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INFINITELY MANY SOLUTIONS FOR A NONLOCAL TYPE PROBLEM WITH SIGN-CHANGING WEIGHT FUNCTION

ELHOUSSINE AZROUL, ABDELMOUJIB BENKIRANE,
MOHAMMED SRATI, CÉSAR TORRES

Abstract. In this article, we study the existence of weak solutions for a fractional type problem driven by a nonlocal operator of elliptic type

$$\begin{gathered}(-\Delta)^s_{a_1}u-\lambda a_2(|u|)u=f(x,u)+g(x)|u|^{q(x)-2}u \quad \text{in } \Omega\\ u=0 \quad \text{in } \mathbb{R}^N\setminus\Omega.\end{gathered}$$

Our approach is based on critical point theorems and variational methods.

1. Introduction

The aim of this article is to study the existence of weak solutions of the nonlocal problem

$$\begin{gathered}(-\Delta)^s_{a_1}u-\lambda a_2(|u|)u=f(x,u)+g(x)|u|^{q(x)-2}u \quad \text{in } \Omega\\ u=0 \quad \text{in } \mathbb{R}^N\setminus\Omega,\end{gathered} \tag{1.1}$$

where Ω is a Lipschitz open bounded subset of $\mathbb{R}^N$, $N\geq 1$, $q:\overline{\Omega}\to(1,+\infty)$ is a bounded continuous function, $s\in(0,1)$, λ is a positive real parameters, $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function with a subcritical growth conditions and $(-\Delta)^s_{a_1}$ is a nonlocal integro-differential operator of elliptic type defined as

$$(-\Delta)^s_{a_1}u(x)=2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}a_1\Big(\frac{|u(x)-u(y)|}{|x-y|^s}\Big)\frac{u(x)-u(y)}{|x-y|^s}\frac{dy}{|x-y|^{N+s}}\,,$$

for $x\in\mathbb{R}^N$, and $a_1:\mathbb{R}\to\mathbb{R}$ to be specified later.

In the previous decade, great attention has been devoted to the study of nonlinear problems in modular spaces, in particular quasilinear problems involving the Φ-Laplacian operator, see for instance [6, 7, 9, 19]. These kind of operator appear in several problems in Mathematical physics, for example in nonlinear elasticity [23] when

$$\Phi(t)=(1+t^2)^\alpha-1,\quad \alpha\in(1,\frac{N}{N-2}).$$

In plasticity [24] when

$$\Phi(t)=t^p\ln(1+t),\quad p\in\Big(\frac{-1+\sqrt{1+4N}}{2},N-1\Big),\ N\geq 3.$$

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In biophysics and physics of plasmas [25] when

$$\Phi(t)=\frac{1}{p}|t|^p+\frac{1}{q}|t|^q, \quad 1<p<q<N,\ q\in(p,p^*).$$

Recently we have seen an increasing development of the theory of nonlocal operators; such operators arise naturally in the context of stochastic Lévy processes with jumps and have been studied thoroughly both from the point of view of Probability and Analysis as they proved to be accurate models to describe different phenomena in physics, finance, image processing, and ecology; see for instance [2, 3, 4, 14, 17, 31, 21] and references therein.

When $q(x)=q$ a positive constant for all $x\in\overline{\Omega}$, $a_1(t)=t^{p_1-2}$ and $a_2(t)=t^{p_2-2}$, the problem (1.1) is the well known fractional p_1-Laplacian problem

$$\begin{gathered}
(-\Delta)^s_{p_1}u-\lambda|u|^{p_2-2}u=f(x,u)+g(x)|u|^{q-2}u \quad \text{in } \Omega\\
u=0 \quad \text{in } \mathbb{R}^N\setminus\Omega,
\end{gathered} \tag{1.2}$$

where $(-\Delta)^s_{p_1}$ is the fractional p-Laplacian operator defined as

$$(-\Delta)^s_{p_1}u(x)=2\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{|u(x)-u(y)|^{p_1-2}(u(x)-u(y))}{|x-y|^{N+sp_1}}dy\,.$$

Problem (1.2) have been extensively investigated in recent years and many existence results have been obtained under general hypotheses [10, 11, 12, 18, 27, 29, 32, 33]. Bartolo and Bisci [12] studied the multiplicity of weak solutions to problem (1.2) with $g(x)=0$. Mosconi et al. [32] studied the existence of weak solution of (1.2) with $g(x)=0$ and $f(x,u)=|u|^{p^*_s-2}u$. We also mention the recent work by Bueno et al. [18], where they have considered the existence and multiplicity of solution for problem (1.2) with exponential growth.

Inspired by the previous works, the aim of this paper is to study the existence of a weak solutions for (1.1). Before stating our results let us introduce the main ingredients involved in our approach. Regarding the functions f, g, and ϕ we assume that:

- (A1) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying:
$$\begin{gathered}
f(x,t)=o(a_1(|t|)t) \quad \text{as } t\to 0 \text{ uniformly in } x,\\
f(x,t)=o(a_2(|t|)t) \quad \text{as } t\to \infty \text{ uniformly in } x\,;
\end{gathered}$$
- (A2) there exists $\mu\in(\phi_1^+,\phi_2^-)$ such that $tf(x,t)\geq\mu F(x,t)>0$ for all $|t|\geq 0$ and a.e. $x\in\Omega$, where $F(x,t):=\int_0^t f(x,\tau)\,d\tau$ and ϕ_1^+,ϕ_2^- are given in (A1);
- (A3) $f(x,-t)=-f(x,t)$ for all $(x,t)\in\Omega\times\mathbb{R}$;
- (A4) $0\leq g\in L^\infty(\Omega)$;
- (A5) $\lim_{t\to\infty}\frac{\Phi_2(kt)}{(\Phi_1)_*(t)}=0$ for all $k>0$.

