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ASYMPTOTIC BEHAVIOR FOR A QUASI-AUTONOMOUS GRADIENT SYSTEM OF EXPANSIVE TYPE GOVERNED BY A QUASICONVEX FUNCTION

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Abstract. We consider the quasi-autonomous first-order gradient system

$$\dot{u}(t) = \nabla\phi(u(t)) + f(t), \quad t \in [0, +\infty)$$
$$u(0) = x_0 \in H,$$

where $\phi : H \to \mathbb{R}$ is a differentiable quasiconvex function such that $\nabla\phi$ is Lipschitz continuous. We study the asymptotic behavior of solutions to this system in continuous and discrete time. We show that each solution either approaches infinity in norm or converges weakly to a critical point of ϕ. This further concludes that the existence of bounded solutions and implies that ϕ has a nonempty set of critical points. Some strong convergence results, as well as numerical examples, are also given in both continuous and discrete cases.

1. Introduction

Let H be a real Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. By $\to$ and $\rightharpoonup$ we denote strong and weak convergence, respectively, in H. The study of existence and asymptotic behavior of solutions to first-order evolution equations of the form

$$\dot{u}(t) + Au(t) \ni 0 \quad \text{a.e } t \in [0, +\infty), \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a possibly multivalued maximal monotone operator, goes back to the 1970s; see [5]. In [1, 3], the authors proved that if $A^{-1}(0) \neq \emptyset$, then the mean of solutions to (1.1) converges weakly to an element of $A^{-1}(0)$. But in general, the solutions to (1.1) are not strongly convergent [2]. Bruck [6] proved the weak convergence of solutions to (1.1) under an additional condition on the monotone operator A, which is called demipositivity. By introducing the notion of nonexpansive and almost nonexpansive curves in H, Djafari Rouhani [9, 8] studied the asymptotic behavior of the mean of bounded solutions to the following quasi-autonomous system of monotone type

$$\dot{u}(t) + Au(t) \ni f(t), \quad t \in [0, +\infty). \tag{1.2}$$

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An important example of a maximal monotone operator is the subdifferential of a proper, convex and lower semicontinuous function. Inspired by its applications in economics, the study of functions that are not convex but have convex sublevel sets have received a particular attention; see [7, 16] and the references therein. The functions with convex sublevel sets are called *quasiconvex*. Here is a formal definition: a function $\phi : H \to (-\infty, +\infty]$ is called quasiconvex if

$$\phi(\lambda x + (1-\lambda)y) \le \max\{\phi(x), \phi(y)\}, \quad \forall x, y \in H,\ \forall \lambda \in [0,1].$$

We say that ϕ is *strongly quasiconvex* if there exists $\alpha > 0$ such that

$$\phi(\lambda x + (1-\lambda)y) \le \max\{\phi(x), \phi(y)\} - \alpha\lambda(1-\lambda)\|x-y\|^2, \quad \forall x, y \in H,\ \forall \lambda \in [0,1].$$

There are have been many attempts to generalize the notion of subdifferential for non convex functions; see [14] and the references therein. However in any circumstance, the subdifferential of a quasiconvex function is not monotone. On the other hand, if $\phi : H \to \mathbb{R}$ is Gâteaux differentiable, then the following characterization for a quasiconvex function ϕ holds:

> ϕ is quasiconvex on H if and only if for all $x, y \in H$: $\phi(y) \le \phi(x)$ implies $\langle \nabla\phi(x), x - y\rangle \ge 0$.

The above characterization may prove to be useful given the lack of monotonicity. Applying this fact, Goudou and Munier [13] studied (1.2) for the case where A is replaced by $\nabla\phi$ where $\phi : H \to \mathbb{R}$ is a differentiable quasiconvex function with a nonempty set of minimizers. Generally, even the monotone type systems of the form

$$\dot{u}(t) \in Au(t), \quad t \in [0, +\infty), \tag{1.3}$$

are "strongly ill-posed". For example, consider the simple linear case of $A = -\Delta$ with Dirichlet boundary conditions, which yields the heat equation with a final Cauchy data and is not generally solvable. Djafari Rouhani [10, 11] introduced the notion of almost expansive curves, and studied their ergodic and asymptotic properties. Then he applied these results to study the asymptotic behavior of possible solutions to (1.3). In [12], by considering the explicit discretization of (1.3), the authors studied the asymptotic behavior and periodicity of the generated sequence.

In this article, we consider the differential equation

$$\dot{u}(t) = \nabla\phi(u(t)) + f(t), \quad t \in [0, +\infty), \tag{1.4}$$

where $\phi : H \to \mathbb{R}$ is a differentiable quasiconvex function such that $\nabla\phi$ is Lipschitz continuous and $f \in W^{1,1}((0, +\infty); H)$. The Lipschitz continuity of $\nabla\phi$ implies that the system (1.4) with an initial condition has a unique solution $u(t)$. To study the asymptotic behavior of such a solution, we define

$$L(u) = \{y \in H : \exists T > 0 \text{ s.t. } \phi(y) \le \phi(u(t))\, \forall t \ge T\}.$$

Since we are only concerned about the asymptotic behavior of the trajectories, for simplicity and without loss of generality, in our calculations we always take $T = 0$. The set of all global minimizers of ϕ is denoted by $\operatorname{argmin}\phi$. Clearly, $\operatorname{argmin}\phi \subset L(u)$.

In Section 2, we show that if (1.4) has a solution u such that $\liminf_{t\to+\infty} \|u(t)\| < +\infty$, then $L(u) \ne \emptyset$, and we prove that every such solution converges weakly to some element in $(\nabla\phi)^{-1}(0)$ and if this element does not belong to $\operatorname{argmin}\phi$, then the convergence is strong. We also show that if $u(t)$ is an unbounded solution, then $\|u(t)\|$ approaches infinity as $t \to +\infty$. Some strong convergence results are obtained as well. Section 3 is devoted to the study of the explicit discretization of (1.4). In that section, we prove similar convergence results as in Section 2. This provides an algorithm to approximate an element of $(\nabla\phi)^{-1}(0)$. A numerical example shows that by choosing suitable step sizes, the convergence of the scheme can be fast. Our results in this paper extend and improve our previous results in [10, 11, 12].

