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EXISTENCE OF SOLUTIONS FOR CRITICAL FRACTIONAL p-LAPLACIAN EQUATIONS WITH INDEFINITE WEIGHTS

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Abstract. This article concerns the critical fractional p-Laplacian equation with indefinite weights

$$(-\Delta_p)^s u = \lambda g(x)|u|^{p-2}u + h(x)|u|^{p_s^*-2}u \quad \text{in } \mathbb{R}^N,$$

where $0 < s < 1 < p < \infty$, $N > sp$ and $p_s^* = Np/(N-sp)$, the weight functions g may be indefinite, and h changes sign. Specifically, based on the results of asymptotic estimates for an extremal in the fractional Sobolev inequality and the discrete spectrum of fractional p-Laplacian operator, we establish an existence criterion for a nontrivial solution to this problem.

1. Introduction

The purpose of this article is to study the existence of nontrivial solutions for the critical fractional p-Laplacian equation

$$(-\Delta_p)^s u = \lambda g(x)|u|^{p-2}u + h(x)|u|^{p_s^*-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $0 < s < 1 < p < \infty$, $N > sp$, $p_s^* = Np/(N-sp)$ and $(-\Delta_p)^s$ is the fractional p-Laplacian operator which is defined as

$$(-\Delta_p)^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy, \quad \forall x \in \mathbb{R}^N,$$

the weight functions g may be indefinite and h changes sign.

In recent years, there has been an increasing attention to nonlocal diffusion problems, in particular to the ones driven by the fractional Laplacian operator [15]. One of the reasons for this comes from the fact that this operator naturally arises in several physical phenomena like flames propagation [9] and anomalous diffusion [22], in geophysical fluid dynamics [11], or in mathematical finance [13] and so on. It also provides a simple model to describe certain jump Lévy processes in probability theory [3].

For fractional Laplacian problems on bounded domains, an approach was proposed by Caffarelli and Silvestre in [10], which allows to transform a nonlocal problem into a local problem via the Dirichlet-Neumann map. By employing the harmonic extension, a large number of existence results have been obtained for Dirichlet problems involving the fractional Laplacian in bounded domain. In particular, to

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deal with the critical nonlinearity, we refer to [4, 5, 12] and the references therein. Furthermore, Servadei and Valdinoci [25] studied the Brezis-Nirenberg problem

$$(-\Delta_p)^s u = \lambda |u|^{p-2}u + |u|^{p_s^*-2}u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega,$$

and proved the existence of solutions for the above problem with $p = 2$. Mosconi et al. [23] deduced the existence of nontrivial solution for the above problem under different conditions of N, s, p and λ, by utilizing the asymptotic estimates for the minimizers obtained in [7] and an abstract linking theorem based on the cohomological index [29]. It is worth noting that these results were established for a bounded domain.

There are a lot of novel researches and interesting results of fractional Laplacian in the whole space, see [1, 2, 6, 8, 16, 18, 20, 24, 26, 30] and the references therein. For instance, Dipierro et al. [16] considered the existence of a positive solution for the problem

$$(-\Delta)^s u = \varepsilon f(x) u^q + u^{2_s^*-1} \quad \text{in } \mathbb{R}^N,$$

where $0 \le q < 2_s^* - 1$, $\varepsilon > 0$ is a small parameter and f is a continuous and compactly supported function. Due to the lack of regularity of the associated energy functional, the case $0 < q < 1$ is particularly difficult. Moreover, in [8], Bucur and Medina also investigated the above problem when $1 \le q < 2_s^* - 1$ by using different methods from [16]. Bonder et al. [6] dealt with the following critical equation involving the fractional p-Laplacian

$$(-\Delta_p)^s u = \lambda f(x)|u|^{q-2}u + K(x)|u|^{p_s^*-2}u \quad \text{in } \mathbb{R}^N,$$

where $p \le q < p_s^*$, $0 \le f \in L^1_{\rm loc}(\mathbb{R}^N)$ is such that the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(f dx; \mathbb{R}^N)$ is compact and K is nonnegative, bounded and has a limit at ∞. By the concentration-compactness principle and mountain pass theorem, they obtained some existence results for two cases: $q = p$ and $p < q < p_s^*$, respectively. Recently, Ambrosio et al. [1] investigated the existence and concentration of positive solutions for the p-fractional Schrödinger equation

$$\varepsilon^{sp}(-\Delta_p)^s u + V(x)|u|^{p-2}u = f(x) + \gamma |u|^{p_s^*-2}u \quad \text{in } \mathbb{R}^N,$$

where ε is a small parameter, $\gamma \in \{0, 1\}$, V is a continuous positive potential having a local minimum and f is a superlinear continuous function with subcritical growth. The main results were deduced via penalization techniques and suitable variational arguments. However, it is worth emphasizing that, we can find that weight functions were all assumed to be positive in these above mentioned literatures.

Motivated by the above analysis, we study the existence of nontrivial solution for problem (1.1), which is closely related to the principle eigenvalue of the problem

$$(-\Delta_p)^s u = \lambda g(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

In [14], we established the existence of a sequence of eigenvalues which converges to infinity for problem (1.2), and the principle eigenvalue is simple. The corresponding eigenfunction may be positive in $\mathbb{R}^N$ (see Lemma 2.1 below), which is a key step to prove the $(PS)_c$ condition. In this article we extend the results in [17] for critical classical Laplacian problem to the fractional setting.

It is worth mentioning that, as far as we know, there is no result for fractional p-Laplacian in $\mathbb{R}^N$ with indefinite weights. This problem will present us with two main difficulties: firstly, the lack of an explicit formula of the extremal for the

Sobolev embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$. For $p=2$, we know that the extremal is of the explicit form $cU(\frac{|x-x_0|}{\varepsilon})$ with

$$U(x) = \frac{1}{(1+|x|^{p'})^{\frac{N-sp}{p}}}, \quad x \in \mathbb{R}^N,$$

where $p' = p/(p-1)$ is the Hölder conjugate of p, $c \neq 0$, $x_0 \in \mathbb{R}^N$ and $\varepsilon > 0$, which was firstly proved by Lieb in [19] and can be devoted to proving the existence of solution. Although it has been conjectured that this extremal has a similar explicit form for the general case, to the best of our knowledge, it is still open until now. However, we can verify an existence of nontrivial solution by utilizing the asymptotic estimates for the extremal, which was obtained by Brasco et al. [7]. Secondly, the nonlinearities contain some indefinite weights, which give us further difficulty. Indeed, the indefinite weight case is more complicated than that of the positive one (see proof of Lemma 3.1), when we consider the convergence in the corresponding topology space, although we can define some spaces, their topology (or norm) are naturally related to indefinite weight functions, to give some technical assistance. Concerning the whole space, it causes the lack of the compact embedding, we will overcome the difficulty by the new concentration-compactness principle obtained in Bonder et al. [6] recently.

In this article, we assume that the weight functions g and h satisfy the following conditions: let $g = g^+ - g^-$ with $g^+, g^- \geq 0$, $g^+ \in L^\infty(\mathbb{R}^N) \cap L^{\frac{N}{sp}}(\mathbb{R}^N)$, $g^- \in L^\infty(\mathbb{R}^N)$, and

- (A1) $h \in L^\infty(\mathbb{R}^N)$ and $h^+ \not\equiv 0$;
- (A2) there exist constants $\rho > 0$ and $\theta > 1$ such that $h(x) = h(0) + o(|x|^{\frac{N}{p}})$ for $x \in B(0, \rho\theta)$;
- (A3) $h(0) = \|h\|_\infty$ and $h(x) > 0$ for $x \in B(0, \rho\theta)$;
- (A4) $g(x) \geq g_0 > 0$ in $B(0, \rho\theta)$.

