DYNAMICS OF FLOCKING MODELS WITH TWO SPECIES

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ABSTRACT. This article studies the flocking behavior of self-organized agents in two species. First, referring to the work of Olfati-Saber and the classical Cucker-Smale model, we establish a discrete system describing the flocking dynamic of the agents in two species. Second, by using the LaSalle's invariance principle, we show that the system with global interaction will achieve unconditional time-asymptotic flocking, and the system with local interaction has a time-asymptotic flocking under certain assumptions. Moreover, we investigate the local asymptotic stability of a class of flocking solutions. Finally, some numerical simulations and qualitative results are presented.

1. INTRODUCTION

The self-organized behavior, illustrating that some agents interact with each other to reach a position and velocity balance, wildly exist in the natural world and human society. Scientists have established many models of various phenomena to describe these behaviors, such as birds flocking [4, 10], fishes swarming [2], complex network [6] and the price monitor in the market [1, 18]. The corresponding models have also been applied in the field of systems science and control engineering, such as the flying control for space flight [26, 27, 34] and the formation control for mobile robots team [29].

In 1986, Reynolds [28] introduced three heuristic rules showing the basic requirements on the collective behavior of self-organized agents, which can be summarized as follows.

(1) cohesion: attempt to stay close to nearby flockmates;
(2) separation: avoid collisions with nearby flockmates;
(3) alignment: attempt to match velocity with nearby flockmates.

Following the above rules, scientists have established many standard models. One class of these works is the flocking models, which mainly describe a final state of alignment under some principles of interaction between agents. In detail, let $x_i(t) \in \mathbb{R}^d$ and $v_i(t) \in \mathbb{R}^d$ be the position and velocity of the i-th agent in a group $\mathcal{N}$ with n agents, $i = 1, \ldots, n$. Then mathematically, the agents in $\mathcal{N}$ achieve flocking if for any $1 \leq i, j \leq n$, the following two properties hold:

(1) $\|v_j(t) - v_i(t)\| = 0$, as $t \to \infty$.
(2) there exists a constant $C > 0$ such that for all $t \geq 0$, $\|x_j(t) - x_i(t)\| \leq C$.

In 2006, Olfati-Saber [24] studied the following flocking model in which the interactions between agents are defined as the addition of an action function of distances and a matching function of velocities,

$$\dot{x}_i(t) = v_i(t),$$

$$\dot{x}_i(t) = \sum_{j=1}^{n} \phi(\|x_j(t) - x_i(t)\| - d)\vec{n}_{ij} + c(v_j(t) - v_i(t)), \tag{1.1}$$

where the interaction function ϕ is a monotone increasing function with $\phi(0) = 0$, $\vec{n}_{ij} = \frac{x_j(t) - x_i(t)}{\|x_j(t) - x_i(t)\|}$, and c, d are positive constants called the velocity matching coefficient and the balance distance, respectively. Under the condition that the corresponding graph of the agents keeps connected, the author obtained that the agents achieve flocking.

In 2007, Cucker and Smale [12] considered the model (C-S model):

$$\dot{x}_i(t) = v_i(t),$$

$$\dot{v}_i(t) = \sum_{j=1}^{n} a_{ij}(x(t))(v_j(t) - v_i(t)), \tag{1.2}$$

with the velocity matching coefficient function

$$a_{ij} = \frac{H}{(1 + \|x_j(t) - x_i(t)\|^2)^\beta},$$

where $H > 0$, $\beta > 0$ are parameters. They concluded that the agents reach a final state as unconditional flocking if $\beta < 1/2$, yet for $\beta \geq 1/2$, it is necessary to have some restrictions on the initial values.

Following the C-S model, scientists have developed various models and gained many results. In 2008, Ha and Tadmor used the BBGKY hierarchy to turn the C-S model into a Vlasov type mean-field model and proved that for $\beta = \frac{1}{2}$, the system still has unconditional flocking [32]. In 2009, Carrillo extended the C-S model to a continuous kinetic version model with a Boltzmann-type equation [3], and proved that the solutions will exponentially concentrate their velocities to their mean and they will converge towards a translational flocking solution. In 2014, Motsch and Tadmor presented a generalized C-S model in [23] and proved the flocking of the system under a row-stochastic assumption and some requirements on the interaction between agents. There are many other interesting results related to the above models, such as flocking with random influence of white noise [11, 17], randomly switching topologies [15], stochastic mean-field limit [9], non-symmetric interaction or leadership [13, 30], collision avoidance of obstacle [5, 35], and time-delay [7, 14, 21, 25, 31], etc.

One can notice that nearly all the above research works studied only the flocking behaviors of agents in one species, that is to say, all the agents follow the same interaction function and velocity matching coefficient. While, no matter in the natural world or human society, the interactions and gathering of multi-species agents are more widespread situations, such as several kinds of birds migrate together, various vehicles move together on the road, and peoples live together in a multi-ethnic society. In these circumstances, it can always be observed that two or more species of agents with different influence disciplines tend to form a whole cluster and move in the same way.

As it was pointed out in [8], the mathematical study of two-species behaviors is a hot topic in the research area of flocking behaviors. Several continuous models on two-species flocking have been established and analysed, see [8, 20, 33] for examples. The main goal of this paper is to establish a discrete model of some self-organized agents from two species and to investigate the flocking phenomena and related dynamics of the model. Furthermore, we will present several numerical patterns of flocking dynamics to illustrate our results and some other phenomena related to the models.

The outline of this article is as follows: in section 2, we establish the two-species flocking model. In section 3, the model is considered to have global interaction functions and is proved to have unconditional flocking. In section 4, we carry out a local model by adding a restriction on the interaction range. Then we use LaSalle's invariance principle and the invariant manifold theory to analyze the stability of the solutions. Finally, some numerical simulations and qualitative results will be concluded in section 5.

2. Flocking model for two groups

In this section, we will establish a flocking model of two species based on the Reynolds three rules and the models (1.1) and (1.2) in section 1. Before the modeling work, let us recall some concepts of graph theory. A graph G is defined as a pair $(\mathcal{V}, \mathcal{E})$ with a set of vertices $\mathcal{V} = \{1, 2, \ldots, n\}$ and a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Let $\|\mathcal{V}\|$ and $\|\mathcal{E}\|$ be the numbers of vertices and edges in graph G, respectively, then it is easy to find that $\|\mathcal{V}\| = n$ and $\|\mathcal{E}\| \leq n^2$. The set of neighbours of vertex i is defined by

$$N_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}.$$

If for any $i, j \in \mathcal{V}$, there has a sequence of vertices $\{i, v_1, \ldots, v_n, j\}$ such that $v_1 \in N_i$, $j \in N_{v_n}$, and $v_{k+1} \in N_{v_k}, k = 1, \ldots, n-1$, then we say that the graph G is connected and this sequence is called a walk between i and j. If G is disconnected, then $\mathcal{V}$ can be decompose into a pair of separated sets $\mathcal{V}_1$ and $\mathcal{V}_2$, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, and $(i, j) \notin \mathcal{E}$ for any $i \in \mathcal{V}_1$, $j \in \mathcal{V}_2$.

Definition 2.1. Let $G = (\mathcal{V}, \mathcal{E})$ be a graph. A subset $\hat{\mathcal{V}} \subseteq \mathcal{V}$ is called a cluster if the corresponding graph $\hat{G} = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$ is a connected graph with $\hat{\mathcal{E}} = \hat{\mathcal{V}} \times \hat{\mathcal{V}} \cap \mathcal{E}$.

An $n \times n$ matrix $A(x) = [a_{ij}(x)]$ can be viewed as a weighted adjacency matrix of graph $G = (\mathcal{V}, \mathcal{E})$, i.e. for any $i, j \in \mathcal{V}$, $a_{ij}(x) \neq 0$ if and only if $(i, j) \in \mathcal{E}$. The degree matrix $D(x)$ of G is defined as a diagonal matrix with diagonal elements $\sum_{i=1}^{n} a_{ii}(x)$. Then, matrix $L(x) = [l_{ij}(x)]$ induced by

$$L(x) = D(x) - A(x) = \begin{pmatrix} \sum_{j \neq 1} a_{1j} & -a_{12} & \ldots & -a_{1n} \\ -a_{21} & \sum_{j \neq 2} a_{2j} & \ldots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \ldots & \sum_{j \neq n} a_{nj} \end{pmatrix}, \quad (2.1)$$

is called the Laplacian matrix of G.

It is easy to find that in the matrix $L(x)$, the elements of each row have a sum of zero. This fact implies that $L(x)$ has a eigenvalue $\lambda_1 = 0$ with the related eigenvector $\mathbf{1}^n = (1, \ldots, 1)^\top$. Furthermore, the Laplacian matrix $L(x)$ has the following properties [16, 22]:

(1) If $A(x)$ is a non-negative matrix, then the eigenvalues of the corresponding $L(x)$ satisfy $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

(2) If $A(x)$ is a non-negative symmetric matrix, then the corresponding $L(x)$ is a positive semi-definite matrix which satisfies

$$z^\top L z = \frac{1}{2} \sum_{a_{ij} \neq 0} a_{ij}(z_j - z_i)^2, \quad z \in \mathbb{R}^n. \tag{2.2}$$

(3) If $A(x)$ is the adjacency matrix of a graph G, then the Fiedler number

$$\lambda_2 = \min_{z \perp \mathbf{1}^n} \frac{z^\top L z}{\|z\|^2} > 0,$$

if and only if the graph G is connected.

In the follow-up post, we consider two groups of agents which are divided according to the species. The two groups are indicated by the sets $\mathcal{N}_1$, $\mathcal{N}_2$ with $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$ and $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{N}$. For the sake of convenience, here we give some notations which will be used. Let

n_i: the number of agents in $\mathcal{N}_i$, $i = 1, 2$;

n: the total number of all the agents in $\mathcal{N}$, i.e., $n = n_1 + n_2$;

$x_l(t) \in \mathbb{R}^d$: the position of the l-th agent in $\mathcal{N}_1$ at time t;

$u_l(t) \in \mathbb{R}^d$: the velocity of the l-th agent in $\mathcal{N}_1$ at time t;

$y_i(t) \in \mathbb{R}^d$: the position of the i-th agent in $\mathcal{N}_2$ at time t;

$v_i(t) \in \mathbb{R}^d$: the velocity of the i-th agent in $\mathcal{N}_2$ at time t.

Therefore, we can use the pairs $(x_l, u_l) \in \mathcal{N}_1$ and $(y_i, v_i) \in \mathcal{N}_2$ as the l-th agent in $\mathcal{N}_1$ and i-th agent in $\mathcal{N}_2$, respectively.

Based on the Newton's second law, we illustrate the interactions of the agents in two groups by the kinetic model

$$\begin{aligned} \dot{x}_l &= u_l, \\ \dot{u}_l &= f_l^1 + g_l^1, \quad l = 1, \ldots, n_1, \\ \dot{y}_i &= v_i, \\ \dot{v}_i &= f_i^2 + g_i^2, \quad i = 1, \ldots, n_2, \end{aligned} \tag{2.3}$$

here, for $k = 1, 2$ and $l \in \{1, \ldots, n_k\}$, f_l^k represents the influence acting on the l-th agent in $\mathcal{N}_k$ from other agents in $\mathcal{N}_k$ and g_l^k represents that from all agents in $\mathcal{N}/\mathcal{N}_k$.