Lemma 1.1 ([13]). *Let $\phi : H \to \mathbb{R}$ be a continuously differentiable and quasiconvex function and $u : [0, +\infty) \to H$ be a curve such that there is some point $\tilde{x}$ in H and a number $r > 0$ and $T \geq 0$ satisfying*

$$\phi(y) \leq \phi(u(t)) \quad \forall t \geq T,\ \forall y \in \overline{\mathbb{B}}(\tilde{x}, r).$$

Then for all $t \geq T$, we have

$$r\|\nabla\phi(u(t))\| \leq \langle \nabla\phi(u(t)), u(t) - \tilde{x}\rangle.$$

2. Continuous case

Let $\phi : H \to \mathbb{R}$ be a quasiconvex function such that $\nabla\phi$ is Lipschitz continuous. We consider the Cauchy problem

$$\begin{gathered} \dot{u}(t) = \nabla\phi(u(t)) + f(t), \quad t \in [0, +\infty), \\ u(0) = x \in H, \end{gathered} \tag{2.1}$$

where $f \in W^{1,1}((0, +\infty), H)$. Since $\nabla\phi$ is Lipschitz continuous, the Cauchy-Lipschitz theorem guatentees the existence of a unique solution $u \in C^1([0, +\infty); H)$ to (2.1).

Proposition 2.1. *Assume that $u(t)$ is a solution to* (2.1)*. For an arbitrary interval $[a, b]$, where $b \geq a \geq 0$, and each $y \in L(u)$, we have*

$$\|u(a) - y\| \leq \|u(b) - y\| + \int_a^b \|f(t)\| dt, \tag{2.2}$$

and therefore $\lim_{t\to+\infty} \|u(t) - y\|$ *exists (it may be infinite).*

Proof. Let $y \in L(u)$ be arbitrary and fixed. Multiplying both sides of (2.1) by $(u(t) - y)$, we obtain

$$\langle \dot{u}(t), u(t) - y\rangle = \langle \nabla\phi(u(t)), u(t) - y\rangle + \langle f(t), u(t) - y\rangle,$$

which together with the characterization of quasiconvex functions imply that

$$\langle \dot{u}(t), u(t) - y\rangle - \langle f(t), u(t) - y\rangle \geq 0. \tag{2.3}$$

Since $u(t)$ is absolutely continuous, $\|u(t) - y\|$ is also absolutely continues, and hence $\frac{d}{dt}\|u(t) - y\|$ exists almost everywhere on $[a, b]$. Thus, we have

$$\langle \dot{u}(t), u(t) - y\rangle = \frac{1}{2}\frac{d}{dt}\|u(t) - y\|^2 = \|u(t) - y\|\frac{d}{dt}\|u(t) - y\| \quad \text{a.e. } t \in [a, b]. \tag{2.4}$$

Combining (2.3) and (2.4) and using the Cauchy-Schwarz inequality, we obtain

$$\|u(t) - y\|\Big(\frac{d}{dt}\|u(t) - y\| + \|f(t)\|\Big) \geq 0 \quad \text{a.e. } t \in [a, b]. \tag{2.5}$$

If there exists $t_0 \in [a, b]$ such that $\frac{d}{dt}\|u(t) - y\|\big|_{t=t_0} + \|f(t_0)\| < 0$, then $\frac{d}{dt}\|u(t) - y\|\Big|_{t=t_0} < 0$, and by (2.5), $u(t_0) = y$. Also, by the definition of the derivative, there exists some sufficiently small $h > 0$, such that $\|u(t_0 + h) - y\| < \|u(t_0) - y\| = 0$ which is impossible. Hence

$$\frac{d}{dt}\|u(t) - y\| + \|f(t)\| \geq 0 \quad \text{a.e. } t \in [a, b]. \tag{2.6}$$

Integrating the above inequality on $[a, b]$, we obtain

$$\|u(a) - y\| \leq \|u(b) - y\| + \int_a^b \|f(t)\|dt.$$

Now we conclude the result by taking $\liminf$ as $b \to +\infty$, and then taking $\limsup$ as $a \to +\infty$ in the above inequality. $\square$

Proposition 2.2. *Let $u(t)$ be a solution to (2.1) such that $\liminf_{t\to+\infty} \|u(t)\| < +\infty$. Then $\lim_{t\to+\infty} \nabla\phi(u(t)) = 0$ and $\lim_{t\to+\infty} \phi(u(t))$ exists and is finite.*

Proof. Multiplying both sides of (2.1) by $\dot{u}(t)$, we obtain

$$\|\dot{u}(t)\|^2 = \langle \nabla\phi(u(t)), \dot{u}(t)\rangle + \langle f(t), \dot{u}(t)\rangle. \tag{2.7}$$

Applying the Cauchy-Schwarz inequality to the above equation, we obtain

$$\|\dot{u}(t)\|^2 \leq \frac{d}{dt}\phi(u(t)) + \|f(t)\|\|\dot{u}(t)\|. \tag{2.8}$$

Since $\liminf_{t\to+\infty} \|u(t)\| < +\infty$ and ϕ is bounded on bounded sets, there is a sequence $u(t_n)$ such that the sequence $\phi(u(t_n))$ is bounded. Integrating the above inequality on $[0, t_n]$, we obtain

$$\int_0^{t_n} \|\dot{u}(t)\|^2 dt \leq \phi(u(t_n)) - \phi(u(0)) + \int_0^{t_n} \|f(t)\|\|\dot{u}(t)\|dt. \tag{2.9}$$