To simplify the notation, we put

$$D^su:=\frac{u(x)-u(y)}{|x-y|^s}, \quad d\mu=\frac{dx\,dy}{|x-y|^N} \quad \forall (x,y)\in\Omega\times\Omega,$$

and for any $i=1,2$, we put

$$\xi_i^m(t)=\min\{t^{\phi_i^-},t^{\phi_i^+}\}, \quad \xi_i^l(t)=\max\{t^{\phi_i^-},t^{\phi_i^+}\},$$

and we put

$$\xi_3^m(t)=\min\{t^{q^-},t^{q^+}\}, \quad \xi_3^l(t)=\max\{t^{q^-},t^{q^+}\},$$

So we say that $u \in W_0^s L_{\Phi_1}(\Omega)$ is a weak solution of problem (1.1) if,

$$\begin{aligned} &\int_\Omega \int_\Omega a_1(|D^s u|) D^s u D^s v \, d\mu - \lambda \int_\Omega a_2(|u) uv \, dx \\ &- \int_\Omega f(x,u) v \, dx - \int_\Omega g(x) |u|^{q(x)-2} uv \, dx = 0, \quad \forall v \in W_0^s L_{\Phi_1}(\Omega). \end{aligned}$$

Now we are in position to states our main results.

Theorem 1.1. *Assume* $\lambda = 0$ *and* (A1), (A2), (A4), (A5) *hold. Then there exists* $\lambda_0 > 0$ *such that for* $\|g\|_{\widehat{q}(x)} < \lambda_0$, *problem* (1.1) *has at lest one nontrivial solution.*

Theorem 1.2. *Assume* $g(x) = 0$ *and* (A1)–(A3) *hold. Then problem* (1.1) *has infinitely many nontrivial solutions.*

This article is organized as follows: In the second section, we collect some preliminaries on fractional Orlicz-Sobolev spaces that will be used later. In the third section, by using the mountain pass theorem and Fountain Theorem in critical point theory, we obtain the existence and multiplicity of nontrivial solutions of problem (1.1).

2. Preliminaries results

We introduce the fractional Orlicz-Sobolev space to investigate problem (1.1). Let us recall the definitions and some elementary properties of this spaces. We refer the reader to [1, 8, 21] for further reference and for some of the proofs of the results in this section.

Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$. For $i = 1, 2$, we assume that $a_i : \mathbb{R} \to \mathbb{R}$ in (1.1) is such that $\phi_i : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi_i(t) = \begin{cases} a_i(|t|)t & \text{for } t \neq 0, \\ 0 & \text{for } t = 0, \end{cases}$$

is increasing homeomorphism from $\mathbb{R}$ onto itself. Let

$$\Phi_i(t) = \int_0^t \phi_i(\tau) \, d\tau \quad \text{for all } t \in \mathbb{R}, \ i = 1, 2.$$

Then, for any $i = 1, 2$, Φ_i is N-function, i.e. $\Phi_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, convex, increasing function, with $\frac{\Phi_i(t)}{t} \to 0$ as $t \to 0$ and $\frac{\Phi_i(t)}{t} \to \infty$ as $t \to \infty$, see [1].

For the functions Φ_i $i = 1, 2$, we introduce the Orlicz space,

$$L_{\Phi_i}(\Omega) = \big\{ u : \Omega \to \mathbb{R} \text{ measurable } \int_\Omega \Phi_i(\lambda |u(x)|) \, dx < \infty \text{ for some } \lambda > 0 \big\},$$

which are Banach spaces endowed with the Luxemburg norm

$$\|u\|_{\Phi_i} = \inf\{\lambda > 0 : \int_\Omega \Phi_i\big(\frac{|u(x)|}{\lambda}\big) dx \leq 1\}.$$

The conjugate N-function of Φ_i $i = 1, 2$, is defined by

$$\overline{\Phi_i}(t) = \int_0^t \overline{\phi_i}(\tau) \, d\tau,$$

where $\overline{\phi_i} : \mathbb{R} \to \mathbb{R}$ $i = 1, 2$, is given by $\overline{\phi_i}(t) = \sup\{s : \phi_i(s) \le t\}$. Furthermore, it is possible to prove a Hölder type inequality,

$$\Big| \int_\Omega uv \, dx \Big| \le 2\|u\|_{\Phi_i} \|v\|_{\overline{\Phi_i}} \quad \forall u \in L_{\Phi_i}(\Omega),\ v \in L_{\overline{\Phi_i}}(\Omega),\ i = 1, 2. \tag{2.1}$$

Throughout this paper, we assume that

$$1 < \phi_i^- := \inf_{t \ge 0} \frac{t\phi_i(t)}{\Phi_i(t)} \le \phi_i^+ := \sup_{t \ge 0} \frac{t\phi_i(t)}{\Phi_i(t)} < +\infty, \quad i = 1, 2. \tag{2.2}$$

The above relation implies that for any $i = 1, 2$, $\Phi_i \in \Delta_2$ i.e. Φ_i satisfies the global Δ_2-condition:

$$\Phi_i(2t) \le K_i \Phi_i(t) \text{ for all } t \ge 0,\ i = 1, 2,$$

where K_i $i = 1, 2$, are positive constants, for more details see [30].

Furthermore, for $i = 1, 2$, we assume that

$$\text{the function } [0, \infty) \ni t \mapsto \Phi_i(\sqrt{t}) \text{ is convex.} \tag{2.3}$$

From this condition we have that $L_{\Phi_i}(\Omega)$ is a uniformly convex space (see [30]).

Lemma 2.1 ([15])**.** *Assume that $\Phi_i \in \Delta_2$ for $i = 1, 2$. Then*

$$\overline{\Phi_i}(\phi_i(t)) \le c\Phi_i(t) \quad \textit{for all } t \ge 0,\ i = 1, 2, \tag{2.4}$$

where c is a positive constant.