First, according to the cohesion and separation rules, when the agents achieve flocking, the distance between every two agents will tend to reach a proper value. This ideal value under the flocking state is called the balance distance. In the model of two groups, we introduce three different balance distances to show the distinction between the two groups. In details, for agents in $\mathcal{N}_1$ and $\mathcal{N}_2$, the balance distances are respectively denoted by d_1, d_2, and for agents from different groups, the balance distance is d_0 which should be not less than d_1, d_2. Hence, we utilize the idea of Ofati-Saber [24] presented in model (1.1) and give a collective potential function V as follows

$$V(x, y) = V_1(x) + V_2(y) + V_0(x, y), \tag{2.4}$$

with

$$V_1(x) = \frac{1}{2} \sum_{k=1}^{n_1} \sum_{l=1}^{n_1} \psi(\|x_k - x_l\| - d_1),$$

$$V_2(y) = \frac{1}{2} \sum_{j=1}^{n_2} \sum_{i=1}^{n_2} \psi(\|y_j - y_i\| - d_2),$$

$$V_0(x,y) = \frac{1}{2} \sum_{k=1}^{n_1} \sum_{i=1}^{n_2} \psi(\|x_k - y_i\| - d_0),$$

where $x = (x_1, \ldots, x_{n_1})$, $y = (y_1, \ldots, y_{n_2})$, and $\psi(z) : \mathbb{R} \to \mathbb{R}$ is a nonnegative function with its strict minimum at the original point. This property of $\psi(z)$ reflects the attractive and repulsive behaviors between agents and these behaviors ensure that the agents can always tend to their balance distances.

Next, to depict the alignment rule, it is natural to apply the velocity matching idea of the C-S model [12] as it is presented in (1.2). Specifically, we can give the expression of the influence functions,

$$f_l^1 = \sum_{k=1}^{n_1} \phi(\|x_k - x_l\| - d_1)\vec{e}(x_k, x_l) + \sum_{k=1}^{n_1} a_1(x_k, x_l)(u_k - u_l),$$

$$g_l^1 = \sum_{j=1}^{n_2} \phi(\|y_j - x_l\| - d_0)\vec{e}(y_j, x_l) + \sum_{j=1}^{n_2} a_0(y_j, x_l)(v_j - u_l),$$

$$f_i^2 = \sum_{j=1}^{n_2} \phi(\|y_j - y_i\| - d_2)\vec{e}(y_j, y_i) + \sum_{j=1}^{n_2} a_2(y_j, y_i)(v_j - v_i),$$

$$g_i^2 = \sum_{k=1}^{n_1} \phi(\|x_k - y_i\| - d_0)\vec{e}(x_k, y_i) + \sum_{k=1}^{n_1} a_0(x_k, y_i)(u_k - v_i),$$

where the interaction function $\phi(z)$ is the derivation of $\psi(z)$, $\vec{e}(z_1, z_2)$ is a unit vector along the line of the two agents, expressed as

$$\vec{e}(z_1, z_2) = \frac{z_1 - z_2}{\|z_1 - z_2\|},$$

and the functions a_1, a_2, a_0 are the velocity matching coefficients.

Now, conclusively, we present the system of agents in two groups which will be investigated in this article,

$$\dot{x}_l = u_l,$$

$$\dot{u}_l = \sum_{k=1}^{n_1} \phi(\|x_k - x_l\| - d_1)\vec{e}(x_k, x_l) + \sum_{k=1}^{n_1} a_1(x_k, x_l)(u_k - u_l)$$

$$+ \sum_{j=1}^{n_2} \phi(\|y_j - x_l\| - d_0)\vec{e}(y_j, x_l) + \sum_{j=1}^{n_2} a_0(y_j, x_l)(v_j - u_l),$$

$$\dot{y}_i = v_i, \tag{2.5}$$

$$\dot{v}_i = \sum_{j=1}^{n_2} \phi(\|y_j - y_i\| - d_2)\vec{e}(y_j, y_i) + \sum_{j=1}^{n_2} a_2(y_j, y_i)(v_j - v_i)$$

$$+ \sum_{k=1}^{n_1} \phi(\|x_k - y_i\| - d_0)\vec{e}(x_k, y_i) + \sum_{k=1}^{n_1} a_0(x_k, y_i)(u_k - v_i).$$

Remark 2.2. System (2.5) naturally corresponds to a graph $G = (\mathcal{N}, \mathcal{E}_\mathcal{N})$ with a block adjacency matrix induced by the velocity matching coefficients.

According to the definition of flocking given by Cucker and Smale [12], we carry out a new definition of two-groups flocking.

Definition 2.3. For agents in the two groups $\mathcal{N}_1$ and $\mathcal{N}_2$, they have a time-asymptotic flocking if and only if for any $l = 1, \ldots, n_1$ and $i = 1, \ldots, n_2$, the vector $(x_l(t), u_l(t), y_i(t), v_i(t))$ satisfies

(1) $\|u_l(t) - v_i(t)\| = 0$, as $t \to \infty$,
(2) there exists a constant C such that $\|x_l(t) - y_i(t)\| \leq C$ for all $t \geq 0$.

Same as the case of one-group flocking, the above definition implies that if agents in the two groups achieve flocking, their velocities are asymptotically equal, and in the whole process, there is no agent get away from the groups.

For the convenience of studying, we then consider the translation invariance of system (2.5). We translate the system as:

$$\tilde{x}_l = x_l - X_c, \quad \tilde{y}_i = y_i - X_c,$$
$$\tilde{u}_l = u_l - V_c, \quad \tilde{v}_i = v_i - V_c,$$

where

$$X_c = \frac{1}{n}\Big(\sum_{l=1}^{n_1} x_l + \sum_{i=1}^{n_2} y_i\Big), \quad V_c = \frac{1}{n}\Big(\sum_{l=1}^{n_1} u_l + \sum_{i=1}^{n_2} v_i\Big).$$

Noting that $\dot{X}_c = V_c$ and $\dot{V}_c = 0$, which means

$$X_c = V_c(0)t + X_c(0), \quad V_c \equiv V_c(0),$$

we can obtain that the translation of the system has invariance. Thus, without loss of generality, we may assume that $V_c = X_c = 0$ and this assumption can be realized by the above translation. In other word, we only need to investigate system (2.5) restricted on the invariant manifold $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ with

$$\mathcal{M}_1 = \Big\{(x, y) : \sum_{l=1}^{n_1} x_l + \sum_{i=1}^{n_2} y_i = 0\Big\}, \quad \mathcal{M}_2 = \Big\{(u, v) : \sum_{l=1}^{n_1} u_l + \sum_{i=1}^{n_2} v_i = 0\Big\},$$

where $u = (u_1, \ldots, u_{n_1})$, $v = (v_1, \ldots, v_{n_2})$.

With the definition of two-groups flocking, it is easy to discover that if the system restricted on $\mathcal{M}$ achieves flocking, the velocities will asymptotically equal to 0. Thus the flocking problem of system (2.5) on $\mathbb{R}^{dn}$ turns to be the stability problem of fixed points on the manifold $\mathcal{M}$.

3. COLLECTIVE DYNAMIC UNDER GLOBAL INTERACTION

In this section, we consider the case when system (2.5) has global interaction, which means any two agents in $\mathcal{N}$ interact with each other at any time. We first present several assumptions on the interaction function and velocity matching coefficients.

(A1) The interaction function $\phi(z) : \mathbb{R} \to \mathbb{R}$ is a C^1-smooth monotonically increasing function with $\phi(0) = 0$ and there exists a constant C_0 such that for all $z \in \mathbb{R}$, $|\phi(z)| \leq C_0$.

(A2) The potential function

$$\psi(z) = \int_0^z \phi(s)\, ds \tag{3.1}$$

satisfies $\psi(z) \to \infty$ as $z \to \infty$.

(A3) The velocity matching coefficients $a_1 \equiv c_1, a_2 \equiv c_2, a_0 \equiv c_0$, where $c_1, c_2 \geq 0$ and $c_0 > 0$.

An example fulfilling (A1) and (A2) is the action function presented in [24]. It is an s-type function defined by

$$\phi(z) = \frac{a+b}{2}\sigma(z+c) + \frac{a-b}{2}, \tag{3.2}$$

where $\sigma(z) = z/\sqrt{1+z^2}$, $b \geq a > 0$, and $c = |a-b|/(2\sqrt{ab})$. The corresponding ψ is

$$\psi(z) = \frac{a+b}{2}\left(\sqrt{1+(z+c)^2} - \sqrt{1+c^2}\right) + \frac{a-b}{2}z.$$

By choosing the parameters as $a = 2, b = 3$, the graphs of $\phi(z)$ and $\psi(z)$ can be illustrated as following figures.
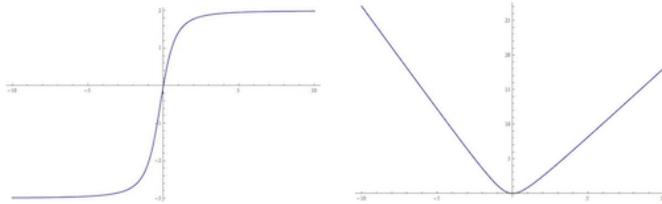
FIGURE 1. Functions $\phi(z)$ (left), and ψ (right)

Theorem 3.1. *Let* (A1)–(A3) *hold for system* (2.5). *Then the agents in groups* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *have a time-asymptotic flocking.*

Proof. The structural energy of system (2.5) is indicated by a Hamiltonian

$$H(x, u, y, v) = V(x, y) + K(u, v), \tag{3.3}$$

where $V(x, y)$ is the potential energy induced by the differences of positions which is defined by (2.4) and

$$K(u, v) = \frac{1}{2}\sum_{l=1}^{n_1}\|u_l\|^2 + \frac{1}{2}\sum_{i=1}^{n_2}\|v_i\|^2 \tag{3.4}$$

is the kinetic energy induced by velocities.