Next, we state the main result of this article.

Theorem 1.1. *Assume that* (A1)–(A4) *hold, then for any* $\lambda \in (0, \lambda_1^+)$*, problem* (1.1) *admits at least a nontrivial solution.*

The remainder of this paper is organized as follows. In Section 2, we collect some notation and known results, and we prove some preliminaries. In Section 3, by using the concentration-compactness principle and mountain pass theorem, we prove Theorem 1.1.

2. Preliminaries

We recall that for any $0 < s < 1$, the fractional Sobolev space is defined by

$$D^{s,p}(\mathbb{R}^N) = \{u \in L^{p_s^*}(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

where the term

$$[u]_{s,p} = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^{1/p}$$

is the so-called Gagliardo seminorm of u. With the induced norm $\|u\|_{D^{s,p}(\mathbb{R}^N)} = [u]_{s,p}$, the space $D^{s,p}(\mathbb{R}^N)$ is a uniformly convex Banach space, and there exists a positive constant $C_{p_s^*}$ such that

$$\|u\|_{p_s^*} \leq C_{p_s^*}[u]_{s,p} \quad \text{for } u \in D^{s,p}(\mathbb{R}^N)\,.$$

Let the best Sobolev constant in this inequality be

$$S=\inf_{u\in D^{s,p}(\mathbb{R}^N)\setminus\{0\}}\frac{[u]_{s,p}^p}{\|u\|_{p_s^*}^p}\,. \tag{2.1}$$

Next, we define the fractional (s,p)-gradient of a function $u\in D^{s,p}(\mathbb{R}^N)$ as (see [6])

$$|D^s u(x)|^p=\int_{\mathbb{R}^N}\frac{|u(x+t)-u(x)|^p}{|t|^{N+sp}}\,dt.$$

Observe that this (s,p)-gradient is well defined in $\mathbb{R}^N$ and $|D^s u(x)|\in L^p(\mathbb{R}^N)$.

We introduce a weight function

$$\omega(x)=\frac{1}{(1+|x|)^{sp}},\quad x\in\mathbb{R}^N,$$

and define $w(x)=\max\{g^-(x),\omega(x)\}$. Let X be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_X=\Big([u]_{s,p}^p+\int_{\mathbb{R}^N}w|u|^p\,dx\Big)^{1/p},$$

then X is a uniformly convex Banach space [14, Lemma 2.1]. Obviously, note that the embedding $X\hookrightarrow D^{s,p}(\mathbb{R}^N)$ is continuous, and since $D^{s,p}(\mathbb{R}^N)\hookrightarrow L^\sigma_{\rm loc}(\mathbb{R}^N)$ is compact for $\sigma\in[1,p_s^*)$, we deduce that the embedding $X\hookrightarrow L^\sigma_{\rm loc}(\mathbb{R}^N)$ is compact for $\sigma\in[1,p_s^*)$.

Weak solutions of (1.1) coincide with critical points of the C^1-functional $J_\lambda: X\to\mathbb{R}$,

$$J_\lambda(u)=\frac{1}{p}[u]_{s,p}^p-\frac{\lambda}{p}\int_{\mathbb{R}^N}g|u|^p\,dx-\frac{1}{p_s^*}\int_{\mathbb{R}^N}h|u|^{p_s^*}\,dx,\quad u\in X. \tag{2.2}$$

Lemma 2.1 ([14, Theorem 1.1]). *Suppose that $g^+\in L^{\frac{N}{sp}}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $g^-\in L^\infty(\mathbb{R}^N)$ and g^+, $g^-\geq 0$. Then there exists a simple eigenvalue $\lambda_1^+>0$ such that the eigenvalue problem (1.2) has a positive eigenfunction $u_1^+\in X$ associated with λ_1^+. Moreover, for any $\lambda\in(0,\lambda_1^+]$, we have*

$$\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy-\lambda\int_{\mathbb{R}^N}g|u|^p\,dx\geq 0,\quad u\in X.$$

The following lemma provides the concentration-compactness principle for fractional Laplacian operator in unbounded domains.

Lemma 2.2 ([6, Theorem 1.1]). *Let $\{u_k\}_{k\in\mathbb{N}}\subset D^{s,p}(\mathbb{R}^N)$ be a weakly convergent sequence with weak limit u. Then there exist two bounded measures μ and ν, an at most enumerable set of indices I, $x_i\in\mathbb{R}^N$, and positive real numbers μ_i, ν_i, $i\in I$, such that the following convergence hold weakly* in the sense of measures,*

$$|D^s u_k|^p\,dx\rightharpoonup\mu\geq|D^s u|^p\,dx+\sum_{i\in I}\mu_i\delta_{x_i},$$

$$|u_k|^{p_s^*}\,dx\rightharpoonup\nu=|u|^{p_s^*}\,dx+\sum_{i\in I}\nu_i\delta_{x_i},$$

$$S\nu_i^{p/p_s^*}\leq\mu_i,\quad \textit{for all } i\in I.$$

Moreover, if we define

$$\mu_\infty=\lim_{R\to\infty}\limsup_{k\to\infty}\int_{|x|>R}|D^s u_k|^p\,dx,\quad \nu_\infty=\lim_{R\to\infty}\limsup_{k\to\infty}\int_{|x|>R}|u_k|^{p_s^*}\,dx,$$

then

$$\limsup_{k\to\infty}\int_{\mathbb{R}^N}|D^s u_k|^p\,dx=\mu(\mathbb{R}^N)+\mu_\infty,$$
$$\limsup_{k\to\infty}\int_{\mathbb{R}^N}|u_k|^{p_s^*}\,dx=\nu(\mathbb{R}^N)+\nu_\infty,$$
$$S\nu_\infty^{p/p_s^*}\le\mu_\infty.$$

Now, we estimate the decay of this nonlocal gradient and a compactness conclusion with weights.

Lemma 2.3 ([6, Coroll. 2.3])**.** *Let $\phi\in W^{1,\infty}(\mathbb{R}^N)$ be such that $supp(\phi)\subset B(0,1)$, given $r>0$ and $x_0\in\mathbb{R}^N$, we define $\phi_{r,x_0}(x)=\phi(\frac{x-x_0}{r})$. Then*

$$|D^s\phi_{r,x_0}(x)|^p\le C\min\Big\{r^{-sp},r^N|x-x_0|^{-(N+sp)}\Big\},$$

where $C>0$ depends on N,s,p and $\|\phi\|_{W^{1,\infty}(\mathbb{R}^N)}$.

Lemma 2.4 ([6, Lemma 2.4])**.** *If $a\in L^\infty(\mathbb{R}^N)$, there exist $\alpha>0$ and $C>0$ such that*

$$0\le a(x)\le C|x|^{-\alpha}.$$

Then, if $\alpha>sp$, the embedding $D^{s,p}(\mathbb{R}^N)\hookrightarrow L^p(adx;\mathbb{R}^N)$ is compact. That is, for any bounded sequence $\{u_k\}_{k\in\mathbb{N}}\subset D^{s,p}(\mathbb{R}^N)$, there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}\subset\{u_k\}_{k\in\mathbb{N}}$ and a function $u\in D^{s,p}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N}a|u_{k_j}-u|^p\,dx\to0\quad\text{as } j\to\infty.$$

To verify the existence of solution for (1.1), we first need to show some preliminary results.