Proposition 2.2 ([30])**.** *Suppose that* (2.2) *holds. Then*

$$\|u\|_{\Phi_i}^{\phi_i^-} \le \int_\Omega \Phi_i(|u|)\, dx \le \|u\|_{\Phi_i}^{\phi_i^+}, \quad \forall u \in L_{\Phi_i}(\Omega) \textit{ with } \|u\|_{\Phi_i} > 1,\ i = 1, 2,$$

$$\|u\|_{\Phi_i}^{\phi_i^+} \le \int_\Omega \Phi_i(|u|)\, dx \le \|u\|_{\Phi_i}^{\phi_i^-}, \quad \forall u \in L_{\Phi_i}(\Omega) \textit{ with } \|u\|_{\Phi_i} < 1,\ i = 1, 2.$$

Definition 2.3. Let A, B be two N-functions. A is stronger (resp. essentially stronger) than B, $A \succ B$ (resp. $A \succ\succ B$), if

$$B(x) \le A(ax), \quad x \ge x_0 \ge 0,$$

for some (resp. for each) $a > 0$ and x_0 (depending on a).

Remark 2.4 ([1, Section 8.5])**.** $A \succ\succ B$ is equivalent to

$$\lim_{x \to \infty} \frac{B(\lambda x)}{A(x)} = 0,$$

for all $\lambda > 0$.

Now, we define the fractional Orlicz-Sobolev space

$$W^s L_{\Phi_1}(\Omega) = \Big\{ u \in L_{\Phi_1}(\Omega) : \int_\Omega \int_\Omega \Phi_1\Big(\frac{|u(x) - u(y)|}{|x - y|^s} \Big) \frac{dx\, dy}{|x - y|^N} < \infty \Big\}.$$

This space is equipped with the norm

$$\|u\|_{s,\Phi_1} = \|u\|_{\Phi_1} + [u]_{s,\Phi_1}, \tag{2.5}$$

where $[\cdot]_{s,\Phi_1}$ is the Gagliardo seminorm

$$[u]_{s,\Phi_1} = \inf \Big\{ \lambda > 0 : \int_\Omega \int_\Omega \Phi_1\Big(\frac{|u(x) - u(y)|}{\lambda |x - y|^s} \Big) \frac{dx\, dy}{|x - y|^N} \le 1 \Big\}.$$

The space $W^sL_{\Phi_1}(\Omega)$ is a separable Banach space (resp. reflexive) space if and only if $\Phi_1 \in \Delta_2$ (resp. $\Phi_1 \in \Delta_2$ and $\overline{\Phi_1} \in \Delta_2$). Furthermore if $\Phi_1 \in \Delta_2$ and $\Phi_1(\sqrt{t})$ is convex, then the space $W^sL_{\Phi_1}(\Omega)$ is uniformly convex, see [15].

Let $W_0^sL_{\Phi_1}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{s,\Phi_1}$ defined in (2.5). Then we have the following result.

Theorem 2.5 (Generalized Poincaré inequality [8]). *Let Ω be a bounded open subset of $\mathbb{R}^N$, and let $s \in (0,1)$. Then there exists a positive constant μ such that*

$$\|u\|_{\Phi_1} \le \mu[u]_{s,\Phi_1} \quad \text{for all } u \in W_0^sL_{\Phi_1}(\Omega).$$

Therefore, if Ω is bounded, then $\|\cdot\| := [\cdot]_{s,\Phi_1}$ is a norm of $W_0^sL_{\Phi_1}(\Omega)$ equivalent to $\|\cdot\|_{s,\Phi_1}$.

Remark 2.6. By Theorem 2.5, there exists a positive constant λ_1 such that

$$\int_\Omega \Phi_1(|u|)\,dx \le \lambda_1 \int_\Omega\int_\Omega \Phi_1\Big(\frac{|u(x)-u(y)|}{|x-y|^s}\Big)\frac{dx\,dy}{|x-y|^N}, \tag{2.6}$$

for all $u \in W_0^sL_{\Phi_1}(\Omega)$.

Proposition 2.7 ([8]). *Assume (2.2) is satisfied. Then*

$$[u]_{s,\Phi_1}^{\phi_1^-} \le \Psi(u) \le [u]_{s,\Phi_1}^{\phi_1^+}, \quad \forall u \in W^sL_{\Phi_1}(\Omega) \text{ with } [u]_{s,\Phi_1} > 1, \tag{2.7}$$

$$[u]_{s,\Phi_1}^{\phi_1^+} \le \Psi(u) \le [u]_{s,\Phi_1}^{\phi_1^-}, \quad \forall u \in W^sL_{\Phi_1}(\Omega) \text{ with } [u]_{s,\Phi_1} < 1, \tag{2.8}$$

where

$$\Psi(u) = \int_\Omega\int_\Omega \Phi_1\Big(\frac{|u(x)-u(y)|}{|x-y|^s}\Big)\frac{dx\,dy}{|x-y|^N}.$$

Theorem 2.8 ([8]). *Let Ω be a bounded open subset of $\mathbb{R}^N$. Then*

$$C_0^\infty(\Omega) \subset C_0^2(\Omega) \subset W_0^sL_{\Phi_i}(\Omega)\ i=1,2.$$

In this article, we assume the following two conditions:

$$\int_0^1 \frac{\Phi_1^{-1}(\tau)}{\tau^{\frac{N+s}{N}}}\,d\tau < \infty, \tag{2.9}$$

$$\int_1^\infty \frac{\Phi_1^{-1}(\tau)}{\tau^{\frac{N+s}{N}}}\,d\tau = \infty. \tag{2.10}$$

We define the inverse Sobolev conjugate N-function of Φ_1 as follows,

$$(\Phi_1)_*^{-1}(t) = \int_0^t \frac{\Phi_1^{-1}(\tau)}{\tau^{\frac{N+s}{N}}}\,d\tau. \tag{2.11}$$

Theorem 2.9 ([8]). *Let Ω be a bounded open subset of $\mathbb{R}^N$ with $C^{0,1}$-regularity and bounded boundary. If (2.10), (2.11) and (2.2) hold, then*

$$W^sL_{\Phi_1}(\Omega) \hookrightarrow L_{(\Phi_1)_*}(\Omega). \tag{2.12}$$

Theorem 2.10 ([8]). *Let Ω be a bounded open subset of $\mathbb{R}^N$ with $C^{0,1}$-regularity and bounded boundary. If (2.10), (2.11) and (2.2) hold, then*

$$W^sL_{\Phi_1}(\Omega) \hookrightarrow L_B(\Omega), \tag{2.13}$$

is compact for all $B \prec\prec (\Phi_1)_*$.