Let $(x(t), u(t), y(t), v(t))$ be the solution of system (2.5) with the initial value $(x(0), u(0), y(0), v(0)) \in \mathcal{M}$. With the assumptions (A1) and (A3), by differentiating $H(x, u, y, v)$ along the solution with the respect of time, we have

$$
\begin{aligned}
\dot{H} =& \frac{1}{2}\sum_{k \neq l}\phi(\|x_k - x_l\| - d_1)\langle \vec{e}(x_k, x_l), u_k - u_l\rangle + \sum_{l=1}^{n_1}\langle \frac{d}{dt}u_l, u_l\rangle \\
&+ \frac{1}{2}\sum_{i \neq j}\phi(\|y_j - y_i\| - d_2)\langle \vec{e}(y_j, y_i), v_j - v_i\rangle + \sum_{i=1}^{n_2}\langle \frac{d}{dt}v_i, v_i\rangle \\
&+ \sum_{l=1}^{n_1}\sum_{i=1}^{n_2}\phi(\|x_l - y_i\| - d_0)\langle \vec{e}(x_l, y_i), u_l - v_i\rangle \\
=& c_1\sum_{k=1}^{n_1}\sum_{l=1}^{n_1}\langle u_l, u_k - u_l\rangle + c_2\sum_{j=1}^{n_2}\sum_{i=1}^{n_2}\langle v_i, v_j - v_i\rangle + c_0\sum_{i=1}^{n_2}\sum_{l=1}^{n_1}\langle u_l, v_i - u_l\rangle
\end{aligned}
$$

$$= c_1 \sum_{k=1}^{n_1} \sum_{l=1}^{n_1} \langle u_l, u_k - u_l \rangle + c_2 \sum_{j=1}^{n_2} \sum_{i=1}^{n_2} \langle v_i, v_j - v_i \rangle + c_0 \sum_{i=1}^{n_2} \sum_{l=1}^{n_1} \langle u_l, v_i - u_l \rangle$$

$$= -\frac{1}{2} c_1 \sum_{k,l} \|u_k - u_l\|^2 - \frac{1}{2} c_2 \sum_{j,i} \|v_j - v_i\|^2 - \frac{1}{2} c_0 \sum_{i,l} \|v_i - u_l\|^2 \le 0,$$

and

$$\dot{H}(x, u, y, v) = 0 \iff u_l = v_i, \quad l = 1, \ldots, n_1, \ i = 1, \ldots, n_2. \tag{3.5}$$

Therefore,

$$H(x(t), u(t), y(t), v(t)) \le H_0 := H(x(0), u(0), y(0), v(0)), \quad t \ge 0.$$

Let

$$\Omega_0 = \{(x, u, y, v) \in \mathcal{M} : H(x, u, y, v) \le H_0\}.$$

Then Ω_0 is a positive invariant set of system (2.5). Furthermore, Ω_0 is bounded. In fact, for any $(x, u, y, v) \in \Omega_0$, it is clear that $K(u, v)$, $V_1(y)$, $V_2(x)$ are all less than H_0, so $\|u_l\|$ and $\|v_i\|$ are bounded, and by (A2), we deduce that $\|x_k - x_l\|$ and $\|y_j - y_i\|$ are bounded. As the fact that $(x, u, y, v) \in \mathcal{M}$, we obtain that $\|x_l\|$ and $\|y_i\|$ are bounded. Therefore, Ω_0 is a bounded positive invariant manifold.

From the LaSalle's invariance principle [19], $(x(t), u(t), y(t), v(t))$ converges to the largest invariant set in

$$S = \{(x, u, y, v) \in \Omega_0 : \dot{H}(x, u, y, v) = 0\}.$$

By (3.5), we obtain that for any $l = 1, \ldots, n_1, i = 1, \ldots, n_2$,

$$\|u_l(t) - v_i(t)\| \to 0, \quad \text{as } t \to \infty.$$

From this and the boundedness of positions, it follows that the agents have a time-asymptotic flocking. □

Theorem 3.1 shows that the two groups of agents with global interaction fulfilling the assumptions will unconditionally achieve flocking. However, in the real world, one agent may not pay attention to all other agents at the same time. Therefore, we will develop a more realistic model in the next section.

4. Collective dynamic under local interaction

In this section, we investigate the flocking system with local interaction, which means the interaction between any two agents is considered to have finite cut-off at a certain distance. As it is proposed in the above section, we used three different distances d_1, d_2, and d_0 to represent the balance distances of agents in $\mathcal{N}_1$, agents in $\mathcal{N}_2$, and agents from different groups. In the same way, to realize the cut-off of interaction between agents, we introduce three relevant non-influence distances r_1, r_2, and r_0 which are called the interaction radii. Similarly, r_0 is not less that r_1 and r_2. In addition, for the sake of convenience, we reorder the positions and velocities of agents in $\mathcal{N}$ as $q = (q_1, \ldots, q_n) = (x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2})$ and $p = (p_1, \ldots, p_n) = (u_1, \ldots, u_{n_1}, v_1, \ldots, v_{n_2})$.

With the above arguments, we carry out the following local interaction functions of agents in the two groups,

$$\phi_{ij} = \begin{cases} \phi_1(\|q_j - q_i\|) = \rho^1(\|q_j - q_i\|)\widehat{\phi}(\|q_j - q_i\| - d_1), & i,j = 1, \ldots, n_1, \\ \phi_2(\|q_j - q_i\|) = \rho^2(\|q_j - q_i\|)\widehat{\phi}(\|q_j - q_i\| - d_2), & i,j = n_1 + 1, \ldots, n, \\ \phi_0(\|q_j - q_i\|) = \rho^0(\|q_j - q_i\|)\widehat{\phi}(\|q_j - q_i\| - d_0), & \text{otherwise,} \end{cases} \tag{4.1}$$

where $\widehat{\phi}$ is a function fulfilling the assumptions (A1) and (A2), and

$$\rho^\alpha(s) = \rho\left(\frac{s}{r_\alpha}\right), \quad \alpha = 0, 1, 2,$$

are three bump functions with $\rho(z)$ a C^1-smooth, uniformly bounded, and monotonic decreasing function satisfying that $\rho(z) = 0$, for all $z \geq 1$ and $\rho(z) = 1$, for all $z \leq 1 - \varepsilon$, here $0 < \varepsilon \ll 1$ is a sufficiently small parameter. One suitable choice is the following function introduced in [24],

$$\rho(z) = \begin{cases} 1, & z \in (0, h), \\ \frac{1}{2}\left(1 + \cos(\pi\frac{z-h}{1-h})\right), & z \in [h, 1], \\ 0, & \text{otherwise,} \end{cases}$$

where $h = 1 - \varepsilon$. The use of these bump functions ensures that the interactions between the agents smoothly vanish as their distances go beyond the interaction radii.

Then, on the basis of the above discussion, we establish the following system to describe the dynamic of agents in two groups under local interaction,

$$\dot{q}_i = p_i,$$
$$\dot{p}_i = \sum_{j=1}^{n} \phi_{ij}\vec{e}(q_j, q_i) + \sum_{j=1}^{n} a_{ij}(p_j - p_i) \tag{4.2}$$

where

$$a_{ij} = a(\|q_j - q_i\|) = \begin{cases} c_1\rho^1(\|q_j - q_i\|), & i,j = 1, \ldots, n_1, \\ c_2\rho^2(\|q_j - q_i\|), & i,j = n_1 + 1, \ldots, n, \\ c_0\rho^0(\|q_j - q_i\|), & \text{otherwise,} \end{cases} \tag{4.3}$$

with c_1, c_2, and c_0 are positive constants.

Similar to the remark in section 2, the above system (4.2) naturally corresponds to a graph $G_{\mathcal{N}}(q) = (\mathcal{N}, \mathcal{E}_{\mathcal{N}}(q))$ with $(i,j) \in \mathcal{E}_{\mathcal{N}}(q)$ if and only if $a_{ij} = a(\|q_j - q_i\|) \neq 0$, $i,j = 1, \ldots, n$. Then, the structural energy of system (4.2) is

$$\widehat{H}(q, p) = \widehat{V}(q) + K(p),$$

with

$$\widehat{V}(q) = \frac{1}{2}\sum_{i,j=1}^{n_1} \psi_1(\|q_j - q_i\|) + \frac{1}{2}\sum_{i,j=n_1+1}^{n} \psi_2(\|q_j - q_i\|) \\ + \sum_{j=1}^{n_1}\sum_{i=n_1+1}^{n} \psi_0(\|q_j - q_i\|), \tag{4.4}$$

where

$$\psi_\alpha(z) = \int_{d_\alpha}^{z} \phi_\alpha(s)\,ds, \quad \alpha = 0, 1, 2. \tag{4.5}$$

We remark that $\psi_\alpha(z)$ has the following properties:

$$\psi_\alpha(z) \equiv h_\alpha := \psi_\alpha(r_\alpha), \quad z \geq r_\alpha,$$
$$\psi_\alpha(z) < h_\alpha, \quad d_\alpha \leq z < r_\alpha, \quad \alpha = 0, 1, 2. \tag{4.6}$$

Now we can state the following conditional flocking result for the local model.

Theorem 4.1. *Let ϕ_{ij} and a_{ij} in system (4.2) be given by (4.1) and (4.3), respectively. If for any $t \geq 0$, $\mathcal{N}$ is a cluster with connected graph $G_\mathcal{N} = (\mathcal{N}, \mathcal{E}_\mathcal{N})$, then the agents have a time-asymptotic flocking.*

Proof. Let $(q(t), p(t))$ be the solution of system (4.2) with $(q(0), p(0)) \in \mathcal{M}$. By differentiating $\widehat{H}(q, p)$ along the solution with the respect of time, we have

$$
\begin{aligned}
\dot{\widehat{H}} =& \frac{1}{2} \sum_{i,j=1}^{n_1} \phi_1(\|q_j - q_i\|)\langle \vec{e}(q_j, q_i), p_j - p_i \rangle \\
&+ \frac{1}{2} \sum_{i,j=n_1+1}^{n} \phi_2(\|q_j - q_i\|)\langle \vec{e}(q_j, q_i), p_j - p_i \rangle \\
&+ \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n} \phi_0(\|q_j - q_i\|)\langle \vec{e}(q_j, q_i), p_j - p_i \rangle + \sum_{i=1}^{n} \langle \frac{d}{dt} p_i, p_i \rangle \\
=& -\frac{1}{2} \sum_{i,j=1}^{n_1} c_1 \rho^1(\|q_j - q_i\|)\langle p_j - p_i, p_j - p_i \rangle \\
&- \frac{1}{2} \sum_{i,j=n_1+1}^{n} c_2 \rho^2(\|q_j - q_i\|)\langle p_j - p_i, p_j - p_i \rangle \\
&- \frac{1}{2} \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n_2} c_0 \rho^0(\|q_j - q_i\|)\langle p_j - p_i, p_j - p_i \rangle \\
=& -\frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \|p_j - p_i\|^2 \leq 0,
\end{aligned}
\tag{4.7}
$$

therefore,

$$\widehat{H}(q(t), p(t)) \leq \widehat{H}_0 := \widehat{H}(q(0), p(0)). \tag{4.8}$$

We claim that if $\dot{\widehat{H}}(q, p) = 0$, then $p_j - p_i = 0, i, j = 1, \ldots, n$. For any agents $(q_j, p_j), (q_i, p_i) \in \mathcal{N}$, by the connectivity of $G_\mathcal{N}(q) = (\mathcal{N}, \mathcal{E}_\mathcal{N}(q))$, there exists a walk $(k_1, \ldots, k_m)$ which goes through agents $(q_{k_1}, p_{k_1}), \ldots, (q_{k_m} p_{k_m})$ in $\mathcal{N}$ with $(q_{k_1}, p_{k_1}) = (q_j, p_j)$, $(q_{k_m}, p_{k_m}) = (q_i, p_i)$, and $a_{k_i k_{i-1}} > 0$, $i = 1, \ldots, m$. By (4.7), we can deduce that $p_{k_1} - p_{k_2} = p_{k_2} - p_{k_3} = \cdots = p_{k_{m-1}} - p_{k_m} = 0$, i.e., $p_j - p_i = 0$.

Let $\widehat{\Omega} = \widehat{\Omega}_0 \cap \Sigma$ with

$$\widehat{\Omega}_0 = \{(q, p) \in \mathcal{M} : \widehat{H}(q, p) \leq \widehat{H}_0\}, \quad \Sigma = \{(q, p) \in \mathcal{M} : \{q_1, \ldots, q_n\} \text{ is a cluster}\}.$$

Since for any $t > 0$, $G_\mathcal{N}$ is connected, combining (4.8), we know that $\widehat{\Omega}$ is a positive invariant manifold of system (4.2). Furthermore, for any $(q, p) \in \widehat{\Omega}$, we have $K(p) < \widehat{H}_0$ and $\|p\|$ is bounded, and $\{q_1, \ldots, q_n\}$ is a cluster, which leads to $\|q\|$ is bounded. Thus, $\widehat{\Omega}$ is a bounded positive invariant manifold of (4.2).