Lemma 2.5. *Assume that $u_n\rightharpoonup u$ in $D^{s,p}(\mathbb{R}^N)$. Then for $f\in L^\infty(\mathbb{R}^N)$, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that for any $\varphi\in C_0^\infty(\mathbb{R}^N)$,*

(i) $\int_{\mathbb{R}^N}f|u_n|^{p_s^*-2}u_n\varphi\,dx\to\int_{\mathbb{R}^N}f|u|^{p_s^*-2}u\varphi\,dx$;
(ii) $\int_{\mathbb{R}^N}f|u_n|^{p-2}u_n\varphi\,dx\to\int_{\mathbb{R}^N}f|u|^{p-2}u\varphi\,dx$.

Proof. (i) Since $u_n\rightharpoonup u$ in $D^{s,p}(\mathbb{R}^N)$ and the embedding $D^{s,p}(\mathbb{R}^N)\hookrightarrow L^\sigma_{\rm loc}(\mathbb{R}^N)$ is compact for $\sigma\in[1,p_s^*)$, we can assume that

$$u_n\to u\text{ a.e. in }\mathbb{R}^N,$$

so $f|u_n|^{p_s^*-2}u_n\to f|u|^{p_s^*-2}u$ a.e. in $\mathbb{R}^N$. Since $\{u_n\}$ is bounded in $L^{p_s^*}(\mathbb{R}^N)$, it follows from $f\in L^\infty(\mathbb{R}^N)$ that $\{f|u_n|^{p_s^*-2}u_n\}$ is bounded in $L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N)$. Thus, we can get that $f|u_n|^{p_s^*-2}u_n\rightharpoonup f|u|^{p_s^*-2}u$ in $L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N)$ (see [27]). For any $\varphi\in C_0^\infty(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N}f|u_n|^{p_s^*-2}u_n\varphi\,dx\to\int_{\mathbb{R}^N}f|u|^{p_s^*-2}u\varphi\,dx.$$

(ii) It can be proved by a similar method as (i), so we omit it here. □

Lemma 2.6. *Suppose that $u_n\rightharpoonup u$ in $D^{s,p}(\mathbb{R}^N)$, then there exists a subsequence of $\{u_n\}$, denoted still by $\{u_n\}$, such that*

$$\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{\frac{N+sp}{p'}}}\rightharpoonup\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{\frac{N+sp}{p'}}}$$

in $L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$, *where* $p' = \frac{p}{p-1}$ *is the Hölder conjugate of* p.

Proof. For simplicity, we define

$$\xi_n(x,y) = \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{\frac{N+sp}{p'}}}.$$

Since $\{u_n\}$ is bounded in $D^{s,p}(\mathbb{R}^N)$, we obtain that $\{\xi_n\}$ is bounded in $L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$, hence there exists $\xi \in L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ such that $\xi_n \rightharpoonup \xi$ in $L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ up to a subsequence.

In addition, since $u_n \rightharpoonup u$ in $D^{s,p}(\mathbb{R}^N)$ and the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^\sigma_{\rm loc}(\mathbb{R}^N)$ is compact for $\sigma \in [1, p_s^*)$, we can assume that $u_n \to u$ a.e. in $\mathbb{R}^N$. Then it follows that $\xi_n(x,y) \to \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{\frac{N+sp}{p'}}}$ a.e. in $\mathbb{R}^N \times \mathbb{R}^N$. Combining this with the boundedness of $\{\xi_n\}$, we obtain that

$$\xi_n(x,y) \rightharpoonup \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{\frac{N+sp}{p'}}} \quad \text{in } L^{p'}(\mathbb{R}^N \times \mathbb{R}^N).$$

Thus

$$\xi(x,y) = \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{\frac{N+sp}{p'}}}.$$

The proof is complete. $\square$

Lemma 2.7. *Let* $\{u_n\} \subset D^{s,p}(\mathbb{R}^N)$ *be a bounded sequence such that* $J_\lambda(u_n) \to c$ *and* $J'_\lambda(u_n) \to 0$ *as* $n \to \infty$, *and let* μ_i, ν_i *be as in Lemma 2.2. Then we have the following estimates:*

$$\nu_i \geq S^{\frac{N}{sp}} h(x_i)^{-\frac{N}{sp}}, \quad \mu_i \geq S^{\frac{N}{sp}} h(x_i)^{1-\frac{N}{sp}} \quad \textit{if } h(x_i) > 0,$$
$$\nu_i = \mu_i = 0 \quad \textit{if } h(x_i) = 0.$$

Proof. The proof closely follows the technique of [6, Lemma 3.6]. We first show that the set I in Lemma 2.2 is finite. Fix a concentration point x_i, let $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that

$$\phi(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 2, \end{cases}$$

and let $\phi_\delta(x) = \phi(\frac{x-x_i}{\delta})$ for $\delta > 0$. According to $J'_\lambda(u_n) \to 0$ as $n \to \infty$, taking the test function $\varphi = u_n\phi_\delta$, then we have

$$\begin{aligned} &\lim_{n\to\infty} \Big(\langle (-\Delta_p)^s u_n, u_n\phi_\delta\rangle - \lambda \int_{\mathbb{R}^N} |u_n|^p \phi_\delta \,dx - \int_{\mathbb{R}^N} h|u_n|^{p_s^*}\phi_\delta \,dx \Big) \\ &= \lim_{n\to\infty} \langle J'_\lambda(u_n), u_n\phi_\delta\rangle = 0. \end{aligned}$$

Thus, it follows from Lemma 2.2, $g, h \in L^\infty(\mathbb{R}^N)$ and $u_n \to u$ in $L^p_{\rm loc}(\mathbb{R}^N)$ that

$$\lim_{n\to\infty} \langle (-\Delta_p)^s u_n, u_n\phi_\delta\rangle = \lambda \int_{\mathbb{R}^N} g|u|^p \phi_\delta \,dx + \int_{\mathbb{R}^N} h\phi_\delta d\nu. \tag{2.3}$$

Next, we verify that

$$\lim_{\delta\to 0}\lim_{n\to\infty} \langle (-\Delta_p)^s u_n, u_n\phi_\delta\rangle = \mu_i. \tag{2.4}$$

We can write

$$\langle (-\Delta_p)^s u_n, u_n\phi_\delta\rangle$$

$$
\begin{aligned}
&= \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(u_n(x)\phi_\delta(x)-u_n(y)\phi_\delta(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p}\phi_\delta(x)}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi_\delta(x)-\phi_\delta(y))u_n(y)}{|x-y|^{N+sp}}\,dx\,dy \\
&= I_1 + I_2.
\end{aligned}
$$