Remark 2.11. We have some examples of functions $\phi : \mathbb{R} \to \mathbb{R}$ which are odd, increasing homeomorphisms from $\mathbb{R}$ into $\mathbb{R}$ and satisfy conditions (2.2) (see [20]).

(1) Let $\phi(t) = p|t|^{p-2}t$ for $t \in \mathbb{R}$, with $p > 1$. For this function it can be proved that $\phi^- = \phi^+ = p$. Furthermore, in this particular case the corresponding Orlicz space $L_\Phi(\Omega)$ is the classical Lebesgue space $L^p(\Omega)$ while the fractional Orlicz-Sobolev space $W^sL_\Phi(\Omega)$ is the fractional Sobolev spaces in this particular case.

(2) Consider

$$\phi(t) = \log(1 + |t|)|t|^{p-2}t, \ \forall t \in \mathbb{R}$$

with $p > 1$. In this case it can be proved that $\phi^- = p$ and $\phi^+ = p + 1$.

(3) Let

$$\phi(t) = \frac{|t|^{p-2}t}{\log(1 + |t|)}, \quad \text{if } t \neq 0, \ \phi(0) = 0$$

with $p > 2$. In this case we have $\phi^- = p - 1$ and $\phi^+ = p$.

Next, we recall some useful properties of variable exponent spaces. For more details we refer the reader to [22, 26], and the references therein. Consider the set

$$C_+(\overline{\Omega}) = \{q \in C(\overline{\Omega}) : q(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For $q \in C_+(\overline{\Omega})$, we define

$$q^+ = \sup_{x \in \overline{\Omega}} q(x) \quad \text{and} \quad q^- = \inf_{x \in \overline{\Omega}} q(x).$$

For any $q \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{q(x)}(\Omega) = \big\{u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^{q(x)}dx < +\infty\big\}.$$

This vector space endowed with the Luxemburg norm

$$\|u\|_{L^{q(x)}(\Omega)} = \inf\big\{\lambda > 0 : \int_\Omega \big|\frac{u(x)}{\lambda}\big|^{q(x)}dx \leq 1\big\}$$

is a separable reflexive Banach space.

Let $\hat{q} \in C_+(\overline{\Omega})$ be the conjugate exponent of q, that is, $\frac{1}{q(x)} + \frac{1}{\hat{q}(x)} = 1$. Then we have the following Hölder-type inequality.

Lemma 2.12 (Hölder inequality)**.** *If $u \in L^{q(x)}(\Omega)$ and $v \in L^{\hat{q}(x)}(\Omega)$, then*

$$\big|\int_\Omega uv dx\big| \leq \Big(\frac{1}{q^-} + \frac{1}{\hat{q}^-}\Big)\|u\|_{L^{q(x)}(\Omega)}\|v\|_{L^{\hat{q}(x)}(\Omega)} \leq 2\|u\|_{L^{q(x)}(\Omega)}\|v\|_{L^{\hat{q}(x)}(\Omega)}.$$

A very important role in manipulating the generalized Lebesgue spaces with variable exponent is played by the modular of the $L^{q(x)}(\Omega)$ space, which defined by $\rho_{q(\cdot)} : L^{q(x)}(\Omega) \to \mathbb{R}$

$$u \mapsto \rho_{q(\cdot)}(u) = \int_\Omega |u(x)|^{q(x)}dx.$$

Proposition 2.13. *Let $u \in L^{q(x)}(\Omega)$, then we have*

(1) $\|u\|_{L^{q(x)}(\Omega)} < 1$ *(resp. $= 1, > 1$) if and only if* $\rho_{q(\cdot)}(u) < 1$ *(resp. $= 1, > 1$);*
(2) $\|u\|_{L^{q(x)}(\Omega)} < 1$ *implies* $\|u\|^{q+}_{L^{q(x)}(\Omega)} \leq \rho_{q(\cdot)}(u) \leq \|u\|^{q-}_{L^{q(x)}(\Omega)}$,
(3) $\|u\|_{L^{q(x)}(\Omega)} > 1$ *implies* $\|u\|^{q-}_{L^{q(x)}(\Omega)} \leq \rho_{q(\cdot)}(u) \leq \|u\|^{q+}_{L^{q(x)}(\Omega)}$.

Proposition 2.14. *If $u, u_k \in L^{q(x)}(\Omega)$ and $k \in \mathbb{N}$, then the following assertions are equivalent*

(1) $\lim_{k\to+\infty} \|u_k - u\|_{L^{q(x)}(\Omega)} = 0$,
(2) $\lim_{k\to+\infty} \rho_{q(\cdot)}(u_k - u) = 0$,
(3) $u_k \to u$ *in measure in* Ω *and* $\lim_{k\to+\infty} \rho_{q(\cdot)}(u_k) = \rho_{q(\cdot)}(u)$.

We conclude this section by recalling a version of the mountain pass Theorem and Fountain Theorem.

Theorem 2.15 ([5, 28])**.** *Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$ satisfies the $(PS)_c$ with $I(0) = 0$. Suppose that the following conditions hold:*

(A6) *There exists $\rho > 0$ and $r > 0$ such that $I(u) \geq r$ for $\|u\| = \rho$.*
(A7) *There exists $e \in X$ with $\|e\| > \rho$ such that $I(e) \leq 0$.*

Let $\Gamma = \{\gamma \in C([0,1], X); \gamma(0) = 0, \gamma(1) = e\}$. Then

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) > r$$

is a critical value of I.

Let X be a reflexive and separable Banach space and X^* its dual space, then from [34] there are $\{\phi_n\}_{n\in\mathbb{N}} \subset X$ and $\{\phi_n^*\}_{n\in\mathbb{N}} \subset X^*$ such that

$$X = \overline{\operatorname{span}\{\phi_n, n \in \mathbb{N}\}},$$
$$X^* = \overline{\operatorname{span}\{\phi_n^*,\ n \in \mathbb{N}\}},$$
$$\langle \phi_n, \phi_m \rangle = \begin{cases} 1 & n = m \\ 0 & n \neq m. \end{cases}$$

For $k = 1, 2, \dots$, let $Y_k = \overline{\operatorname{span}\{\phi_1, \dots, \phi_k\}}$ and $Z_k = \overline{\operatorname{span}\{\phi_k, \phi_{k+1} \dots\}}$.