Then, from LaSalle's invariance principles, the solution $(q(t), p(t))$ of (4.2) starting in $\widehat{\Omega}$ converges to the largest invariant set in

$$S = \{(q, p) \in \widehat{\Omega} | \dot{\widehat{H}}(q, p) = 0\}.$$

From (4.7), we can deduce that for $i, j = 1, \ldots, n$,

$$\|p_j(t) - p_i(t)\| \to 0, \quad \text{as } t \to \infty.$$

The boundedness of $\|q\|$ for all time t can be deduced by the connectivity of $G_{\mathcal{N}}(q)$, consequently, the agents have a time-asymptotic flocking. $\qquad\square$

Remark 4.2. Because of the dissipation of $\widehat{H}$, the assumption that $\mathcal{N}$ is a cluster at any time could be replaced by proposing a constraint on the initial value of the energy function, for example, $\widehat{H}_0 \leq \min\{h_\alpha, \psi_\alpha(0), \alpha = 0, 1, 2\}$.

Besides the above conditional flocking consideration, we investigate the stability of a class of flocking solutions which can be, in fact, entirely viewed as an invariant manifold of system (4.2). We first present the following propositions.

Proposition 4.3. *For $q \in \mathcal{M}_1$, if $G_{\mathcal{N}}(q)$ is connected, then $q \in B := \{q \in \mathcal{M}_1 | \|q\| \leq n^2 r_0\}$.*

Proof. If $G_{\mathcal{N}}(q)$ is connected, then $\max\{\|q_j - q_i\|, i, j = 1, \ldots, n\} \leq n r_0$, thus

$$\|q_i\| = \frac{1}{n}\|n q_i\| = \frac{1}{n}\|(n-1)q_i - \sum_{k \neq i} q_k\| \leq \frac{1}{n} \sum_{k \neq i} \|q_k - q_i\| \leq (n-1)r_0.$$

Consequently, $\|q\| \leq n(n-1)r_0 < n^2 r_0$ and this leads to $q \in B$. $\qquad\square$

Proposition 4.4. *For any $q \in \mathcal{M}_1$, if $G_{\mathcal{N}}(q)$ is disconnected, then there exist a $q^* \in \mathcal{M}_1$ such that $G_{\mathcal{N}}(q^*)$ is connected and $\widehat{V}(q^*) < \widehat{V}(q)$.*

Proof. Since $G_{\mathcal{N}}(q)$ is disconnected, the agents in $\mathcal{N}$ can be decomposed into two sets $\mathcal{V}_1$, $\mathcal{V}_2$, where $\mathcal{V}_1$ is a cluster, as well, $\mathcal{V}_1$ and $\mathcal{V}_2$ are a pair of separated sets. Let $\mathcal{V}_{ij} = \mathcal{V}_i \cap \mathcal{N}_j$, $i, j = 1, 2$. Then $\mathcal{V}_1 = \mathcal{V}_{11} \cup \mathcal{V}_{12}$, $\mathcal{V}_2 = \mathcal{V}_{21} \cup \mathcal{V}_{22}$, and we note that

$$(q^i, p^i) \in \mathcal{V}_i, \quad (q^{ij}, p^{ij}) = ((q_1^{ij}, p_1^{ij}), \ldots, (q_{n_{ij}}^{ij}, p_{n_{ij}}^{ij})) \in \mathcal{V}_{ij}, \quad i, j = 1, 2,$$

where n_{ij} is the number of agents in $\mathcal{V}_{ij}$. Then, by (4.4), we derive

$$\widehat{V}(q) = V^1(q^1) + V^2(q^2) + n_{11}n_{21}h_1 + n_{12}n_{22}h_2 + n_{12}n_{21}h_0 + n_{11}n_{22}h_0,$$

where

$$V^1(q^1) = \frac{1}{2}\sum_{i,j=1}^{n_{11}} \psi_1(\|q_j^{11} - q_i^{11}\|) + \frac{1}{2}\sum_{i,j=1}^{n_{12}} \psi_2(\|q_j^{12} - q_i^{12}\|) + \sum_{j=1}^{n_{11}}\sum_{i=1}^{n_{12}} \psi_0(\|q_j^{11} - q_i^{12}\|),$$

$$V^2(q^2) = \frac{1}{2}\sum_{i,j=1}^{n_{21}} \psi_1(\|q_j^{21} - q_i^{21}\|) + \frac{1}{2}\sum_{i,j=1}^{n_{22}} \psi_2(\|q_j^{22} - q_i^{22}\|) + \sum_{j=1}^{n_{21}}\sum_{i=1}^{n_{22}} \psi_0(\|q_j^{21} - q_i^{22}\|).$$

As $\mathcal{V}_1$ and $\mathcal{V}_2$ are separated sets, we have

$$\delta_1 := d(\mathcal{V}_{11}, \mathcal{V}_{22}) - r_0 > 0, \quad \delta_2 := d(\mathcal{V}_{11}, \mathcal{V}_{21}) - r_1 > 0,$$
$$\delta_3 := d(\mathcal{V}_{12}, \mathcal{V}_{21}) - r_0 > 0, \quad \delta_4 := d(\mathcal{V}_{12}, \mathcal{V}_{22}) - r_2 > 0.$$

For $A_1, A_2 \subset \mathcal{N}$, we define

$$d(A_1, A_2) = \min\{\|q_j - q_i\|, \ (q_i, p_i) \in A_1, (q_j, p_j) \in A_2\}.$$

Without loss of generality, we suppose that $\delta_1 = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$. Then there exist $(q_{i_0}^{11}, p_{i_0}^{11}) \in \mathcal{V}_{11}, (q_{j_0}^{22}, p_{j_0}^{22}) \in \mathcal{V}_{22}$, such that

$$\|q_{i_0}^{11} - q_{j_0}^{22}\| - r_0 = \delta_1 > 0. \tag{4.9}$$

Let $q^* = (q^{1*}, q^{2*})$ with

$$q^{1*} = q^1 - \vec{e}(q_{i_0}^{11}, q_{j_0}^{22})\delta, \quad q^{2*} = q^2 + \vec{e}(q_{i_0}^{11}, q_{j_0}^{22})\frac{n_{11} + n_{12}}{n_{21} + n_{22}}\delta,$$

where δ is a positive constant satisfying

$$(1 + \frac{n_{11} + n_{12}}{n_{21} + n_{22}})\delta = \delta_1 + \epsilon$$

for some sufficiently small $\epsilon > 0$. Then we have

$$\|q_j^{11*} - q_i^{11*}\| = \|q_j^{11} - q_i^{11}\|, \quad i, j = 1, \ldots, n_{11},$$
$$\|q_j^{12*} - q_i^{12*}\| = \|q_j^{12} - q_i^{12}\|, \quad i, j = 1, \ldots, n_{12},$$
$$\|q_j^{21*} - q_i^{21*}\| = \|q_j^{21} - q_i^{21}\|, \quad i, j = 1, \ldots, n_{21},$$
$$\|q_j^{22*} - q_i^{22*}\| = \|q_j^{22} - q_i^{22}\|, \quad i, j = 1, \ldots, n_{22};$$

therefore,

$$V^1(q^{1*}) = V^1(q^1), \quad V^2(q^{2*}) = V^2(q^2). \tag{4.10}$$

Meanwhile, we infer that

$$\begin{aligned}
\|q_{i_0}^{11*} - q_{j_0}^{22*}\| &= \left\|\left(\|q_{i_0}^{11} - q_{j_0}^{22}\| - \delta - \frac{n_{11} + n_{12}}{n_{21} + n_{22}}\delta\right)\vec{e}(q_{i_0}^{11}, q_{j_0}^{22})\right\| \\
&= \|q_{i_0}^{11} - q_{j_0}^{22}\| - (\delta_1 + \epsilon) \\
&= r_0 - \epsilon.
\end{aligned} \tag{4.11}$$

Then as $\delta_2 \geq \delta_1$,

$$\begin{aligned}
\|q_i^{11*} - q_j^{21*}\| &= \|q_i^{11} - q_k^{21} - \left(\delta + \frac{n_{11} + n_{12}}{n_{21} + n_{22}}\delta\right)\vec{e}(q_{i_0}^{11}, q_{j_0}^{22})\| \\
&\geq \|q_i^{11} - q_k^{21}\| - (\delta_1 + \epsilon) \\
&\geq \delta_2 + r_1 - (\delta_1 + \epsilon) \\
&\geq r_1 - \epsilon, \quad i = 1, \ldots, n_{11}, \ k = 1, \ldots, n_{21}.
\end{aligned}$$

Similarly,

$$\|q_j^{22*} - q_l^{12*}\| \geq r_2 - \epsilon, \quad j = 1, \ldots, n_{22}, \ l = 1, \ldots, n_{12},$$
$$\|q_k^{21*} - q_l^{12*}\| \geq r_0 - \epsilon, \quad k = 1, \ldots, n_{21}, \ l = 1, \ldots, n_{12}.$$

By (4.9), for any $i \neq i_0$ and $j \neq j_0$,

$$\|q_i^{11*} - q_j^{22*}\| \geq \|q_i^{11} - q_j^{22}\| - (\delta_1 + \epsilon) \geq \|q_{i_0}^{11} - q_{j_0}^{22}\| - (\delta_1 + \epsilon) = r_0 - \epsilon. \tag{4.12}$$

Combining (4.11)-(4.12), with the properties shown in (4.6), we have

$$\sum_{i=1}^{n_{11}}\sum_{j=1}^{n_{22}} \psi_0(\|q_j^{22*} - q_i^{11*}\|) < n_{11}n_{22}h_0, \quad \sum_{i=1}^{n_{11}}\sum_{j=1}^{n_{21}} \psi_1(\|q_j^{21*} - q_i^{11*}\|) \leq n_{11}n_{21}h_1,$$
$$\sum_{i=1}^{n_{12}}\sum_{j=1}^{n_{22}} \psi_2(\|q_j^{22*} - q_i^{12*}\|) \leq n_{12}n_{22}h_2, \quad \sum_{i=1}^{n_{12}}\sum_{j=1}^{n_{21}} \psi_0(\|q_j^{21*} - q_i^{12*}\|) \leq n_{12}n_{21}h_0.$$

Adding the above expression and (4.10), we have $\widehat{V}(q^*) < \widehat{V}(q)$.

If $G_{\mathcal{N}}(q^*)$ is still disconnected, we can reuse the above process to get an new q^* such that $G_{\mathcal{N}}(q^*)$ is connected and $\widehat{V}(q^*) < \widehat{V}(q)$.   □

Let $M = M_q \times M_p$ with

$$M_q = \big\{q \in \mathcal{M}_1 : \widehat{V}(q) = \min_{y \in \mathcal{M}_1}\{\widehat{V}(y)\}\big\}, \quad M_p = \{p \in \mathcal{M}_2 | p_i = 0\}.$$

We then state the following properties of M_q.

Lemma 4.5. M_q *is a nonempty compact subset of* $\mathcal{M}_1$.

Proof. For any $q \in \mathcal{M}_1$ with $G_{\mathcal{N}}(q)$ disconnected, by Proposition 4.3, there exist a $q^* \in \mathcal{M}_1$ such that $G_{\mathcal{N}}(q^*)$ is connected and $\widehat{V}(q^*) < \widehat{V}(q)$. By Proposition 4.1, we know that $q^* \in B$. Thus, the minimum of $\widehat{V}(q)$ in $\mathcal{M}_1$ is just the minimum in B.