Again by applying the Cauchy-Schwarz inequality, we have

$$\int_0^{t_n} \|\dot{u}(t)\|^2 dt \leq C_1 + C_2\Big(\int_0^{t_n} \|\dot{u}(t)\|^2 dt\Big)^{1/2}, \tag{2.10}$$

where $C_1 = \sup_{n\geq 0}\{\phi(u(t_n)) - \phi(0)\}$, and $C_2 = \big(\int_0^{+\infty} \|f(t)\|^2 dt\big)^{1/2}$. This shows that $\dot{u} \in L^2((0, +\infty); H)$. On the other hand,

$$\|u(t) - u(s)\| \leq \int_s^t \|\dot{u}(\tau)\|d\tau \leq (t - s)^{1/2}\Big(\int_s^t \|\dot{u}(\tau)\|^2 d\tau\Big)^{1/2}$$

which implies that u is uniformly continuous. This, the Lipschitz continuity of $\nabla\phi$, and the fact that $f \in W^{1,1}((0, +\infty); H)$ imply that $\dot{u}$ is uniformly continuous. Now, since $\dot{u}(t)$ is uniformly continuous and belongs to $L^2((0, +\infty), H)$, we have $\lim_{t\to+\infty} \dot{u}(t) = 0$. Therefore, since $f \in W^{1,1}((0, +\infty); H)$, we have

$$\lim_{t\to+\infty} \nabla\phi(u(t)) = 0.$$

From (2.7), by applying the Cauchy-Schwarz inequality, we also have

$$\frac{d}{dt}\phi(u(t)) \leq \|\dot{u}(t)\|^2 + \|\dot{u}\|\|f(t)\|.$$

Integrating on $[s, t]$, we obtain

$$\phi(u(t)) \leq \phi(u(s)) + \int_s^t \|\dot{u}(\tau)\|^2 d\tau + \sup_{t\geq 0} \|\dot{u}(t)\| \int_s^t \|f(\tau)\|d\tau.$$

Taking $\limsup$ as $t \to +\infty$, then taking $\liminf$ as $s \to +\infty$ in the above inequality, we conclude that $\lim_{t\to+\infty} \phi(u(t))$ exists and is finite. $\square$

Proposition 2.3. *If $u(t)$ is a solution to* (2.1) *such that* $\liminf_{t\to+\infty}\|u(t)\|<+\infty$, *then $L(u)\neq\emptyset$ and u is bounded.*

Proof. Assume by contradiction that $\liminf_{t\to+\infty}\|u(t)\|<+\infty$, and $L(u)=\emptyset$. Then for each $z\in H$ there exists a sequence t_n^z such that $\phi(z)>\phi(u(t_n^z))$. On the other hand, by Proposition 2.2, we know that $\lim_{t\to+\infty}\phi(u(t))$ exists and is finite. Hence, we have

$$\lim_{t\to+\infty}\phi(u(t))=\lim_{n\to+\infty}\phi(u(t_n^z))\le\phi(z)\quad\forall z\in H,$$

which implies that $\lim_{t\to+\infty}\phi(u(t))=\inf\phi$. Since $\liminf_{t\to+\infty}\|u(t)\|<+\infty$, there exists a bounded subsequence of $u(t)$, say $u(t_n)$. Since ϕ is bounded on bounded sets, then $\inf\phi=\lim_{n\to+\infty}\phi(u(t_n))>-\infty$. Also, the boundedness of $u(t_n)$ implies that there exist a subsequence of $u(t_n)$, which we denote again by $u(t_n)$, and some $p\in H$ such that $u(t_n)\rightharpoonup p$. Now using the lower semicontinuity of ϕ for the weak topology, we obtain

$$\phi(p)\le\lim_{n\to+\infty}\phi(u(t_n))=\lim_{t\to+\infty}\phi(u(t))=\inf\phi.$$

This yields that $p\in\operatorname{argmin}\phi$, which contradicts $L(u)=\emptyset$. Now Proposition 2.1 implies that u is bounded. □

Theorem 2.4. *Let $u(t)$ be a solution to* (2.1). *If* $\liminf_{t\to+\infty}\|u(t)\|<+\infty$, *then there exists some $p\in(\nabla\phi)^{-1}(0)$ such that $u(t)\rightharpoonup p$ as $t\to+\infty$, and if $p\notin$* $\operatorname{argmin}\phi$, *the convergence is strong. If $u(t)$ is unbounded, then $\|u(t)\|\to+\infty$ as $t\to+\infty$.*

Proof. If $\liminf_{t\to+\infty}\|u(t)\|<+\infty$, then by Proposition 2.3, u is bounded and $L(u)\neq\emptyset$. Let $y\in L(u)$. By Proposition 2.1, we know that $\lim_{t\to+\infty}\|u(t)-y\|$ exists and is finite, and by Proposition 2.2, we know that $\lim_{t\to+\infty}\phi(u(t))$ exists and is finite too. Let q be a weak cluster point of $u(t)$. There exists a sequence t_n such that $t_n\to+\infty$ as $n\to+\infty$, and $u(t_n)\rightharpoonup q$ as $n\to+\infty$. If $\lim_{t\to+\infty}\phi(u(t))=\phi(y)$, using the fact that ϕ is lower semicontinuous for the weak topology, we have

$$\phi(q)\le\lim_{n\to+\infty}\phi(u(t_n))=\lim_{t\to+\infty}\phi(u(t))=\phi(y).$$

Also since $y\in L(u)$, we have $q\in L(u)$. Now an easy application of the Opial lemma [15] shows that there exists $p\in L(u)$ such that $u(t)\rightharpoonup p$ as $t\to+\infty$. If $p\in\operatorname{argmin}\phi$, then the conclusion follows. Otherwise, there is an element in $L(u)$ which we denote again by y, such that $\lim_{t\to+\infty}\phi(u(t))>\phi(y)$. Hence there exist $t_0\ge 0$ and $r>0$, such that

$$\phi(z)\le\phi(u(t)),\quad\forall t\ge t_0,\quad\forall z\in\overline{\mathbb{B}}(y,r).$$

Thus by Lemma 1.1, we have

$$r\|\nabla\phi(u(t))\|\le\langle\nabla\phi(u(t)),u(t)-y\rangle\quad t\ge t_0.\tag{2.11}$$

Replacing $\nabla\phi(u(t))$ from (2.1) on the right hand side of the above inequality and then using the Cauchy-Schwarz inequality, we obtain

$$r\|\nabla\phi(u(t))\|\le\frac{1}{2}\frac{d}{dt}\|u(t)-y\|^2+\|u(t)-y\|\|f(t)\|\quad t\ge t_0.$$