Theorem 2.16 ([13])**.** *Assume that the even functional $I \in C^1(X,\mathbb{R})$ satisfies the $(PS)_c$ condition and for almost every $k \in \mathbb{N}$, there exists ρ_k, r_k >0 such that*

(a) $b_k := \inf_{u\in Z_k, \|u\|=r_k} I(u) \to \infty$ *as* $k \to \infty$
(b) $a_k := \max_{u\in Y_k, \|u\|=\rho_k} I(u) \leq 0$.

Then I has a sequence of critical points $\{u_n\}$ such that $I(u_k) \to \infty$ as $k \to \infty$.

3. Main results

In this section, we prove Theorems 1.1 and 1.2. We start our analysis with the following Remark.

Remark 3.1. (1) By (A5), we can apply Theorem 2.10 and obtain that $W_0^s L_{\Phi_1}(\Omega)$ is compactly embedded in $L_{\Phi_2}(\Omega)$; that is,

$$\|u\|_{\Phi_2} \leq c_1 \|u\| \tag{3.1}$$

where $c_1 > 0$. On the other hand, since $q^+ \leq \phi_2^-$, that is

$$q^+ \leq \frac{t\phi_2(t)}{\Phi_2(t)} \quad \forall t > 0,$$

this implies that for $t > t_0 > 0$,

$$|t|^{q^+} \leq c\Phi_2(t)\ \forall t > t_0$$

then we obtain that $W_0^s L_{\Phi_1}(\Omega)$ is compactly embedded in $L^{q^+}(\Omega)$ on particular in $L^{q(x)}(\Omega)$, that is,

$$\|u\|_{q(x)} \leq c_2 \|u\| \tag{3.2}$$

where $c_2 > 0$. Thus, a solution for a problem of type (1.1) will be sought in $W_0^s L_{\Phi_1}(\Omega)$.

(2) The dual space of $(W_0^s L_{\Phi_1}(\Omega), \|\cdot\|)$ is denoted by $\big((W_0^s L_{\Phi_1}(\Omega))^*, \|\cdot\|_*\big)$.

To prove our main results, we introduce the functional J in $W_0^s L_{\Phi_1}(\Omega)$ defined by

$$J_\lambda(u) = \Psi(u) - \lambda \int_\Omega \Phi_2(u)\,dx - \int_\Omega F(x,u)\,dx - \frac{1}{q(x)} \int_\Omega g(x)|u|^{q(x)} dx$$

with

$$\Psi(u) = \int_\Omega \int_\Omega \Phi_1\Big(\frac{|u(x)-u(y)|}{|x-y|^s}\Big) \frac{dx\,dy}{|x-y|^N}.$$

By a standard argument [9, Lemma 3] and [8, Lemma 6], we can use (A1) to show that $J_\lambda \in C^1(W_0^s L_{\Phi_1}(\Omega), \mathbb{R})$ and

$$\begin{aligned} \langle J_\lambda'(u), v\rangle = \int_\Omega \int_\Omega a_1(|D^s u|) D^s u D^s v\,d\mu - \lambda \int_\Omega a_2(|u)uv\,dx - \int_\Omega f(x,u)v\,dx \\ - \int_\Omega g(x)|u|^{q(x)-2} uv\,dx, \end{aligned}$$

for all $u, v \in W_0^s L_{\Phi_1}(\Omega)$. Therefore, the critical points of J are weak solution of (1.1).

Lemma 3.2. *Suppose that* (A1), ((A5) *hold, and let $u_n \rightharpoonup u$ weakly in $W_0^s L_{\Phi_1}(\Omega)$. Then, up to a subsequence, we have*

$$\int_\Omega a_2(|u_n|) u_n (u_n - u)\,dx \to 0, \tag{3.3}$$

$$\int_\Omega g(x)|u_n|^{q(x)-2} u_n (u_n - u)\,dx \to 0, \tag{3.4}$$

$$\int_\Omega f(x,u_n)(u_n - u)\,dx \to 0. \tag{3.5}$$

Proof. From Remark 3.1, up to a subsequence, we see that $u_n \to u$ strongly in $L_{\Phi_2}(\Omega)$, and by dominated convergence theorem, there exists a subsequence, still denoted by u_n, and $h_1 \in L_{\Phi_2}(\Omega)$, such that, for almost every x on Ω,

$$\begin{gathered} |u_n(x)| \le h_1(x) \quad \forall n \in \mathbb{N}, \\ u_n(x) \to u(x)\ \forall n \in \mathbb{N}. \end{gathered}$$

From the Hölder inequality and Lemma 2.1, we have

$$\begin{aligned} \Big|\int_\Omega a_2(|u_n|) u_n (u_n - u)\,dx\Big| &\le \int_\Omega |a_2(|h_1|) h_1||u_n - u| dx \\ &\le \|a_2(h_1)|h_1|\,\|_{\overline{\Phi_2}} \|u_n - u\|_{\Phi_1} \to 0. \end{aligned}$$

Next, we have $W_0^s L_{\Phi_1}(\Omega)$ is compactly embedded in $L^{q(x)}(\Omega)$ passing the a subsequence if necessary, to obtain

$$u_n \to u \quad \text{strongly in } L^{q(x)}(\Omega)$$

and by the dominated convergence theorem, there exists a subsequence, still denoted by u_n, and $h_2 \in L^{q(x)}(\Omega)$, such that, for almost everywhere on Ω,

$$|u_n(x)| \le h_2(x)\ \forall n \in \mathbb{N}$$

and $u_n(x) \to u(x)$ for all $n \in \mathbb{N}$. From the Hölder inequality, we have

$$\begin{aligned}
\Big| \int_\Omega g(x)|u_n|^{q(x)-2}u_n|u_n-u|\,dx\Big| &\le \int_\Omega g(x)|h_2|^{q(x)-1}|u_n-u|\,dx \\
&\le \|g\|_\infty \int_\Omega |h_2|^{q(x)-1}|u_n-u|\,dx \\
&\le \|g\|_\infty \|h_1\|_{\widehat{q}(x)}^{q(x)-1}\|u_n-u\|_{q(x)} \to 0.
\end{aligned}$$