Then from (4.4), $\widehat{V}(q)$ is a continuous function which must have minimum in B. Thus, M_q is not empty. Meanwhile, $M_q \subseteq B$, therefore M_q is compact.   □

By Lemma 4.5, M is nonempty. Moreover, for any $(q,p) \in M$, it is obviously that (q,p) is a flocking solution of system (4.2), thus, M is an invariant manifold of (4.2) and $\dim(M) = \dim(M_q)$.

To analyze the local stability of M, we ought to investigate the Jacobian matrix of the system (4.2) calculated on $(q,p) \in M$. Noting that

$$J = \begin{pmatrix} 0 & I \\ \mathcal{G} & -\mathcal{L} \end{pmatrix}, \tag{4.13}$$

where $\mathcal{G} = -\nabla^2 \widehat{V}(q)$, $\mathcal{L} = L \otimes I$ with L is the Laplacian matrix defined by (2.1) and I is unit matrices with corresponding order, we have the following observations.

Lemma 4.6. $\ker(\mathcal{L}) \subset \ker(\mathcal{G})$.

Proof. Noting that $\ker(\mathcal{L}) \subset \ker(\mathcal{G})$ is just equivalent to that if $\mathcal{L}\xi = \mathbf{0}$ for some $\xi \in \mathbb{R}^d$, then $\mathcal{G}\xi = \mathbf{0}$. Meanwhile, one can induce by (2.1) that if $L\xi = \mathbf{0}$, then $\xi = \xi^0 \mathbf{1}^d$ with $\xi^0 \in \mathbb{R}$. Hence, we only need to prove that $\mathcal{G}\mathbf{1}^d = \mathbf{0}$.

We observe that $\mathcal{G} := [\mathcal{G}_{ij}]_{n \times n}$, where $\mathcal{G}_{ij} = -\nabla_j \nabla_i \widehat{V}(q)$ is a $d \times d$ matrix. Therefore, $\mathcal{G}\mathbf{1}^d = \mathbf{0}$ if and only if

$$\sum_{j=1}^{n} \mathcal{G}_{ij} = O, \quad i = 1, \ldots, n, \tag{4.14}$$

where O is a $d \times d$ zero matrix.

To prove (4.14), we rewrite $\widehat{V}(q)$ as

$$\widehat{V}(q) = \frac{1}{2}\sum_{k,l=1}^{n_1} U_1(\|q_k - q_l\|^2) + \frac{1}{2}\sum_{i,j=n_1+1}^{n} U_2(\|q_j - q_i\|^2)$$

$$+ \sum_{k=1}^{n_1}\sum_{i=n_1+1}^{n} U_0(\|q_k - q_i\|^2),$$

where $U_\alpha(s) = \psi_\alpha(s^{1/2})$, $s \geq 0$, $\alpha = 0, 1, 2$.

For $i \in \{n_1 + 1, \ldots, n\}$, $k \in \{1, \ldots, n_1\}$ and $j \in \{n_1 + 1, \ldots, n\}$, we have

$$\nabla_i \widehat{V}(q) = \nabla_i \sum_{j=n_1+1}^{n} U_2(\|q_j - q_i\|^2) + \nabla_i \sum_{k=1}^{n_1} U_0(\|q_k - q_i\|^2)$$

$$= 2 \sum_{j=n_1+1}^{n} \frac{\partial}{\partial q_i} U_2(\|q_j - q_i\|^2)(q_j - q_i) + 2 \sum_{k=1}^{n_1} \frac{\partial}{\partial q_i} U_0(\|q_k - q_i\|^2)(q_k - q_i),$$

$$\nabla_i^2 \widehat{V}(q) = 2 \Big(\sum_{j=n_1+1}^{n} \frac{\partial}{\partial q_i} U_2(\|q_j - q_i\|^2) \Big) I_d + 2 \Big(\sum_{k=1}^{n} \frac{\partial}{\partial q_i} U_0(\|q_k - q_i\|^2) \Big) I_d$$

$$+ 4 \sum_{j=n_1+1}^{n} \frac{\partial^2}{\partial q_i \partial q_j} U_2(\|q_j - q_i\|^2) \|q_j - q_i\|^2$$

$$+ 4 \sum_{k=1}^{n} \frac{\partial^2}{\partial q_i \partial q_k} U_0(\|q_k - q_i\|^2) \|q_k - q_i\|^2,$$

$$\nabla_j \nabla_i \widehat{V}(q) = -2 \Big(\frac{\partial}{\partial q_i} U_2(\|q_j - q_i\|^2) \Big) I_d - 4 \frac{\partial^2}{\partial q_i \partial q_j} U_2(\|q_j - q_i\|^2) \|q_j - q_i\|^2,$$

$$\nabla_k \nabla_i \widehat{V}(q) = -2 \Big(\frac{\partial}{\partial q_i} U_0(\|q_k - q_i\|^2) \Big) I_d - 4 \frac{\partial^2}{\partial q_i \partial q_k} U_0(\|q_k - q_i\|^2) \|q_k - q_i\|^2,$$

which leads to

$$\sum_{j=1}^{n} \mathcal{G}_{ij} = \mathcal{G}_{ii} + \sum_{k=1}^{n_1} \mathcal{G}_{ik} + \sum_{j=n_1+1}^{n} \mathcal{G}_{ij} = O.$$

In the similar way, (4.14) also holds for $i \in \{1, \ldots, n_1\}$, $k \in \{1, \ldots, n_1\}$ and $j \in \{n_1 + 1, \ldots, n\}$. This completes the proof. $\square$

Proposition 4.7. $\dim(\ker(\mathcal{G})) \geq \dim(M_q)$.

Proof. For any smooth curve $\Gamma(s) \in M_q$ with $\Gamma(0) = q \in M_q$, we have

$$\nabla \widehat{V}(\Gamma(s)) = 0, \quad \nabla^2 \widehat{V}(\Gamma(s)) \frac{d}{ds} \Gamma(s) = 0,$$

this means that $\frac{d}{ds}\Gamma(s)\big|_{s=0}$, the tangent vector of M_q at q, belongs to $\ker(\mathcal{G})$. Therefore, the dimension of $\ker(\mathcal{G})$ is not less than that of the tangent space of M_q. Thus $\dim(\ker(\mathcal{G})) \geq \dim(M_q)$. $\square$

Now we can state and prove a result about the local stability of two-groups flocking solutions under local interaction.

Theorem 4.8. *Let ϕ_{ij} and a_{ij} in system (4.2) be given by (4.1) and (4.3), respectively. Assuming that $\dim(\ker(\mathcal{G})) = \dim(M_q)$, then the manifold M is locally asymptotically stable.*

Proof. It is not difficult to see that $(u, v)^\top \in \ker(J)$ if and only if

$$\begin{aligned} v &= 0, \\ \mathcal{G}u - \mathcal{L}v &= 0, \end{aligned} \tag{4.15}$$

or equivalently, $u \in \ker(\mathcal{G})$ and $v = 0$, so $\dim(\ker(J)) = \dim(\ker(\mathcal{G}))$. Combining this with the assumption that $\dim(\ker(\mathcal{G})) = \dim(M_q)$, we obtain

$$\dim(\ker(J)) = \dim(M_q) = \dim(M).$$

Now we show that if $\lambda \neq 0$ is an eigenvalue of J, then $\mathrm{Re}(\lambda) < 0$. Let $(\xi, \eta)^\top$ be the eigenvector corresponding to λ, then

$$\eta = \lambda \xi,$$
$$\mathcal{G}\xi - \mathcal{L}\eta = \lambda \eta,$$

which leads to

$$\lambda^2 \xi + \lambda \mathcal{L}\xi - \mathcal{G}\xi = 0.$$

premultiplying by $\xi^\top$, we solve the eigenvalue λ as

$$\lambda = \frac{-\xi^\top \mathcal{L}\xi \pm \sqrt{(\xi^\top \mathcal{L}\xi)^2 + 4\|\xi\|^2 \xi^\top \mathcal{G}\xi}}{2\|\xi\|^2}.$$

From Lemma 4.6 and the property of L in (2.2), we conclude that $\xi^\top \mathcal{L}\xi > 0$. Moreover, since $q \in M_q$ is a minimum point, $\xi^\top \mathcal{G}\xi \leq 0$. As mentioned above, we get $\mathrm{Re}(\lambda) < 0$.

Combining this with Lemma 4.5, we state that M is a normally hyperbolic invariant manifolds on $\mathcal{M}$, which induce that M is locally asymptotically stable. $\square$

Remark 4.9. 1. Theorem 4.8 leads to the flocking solutions in manifold M being locally asymptotically stable.

2. The assumption $\dim(\ker(\mathcal{G})) = \dim(M_q)$ can happen in many situations. For example, when $n_1 = n_2 = 1$, $\phi_\alpha(z) = \rho^\alpha z$, and $d_\alpha < r_\alpha \leq \sqrt{2}d_\alpha$, $\alpha = 0, 1, 2$, we have $\dim(\ker(\mathcal{G})) = \dim(M_q) = d$.

5. Numerical simulations

We have proved in the above sections that the systems of two groups with global and local interaction could both achieve flocking under some proper assumptions. In this section, some numerical results will be presented to show the above conclusions visually. In addition, we would like to remark that the relationship between the parameters and the final configuration of the flocking solutions is also a notable problem, and we will display a few numerical results to show the related phenomena in subsection 5.3.

In all the following figures, the positions and the velocities of agents in $\mathcal{N}_1$ are showed by red stars and red arrows, while in $\mathcal{N}_2$ are showed by blue squares and blue arrows. For the two groups of agents with global interaction, they follow system (2.5) in section 3, where ϕ is chosen as (3.2). For agents with local interaction, they follow system (4.2) in section 4 with $\hat{\phi}$ chosen as (3.2). Both the parameters in ϕ and $\hat{\phi}$ are chosen as $a = 1, b = 7$. For the velocity matching coefficients, we find that their differences do not have much effect on the final configurations, thus for convenience, we choose $c_0 = c_1 = c_2 = 1$.

5.1. **One group versus two groups.** We present four sets of figures showing the time-asymptotical flocking of one group and two groups of agents under global and local interactions, respectively. The number of agents is $n = 30$ with $n_1 = n_2 = 15$. It is clear that if the balance distances are chosen as $d_0 = d_1 = d_2$, the two-groups system turns into a one-group system. The initial positions of all the four sets of figures are chosen at random from $[0, 160]^2$ and the initial velocities are chosen at random from $[0, 10]^2$.