Integrating the above inequality on $[t_0,t]$ and then letting $t\to+\infty$ implies that $\nabla\phi(u(t))\in L^1((0,+\infty),H)$. Therefore $\dot u(t)\in L^1((0,+\infty),H)$. Hence $u(t)\to p$ as $t\to+\infty$. Since $\nabla\phi$ is continuous and by Proposition 2.2, $\lim_{t\to+\infty}\nabla\phi(u(t))=$

0, we obtain $p \in (\nabla\phi)^{-1}(0)$. If $u(t)$ is unbounded and $\liminf_{t\to+\infty} \|u(t)\| < +\infty$, then by Proposition 2.3, u is bounded which is a contradiction. Therefore $\lim_{t\to+\infty} \|u(t)\| = +\infty$, if u is unbounded. □

Theorem 2.4 shows that if $(\nabla\phi)^{-1}(0) = \emptyset$, then for any solution to (2.1), we have $\lim_{t\to+\infty} \|u(t)\| = +\infty$. We illustrate Theorem 2.4 by the following example.

Example 2.5. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(x) = \arctan(x^3)$. Then ϕ is a quasiconvex function such that $\nabla\phi$ is Lipschitz continuous and $(\nabla\phi)^{-1}(0) = \{0\}$ and $\operatorname{argmin}\phi = \emptyset$. Considering (2.1), where $\phi(x) = \arctan(x^3)$, we have the following Cauchy problem

$$\begin{gathered}\dot{u}(t) = \frac{3u(t)^2}{1+u(t)^6}, \quad t \in [0,+\infty), \\ u(0) = u_0 \in \mathbb{R}.\end{gathered} \tag{2.12}$$

By the Cauchy-Lipschitz theorem, the initial value problem (2.12) has a unique solution. Therefore if $u(0) = 0$, then $u(t) \equiv 0$ is the unique solution to (2.12). Hence if $u(t_0) = 0$, for some $t_0 \geq 0$, then by the uniqueness of solutions to (2.12) $u(t) = 0$ for all $t \geq t_0$. Now we assume that $u(t) \neq 0$ for all $t \geq 0$. Using (2.12), by a simple calculation, we obtain

$$t + c = \frac{-1}{3u(t)} + \frac{u(t)^5}{15}, \quad t \geq 0, \tag{2.13}$$

where c is a constant. Letting $t \to +\infty$, from (2.13), we see that $u(t) \to 0^-$ or $u(t) \to +\infty$. This also shows that in each case $L(u) \neq \emptyset$ for (2.12). In fact, here $L(u)$ contains the set $\{u(t) : t \geq 0\}$ because in our case $\phi(u(t))$ is increasing.

Remark 2.6. Theorem 2.4 shows that the existence of a bounded solution to (2.1) implies that $(\nabla\phi)^{-1}(0)$ is nonempty. However, Example 2.5 shows that the converse is not true. In fact, this is because if for example we choose $u(0) = 1$ in (2.1), then since $u(t)$ is increasing, the unique solution of (2.1) must tend to $+\infty$, because it cannot tend to 0^- as $t \to +\infty$.

Theorem 2.7. *If either one of the following assumptions is satisfied, then bounded solutions to* (2.1) *converge strongly to some point in* $(\nabla\phi)^{-1}(0)$*:*

(i) *Sublevel sets of* ϕ *are compact.*
(ii) $\operatorname{int} L(u) \neq \emptyset$.

Proof. (i) By Theorem 2.4, it suffices to consider the case $\lim_{t\to+\infty} \phi(u(t)) = \inf\phi$. That is when $u(t) \rightharpoonup p \in \operatorname{argmin}\phi$ as $t \to +\infty$. Let $\psi(t) = \phi(u(t)) - \inf\phi$. Clearly $\psi(t) \geq 0$, and $\lim_{t\to+\infty} \psi(t) = 0$. If there exists a sequence $t_n \subset [0,+\infty)$ such that $t_n \uparrow +\infty$, and $\psi(t_n) = 0$ for all $n \geq 0$, then $\phi(u(t_n)) = \inf\phi \leq \phi(u(t))$ for all $t \geq 0$ and in particular for all $t \in [t_0, t_n]$ and for all $n \geq 0$. Otherwise, there exists some $t_0 > 0$ such that $\psi(t) > 0$ for all $t \geq t_0$. Let $n \geq t_0$. Since ψ is continuous and $[t_0, n]$ is compact, then ψ takes its minimum on $[t_0, n]$. We define the sequence t_n as the smallest element of $\operatorname{argmin}\psi|_{[t_0,n]}$. Therefore for all $n \in \mathbb{N}$ if $t \in [t_0, t_n]$, then $\psi(t_n) \leq \psi(t)$ and hence $\phi(u(t_n)) \leq \phi(u(t))$. Now either t_n has a subsequence t_{n_k} such that $t_{n_k} \uparrow +\infty$ as $k \to +\infty$, or there exists $N \in \mathbb{N}$ such that $t_n < N$ for all n. In the first case, by going to a subsequence, we conclude that $t_n \uparrow +\infty$. In the second case, for all $n \geq N$ the minimizer of ψ on $[t_0, t_n]$ is not greater than N. Therefore $t_n = t_N$ for all $n \geq N$. This implies that $\lim_{n\to+\infty} \psi(t_n) = \psi(t_N) \neq 0$

which is a contradiction. Therefore the second case never happens. Thus we showed that we always have $u(t_n) \in \{x : \phi(x) \le \phi(u(t_0))\}$ for all n. Since sublevel sets of ϕ are compact, then the sequence $u(t_n)$ has a strong limit point, say p, which by going to a subsequence we may assume $u(t_n) \to p$ as $n \to +\infty$. Since ϕ is continuous and $\lim_{t\to+\infty} \phi(u(t)) = \inf \phi$, we obtain $p \in \operatorname{argmin} \phi$. By Proposition 2.1, we know that $\lim_{t\to+\infty} \|u(t) - p\|^2$ exists which implies that $u(t) \to p$ as $t \to +\infty$.