Clearly, $I_1 = \int_{\mathbb{R}^N} |D^s u_n|^p \phi_\delta\,dx$, then by using $u_n \rightharpoonup u$ in $D^{s,p}(\mathbb{R}^N)$ and Lemma 2.2, we have

$$
\lim_{n\to\infty} I_1 = \int_{\mathbb{R}^N} \phi_\delta d\mu.
$$

Since $\mu - \sum_{i\in I}\mu_i\delta_{x_i}$ has no atoms and $\phi_\delta \to 0$ as $\delta \to 0$ for any $x \neq x_i$, $i \in I$, we conclude that $\lim_{\delta\to 0}\lim_{n\to\infty} I_1 = \mu_i$. Now,we estimate I_2 using Hölder inequality we have

$$
\begin{aligned}
I_2 &\le \int_{\mathbb{R}^N} (|D^s u_n|^p)^{\frac{p-1}{p}} (|D^s\phi_\delta|^p)^{1/p}|u_n|dy \\
&\le \|D^s u_n\|_p^{p-1}\Big(\int_{\mathbb{R}^N} |u_n|^p|D^s\phi_\delta|^p\,dx\Big)^{1/p} \\
&\le c\Big(\int_{\mathbb{R}^N} |u_n|^p|D^s\phi_\delta|^p\,dx\Big)^{1/p}.
\end{aligned}
$$

Then by Lemmas 2.3 and 2.4, we know that

$$
\limsup_{n\to\infty} I_2 \le c\Big(\int_{\mathbb{R}^N} |u|^p|D^s\phi_\delta|^p\,dx\Big)^{1/p}. \tag{2.5}
$$

Furthermore, we check that $\lim_{\delta\to 0}\int_{\mathbb{R}^N} |u|^p|D^s\phi_\delta|^p\,dx = 0$. In fact, thanks to Lemma 2.3 and Hölder inequality, we deduce that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |u|^p|D^s\phi_\delta|^p\,dx \\
&\le C\Big(\delta^{-sp}\int_{|x|<\delta}|u|^p\,dx + \delta^N\int_{|x|\ge\delta}\frac{|u|^p}{|x|^{N+sp}}\,dx\Big) \\
&\le C\delta^{-sp}\Big(\int_{|x|<\delta}|u|^{p_s^*}\,dx\Big)^{p/p_s^*}|B_\delta|^{\frac{sp}{N}} + C\delta^N\sum_{k=0}^\infty\int_{2^k\delta\le|x|\le 2^{k+1}\delta}\frac{|u|^p}{|x|^{N+sp}}\,dx \\
&\le c_1\Big(\int_{|x|<\delta}|u|^{p_s^*}\,dx\Big)^{p/p_s^*} \\
&\quad + C\sum_{k=0}^\infty 2^{-k(N+sp)}\delta^{-sp}\Big(\int_{|x|\le 2^{k+1}\delta}|u|^{p_s^*}\,dx\Big)^{p/p_s^*}|B_{2^{k+1}\delta}|^{\frac{sp}{N}} \\
&= c_1\Big(\int_{|x|<\delta}|u|^{p_s^*}\,dx\Big)^{p/p_s^*} + c_2\sum_{k=0}^\infty 2^{-kN}\Big(\int_{|x|\le 2^{k+1}\delta}|u|^{p_s^*}\,dx\Big)^{p/p_s^*}.
\end{aligned}
$$

Since $u \in L^{p_s^*}(\mathbb{R}^N)$, then it is easy to see that $\lim_{\delta\to 0}\int_{|x|<\delta}|u|^{p_s^*}\,dx = 0$. Given $\varepsilon > 0$, taking $k_0 \in \mathbb{N}$ such that $c_2\sum_{k=k_0+1}^{\infty} 2^{-kN} < \varepsilon$, we obtain

$$
\begin{aligned}
&c_2\sum_{k=0}^{\infty} 2^{-kN}\Big(\int_{|x|\le 2^{k+1}\delta}|u|^{p_s^*}\,dx\Big)^{p/p_s^*}\\
&\le \varepsilon\|u\|_{p_s^*}^p + c_2\sum_{k=0}^{k_0} 2^{-kN}\Big(\int_{|x|\le 2^{k_0+1}\delta}|u|^{p_s^*}\,dx\Big)^{p/p_s^*}.
\end{aligned}
$$

Then,

$$
\limsup_{\delta\to 0} c_2\sum_{k=0}^{\infty} 2^{-kN}\Big(\int_{|x|\le 2^{k+1}\delta}|u|^{p_s^*}\,dx\Big)^{p/p_s^*} \le \varepsilon\|u\|_{p_s^*}^p.
$$

Hence,

$$
\lim_{\delta\to 0}\int_{\mathbb{R}^N}|u|^p|D^s\phi_\delta|^p\,dx = 0,
$$

and from (2.5) it follows that $\lim_{\delta\to 0}\lim_{n\to\infty} I_2 = 0$. So, the limit equality (2.4) holds. On the other hand, by applying the property of the function ϕ_δ, we know that

$$
\lim_{\delta\to 0}\int_{\mathbb{R}^N} g|u|^p\phi_\delta\,dx = 0 \quad\text{and}\quad \lim_{\delta\to 0}\int_{\mathbb{R}^N} h\phi_\delta d\nu = h(x_i)\nu_i.
$$

Then, in view of (2.3), we have $h(x_i)\nu_i = \mu_i$ for any $i \in I$, which implies that $h(x_i) \ge 0$. So, it follows from Lemma 2.2 that

$$
\begin{gathered}
\nu_i \ge S^{\frac{N}{sp}} h(x_i)^{-\frac{N}{sp}}, \quad \mu_i \ge S^{\frac{N}{sp}} h(x_i)^{1-\frac{N}{sp}} \quad \text{if } h(x_i) > 0,\\
\nu_i = \mu_i = 0 \quad \text{if } h(x_i) = 0.
\end{gathered}
$$

This completes the proof. □

3. Main results

In this section, we prove the existence of solutions for critical fractional p-Laplacian with indefinite weights of problem (1.1). By using the concentration-compactness principle and mountain pass theorem, we prove Theorem 1.1. Firstly, we prove that the functional J_λ satisfies the $(PS)_c$ condition for small energy levels.

Lemma 3.1. *For any $\lambda \in (0, \lambda_1^+)$, the functional J_λ satisfies the $(PS)_c$ condition for all $c < \frac{s}{N} S^{\frac{N}{sp}}\|h\|_\infty^{1-\frac{N}{sp}}$.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence; that is,

$$
J_\lambda(u_n) \to c \quad\text{and}\quad J_\lambda'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.1}
$$

First of all, we show that the sequence $\{u_n\}$ is bounded in X. If this is not true, we may suppose that, up to a subsequence, still denoted by $\{u_n\}$ such that $\|u_n\|_X \to \infty$ as $n \to \infty$. For $n \in \mathbb{N}$, let $v_n = \frac{u_n}{\|u_n\|_X}$, then we can assume that there exists $v \in X$ such that

$$
v_n \rightharpoonup v \quad \text{in } X, \quad v_n \to v \quad \text{in } L^p_{\rm loc}(\mathbb{R}^N), \quad v_n \to v \quad \text{a.e. in } \mathbb{R}^N. \tag{3.2}
$$

Obviously, by (3.1), we have

$$
c + o(1)\|u_n\|_X = J_\lambda(u_n) - \frac{1}{p_s^*}\langle J_\lambda'(u_n), u_n\rangle = \frac{s}{N}\Big([u_n]_{s,p}^p - \lambda\int_{\mathbb{R}^N} g|u_n|^p\,dx\Big).
$$