Figures 2–5 show that for random initial data fulfilling the connective requirement of the local model, both two systems can get a flocking consensus. Meanwhile,
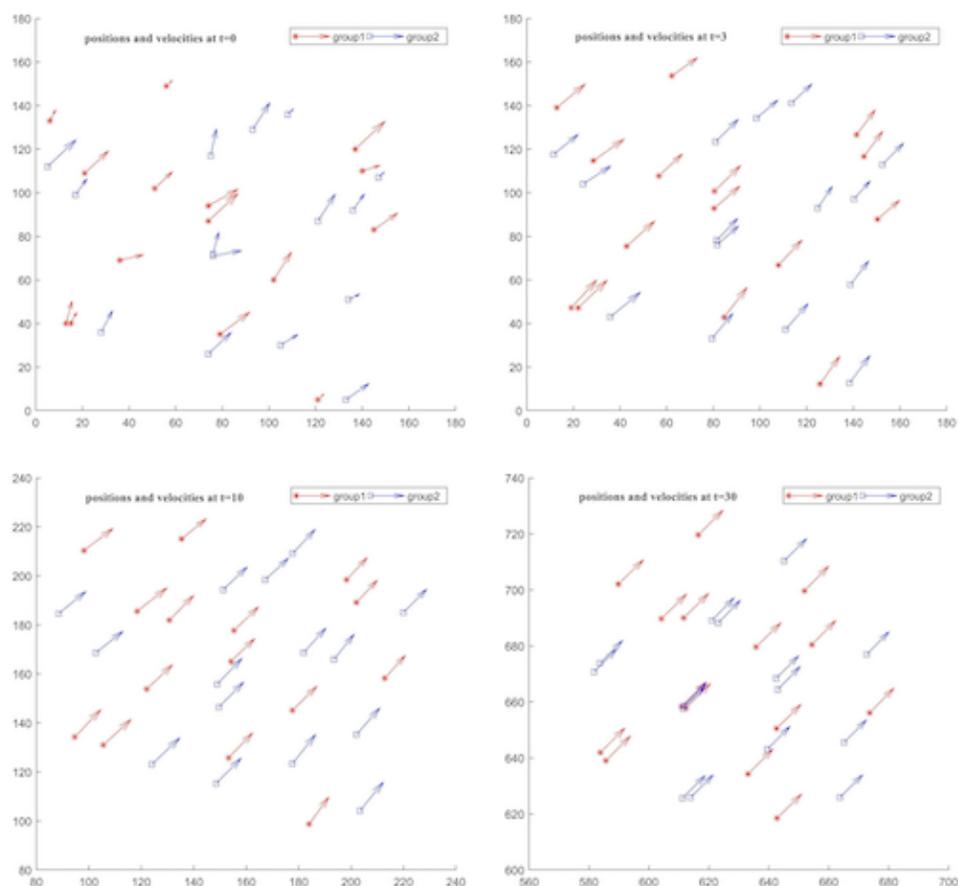
FIGURE 2. One-group system under global interactions with $d_0 = d_1 = d_2 = 40$.

when considering the patterns, there shows to have some differences among these figures: for the one-group system, the agents mix together at the final states, while for the two-groups system, the agents present a layering phenomenon as they tend to get close with agents in their own group. Furthermore, we can also find that under local interactions, the agents have a clear lattice structure, while under global interactions, they do not have a regular pattern.

5.2. **Global interaction versus local interaction.** In this subsection, we present three sets of figures to show the effects on the final flocking states from connectivity of the initial data.

In Figures 6–8, the initial velocities are chosen at random from the set $[-5, 15]^2$ and the initial positions are given in the corresponding captions. The balance distances are chosen as $d_1 = d_2 = 40$, $d_0 = 80$, and in the local model, the interaction radii are chosen as $r_1 = r_2 = 56$, $r_0 = 100$. The flocking states of local and global systems are separately presented in parts (b) and (c) of each figure.
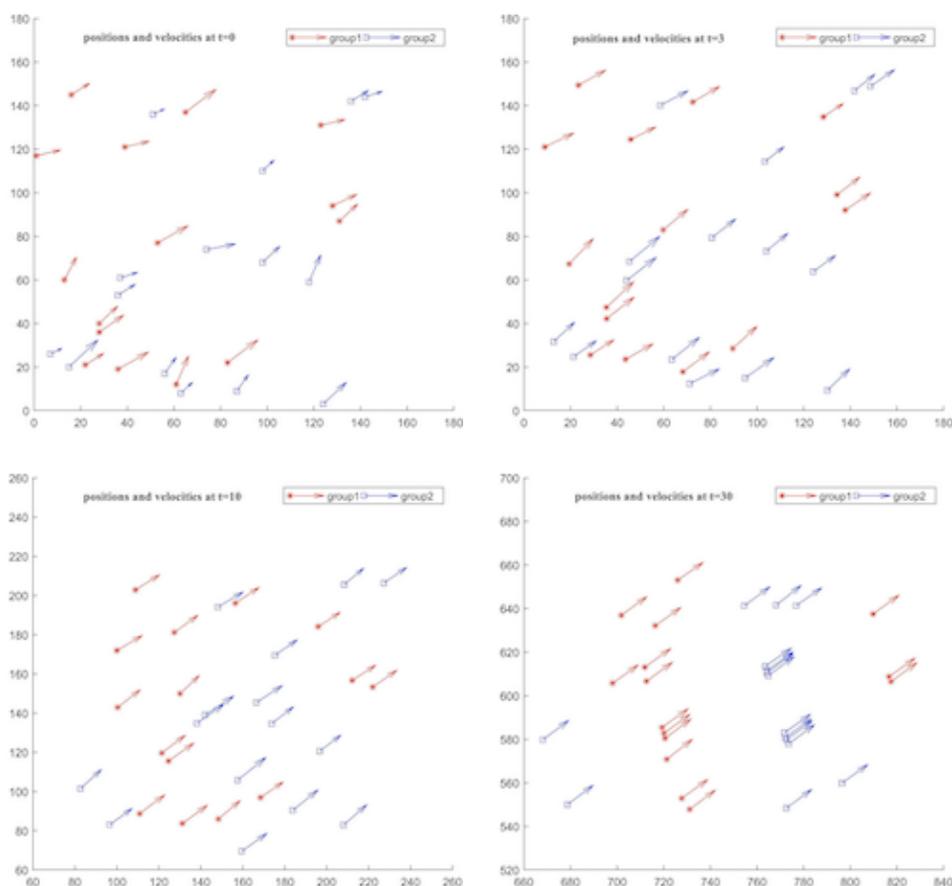
FIGURE 3. Two-group system under global interactions with $d_1 = d_2 = 40$ and $d_0 = 60$.

From the numerical results, we see that under global interactions, no matter the initial data is connected or not, the agents always tend to form a whole cluster and achieve flocking. While under local interactions, the connectivity of initial data could affect the flocking of the agents and cause different final states: Figure 6 shows that the initial data are totally connected and the agents achieve flocking with $d(\mathcal{N}_1, \mathcal{N}_2) = d_0 = 80$; Figure 7 shows that $\mathcal{N}_1$ and $\mathcal{N}_2$ are a pair of separated sets in the initial state and the agents do not achieve flocking all along the time; Figure 8 shows that the initial data of $\mathcal{N}_2$ are not connected, and in the final state, the agents in $\mathcal{N}_2$ separated into two parts, one part form a cluster with $\mathcal{N}_1$ while the other part get away from the cluster. We remark that for some disconnected initial data, the agents do not achieve flocking under local interactions.

5.3. **Interlaced versus separated.** As it has been referred in subsection 5.1, the change of the balance distances and interaction radii could ultimately affect the patterns of the two groups. Hence in this subsection, we will use three sets of figures to perform the effect on the patterns caused by d_0 and r_0. Here, the agents
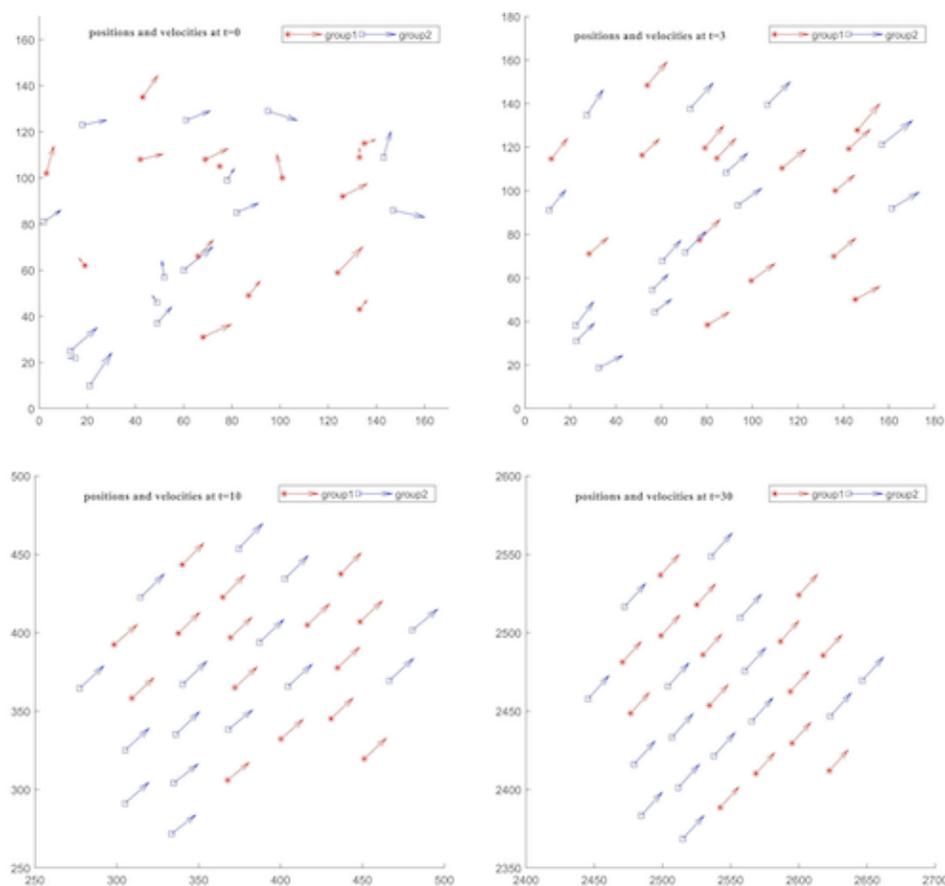
FIGURE 4. One-group system under local interactions with $d_0 = d_1 = d_2 = 40$ and $r_0 = r_1 = r_2 = 56$.

are considered under the local interactions, and three sets of initial data are chosen to show this phenomenon.

To describe the interlacing state of the two groups $\mathcal{N}_1, \mathcal{N}_2$, we introduce the following criterion: if $d(\mathcal{N}_1, \mathcal{N}_2) = d_0$ and for both of the two groups, there exist three agents A, B, C from one groups and one agent O from the other group, such that O inside $\triangle ABC$, then $\mathcal{N}_1, \mathcal{N}_2$ are interlaced. Otherwise, $\mathcal{N}_1, \mathcal{N}_2$ are strongly separated.