(ii) Since $\operatorname{int} L(u) \neq \emptyset$, there exists some $y \in L(u)$ such that for some $r > 0$, we have $\overline{\mathbb{B}}(y, r) \subset L(u)$. Let $z = y + r\frac{\nabla\phi(u(t))}{\|\nabla\phi(u(t))\|}$. Since $\phi(z) \le \phi(u(t))$, using the characterization of quasiconvex functions, we have

$$\langle \nabla\phi(u(t)), u(t) - z\rangle = \langle \nabla\phi(u(t)), u(t) - y\rangle - r\|\nabla\phi(u(t))\| \ge 0.$$

The above inequality is in fact (2.11) which holds for all $t \ge 0$. The rest of the proof is similar to the proof of Theorem 2.4. □

Theorem 2.8. *Assume that $\phi : H \to \mathbb{R}$ is a strongly quasiconvex function and $u(t)$ is a bounded solution to* (2.1). *Then* $\operatorname{argmin} \phi$ *is a singleton and $u(t)$ converges strongly to the unique minimizer of ϕ.*

Proof. By Proposition 2.3, the boundedness of $u(t)$ implies that $L(u) \neq \emptyset$. Let $y \in L(u)$ and $\lambda \in (0, 1)$. Since ϕ is strongly quasiconvex, there exists some $\alpha > 0$ such that

$$\phi(u(t) + \lambda(y - u(t))) - \phi(u(t)) \le -\alpha\lambda(1 - \lambda)\|u(t) - y\|^2,$$

where $\phi(u(t)) = \max\{\phi(u(t)), \phi(y)\}$. Dividing both sides of the above inequality by λ and letting λ tend to zero, we obtain

$$\alpha\|u(t) - y\|^2 \le \langle \nabla\phi(u(t)), u(t) - y\rangle.$$

Replacing $\nabla\phi(u(t))$ from (2.1), we have

$$\alpha\|u(t) - y\|^2 \le \langle \dot{u}(t) - f(t), u(t) - y\rangle \le \frac{1}{2}\frac{d}{dt}\|u(t) - y\|^2 + M\|f(t)\|,$$

where $M = \sup_{t\ge0} \|u(t) - y\|$. Integrating the above inequality on $[0, t]$, we obtain

$$\alpha \int_0^t \|u(\tau) - y\|^2 d\tau \le \frac{1}{2}\|u(t) - y\|^2 - \frac{1}{2}\|u(0) - y\|^2 + M \int_0^t \|f(\tau)\| d\tau.$$

Since $f(t) \in L^1((0, +\infty), H)$, letting $t \to +\infty$, we have $\|u(t)-y\| \in L^2((0, +\infty), H)$. On the other hand, from Proposition 2.1, we know that $\lim_{t\to+\infty} \|u(t) - y\|$ exists, hence $\lim_{t\to+\infty} u(t) = y$. The uniqueness of the limit implies that $L(u)$ is a singleton. Therefore $\operatorname{argmin} \phi = L(u)$. □

3. Discrete case

Consider the following discrete version of (2.1),

$$\begin{gathered} u_{n+1} - u_n = \lambda_n \nabla\phi(u_n) + f_n, \\ u_0 = x \in H, \end{gathered} \tag{3.1}$$

where $f_n \in l^1$, $\lambda_n \ge \varepsilon$ for some $\varepsilon > 0$, and $\phi : H \to \mathbb{R}$ is a differentiable quasiconvex function such that $\nabla\phi$ is Lipschitz continuous with Lipschitz constant K. We start by recalling the following lemma from [4].

Lemma 3.1. *Let U be a nonempty, open and convex subset of H, $K > 0$, and $\phi : U \to \mathbb{R}$ be a Fréchet differentiable function such that $\nabla\phi$ is Lipschitz continuous with Lipschitz constant K on U, and let x and y be in U. Then the following hold:*

$$|\phi(y) - \phi(x) - \langle \nabla\phi(x), y - x\rangle| \le \frac{K}{2}\|y - x\|^2.$$

To study the asymptotic behavior of u_n, we define a discrete version of $L(u)$,

$$L(u_n) = \{y \in H : \exists N > 0 \text{ s.t. } \phi(y) \le \phi(u_n) \quad \forall n \ge N\}.$$

As in the continuous case, without loss of generality we assume that $N = 0$. The following proposition, which is a discrete version of Proposition 2.1, will be needed in the sequel.

Proposition 3.2. *Let u_n be the sequence generated by (3.1). For each $y \in L(u_n)$, and $k < m$, we have*

$$\|u_k - y\| \le \|u_m - y\| + \sum_{n=k}^{m-1} \|f_n\|, \tag{3.2}$$

and consequently $\lim_{n\to+\infty} \|u_n - y\|$ exists (it may be infinite).

Proof. Multiplying (3.1) by $(u_n - y)$ and then using the characterization of quasi-convex functions, we obtain

$$\langle u_{n+1} - u_n, u_n - y\rangle \ge \langle f_n, u_n - y\rangle.$$

Applying the polarization identity and the Cauchy-Schwarz inequality, we obtain

$$\|u_{n+1} - y\|^2 - \|u_n - y\|^2 + 2\|u_n - y\|\|f_n\| \ge 0.$$

If $\|u_{n+1} - y\| + \|u_n - y\| \ne 0$, then

$$\|u_{n+1} - y\| - \|u_n - y\| + 2\frac{\|u_n - y\|}{\|u_{n+1} - y\| + \|u_n - y\|}\|f_n\| \ge 0,$$

which implies that

$$\|u_{n+1} - y\| - \|u_n - y\| + 2\|f_n\| \ge 0. \tag{3.3}$$

If $\|u_{n+1} - y\| + \|u_n - y\| = 0$, the above inequality is clearly true. Summing (3.3) from $n = k$ to $n = m - 1$, we obtain

$$\|u_k - y\| \le \|u_m - y\| + \sum_{n=k}^{m-1} \|f_n\|.$$

Now by taking lim inf as $m \to +\infty$ and then taking lim sup as $k \to +\infty$ in the above inequality, we conclude the result. □