Multiplying both sides by $\|u_n\|_X^{-p}$ and letting $n \to \infty$, we deduce that

$$[v_n]_{s,p}^p - \lambda \int_{\mathbb{R}^N} g|v_n|^p \, dx \to 0. \tag{3.3}$$

Since the embedding $X \hookrightarrow D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$ is continuous and $\|v_n\|_X = 1$, we know that $\{v_n\}$ is bounded in $L^{p_s^*}(\mathbb{R}^N)$, then $\{|v_n|^p\}$ is bounded in $L^{\frac{p_s^*}{p}}(\mathbb{R}^N)$. Combining this with (3.2), we have $|v_n|^p \rightharpoonup |v|^p$ in $L^{\frac{p_s^*}{p}}(\mathbb{R}^N)$. So, in view of $g^+ \in L^{\frac{N}{sp}}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} g^+|v_n|^p \, dx \to \int_{\mathbb{R}^N} g^+|v|^p \, dx. \tag{3.4}$$

For $\lambda \in (0, \lambda_1^+)$, by (3.2)-(3.4), we have

$$\begin{aligned}
0 &\le [v]_{s,p}^p - \lambda \int_{\mathbb{R}^N} g|v|^p \, dx \\
&= [v]_{s,p}^p - \lambda \int_{\mathbb{R}^N} (g^+ - g^-)|v|^p \, dx \\
&\le \liminf_{n\to\infty} \Big([v_n]_{s,p}^p - \lambda \int_{\mathbb{R}^N} g^+|v_n|^p \, dx + \lambda \int_{\mathbb{R}^N} g^-|v_n|^p \, dx \Big) = 0.
\end{aligned} \tag{3.5}$$

Thus, the variational characterization of the principle eigenvalue λ_1^+ (see Lemma 2.1) implies that $v \equiv 0$. Then from (3.4) and (3.5) it follows that

$$\int_{\mathbb{R}^N} g^+|v_n|^p \, dx \to 0, \quad \lim_{n\to\infty} [v_n]_{s,p}^p = 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} g^-|v_n|^p \, dx = 0.$$

By the Hardy-type inequality [21, Theorem 2], we obtain

$$0 \le \int_{\mathbb{R}^N} \omega |v_n|^p \, dx \le C_{N,s,p}[v_n]_{s,p}^p,$$

we then deduce that $\lim_{n\to\infty} \|v_n\|_X = 0$, which contradicts that $\|v_n\|_X = 1$. Therefore, $\{u_n\}$ is bounded in X.

Next, we verify that $u_n \to u$ in X. Since $\{u_n\}$ is bounded in X, we can assume that there exists $u \in X$ such that $u_n \rightharpoonup u$ in X. Then applying Lemmas 2.5 and 2.6 and $g, h \in L^\infty(\mathbb{R}^N)$, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \, dx \, dy \\
&\to \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \, dx \, dy,
\end{aligned}$$

$$\int_{\mathbb{R}^N} g|u_n|^{p-2}u_n\varphi \, dx \to \int_{\mathbb{R}^N} g|u|^{p-2}u\varphi \, dx,$$

$$\int_{\mathbb{R}^N} h|u_n|^{p_s^*-2}u_n\varphi \, dx \to \int_{\mathbb{R}^N} h|u|^{p_s^*-2}u\varphi \, dx.$$

Hence, in view of $\langle J_\lambda'(u_n), \varphi \rangle \to 0$, we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+sp}} \, dx \, dy \\
&- \lambda \int_{\mathbb{R}^N} g|u|^{p-2}u\varphi \, dx - \int_{\mathbb{R}^N} h|u|^{p_s^*-2}u\varphi \, dx = 0,
\end{aligned}$$

that is, $\langle J_\lambda'(u),\varphi\rangle = 0$.

From the proof of Lemma 2.7, we know that for any $i\in I$,

$$h(x_i)\nu_i = \mu_i, \tag{3.6}$$

$$\nu_i = 0 \quad \text{if } h(x_i)=0, \quad \nu_i \ge S^{\frac{N}{sp}} h(x_i)^{-\frac{N}{sp}} \quad \text{if } h(x_i)>0. \tag{3.7}$$

Suppose that $\nu_i \neq 0$ for some $i\in I$. Becasue $u_n \rightharpoonup u$ in X, we may obtain that $u_n \to u$ a.e. in $\mathbb{R}^N$. On the other hand, we can get that $\{|u_n|^p\}$ is bounded in $L^{\frac{p_s^*}{p}}(\mathbb{R}^N)$, since the embedding $X \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$ is continuous. Then, it follows from $g^+ \in L^{\frac{N}{sp}}(\mathbb{R}^N)$ that

$$\int_{\mathbb{R}^N} g^+|u_n|^p\,dx \to \int_{\mathbb{R}^N} g^+|u|^p\,dx. \tag{3.8}$$

Thus, by (3.1), Lemma 2.2 and (3.8), we deduce that

$$\begin{aligned}
c+o(1)\|u_n\|_X &= J_\lambda(u_n) - \frac{1}{p_s^*}\langle J_\lambda'(u_n),u_n\rangle\\
&= \frac{s}{N}\Big([u_n]_{s,p}^p - \lambda\int_{\mathbb{R}^N} g|u_n|^p\,dx\Big)\\
&= \frac{s}{N}\Big([u_n]_{s,p}^p - \lambda\int_{\mathbb{R}^N} g^+|u_n|^p\,dx + \lambda\int_{\mathbb{R}^N} g^-|u_n|^p\,dx\Big)\\
&\ge \frac{s}{N}\Big([u]_{s,p}^p - \lambda\int_{\mathbb{R}^N} g|u|^p\,dx + \sum_{i\in I}\mu_i\Big) + o(1).
\end{aligned}$$

Furthermore, combining $J_\lambda'(u)=0$, (3.6) with (3.7), we obtain

$$\begin{aligned}
c+o(1)\|u_n\|_X &\ge \frac{s}{N}\int_{\mathbb{R}^N} h|u|^{p_s^*}\,dx + \frac{s}{N}\sum_{i\in I} h(x_i)\nu_i + o(1)\\
&\ge \frac{s}{N}\int_{\mathbb{R}^N} h|u|^{p_s^*}\,dx + \frac{s}{N}S^{\frac{N}{sp}}\sum_{i\in I} h(x_i)^{1-\frac{N}{sp}} + o(1).
\end{aligned}$$

In view of $c < \frac{s}{N}S^{\frac{N}{sp}}\|h\|_\infty^{1-\frac{N}{sp}}$, we obtain that $\int_{\mathbb{R}^N} h|u|^{p_s^*}\,dx < 0$. However, by the fact $J_\lambda'(u)=0$ and $\lambda\in(0,\lambda_1^+)$, we have

$$\int_{\mathbb{R}^N} h|u|^{p_s^*}\,dx = [u]_{s,p}^p - \lambda\int_{\mathbb{R}^N} g|u|^p\,dx \ge 0,$$

which is a contraction. Thus $\nu_i=\mu_i=0$ for any $i\in I$, then it follows from Lemma 2.2 that

$$\int_{\mathbb{R}^N} |u_n|^{p_s^*}\,dx \to \int_{\mathbb{R}^N} |u|^{p_s^*}\,dx.$$