On the other hand, from (A1), for $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|f(x,t)| \le \varepsilon\phi_1(t) + c_\varepsilon\phi_2(t) \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Since $W_0^sL_{\Phi_1}(\Omega)$ is compactly embedded in $L_{\Phi_1}(\Omega)$ and in $L_{\Phi_2}(\Omega)$. Then $u_n \to u$ strongly in $L_{\Phi_i}(\Omega)$ for any $i = 1,2$, so by dominated convergence theorem, for any $i = 1,2$ there exist $h_i \in L_{\Phi_i}(\Omega)$ such that, for almost everywhere on Ω

$$|u_n(x)| \le h_i(x) \quad \forall n \in \mathbb{N}\ i = 1,2$$

and $u_n(x) \to u(x)$ $\forall n \in \mathbb{N}$. So, we have

$$\begin{aligned}
\Big| \int_\Omega f(x,u_n)(u_n-u)\,dx\Big| &\le \varepsilon \int_\Omega \phi_1(u_n)(u_n-u)\,dx + c_\varepsilon \int_\Omega \phi_2(u_n)(u_n-u)\,dx \\
&\le \varepsilon \int_\Omega \phi_1(h_1)|u_n-u|\,dx + c_\varepsilon \int_\Omega \phi_2(h_2)|u_n-u|\,dx \\
&\le \varepsilon\|\phi_1(h_1)\|_{\overline{\Phi_1}}\|u_n-u\|_{\Phi_1} + c_\varepsilon\|\phi_2(h_2)\|_{\overline{\Phi_2}}\|u_n-u\|_{\Phi_2} \\
&\to 0.
\end{aligned}$$

This completes the proof. □

Definition 3.3. We say that a sequence $\{u_n\} \subset W_0^sL_{\Phi_1}(\Omega)$ is a $(\mathrm{PS})_c$ sequence of J_λ if

$$J_\lambda(u_n) \to c \text{ in } \mathbb{R} \quad \text{and} \quad J'_\lambda(u_n) \to 0 \text{ in } (W_0^sL_{\Phi_1}(\Omega))^*.$$

Lemma 3.4. *There exist positive numbers ρ, δ, λ_0 such that $J_0(u) \ge \delta$ with $\|u\| = \rho$ for all g satisfying $\|g\|_\infty < \lambda_0$.*

Proof. From (A1) for $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|F(x,t)| \le \varepsilon\Phi_1(t) + c_\varepsilon\Phi_2(t) \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{3.6}$$

Then

$$\begin{aligned}
J_0(u) &= \Psi(u) - \int_\Omega F(x,u)\,dx - \frac{1}{q(x)}\int_\Omega g(x)|u|^{q(x)}dx \\
&\ge \Psi(u) - \varepsilon\int_\Omega \Phi_1(u) - c_\varepsilon\int_\Omega \Phi_2(u)\,dx - \frac{1}{q^-}\int_\Omega g(x)|u|^{q(x)}dx \\
&\ge (1-\varepsilon\lambda_1)\Psi(u) - c_\varepsilon\int_\Omega \Phi_2(u)\,dx - \frac{1}{q^-}\int_\Omega g(x)|u|^{q(x)}dx \\
&\ge (1-\varepsilon\lambda_1)\xi_1^m(\|u\|) - c\varepsilon\xi_2^l(\|u\|_{\Phi_2}) - \frac{1}{q^-}\|g\|_\infty\xi_3^l(\|u\|_{q(x)}) \\
&\ge (1-\varepsilon\lambda_1)\xi_1^m(\|u\|) - c\varepsilon\xi_2^l(c_1\|u\|) - \frac{1}{q^-}\|g\|_\infty\xi_3^l(c_2\|u\|).
\end{aligned}$$

For $\rho > 0$ sufficiently small such that $\rho = \|u\| < \min\{1, \frac{1}{c_1}, \frac{1}{c_2}\}$, we have

$$\begin{aligned} J_0(u) &\geq (1-\varepsilon\lambda_1)\|u\|^{\phi_1^-} - c\varepsilon c_1^{\phi_2^+}\|u\|^{\phi_2^+} - \frac{c_2^{q^+}}{q^-}\|g\|_\infty \|u\|^{q^+} \\ &= \|u\|^{q^+}\big[(1-\varepsilon\lambda_1)\|u\|^{\phi_1^- - q^+} - c_\varepsilon c_1^{\phi_2^+}\|u\|^{\phi_2^+ - q^+} - \frac{c_2^{q^+}}{q^-}\|g\|_\infty\big]. \end{aligned}$$

Note that ε may be chosen small enough and $1 < q^+ < \phi_1^- < \phi_2^+$, and we easily obtain $\rho, \delta_0 > 0$ small enough such that

$$(1-\varepsilon\lambda_1)\rho^{\phi_1^- - q^+} - c_\varepsilon c_1^{\phi_2^+}\rho^{\phi_2^+ - q^+} \geq \delta_0.$$

Take $\lambda_0 = \frac{q^-\delta_0}{2c_2^{q^+}}$, then we have

$$J_0(u) \geq \rho^{q^+}\Big(\delta_0 - \frac{c_2^{q^+}}{q^-}\|g\|_\infty\Big) \geq \rho^{q^+}\Big(\delta_0 - \frac{c_2^{q^+}}{q^-}\lambda_0\Big) = \frac{\delta_0}{2}\rho^{q^+},$$

with $\|u\| = \rho$. Therefore, we can choose $\delta = \frac{\delta_0}{2}\rho^{q^+}$ such that the conclusion holds. $\square$