Proposition 3.3. *Let u_n be a solution to (3.1) such that $\liminf_{n\to+\infty} \|u_n\| < +\infty$. Then $L(u_n)$ is nonempty if and only if $\lim_{n\to+\infty} \phi(u_n)$ exists, and in this case u_n is bounded.*

Proof. Let $y \in L(u_n)$. Since $\liminf_{n\to+\infty} \|u_n\| < +\infty$, by (3.2), u_n is bounded. Multiplying both sides of (3.1) by $(u_n - y)$ and then applying the polarization identity and the Cauchy-Schwarz inequality, we obtain

$$\|u_{n+1} - u_n\|^2 \le \|u_{n+1} - y\|^2 - \|u_n - y\|^2 + 2\|u_n - y\|\|f_n\|. \tag{3.4}$$

Summing both sides of (3.4) from $n = 0$ to $n = m$, and then letting $m \to +\infty$, we find that $(u_{n+1} - u_n) \in l^2$. Hence

$$\lim_{n\to+\infty} \nabla\phi(u_n) = \lim_{n\to+\infty}(u_{n+1} - u_n) = 0.$$

Now since $\nabla\phi$ is Lipschitz continuous with Lipschitz constant K, by Lemma 3.1 we have

$$\phi(u_{n+1}) - \phi(u_n) \le \langle \nabla\phi(u_{n+1}), u_{n+1} - u_n\rangle + \frac{K}{2}\|u_{n+1} - u_n\|^2. \tag{3.5}$$

On the other hand, by the polarization identity, we have

$$\begin{aligned}&\langle \nabla\phi(u_{n+1}), u_{n+1} - u_n\rangle \\ &= \frac{1}{2}\Big(\|\nabla\phi(u_{n+1})\|^2 + \|u_{n+1} - u_n\|^2 - \|u_{n+1} - u_n - \nabla\phi(u_{n+1})\|^2\Big).\end{aligned}$$

Substituting from the above identity in (3.5) and then using (3.1), we obtain

$$\phi(u_{n+1}) - \phi(u_n) \le \frac{1}{2\varepsilon^2}\|u_{n+2} - u_{n+1} - f_{n+1}\|^2 + \frac{1+K}{2}\|u_{n+1} - u_n\|^2. \tag{3.6}$$

Summing the above inequality from $n = k$ to $n = m - 1$, we obtain

$$\phi(u_m) - \phi(u_k) \le \frac{1}{2\varepsilon^2}\sum_{n=k+1}^{m}\|u_{n+1} - u_n - f_n\|^2 + \frac{1+K}{2}\sum_{n=k}^{m-1}\|u_{n+1} - u_n\|^2.$$

Since f_n and $(u_{n+1}-u_n)$ belong to l^2, by taking lim sup as $m \to +\infty$ and then taking lim inf as $k \to +\infty$ in the above inequality, we conclude that $\lim_{n\to+\infty}\phi(u_n)$ exists. Moreover u_n is bounded by Proposition 3.2. Conversely, assume by contradiction that $L(u_n) = \emptyset$. Then for each $z \in H$ there exists a subsequence $u^z_{n_k}$ of the sequence u_n such that $\phi(z) > \phi(u^z_{n_k})$ for all $k \ge 1$. Since $\lim_{n\to+\infty}\phi(u_n)$ exists, we have

$$\lim_{n\to+\infty}\phi(u_n) = \lim_{k\to+\infty}\phi(u^z_{n_k}) \le \phi(z) \quad \forall z \in H,$$

hence $\lim_{n\to+\infty}\phi(u_n) = \inf\phi$. Since by assumption $\liminf_{n\to+\infty}\|u_n\| < +\infty$, Proposition 3.3 shows that u_n is bounded. This together with the boundedness of ϕ on bounded sets, and the above inequality imply that $\inf\phi > -\infty$. Also, u_n has a nonempty set of weak cluster points. Hence there exists a subsequence u_{n_k} of u_n that converges weakly to some point $p \in H$. By the lower semicontinuity of ϕ for the weak topology, we obtain

$$\phi(p) \le \lim_{n\to+\infty}\phi(u_{n_k}) = \lim_{n\to+\infty}\phi(u_n) = \inf\phi$$

which implies that $p \in \operatorname{argmin}\phi$, a contradiction with $L(u_n) = \emptyset$. □

Remark 3.4. If ϕ is convex, we can omit the condition that $\nabla\phi$ is Lipschitz continuous, because in this case, (3.5) is satisfied with $K = 0$.

As we have seen in the continuous case, clearly $\operatorname{argmin}\phi \subset L(u_n)$. In the following Proposition, we state some conditions which together with the assumption $\liminf_{n\to+\infty}\|u_n\| < +\infty$, imply that $L(u_n)$ is nonempty.

Proposition 3.5. *Assume that u_n is a solution to* (3.1) *such that $\lim_{n\to+\infty}\|u_n\| < +\infty$. If either one of the following conditions is satisfied, then $L(u_n)$ is nonempty:*

(i) *ϕ is convex and the sequence of step sizes λ_n is bounded above,*
(ii) $\limsup_{n\to+\infty}\lambda_n < 2/K$.

Proof. Since $\nabla\phi$ is Lipschitz continuous, by Lemma 3.1, we have

$$\langle \nabla\phi(u_n), u_{n+1}-u_n\rangle \le \phi(u_{n+1})-\phi(u_n)+\frac{K}{2}\|u_{n+1}-u_n\|^2. \tag{3.7}$$

Note that if ϕ is convex, using the subdifferential inequality, we can omit the Lipschitz continuity of $\nabla\phi$ and take $K=0$ in (3.7). Multiplying both sides of (3.1) by $(u_{n+1}-u_n)$, using (3.7), we obtain

$$\Big(\frac{1}{\lambda_n}-\frac{K}{2}\Big)\|u_{n+1}-u_n\|^2 \le \phi(u_{n+1})-\phi(u_n)+\frac{1}{\varepsilon}\|f_n\|\|u_{n+1}-u_n\|. \tag{3.8}$$