This together with the weak convergence of $u_n \rightharpoonup u$ in $L^{p_s^*}(\mathbb{R}^N)$ imply that $u_n \to u$ in $L^{p_s^*}(\mathbb{R}^N)$. By $u_n \rightharpoonup u$ in X and (3.1), we derive that

$$\begin{aligned}
&\langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u\rangle \\
&= \iint_{\mathbb{R}^{2N}} \Big(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+sp}} \\
&\quad - \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\Big)(u_n(x)-u_n(y)-u(x)+u(y))\,dx\,dy \\
&\quad - \lambda \int_{\mathbb{R}^N} g(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)dx \\
&\quad - \int_{\mathbb{R}^N} h(|u_n|^{p_s^*-2}u_n - |u|^{p_s^*-2}u)(u_n-u)dx \to 0.
\end{aligned} \tag{3.9}$$

Since $u_n \rightharpoonup u$ in X, $u_n \to u$ in $L^{p_s^*}(\mathbb{R}^N)$, $g^+ \in L^{\frac{N}{sp}}(\mathbb{R}^N)$ and $h \in L^\infty(\mathbb{R}^N)$, we have the convergence

$$\begin{gathered}
\int_{\mathbb{R}^N} g^+(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)dx \to 0, \\
\int_{\mathbb{R}^N} h(|u_n|^{p_s^*-2}u_n - |u|^{p_s^*-2}u)(u_n-u)dx \to 0.
\end{gathered} \tag{3.10}$$

Then, combining (3.9) with (3.10), we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \Big(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+sp}} - \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\Big) \\
&\times (u_n(x)-u_n(y)-u(x)+u(y))\,dx\,dy \\
&+ \lambda \int_{\mathbb{R}^N} g^-(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)dx \to 0.
\end{aligned} \tag{3.11}$$

Using Hölder inequality, we deduce that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \Big(\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+sp}} - \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\Big) \\
&\times (u_n(x)-u_n(y)-u(x)+u(y))\,dx\,dy \\
&\geq [u_n]_{s,p}^p + [u]_{s,p}^p - [u_n]_{s,p}^{p-1}[u]_{s,p} - [u]_{s,p}^{p-1}[u_n]_{s,p} \\
&= \big([u_n]_{s,p}^{p-1} - [u]_{s,p}^{p-1}\big)\left([u_n]_{s,p} - [u]_{s,p}\right)
\end{aligned}$$

and similarly, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} g^-(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)dx \\
&\geq \Big(\Big(\int_{\mathbb{R}^N} g^-|u_n|^p\,dx\Big)^{\frac{p-1}{p}} - \Big(\int_{\mathbb{R}^N} g^-|u|^p\,dx\Big)^{\frac{p-1}{p}}\Big) \\
&\quad \times \Big(\Big(\int_{\mathbb{R}^N} g^-|u_n|^p\,dx\Big)^{1/p} - \Big(\int_{\mathbb{R}^N} g^-|u|^p\,dx\Big)^{1/p}\Big).
\end{aligned}$$

Let $f_b(t) = (t^{b-1} - \beta^{b-1})(t-\beta)$ for $t \in \mathbb{R}^+$, $b > 1$, we know that $f_b(t) \geq 0$. Hence, it follows from (3.11) that

$$\big([u_n]_{s,p}^{p-1} - [u]_{s,p}^{p-1}\big)\left([u_n]_{s,p} - [u]_{s,p}\right) \to 0,$$

$$\Big(\Big(\int_{\mathbb{R}^N} g^-|u_n|^p\,dx\Big)^{\frac{p-1}{p}} - \Big(\int_{\mathbb{R}^N} g^-|u|^p\,dx\Big)^{\frac{p-1}{p}}\Big)$$
$$\times\Big(\Big(\int_{\mathbb{R}^N} g^-|u_n|^p\,dx\Big)^{1/p} - \Big(\int_{\mathbb{R}^N} g^-|u|^p\,dx\Big)^{1/p}\Big) \to 0.$$

The function f_b also has the following property: if $f_b(t_n)\to 0$, then $t_n\to\beta$. Thus,

$$[u_n]_{s,p}^p \to [u]_{s,p}^p \quad\text{and}\quad \int_{\mathbb{R}^N} g^-|u_n|^p\,dx \to \int_{\mathbb{R}^N} g^-|u|^p\,dx \quad\text{as } n\to\infty.$$

By the Hardy-type inequality, we derive that $\int_{\mathbb{R}^N}\omega|u_n|^p\,dx \to \int_{\mathbb{R}^N}\omega|u|^p\,dx$ as $n\to\infty$. So, $\|u_n\|_X\to\|u\|_X$, this together with the weak convergence of $u_n\rightharpoonup u$ in X implies that $u_n\to u$ in X. □

To verify that J_λ has the geometric structure required by the mountain pass theorem, we need to introduce the following results.

Lemma 3.2 ([7, Proposition 3.1]). *Let $1<p<\infty$, $s\in(0,1)$, $N>sp$ and let S be as in* (2.1). *Then*

(i) *there exists a minimizer for S;*
(ii) *for every minimizer $U\in D^{s,p}(\mathbb{R}^N)$, there exist $x_0\in\mathbb{R}^N$ and a constant sign monotone function $u:\mathbb{R}\to\mathbb{R}$ such that $U(x)=u(|x-x_0|)$;*
(iii) *for every minimizer U, there exists $\lambda_U>0$ such that*
$$\iint_{\mathbb{R}^{2N}} \frac{|U(x)-U(y)|^{p-2}(U(x)-U(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$= \lambda_U\int_{\mathbb{R}^N}|U|^{p_s^*-2}U\varphi\,dx,$$
for $\varphi\in D^{s,p}(\mathbb{R}^N)$.

In the following, we fix a radially symmetric nonnegative decreasing minimizer $U=U(r)$ for S. Multiplying U by a positive constant if necessary, we may assume that U is a radial solution of

$$(-\Delta_p)^sU = |U|^{p_s^*-2}U.$$

Taking the test function U and applying (2.1) yield

$$[U]_{s,p}^p = \|U\|_{p_s^*}^{p_s^*} = S^{\frac{N}{sp}}. \tag{3.12}$$

For $\varepsilon>0$, the function

$$U_\varepsilon(|x|) = \varepsilon^{-\frac{N-sp}{p}}U\Big(\frac{|x|}{\varepsilon}\Big)$$

is also a minimizer for S satisfying (3.12).

Lemma 3.3 ([7, Corollary 3.7]). *There exist constants $C_1,C_2>0$ and $\theta>1$ such that for all $r\ge 1$,*

$$\frac{C_1}{r^{\frac{N-sp}{p-1}}} \le U(r) \le \frac{C_2}{r^{\frac{N-sp}{p-1}}} \quad\textit{and}\quad \frac{U(\theta r)}{U(r)}\le\frac12.$$

Now, we give some auxiliary functions and estimate their norms. In what follows, θ is the constant in Lemma 3.3 that depends only on N, s and p. For ε, $\rho>0$ and $\theta>1$, let us set

$$m_{\varepsilon,\rho} = \frac{U_\varepsilon(\rho)}{U_\varepsilon(\rho)-U_\varepsilon(\rho\theta)}.$$