For Figures 9–11, the initial velocities of agents are chosen at random from the set $[0, 20]^2$ and the initial positions are given in their captions. The balance distances and interaction radii inside the two groups are chosen as $d_1 = d_2 = 40$ and $r_1 = r_2 = 56$. Then we gradually increased d_0 and r_0 to find the variation of the configuration. In each set of figures, we present part (b) as a one-group contrast, part (c) as an interlaced pattern and part (d) as a strongly separated pattern. Their relative d_0 and r_0 are placed in the caption.
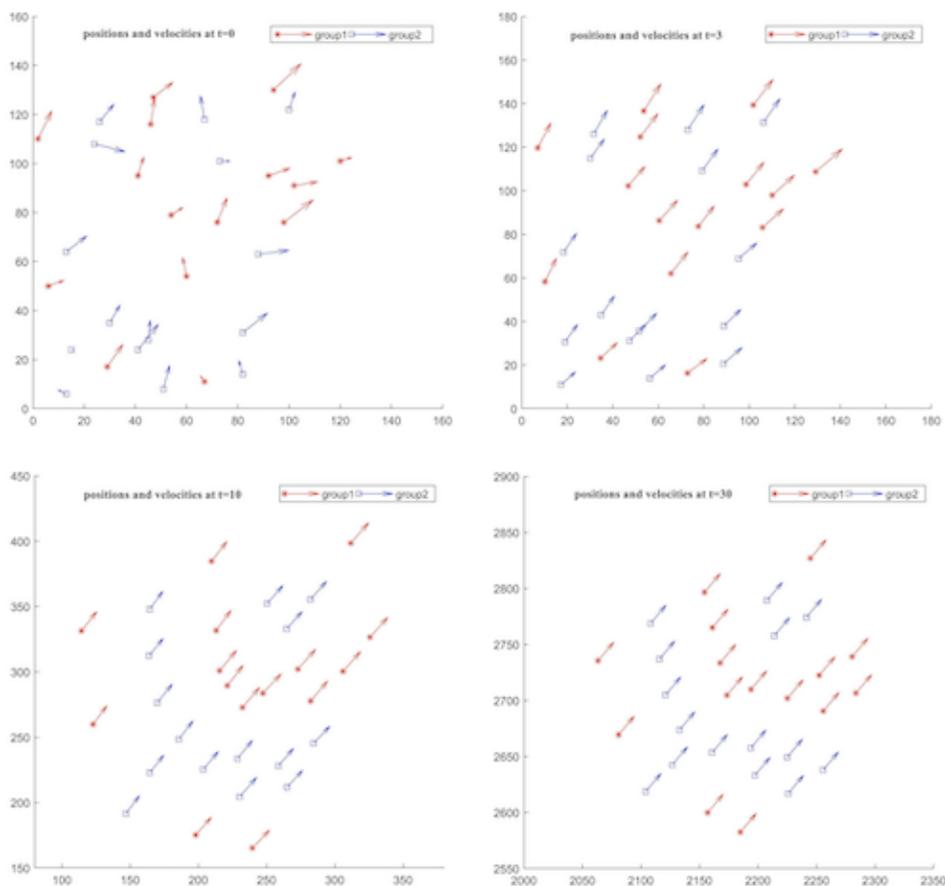
FIGURE 5. Two-group system under local interactions with $d_1 = d_2 = 40$, $d_0 = 60$ and $r_1 = r_2 = 56$ and $r_0 = 84$.

(a) Initial status   (b) Local model   (c) Global model
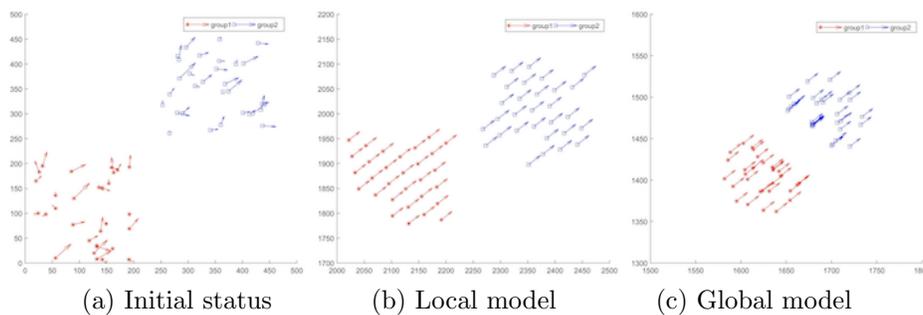
FIGURE 6. The initial positions of agents in $\mathcal{N}_1$ and $\mathcal{N}_2$ are chosen at random from $[0, 200]^2$ and $[250, 450]^2$.

For initial data in Figures 9–11, we can always find a d_0 such that the two groups are interlaced, and another d_0 large enough such that $\mathrm{d}(\mathcal{N}_1, \mathcal{N}_2) = d_0$ and the two

(a) Initial status   (b) Local model   (c) Global model
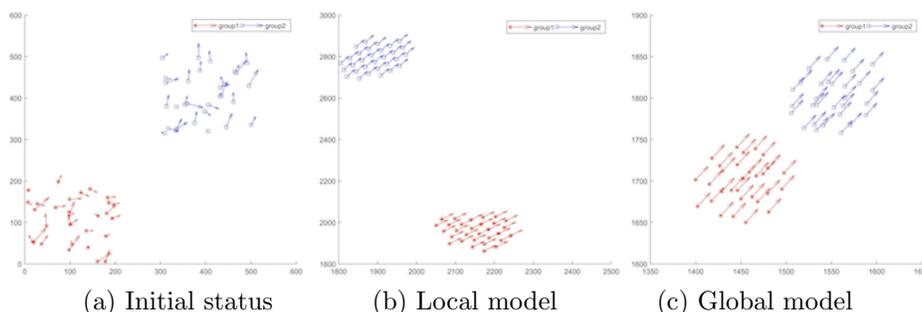
FIGURE 7. The initial positions of agents in $\mathcal{N}_1$ and $\mathcal{N}_2$ are chosen at random from $[0, 200]^2$ and $[300, 450]^2$.

(a) Initial status   (b) Local model   (c) Global model
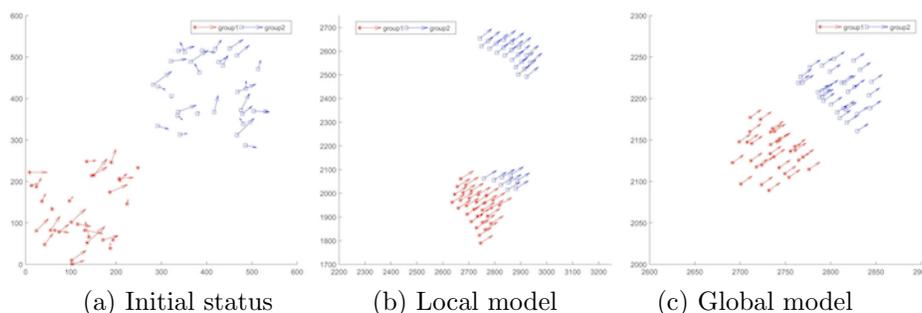
FIGURE 8. The initial positions of agents in $\mathcal{N}_1$ and $\mathcal{N}_2$ are chosen as random from $[0, 250]^2$ and $[300, 550]^2$.

groups are strongly separated. Thus we would like to remark that the larger the difference between d_0 and d_1, d_2, the stronger the tendency of agents in different groups to be separated.

REFERENCES

[1] H. Bae, S. Cho, S. Lee, S. Yun; *A particle model for the herding phenomena induced by dynamic market signals*, Journal of Statistical Physics, **177** (2019), 365–398.

[2] E. Ben-Jacob; *Bacterial self-organization: co-enhancement of complexification and adaptability in a dynamic environment*, Philosophical Transactions of the Royal Society, **361** (2003), 1283–1312.

[3] J. A. Carrillo, M. Fornasier, J. Rosado, G. Toscani; *Asymptotic flocking dynamics for the kinetic cucker smale model*, SIAM Journal on Mathematical Analysis, **42** (2009), 218–236.

[4] J. A. Carrillo, M. Fornasier, G. Toscani, F. Vecil; *Particle, kinetic, and hydrodynamic models of swarming*, Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences, Series: Modelling and Simulation in Science and Technology, Birkhäuser, 2010, pp. 297–336.

(a) Initial status

(b) $d_0 = 40$ and $r_0 = 56$

(c) Interlaced: $d_0 = 54$ and $r_0 = 76$ (d) Strongly separated: $d_0 = 70$ and $r_0 = 98$
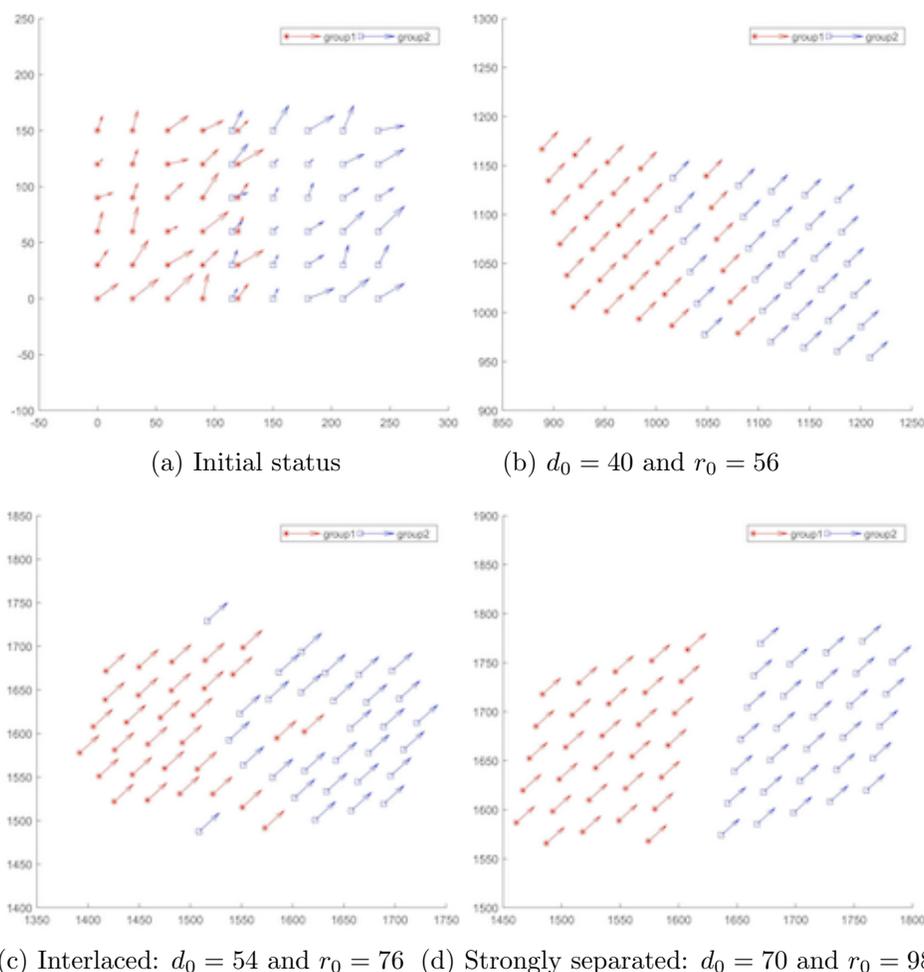
FIGURE 9. The number of agents are $n_1 = n_2 = 30$. The agents have a strict defined initial positions shown in figure(a) where the initial positions of the two groups have one column of intersect.

[5] D. Chang, S. Shadden, J. Marsden, R. Olfati-Saber; *Collision avoidance for multiple agent systems*, Proceedings of the IEEE Conference on Decision and Control, **1** (2003), 539–543.

[6] S. Chen, Y. Qiu, J. Liu, S. Nie; *Fast flocking algorithm of multi-agent network via community division*, Control and Decision, **33** (2018), 1523–1526.

[7] Y. Choi, J. Haskovec; *Cucker-smale model with normalized communication weights and time delay*, Kinetic & Related Models, **2** (2017), 1033–1039.

[8] Y. Choi, B. Kwon; *Two-species flocking particles immersed in a fluid*, Communications in Information and Systems, **13** (2013), 123–149.

[9] Y. Choi, S. Salem; *Cucker-smale flocking particles with multiplicative noises: Stochastic mean-field limit and phase transition*, Kinetic & Related Models, **12** (2019), 573–592.