Lemma 3.5. *There exists $e \in W_0^s L_{\Phi_1}(\Omega)$ with $\|e\| > \rho$ such that $J_0(e) < 0$, where ρ is given in Lemma 3.4.*

Proof. From (A2), we have $F(x, tu) \geq t^\mu F(x, u)$ for all $t \geq 1$ and all $u \in W_0^s L_{\Phi_1}(\Omega)$. By Theorem 2.8, we can fix $u_0 \in C_0^\infty(\Omega)$, such that $\|u_0\| = 1$ and let $t \geq 1$, then

$$\begin{aligned} J_0(tu_0) &= \Psi(tu_0) - \int_\Omega F(x, tu_0)\,dx - \frac{1}{q(x)}\int_\Omega g(x)|tu_0|^{q(x)}dx \\ &\leq \|tu_0\|^{\phi_1^+} - t^\mu \int_\Omega F(x, u_0)\,dx - \frac{t^{q^-}}{q^+}\int_\Omega g(x)|u_0|^{q(x)}dx \\ &\leq t^{\phi_1^+} - t^\mu \int_\Omega F(x, u_0)\,dx - \frac{t^{q^-}}{q^+}\int_\Omega g(x)|u_0|^{q(x)}dx. \end{aligned}$$

Note that $\mu > \phi_1^+ > q^- > 1$, so there exists $t_0 > 0$ large enough such that $\|t_0 u\| > \rho$ and $J_0(t_0 u) < 0$. The proof is completed by taking $e = Tu$ with $T > 0$ large enough. $\square$

Lemma 3.6. *Suppose that $\{u_n\} \subset W_0^s L_{\Phi_1}(\Omega)$ is a $(PS)_c$ sequence of J_0 with $c \neq 0$. Then $\{u_n\}$ has a convergent subsequence in $W_0^s L_{\Phi_1}(\Omega)$.*

To prove the lemma above, we recall the following result.

Lemma 3.7 ([9, 16])**.** *Assume that the sequence $\{u_n\}$ converges weakly to u in $W_0^s L_{\Phi_1}(\Omega)$ and*

$$\limsup_{n\to\infty}\langle \Psi'(u_n), u_n - u\rangle \leq 0. \tag{3.7}$$

Then the sequence $\{u_n\}$ is convergence strongly to u in $W_0^s L_{\Phi_1}(\Omega)$.

Proof of Lemma 3.6. From (A2) we have

$$
\begin{aligned}
& c+1+\|u_n\| \\
&\geq J_0(u_n)-\frac{1}{\mu}\langle J_0'(u_n),u_n\rangle \\
&=\Psi(u_n)-\int_\Omega F(x,u_n)\,dx-\frac{1}{q(x)}\int_\Omega g(x)|u_n|^{q(x)}dx \\
&\quad -\int_\Omega\int_\Omega \phi(|D^s u_n|)D^s u_n\,d\mu+\int_\Omega f(x,u_n)u_n\,dx+\int_\Omega g(x)|u_n|^{q(x)}dx \\
&\geq \Big(1-\frac{\phi_1^-}{\mu}\Big)\Psi(u_n)-\Big(\frac{1}{q^-}-\frac{1}{\mu}\Big)\int_\Omega g(x)|u_n|^{q(x)}dx \\
&\geq \Big(1-\frac{\phi_1^-}{\mu}\Big)\|u_n\|^{\phi_1^-}-\Big(\frac{1}{q^-}-\frac{1}{\mu}\Big)\|g\|_\infty c_2^{q^+}\|u_n\|^{q^+}.
\end{aligned}
\tag{3.8}
$$

Note that $\phi_1^- > q^+ > 1$, so (3.8) implies that $\{u_n\}$ is bounded in $W_0^sL_{\Phi_1}(\Omega)$. Thus, passing to a subsequence, we obtain that $u_n \rightharpoonup u$ in $W_0^sL_{\Phi_1}(\Omega)$ weakly, and we have

$$
\begin{aligned}
\langle J_0'(u_n),u_n-u\rangle &= \langle \Psi'(u_n),u_n-u\rangle-\int_\Omega f(x,u_n)(u_n-u)\,dx \\
&\quad -\int_\Omega g(x)|u_n|^{q(x)-2}u_n(u_n-u)\,dx.
\end{aligned}
$$

Note that, $\langle J_0'(u_n),u_n-u\rangle\to 0$ as $n\to\infty$, from Lemma 3.2, we obtain

$$\langle \Psi'(u_n),u_n-u\rangle\to 0 \text{ as } n\to\infty.$$

Then by Lemma 3.7 and that $u_n \rightharpoonup u$ weakly, we have $u_n \to u$ strongly in $W_0^sL_{\Phi_1}(\Omega)$. □

Proof Theorem 1.1. From Lemmas 3.4–3.6 and by the mountain pass theorem 2.15, J_0 has a positive critical value c, that is, there exists $u\in W_0^sL_{\Phi_1}(\Omega)$, such that $J(u)=c>0$ and $J_0'(u)=0$. Thus u is a solution for (1.1). This completes the Proof. □

Now we prove Theorem 1.2. Since $W_0^sL_{\Phi_1}(\Omega)$ is a separable reflexive Banach space, then from [34] there are $\{\phi_n\}_{n\in\mathbb{N}}\subset W_0^sL_{\Phi_1}(\Omega)$ and $\{\phi_n^*\}_{n\in\mathbb{N}}\subset (W_0^sL_{\Phi_1}(\Omega))^*$ such that

$$
\begin{gathered}
W_0^sL_{\Phi_1}(\Omega)=\overline{\operatorname{span}\{\phi_n,n\in\mathbb{N}\}}, \\
(W_0^sL_{\Phi_1}(\Omega))^*=\overline{\operatorname{span}\{\phi_n^*,\ n\in\mathbb{N}\}}, \\
\langle\phi_n,\phi_m\rangle=\begin{cases}1 & n=m\\ 0 & n\neq m.\end{cases}
\end{gathered}
$$

For $k=1,2,\dots$, let $Y_k=\overline{\operatorname{span}\{\phi_1,\dots,\phi_k\}}$ and $Z_k=\overline{\operatorname{span}\{\phi_k,\phi_{k+1}\dots\}}$. We first given some preliminary lemmas.