Moreover, let us define

$$g_{\varepsilon,\rho}(t)=\begin{cases}0, & \text{if } 0\le t\le U_\varepsilon(\rho\theta),\\ m^p_{\varepsilon,\rho}(t-U_\varepsilon(\rho\theta)), & \text{if } U_\varepsilon(\rho\theta)<t\le U_\varepsilon(\rho),\\ t+U_\varepsilon(\rho)(m^{p-1}_{\varepsilon,\rho}-1), & \text{if } t>U_\varepsilon(\rho),\end{cases}$$

and

$$G_{\varepsilon,\rho}(t)=\int_0^t g'_{\varepsilon,\rho}(\tau)^{1/p}d\tau=\begin{cases}0, & \text{if } 0\le t\le U_\varepsilon(\rho\theta),\\ m_{\varepsilon,\rho}(t-U_\varepsilon(\rho\theta)), & \text{if } U_\varepsilon(\rho\theta)<t\le U_\varepsilon(\rho),\\ t, & \text{if } t>U_\varepsilon(\rho).\end{cases}$$

The functions $g_{\varepsilon,\rho}$ and $G_{\varepsilon,\rho}$ are nondecreasing and absolutely continuous. Consider the radially symmetric non-increasing function

$$u_{\varepsilon,\rho}(r)=G_{\varepsilon,\rho}(U_\varepsilon(r)),$$

which satisfies

$$u_{\varepsilon,\rho}(r)=\begin{cases}U_\varepsilon(r), & \text{if } r\le\rho,\\ 0, & \text{if } r\ge\rho\theta.\end{cases}$$

Then, we have the following estimates for $u_{\varepsilon,\rho}$.

Lemma 3.4 ([23, Lemma 2.7])**.** *There exists a constant $C=C(N,p,s)>0$ such that, for any $0<\varepsilon\le\frac{\rho}{2}$, it holds*

$$[u_{\varepsilon,\rho}]^p_{s,p}\le S^{\frac{N}{sp}}+C\Big(\frac{\varepsilon}{\rho}\Big)^{\frac{N-sp}{p-1}}, \tag{3.13}$$

$$\|u_{\varepsilon,\rho}\|^p_p\ge\begin{cases}\frac{1}{C}\varepsilon^{sp}\log(\frac{\rho}{\varepsilon}), & \text{if } N=sp^2,\\ \frac{1}{C}\varepsilon^{sp}, & \text{if } N>sp^2,\end{cases} \tag{3.14}$$

$$\|u_{\varepsilon,\rho}\|^{p^*_s}_{p^*_s}\ge S^{\frac{N}{sp}}-C\Big(\frac{\varepsilon}{\rho}\Big)^{\frac{N}{p-1}}. \tag{3.15}$$

Now we define the function

$$v_{\varepsilon,\rho}(x)=\frac{u_{\varepsilon,\rho}(x)}{\|u_{\varepsilon,\rho}(x)\|_{p^*_s}}.$$

Lemma 3.5. *There exist ε, ρ, $t_0>0$ such that for $\lambda\in(0,\lambda_1^+)$, $J_\lambda(t_0v_{\varepsilon,\rho})<0$ and*

$$\sup_{t\ge0}J_\lambda(tv_{\varepsilon,\rho})<\frac{s}{N}S^{\frac{N}{sp}}\|h\|_\infty^{1-\frac{N}{sp}}.$$

Proof. For any $\lambda\in(0,\lambda_1^+)$, by (2.1), we have

$$J_\lambda(u)\ge\frac{1}{p}\Big(1-\frac{\lambda}{\lambda_1}\Big)[u]^p_{s,p}-\frac{1}{p^*_s}\|h\|_\infty S^{-\frac{p^*_s}{p}}[u]^{p^*_s}_{s,p},$$

so we can obtain that $J_\lambda(u)\ge c$, when $\|u\|_X$ is sufficiently small. Furthermore, it is easy from (2.2) to see that $\lim_{t\to\infty}J_\lambda(tv_{\varepsilon,\rho})=-\infty$, hence $J_\lambda(tv_{\varepsilon,\rho})$ attains its maximum at some $t_\varepsilon\in(0,\infty)$ with $\psi'_\lambda(t_\varepsilon)=0$, where

$$\psi_\lambda(t)=J_\lambda(tv_{\varepsilon,\rho})=\frac{t^p}{p}[v_{\varepsilon,\rho}]^p_{s,p}-\frac{t^p}{p}\lambda\int_{\mathbb{R}^N}g|v_{\varepsilon,\rho}|^p\,dx-\frac{t^{p^*_s}}{p^*_s}\int_{\mathbb{R}^N}h|v_{\varepsilon,\rho}|^{p^*_s}\,dx.$$

Then, we obtain

$$0=\psi_\lambda'(t_\varepsilon)=t_\varepsilon^{p-1}\Big([v_{\varepsilon,\rho}]_{s,p}^p-\lambda\int_{\mathbb{R}^N}g|v_{\varepsilon,\rho}|^p\,dx\Big)-t_\varepsilon^{p_s^*-1}\int_{\mathbb{R}^N}h|v_{\varepsilon,\rho}|^{p_s^*}\,dx,$$

moreover, combining (A2), (A3) and (A4), we deduce that

$$t_\varepsilon^{p_s^*-p}=\frac{[v_{\varepsilon,\rho}]_{s,p}^p-\lambda\int_{\mathbb{R}^N}g|v_{\varepsilon,\rho}|^p\,dx}{\int_{\mathbb{R}^N}h|v_{\varepsilon,\rho}|^{p_s^*}\,dx}\le\frac{[v_{\varepsilon,\rho}]_{s,p}^p}{h(0)\|v_{\varepsilon,\rho}\|_{p_s^*}^{p_s^*}}. \tag{3.16}$$

Clearly, we have

$$J_\lambda(t_\varepsilon v_{\varepsilon,\rho})=\sup_{t\ge0}J_\lambda(tv_{\varepsilon,\rho})=I_1-I_2, \tag{3.17}$$

where

$$I_1=\frac{t_\varepsilon^p}{p}[v_{\varepsilon,\rho}]_{s,p}^p-\frac{t_\varepsilon^{p_s^*}}{p_s^*}h(0)\int_{\mathbb{R}^N}|v_{\varepsilon,\rho}|^{p_s^*}\,dx,$$
$$I_2=\frac{t_\varepsilon^p}{p}\lambda\int_{\mathbb{R}^N}g|v_{\varepsilon,\rho}|^p\,dx-\frac{t_\varepsilon^{p_s^*}}{p_s^*}\int_{\mathbb{R}^N}(h(0)-h)|v_{\varepsilon,\rho}|^{p_s^*}\,dx.$$

In view of (3.13) and (3.15), we obtain

$$[v_{\varepsilon,\rho}]_{s,p}^p=\frac{[u_{\varepsilon,\rho}]_{s,p}^p}{\|u_{\varepsilon,\rho}(x)\|_{p_s^*}^p}\le S+O\big(\frac{\varepsilon}{\rho}\big)^{\frac{N-sp}{p-1}}.$$