[10] F. Cucker, C. Huepe; *Flocking with informed agents*, Maths in Action, **1** (2008), 1–25.

[11] F. Cucker, E. Mordecki; *Flocking in noisy environments*, Journal De Mathematiques Pures Et Appliquees, **89** (2008), 278–296.

(a) Initial status $d_0 = 40$ and $r_0 = 56$

(c) Interlaced: $d_0 = 48$ and $r_0 = 67$ (d) Strongly separated: $d_0 = 80$ and $r_0 = 113$
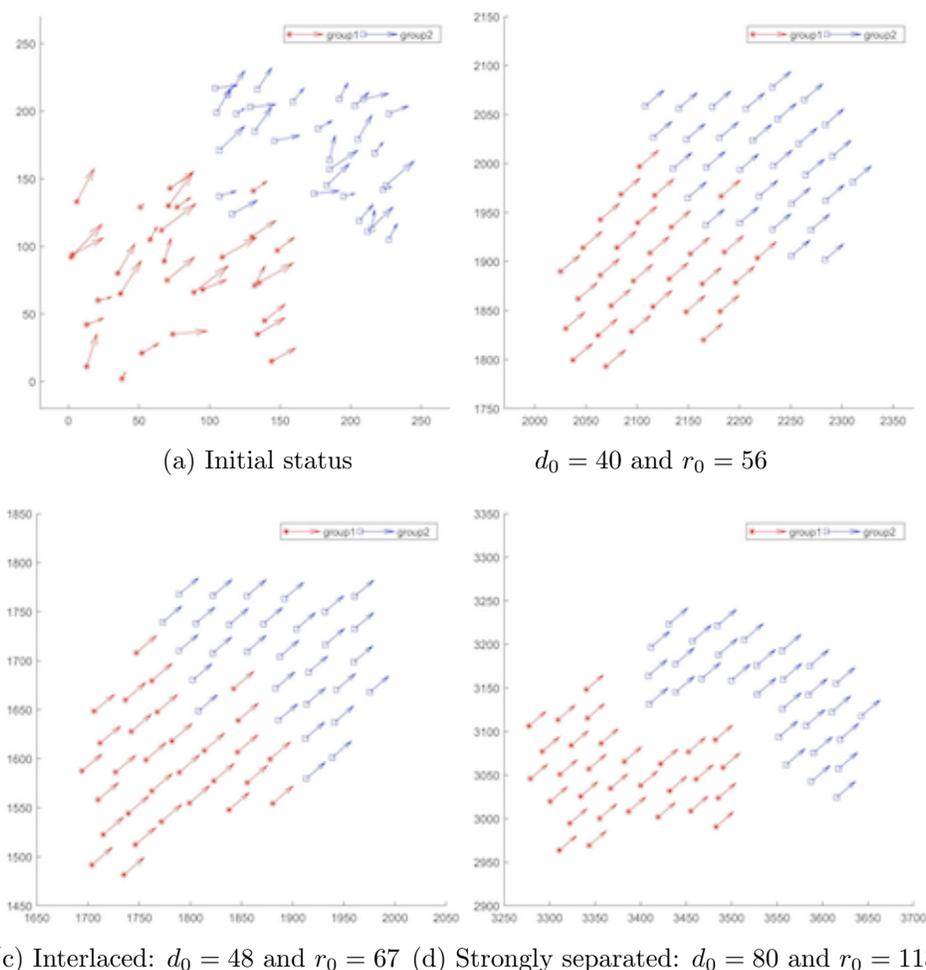
FIGURE 10. The number of agents in the two groups are $n_1 = n_2 = 30$. The initial positions of the agents in $\mathcal{N}_1$ are chosen at random in the set $[0, 150]^2$ and of the agents in $\mathcal{N}_2$, in the set $[100, 250]^2$.

[12] F. Cucker, S. Smale; *On the mathematics of emergence*, Japanese Journal of Mathematics, **2** (2007), 197–227.

[13] F. Dalmao, E. Mordecki; *Cucker-smale flocking under hierarchical leadership and random interactions*, SIAM Journal on Applied Mathematics, **71** (2009), 1307–1316.

[14] J. Dong, S. Ha, D. Kim; *Emergent behaviors of the kuramoto model with a time delay on a general digraph*, SIAM Journal on Applied Dynamical Systems, **19** (2020), 304–328.

[15] J. Dong, S. Ha, D. Kim; *Emergence of mono-cluster flocking in the thermomechanical cucker-smale model under switching topologies*, Analysis and Applications, **19** (2021), 305–342.

[16] C. Godsil, G. F. Royle; *Algebraic graph theory*, Graduate Texts in Mathematics, vol. 207, Springer-Verlag, New York, 2001.

[17] S. Ha, K. Lee, D. Levy; *Emergence of time-asymptotic flocking in a stochastic cucker-smale system*, Communications in Mathematical Sciences, **7** (2009), 453–469.

[18] M. O. Jackson; *Social and economic networks*, Princeton University Press, 2010.

(a) Initial status   (b) $d_0 = 40$ and $r_0 = 56$

(c) Interlaced: $d_0 = 48$ and $r_0 = 67$ (d) Strongly separated: $d_0 = 60$ and $r_0 = 84$
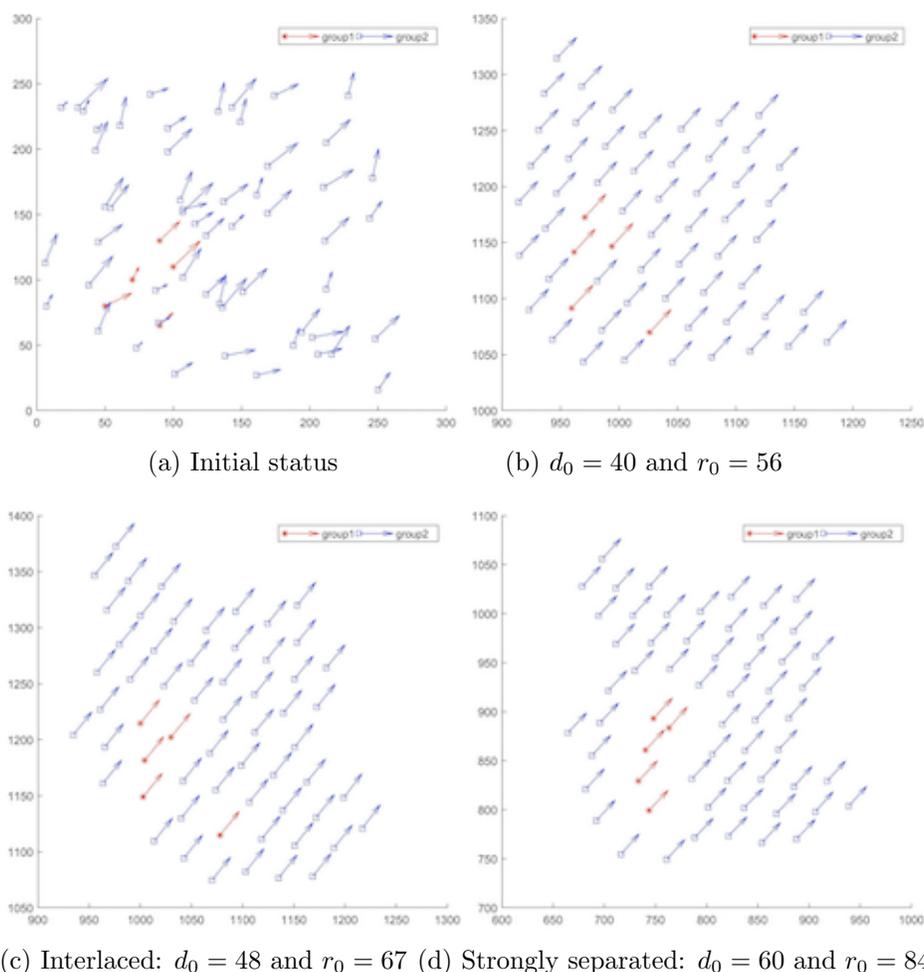
FIGURE 11. The number of the agents are $n_1 = 5$ and $n_2 = 55$. The initial positions of agents $\mathcal{N}_1$ are chosen at random from $[50, 150]^2$ and of agents in $\mathcal{N}_2$ are chosen at random from $[0, 250]^2$.

[19] J. P. LaSalle; *Some extensions of liapunov's second method*, Ire Transactions on Circuit Theory, **7** (1961), 520–527.

[20] Y. Lei and H. Zhao; *Negative sobolev spaces and the two-species vlasov-maxwell-landau system in the whole space*, Journal of Functional Analysis, **267** (2014), 3710–3757.

[21] I. I. Matveeva; *Exponential stability of solutions to nonlinear time-varying delay systems of neutral type equations with periodic coefficients*, Electronic Journal of Differential Equations, **2020** (2020), No. 20, 1-12.

[22] B. Mohar; *The laplacian spectrum of graphs*, Graph Theory, Combinatorics, and Applications, **2** (1991), 871–898.

[23] S. Motsch, E. Tadmor; *Heterophilious dynamics enhances consensus*, SIAM Review, **56** (2014), 577–621.

[24] R. Olfati-Saber; *Flocking for multi-agent dynamic systems: Algorithms and theory*, IEEE Transactions on Automatic Control, **51** (2006), 401–420.

[25] R. Olfati-Saber, R. M. Murray; *Consensus problems in networks of agents with switching topology and time-delays*, IEEE Transactions on Automatic Control, **49** (2004), 1520–1533.

[26] D. Paley, N. E. Leonard, R. Sepulchre, D. Grunbaum, J. K. Parrish; *Oscillator models and collective motion: spatial patterns in the dynamics of engineered and biological networks*, Die Makromolekulare Chemie, **192** (2007), 2293–2305.

[27] L. Perea, P. Elosegui, G. Gomez; *Extension of the cucker-smale control law to space flight formations*, Journal of Guidance Control and Dynamics, **32** (2009), 527–537.

[28] C. W. Reynolds; *Flocks, herds and schools: A distributed behavioral model*, ACM SIGGRAPH Computer Graphics, **21** (1987), 25–34.

[29] K. Saulnier, D. Saldaa, A. Prorok, G. Pappas, V. Kumar; *Resilient flocking for mobile robot teams*, IEEE Robotics and Automation Letters **2** (2017), 1039–1046.

[30] J. Shen; *Cucker-smale flocking under hierarchical leadership*, SIAM Journal on Applied Mathematics, **68** (2007), 694–719.

[31] S. Su, G. Zhang; *Global stability of traveling waves for delay reaction-diffusion systems without quasi-monotonicity*, Electronic Journal of Differential Equations, **2020** (2020), No. 46, 1-18.

[32] E. Tadmor, S. Ha; *From particle to kinetic and hydrodynamic descriptions of flocking*, Kinetic & Related Models **1** (2008), 415–435.

[33] J. Tello, M. Winkler; *Stabilization in a two-species chemotaxis system with a logistic source*, Nonlinearity **25** (2012), 1413–1425.

[34] C. Viragh, G. Vasarhelyi, N. Tarcai, T. Szrenyi, G. Somorjai, T. Nepusz, T. Vicsek; *Flocking algorithm for autonomous flying robots*, Bioinspiration & Biomimetics **9** (2013), 025012.

[35] Q. Xiao, H. Liu, Z. Xu, Z. Ouyang; *On collision avoiding fixed-time flocking with measurable diameter to a cucker smale type self-propelled particle model*, Complexity **11** (2020), 1–12.

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