Since $\liminf_{n\to+\infty}\|u_n\|<+\infty$, the sequence u_n has a bounded subsequence, say u_{n_k}. On the other hand, ϕ is bounded on bounded sets hence summing (3.8) from $n=0$ to $n=n_k-1$, and then using the Cauchy-Schwarz inequality, if either one of the assumptions (i) or (ii) holds, we obtain

$$c\sum_{n=0}^{n_k-1}\|u_{n+1}-u_n\|^2 \le \phi(u_{n_k})-\phi(u_0)+\frac{1}{\varepsilon}\Big(\sum_{n=0}^{n_k-1}\|f_n\|^2\Big)^{1/2}\Big(\sum_{n=0}^{n_k-1}\|u_{n+1}-u_n\|^2\Big)^{1/2},$$

where $c=\inf_{n\ge 0}\Big(\frac{1}{\lambda_n}-\frac{K}{2}\Big)$ is a positive constant. Dividing both sides of the above inequality by $\Big(\sum_{n=0}^{n_k-1}\|u_{n+1}-u_n\|^2\Big)^{1/2}$, and then letting $k\to+\infty$, we obtain $(u_{n+1}-u_n)\in l^2$. By an argument similar to the proof of the first part of Proposition 3.3, we conclude that $\lim_{n\to+\infty}\phi(u_n)$ exists. Now, Proposition 3.3 implies that $L(u_n)\neq\emptyset$. □

Open problem. In the continuous case, Proposition 2.3 shows that condition $\liminf_{t\to+\infty}\|u(t)\|<+\infty$ implies $L(u)\neq\emptyset$. However, in the discrete case, we do not know whether without any additional assumptions, this implication holds.

Theorem 3.6. *Let u_n be the sequence generated by* (3.1) *and $L(u_n)\neq\emptyset$. If $\liminf_{n\to+\infty}\|u_n\|<+\infty$, then there exists some $p\in(\nabla\phi)^{-1}(0)$ such that $u_n\rightharpoonup p$ as $n\to+\infty$ and if $p\notin\operatorname{argmin}\phi$ the convergence is strong. If u_n is unbounded, then $\|u_n\|\to+\infty$, as $n\to+\infty$.*

Proof. Let $y\in L(u_n)$. From Proposition 3.2, we know that $\lim_{n\to+\infty}\|u_n-y\|$ exists. Hence if u_n is unbounded then $\|u_n\|$ goes to infinity as $n\to+\infty$. If u_n is bounded, then $\lim_{n\to+\infty}\|u_n-y\|$ is finite. Let q be a weak cluster point of u_n. There exists a subsequence u_{n_k} such that $u_{n_k}\rightharpoonup q$ as $k\to+\infty$. If $\lim_{n\to+\infty}\phi(u_n)=\phi(y)$, then

$$\phi(q)\le\liminf_{k\to+\infty}\phi(u_{n_k})=\lim_{n\to+\infty}\phi(u_n)=\phi(y),$$

which yields $q\in L(u_n)$. By Opial's lemma [15], there is a $p\in L(u_n)$ such that $u_n\rightharpoonup p$. If $p\notin\operatorname{argmin}\phi$, then there exists an element in $L(u_n)$ which we denote again by y such that $\lim_{n\to+\infty}\phi(u_n)>\phi(y)$. In this case, there are $n_0>0$ and $r>0$, such that

$$\phi(z)\le\phi(u_n),\quad \forall n\ge n_0,\quad \forall z\in\overline{\mathbb{B}}(y,r).$$

Therefore a discrete version of Lemma 1.1 yields

$$r\|\nabla\phi(u_n)\|\le\langle\nabla\phi(u_n),u_n-y\rangle,\quad \forall n\ge n_0. \tag{3.9}$$

Multiplying both sides of the above inequality by λ_n, replacing $\lambda_n\nabla\phi(u_n)$ from (3.1) with $(u_{n+1}-u_n-f_n)$ and then applying the polarization identity and the

Cauchy-Schwarz inequality, we obtain

$$2\lambda_n r\|\nabla\phi(u_n)\| \le \|u_{n+1}-y\|^2 - \|u_n - y\|^2 + 2\|u_n - y\|\|f_n\|.$$

The above inequality implies that $\lambda_n \nabla\phi(u_n) \in l^1$. Hence $(u_{n+1}-u_n) \in l^1$, which implies that u_n is a Cauchy sequence for the strong topology, therefore u_n converges strongly to p. Since $\lim_{n\to+\infty} \nabla\phi(u_n) = 0$ and $\nabla\phi$ is continuous, then $p \in (\nabla\phi)^{-1}(0)$. □

Example 3.7. Assume that ϕ is the same function as in Example 2.5 and consider (3.1) with $\lambda_n = \frac{2}{3}n$ and $f_n \equiv 0$. Summing both sides of (3.1) from $n = 0$ to $n = N-1$, we have

$$u_N = u_0 + \sum_{n=0}^{N-1} \lambda_n \frac{3u_n^2}{1+u_n^6}. \tag{3.10}$$

If the summation in (3.10) converges as $N \to +\infty$, then u_N converges as $N \to +\infty$, and by the continuity of ϕ, $\lim_{N\to+\infty} \phi(u_N)$ exists. By Proposition 3.3, we see that $L(u_n)$ is nonempty in this case. On the other hand, if the summation in (3.10) diverges as $N \to +\infty$, then $u_N \to +\infty$ as $N \to +\infty$. This together with the fact that ϕ is nondecreasing, imply that $L(u_n)$ is nonempty. Therefore all the assumptions of Theorem 3.6 are satisfied. Table 1 compares 1000 iterations of the sequence u_n given by (3.1) with two different initial values $u_0 = -0.5$ and $u_0 = 1$. The numerical results show that for $u_0 = -0.5$, $u_n \to 0 \in (\nabla\phi)^{-1}(0)$ and for $u_0 = 1$, u_n slowly goes to infinity.

