphi \in D^{1,p}(\mathbb{R}^N)$, we obtain

$$\begin{aligned}|\langle \widetilde{I}_\zeta{}'(\bar{u}_n), \varphi\rangle| &= |\langle I'_\zeta(u_n), \bar{\varphi}_n\rangle| \\ &\leqslant \|I'_\zeta(u_n)\|_{D^{-1,p}(\mathbb{R}^N)}\|\bar{\varphi}_n\|_{D^{1,p}(\mathbb{R}^N)} \\ &= o(1)\|\bar{\varphi}_n\|_{D^{1,p}(\mathbb{R}^N)},\end{aligned}$$

where $\bar{\varphi}_n = \sigma_n^{-\frac{N-p}{p}}\varphi(\frac{x-x_n}{\sigma_n})$. Since $\|\bar{\varphi}_n\|_{D^{1,p}(\mathbb{R}^N)} = \|\varphi\|_{D^{1,p}(\mathbb{R}^N)}$, we obtain

$$\widetilde{I}_\zeta{}'(\bar{u}_n) \to 0 \quad \text{as } n \to \infty.$$

Thus there exists $\bar{u}$ such that

$$\bar{u}_n \rightharpoonup \bar{u} \text{ in } D^{1,p}(\mathbb{R}^N), \quad \bar{u}_n \to \bar{u} \text{ a.e. in } \mathbb{R}^N, \quad \bar{u}_n \to \bar{u} \text{ in } L^r_{\rm loc}(\mathbb{R}^N)$$

for all $r \in [p, p^*)$. Then

$$\int_{B(0,1)} |\bar{u}_n(y)|^p \mathrm{d}y = \frac{1}{\sigma_n^p}\int_{B(x_n,\sigma_n)} |u_n(y)|^p \mathrm{d}y \geqslant C_6 > 0.$$

This implies $\bar{u} \not\equiv 0$.

**Step 2.** We now show that $\{x_n/\sigma_n\}$ is bounded. If $x_n/\sigma_n \to \infty$, then for any $\varphi \in D^{1,p}(\mathbb{R}^N)$, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \frac{|\bar{u}_n|^{p-2}\bar{u}_n\varphi}{|x + \frac{x_n}{\sigma_n}|^p}\mathrm{d}x = 0. \tag{3.4}$$

Using that $\langle \widetilde{I_\zeta}'(\bar{u}_n), \varphi\rangle \to 0$ and (3.4), we obtain

$$\langle I_0'(\bar{u}), \varphi\rangle = 0. \tag{3.5}$$

Moreover, from (3.5) and $\bar{u} \not\equiv 0$, we obtain $\bar{u} \in \mathcal{N}^0$. Applying Lemma 3.1, Lemma 3.2, (3.3), (3.4), (3.5) and $\bar{u} \in \mathcal{N}^0$, we obtain

$$\begin{aligned}
\bar{c}_0 > \bar{c}_\zeta &= c_\zeta \\
&= I_\zeta(\bar{u}_n) - \frac{1}{p}\langle I_\zeta'(\bar{u}_n), \bar{u}_n\rangle = \lim_{n\to\infty} K(\bar{u}_n) \\
&\geqslant K(\bar{u}) = I_0(\bar{u}) - \frac{1}{p}\langle I_0'(\bar{u}), \bar{u}\rangle \\
&= I_0(\bar{u}) \geqslant \bar{c}_0,
\end{aligned}$$

which yields a contradiction. Hence, $\{x_n/\sigma_n\}$ is bounded.

**Step 3.** Let $\tilde{u}_n(x) = \sigma_n^{\frac{N-p}{p}} u_n(\sigma_n x)$. Then we can verify that

$$I_\zeta(\tilde{u}_n) = I_\zeta(u_n) \to c_\zeta, \quad I_\zeta'(\tilde{u}_n) \to 0 \quad \text{as } n \to \infty.$$

Arguing as before, we have

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ in } D^{1,p}(\mathbb{R}^N), \quad \tilde{u}_n \to \tilde{u} \text{ a.e. in } \mathbb{R}^N, \quad \tilde{u}_n \to \tilde{u} \text{ in } L^r_{\rm loc}(\mathbb{R}^N)$$

for all $r \in [p, p^*)$. Note that $\{x_n\sigma_n\}$ is bounded. Then there exists $\tilde{R} > 0$ such that

$$\int_{B(0,\tilde{R})} |\tilde{u}_n(y)|^p \mathrm{d}y > \int_{B(\frac{x_n}{\sigma_n},1)} |\tilde{u}_n(y)|^p \mathrm{d}y = \frac{1}{\sigma_n^p} \int_{B(x_n,\sigma_n)} |u_n(y)|^p \mathrm{d}y \geqslant C_5 > 0.$$

As a result, $\tilde{u} \not\equiv 0$. An argument similar to one in Step 2 yields

$$\langle I_\zeta'(\tilde{u}), \varphi\rangle = 0. \tag{3.6}$$

From this equality and $\tilde{u} \not\equiv 0$, we obtain $\tilde{u} \in \mathcal{N}^\zeta$.

**Step 4.** It follows from $\tilde{u} \in \mathcal{N}^\zeta$, (3.3), (3.4) and (3.5) that

$$\begin{aligned}
\bar{c}_\zeta = c_\zeta &= I_\zeta(\tilde{u}_n) - \frac{1}{p}\langle I_\zeta'(\tilde{u}_n), \tilde{u}_n\rangle \\
&= \lim_{n\to\infty} K(\tilde{u}_n) \\
&\geqslant K(\tilde{u}) = I_\zeta(\tilde{u}) - \frac{1}{p}\langle I_\zeta'(\tilde{u}), \tilde{u}\rangle \\
&= I_\zeta(\tilde{u}) \geqslant \bar{c}_\zeta.
\end{aligned}$$

Therefore, the inequalities above have to be equalities. We obtain $I_\zeta(\tilde{u}) = c_\zeta$, which means that $\tilde{u}$ is a ground state solution of problem (1.1) at the energy level $c_\zeta$. We know that $|\tilde{u}| \in D^{1,p}(\mathbb{R}^N)$ and $|\nabla|\tilde{u}|| = |\nabla \tilde{u}|$ a.e. in $\mathbb{R}^N$. Hence, we can choose $\tilde{u} \geqslant 0$. $\square$

**Acknowledgments.** This research was supported by the Key Program of University Natural Science Research Fund of Anhui Province (Grant No. KJ2020A0294).

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