For positive numbers a and b, the maximum of $\hbar(t)=a\frac{t^p}{p}-b\frac{t^{p_s^*}}{p_s^*}$ for $t\ge0$ is attained at $t=(\frac{a}{b})^{\frac{N-sp}{sp^2}}$, then, by the assumption (A3) and above inequality, we can deduce that

$$\begin{aligned}I_1&\le\frac1p\Big(\frac{[v_{\varepsilon,\rho}]_{s,p}^p}{h(0)\int_{\mathbb{R}^N}|v_{\varepsilon,\rho}|^{p_s^*}\,dx}\Big)^{\frac{N-sp}{sp}}[v_{\varepsilon,\rho}]_{s,p}^p\\&\quad-\frac{h(0)}{p_s^*}\Big(\frac{[v_{\varepsilon,\rho}]_{s,p}^p}{h(0)\int_{\mathbb{R}^N}|v_{\varepsilon,\rho}|^{p_s^*}\,dx}\Big)^{\frac{N}{sp}}\int_{\mathbb{R}^N}|v_{\varepsilon,\rho}|^{p_s^*}\,dx\\&=\frac{s}{N}\|h\|_\infty^{1-\frac{N}{sp}}\Big([v_{\varepsilon,\rho}]_{s,p}^p\Big)^{\frac{N}{sp}}\Big(\int_{\mathbb{R}^N}|v_{\varepsilon,\rho}|^{p_s^*}\,dx\Big)^{1-\frac{N}{sp}}\\&\le\frac{s}{N}S^{\frac{N}{sp}}\|h\|_\infty^{1-\frac{N}{sp}}.\end{aligned}$$

Without loss of generality, we can assume that $\{t_\varepsilon\}$ is bounded. In fact, since $\{t_\varepsilon\}$ is bounded from blew, otherwise one concludes easily from (3.17) that $J_\lambda(t_\varepsilon v_{\varepsilon,\rho})\to0$ as $\varepsilon\to0$. In addition, according to (3.16), we know that $\{t_\varepsilon\}$ is bounded from above for $\varepsilon>0$ small. Next, we estimate the L^κ-norm of $u_{\varepsilon,\rho}$, for $\kappa\in[1,\infty)$, Lemma 3.3 yields

$$\begin{aligned}\int_{\mathbb{R}^N}|u_{\varepsilon,\rho}(x)|^\kappa\,dx&\ge\int_{B_\rho(0)}|u_{\varepsilon,\rho}(x)|^\kappa\,dx=\int_{B_\rho(0)}|U_\varepsilon(x)|^\kappa\,dx\\&=\varepsilon^{-\frac{(N-sp)\kappa}{p}}\int_{B_\rho(0)}\big|U\big(\frac{x}{\varepsilon}\big)\big|^\kappa\,dx\\&\ge C_1^\kappa\varepsilon^{\frac{Np-(N-sp)\kappa}{p}}\int_1^{\rho/\varepsilon}r^{-\frac{N-sp}{p-1}\kappa+N-1}dr\,.\end{aligned}$$

Then

$$\int_{\mathbb{R}^N} |u_{\varepsilon,\rho}(x)|^\kappa\,dx \ge c_\kappa \begin{cases} \varepsilon^{N-\frac{N-sp}{p}\kappa}, & \text{if } \kappa > \frac{N(p-1)}{N-sp},\\ \varepsilon^{N-\frac{N-sp}{p}\kappa}|\log\frac{\rho}{\varepsilon}|, & \text{if } \kappa = \frac{N(p-1)}{N-sp},\\ \varepsilon^{\frac{N-sp}{p(p-1)}\kappa}\rho^{N-\frac{N-sp}{p-1}\kappa}, & \text{if } \kappa < \frac{N(p-1)}{N-sp}. \end{cases}$$

Thus, by assumption (A4), we have

$$\int_{\mathbb{R}^N} g|v_{\varepsilon,\rho}|^p\,dx \ge c_p \begin{cases} g_0\varepsilon^{sp}, & \text{if } N > sp^2,\\ g_0\varepsilon^{sp}|\log\frac{\rho}{\varepsilon}|, & \text{if } N = sp^2,\\ g_0\varepsilon^{\frac{N-sp}{p-1}}\rho^{N-\frac{N-sp}{p-1}p}, & \text{if } N < sp^2. \end{cases}$$

Similarly, in view of assumption (A2), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N}(h(0)-h)|v_{\varepsilon,\rho}|^{p_s^*}\,dx &= \frac{1}{\|u_{\varepsilon,\rho}\|_{p_s^*}^{p_s^*}}\int_{\mathbb{R}^N}(h(0)-h)|u_{\varepsilon,\rho}|^{p_s^*}\,dx\\ &\ge \varepsilon^{\frac{N}{p}}\int_1^{\rho/\varepsilon} r^{-\frac{N}{p(p-1)}-1}dr. \end{aligned}$$

Since $\frac{N}{p} \ge sp$ if $N \ge sp^2$ and $\frac{N}{p} > \frac{N-sp}{p-1}$ and $N < sp^2$, then I_2 can be dominated by $\int_{\mathbb{R}^N} g|v_{\varepsilon,\rho}|^p\,dx$. Thus, we conclude that

$$J_\lambda(t_\varepsilon v_{\varepsilon,\rho}) \le \begin{cases} \frac{s}{N}S^{\frac{N}{sp}}\|h\|_\infty^{1-\frac{N}{sp}} - K_1\varepsilon^{sp}, & \text{if } N > sp^2,\\ \frac{s}{N}S^{\frac{N}{sp}}\|h\|_\infty^{1-\frac{N}{sp}} - K_1\varepsilon^{sp}|\log\frac{\rho}{\varepsilon}|, & \text{if } N = sp^2,\\ \frac{s}{N}S^{\frac{N}{sp}}\|h\|_\infty^{1-\frac{N}{sp}} - K_1\varepsilon^{\frac{N-sp}{p-1}}\rho^{N-\frac{N-sp}{p-1}p}, & \text{if } N < sp^2, \end{cases}$$

then

$$J_\lambda(t_\varepsilon v_{\varepsilon,\rho}) < \frac{s}{N}S^{\frac{N}{sp}}\|h\|_\infty^{1-\frac{N}{sp}}$$

if $\varepsilon > 0$ is sufficiently small. This completes the proof. $\square$

Proof of Theorem 1.1. For any $\lambda \in (0,\lambda_1^+)$, by (2.1), we have

$$J_\lambda(u) \ge \frac{1}{p}\Big(1-\frac{\lambda}{\lambda_1}\Big)[u]_{s,p}^p - \frac{1}{p_s^*}\|h\|_\infty S^{-\frac{p_s^*}{p}}[u]_{s,p}^{p_s^*},$$

then it follows that $J_\lambda(u) \ge c > 0$, when $\|u\|_X$ is sufficiently small. And since $J_\lambda(0) = 0$, 0 is a local minimum of J_λ. In addition, noting that

$$J_\lambda(tv_{\varepsilon,\rho}) = \frac{t^p}{p}\Big([v_{\varepsilon,\rho}]_{s,p}^p - \lambda\int_{\mathbb{R}^N} g|v_{\varepsilon,\rho}|^p\,dx\Big) - \frac{t^{p_s^*}}{p_s^*}\int_{\mathbb{R}^N} h|v_{\varepsilon,\rho}|^{p_s^*}\,dx \to -\infty$$

as $t \to +\infty$, fix $t_1 > 0$ so large that $J_\lambda(t_1 v_{\varepsilon,\rho}) < 0$. Now, we construct the set

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = t_1 v_{\varepsilon,\rho}\},$$

and let

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} J_\lambda(\gamma(t)).$$

In view of Lemma 3.5, it is easy to see that

$$c < \frac{s}{N}S^{\frac{N}{sp}}\|h\|_\infty^{1-\frac{N}{sp}},$$

and hence J_λ satisfies the $(PS)_c$ condition by Lemma 3.1. Then, c is a critical level of J_λ via the mountain pass theorem [28], and we obtain that problem (1.1) has a nontrivial solution. □

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