Table 1. Numerical results for (3.10)

n	u_n	u_n
0	-0.5	1
1	-0.00769231	2
10	-0.00404869	3.63765
20	-0.00171074	4.68854
30	-0.0008858	5.46951
40	-0.000533135	6.11128
50	-0.000354164	6.66517
60	-0.000251763	7.15741
70	-0.000187942	7.60348
80	-0.000145564	8.01339
90	-0.000116023	8.39404
100	-0.0000946225	8.7504
1000	-9.94968×10^{-7}	21.8786

Theorem 3.8. *Assume that u_n is a bounded sequence which satisfies* (3.1) *and $L(u_n) \neq \emptyset$. If either one of the following assumptions is satisfied, then u_n converges strongly to some point in $(\nabla\phi)^{-1}(0)$:*

(i) *Sublevel sets of ϕ are compact,*
(ii) $\operatorname{int} L(u_n) \neq \emptyset$.

Proof. (i) By Proposition 3.3, we know that $\lim_{n\to+\infty}\phi(u_n)$ exists. As we have already seen in the proof of Theorem 3.6, if $\lim_{n\to+\infty}\phi(u_n) > \inf\phi$, then the sequence u_n strongly converges to some point in $(\nabla\phi)^{-1}(0)$. Therefore we only need to consider the case where $\lim_{n\to+\infty}\phi(u_n) = \inf\phi$. Let $\psi_n = \phi(u_n) - \inf\phi$. Clearly $\psi_n \geq 0$ and $\lim_{n\to+\infty}\psi_n = 0$. If there exists a subsequence n_k such that $n_k \uparrow +\infty$ and $\psi_{n_k} = 0$ for all $k \geq 0$, then $\phi(u_{n_k}) = \inf\phi \leq \phi(u_n)$ for all $n \geq 0$ and in particular for all $n_0 \leq n \leq n_k$ and for all $k \geq 0$. Otherwise, there exists some $n_0 > 0$ such that $\psi_n > 0$ for all $n \geq n_0$. Let $k \geq n_0$. Choose the subsequence n_k such that $\psi_{n_k} = \min\{\psi_{n_0}, \cdots, \psi_k\}$. Therefore if $n_0 \leq n \leq k$, then $\psi_{n_k} \leq \psi_n$ and hence $\phi(u_{n_k}) \leq \phi(u_n)$. Now either n_k has a subsequence n_{k_l} such that $n_{k_l} \uparrow +\infty$ as $l \to +\infty$, or there exists $N \in \mathbb{N}$ such that $n_k < N$ for all k. In the first case, by going to a subsequence, we conclude that $n_k \uparrow +\infty$ as $k \to +\infty$. In the second case, the minimum of $\psi_{n_0}, \cdots, \psi_k$ is not greater than ψ_{n_N}, for all $k \geq N$. Therefore $n_k = n_N$ for all $k \geq N$. This implies that $\lim_{k\to+\infty}\psi_{n_k} = \psi_{n_N} \neq 0$ which is a contradiction. Therefore the second case never happens. We have $u_{n_k} \in \{x : \phi(x) \leq \phi(u_{n_0})\}$ for all k. Since sublevel sets of ϕ are compact, then the sequence u_{n_k} has a strong cluster point, which is necessarily a weak cluster point of the sequence u_n as well. This together with Theorem 3.6 imply that the set of all strong cluster points of the sequence u_n is the singleton $\{p\}$, where $p \in \operatorname{argmin}\phi$ is the unique weak cluster point of the sequence u_n.

(ii) Since $\operatorname{int} L(u_n) \neq \emptyset$, there exist $y \in L(u_n)$ and $r > 0$ such that $\overline{\mathbb{B}}(y, r) \subset L(u_n)$. Therefore $z = y + r\frac{\nabla\phi(u_n)}{\|\nabla\phi(u_n)\|} \in L(u_n)$. By the characterization of quasiconvex functions, we have

$$\langle\nabla\phi(u_n), u_n - z\rangle = \langle\nabla\phi(u_n), u_n - y\rangle - r\|\nabla\phi(u_n)\| \geq 0.$$

Therefore we have obtained (3.9) for all $n \geq 0$. The rest of the proof follows from the proof of Theorem 3.6. □

Theorem 3.9. *Assume that $\phi : H \to \mathbb{R}$ is a strongly quasiconvex function and u_n is a bounded sequence generated by* (3.1) *with $L(u_n) \neq \emptyset$. Then the sequence u_n converges strongly to the unique minimizer of ϕ.*

Proof. Let $y \in L(u_n)$. Since ϕ is strongly quasiconvexe, there is some $\alpha > 0$ such that

$$\phi(u_n + \lambda(y - u_n)) - \phi(u_n) \leq -\alpha\lambda(1-\lambda)\|u_n - y\|^2 \quad \forall\lambda \in (0, 1).$$

Dividing both sides of the above inequality by λ, and then letting $\lambda \to 0$, we obtain

$$\alpha\|u_n - y\|^2 \leq \langle\nabla\phi(u_n), u_n - y\rangle.$$

Multiplying both sides of the above inequality by λ_n and then replacing $\lambda_n\nabla\phi(u_n)$ from (3.1), we obtain

$$\alpha\varepsilon\|u_n - y\|^2 \leq \langle u_{n+1} - u_n - f_n, u_n - y\rangle \leq \frac{1}{2}\|u_{n+1} - y\|^2 - \frac{1}{2}\|u_n - y\|^2 + M\|f_n\|,$$

where $M = \sup_{n\geq 0}\|u_n - y\|$. Summing the above inequality from $n = 1$ to $n = N$ and then letting $N \to +\infty$, we obtain $\|u_n - y\| \in l^2$. This shows that $L(u_n)$ is a singleton and u_n converges strongly to the unique minimizer of ϕ, which completes the proof. □

4. Conclusions

In this article, we studied the asymptotic behavior of solutions to a quasi-autonomous gradient system of expansive type governed by a differentiable quasi-convex function ϕ, both in continuous and discrete time. In particular, we showed that solutions either blow up and go to infinity in norm, or converge weakly to some critical point of ϕ. Since the gradient of a quasiconvex function is no longer a monotone operator, then compared to the convex case, new methods had to be developed to study this problem. Numerical examples are also given to illustrate our results. Besides the open problem stated in the paper, as a future direction for research, it would be interesting to investigate the possibility of extending the results of the paper to the case where ϕ is not assumed to be differentiable.

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