Lemma 3.8. *Under the assumptions of Theorem* 1.2 *we have*

$$
\begin{gathered}
b_k:=\inf_{u\in Z_k,\|u\|=r_k} J_\lambda(u)\to\infty \quad \text{as } k\to\infty, \\
a_k:=\max_{u\in Y_k,\|u\|=\rho_k} J_\lambda(u)\leq 0.
\end{gathered}
$$

Proof. By (A2), there exist $d_1 > 0$, $M > 0$ such that

$$F(x,t) \geq d_1|t|^{\mu}, \quad \forall |t| \geq M,\ x \in \Omega\,. \tag{3.9}$$

By (A1), for every $\varepsilon > 0$ and all $|t| \leq M$, we have

$$\begin{aligned} |F(x,t)| &\leq \varepsilon\Phi_1(t) + c_\varepsilon\Phi_2(t) \\ &\leq \varepsilon\Phi_1(t) + c_\varepsilon\Phi_2(M) \\ &= \varepsilon\Phi_1(t) + c'_\varepsilon. \end{aligned}$$

Then combining this with (3.9), we have

$$F(x,t) \geq d_1|t|^{\mu} - \varepsilon\Phi_1(t) - c'_\varepsilon, \quad \forall (x,t) \in \Omega \times \mathbb{R}. \tag{3.10}$$

Then, from (3.10) it follows that

$$\begin{aligned} J_\lambda(u) &= \Psi(u) - \lambda\int_\Omega \Phi_2(u)\,dx - \int_\Omega F(x,u)\,dx \\ &\leq \xi_1^l(\|u\|) + \int_\Omega c'_\varepsilon + \varepsilon\Phi_1(u) - d_1|u|^{\mu}dx \\ &\leq \xi_1^l(\|u\|) + c'_\varepsilon|\Omega| + \varepsilon\xi_1^l(\|u\|_{\Phi_1}) - d_1\|u\|^{\mu}_{L^1}. \\ &\leq \|u\|^{\phi_1^+} + c'_\varepsilon|\Omega| + \varepsilon d_2^{\phi_1^+}\|u\|^{\phi_1^+} - d_1 d_3^{\mu}\|u\|^{\mu}, \end{aligned} \tag{3.11}$$

the above inequality is given because all norms are equivalent on the finite dimensional space Y_k. So, since $\mu > \phi_1^+ > 1$, there exists $d_k \geq \max\{1, \frac{1}{d_2}, \frac{1}{d_3}\}$ large enough such that

$$J(u) \leq 0 \quad \text{for every } u \in Y_k \text{ and } \|u\| \geq d_k. \tag{3.12}$$

On the other hand, letting

$$B_1(k) = \sup_{u\in Z_k, \|u\|=1} \|u\|_{\phi_1}, \quad B_2(k) = \sup_{u\in Z_k, \|u\|=1} \|u\|_{\phi_2},$$

we have $B_i(k) \to 0$ as $k \to \infty$. Now for $u \in Z_k$ with $\|u\| = r_k = \frac{1}{B_1(k)+B_2(k)}$, from (3.11), we obtain

$$\begin{aligned} J_\lambda(u) &= \Psi(u) - \lambda\int_\Omega \Phi_2(u)\,dx - \int_\Omega F(x,u)\,dx \\ &\geq \Psi(u) - \lambda\int_\Omega \Phi_2(u)\,dx - \varepsilon\int_\Omega \Phi_1(u)\,dx - c'_\varepsilon\int_\Omega \Phi_2(u)\,dx \\ &\geq \|u\|^{\phi_1^{\mp}} - \lambda\|u\|^{\phi_2^{\mp}}_{\Phi_2} - \varepsilon\|u\|^{\phi_1^{\mp}}_{\Phi_1} - c'_\varepsilon\|u\|^{\phi_2^{\mp}}_{\Phi_2} \\ &\geq \|u\|^{\phi_1^{\mp}} - \lambda B_2^{\phi_2^{\mp}}(k)\|u\|^{\phi_2^{\mp}} - \varepsilon B_1^{\phi_1^{\mp}}(k)\|u\|^{\phi_1^{\mp}} - c'_\varepsilon B_1^{\phi_1^{\mp}}(k)\|u\|^{\phi_2^{\mp}} \\ &\geq r_k^{\phi_1^{\mp}} - \lambda - \varepsilon - c'_\varepsilon \to \infty \quad \text{as } k \to \infty. \end{aligned} \tag{3.13}$$

Hence,

$$b_k := \inf_{u\in Z_k, \|u\|=r_k} J_\lambda(u) \to \infty \quad \text{as } k \to \infty.$$

Combining this with (3.12), we can take $\rho_k = \max\{d_k, r_k + 1\}$ and we have

$$a_k := \max_{u\in Y_k, \|u\|=\rho_k} J_\lambda(u) \leq 0.$$

This completes the proof. $\square$

Lemma 3.9. *Under the assumptions of Theorem 1.2, every $(PS)_c$ sequence has a convergence of subsequence.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence of J_λ. Then $J_\lambda(u_n) \to c$ in $\mathbb{R}$ and $J'_\lambda(u_n) \to 0$ as $n \to \infty$. we claim that $\{u_n\}$ is bounded. Indeed, note that

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{\mu}\langle J'_\lambda(u_n), u_n\rangle = \Psi(u_n) - \frac{1}{\mu}\int_{\Omega\times\Omega} \phi_1(D^s u_n) D^s u_n \, d\mu - \lambda \int_\Omega \Phi_2(u_n)\, dx \\ + \frac{\lambda}{\mu}\int_\Omega \phi_2(u_n) u_n \, dx - \int_\Omega F(x, u_n)\, dx + \frac{1}{\mu} f(x, u_n) u_n \, dx; \end{aligned}$$

consequently

$$\begin{aligned} &\lambda\Big(\frac{\phi_2^-}{\mu} - 1\Big)\int_\Omega \Phi_2(u)\, dx \\ &\le \frac{\lambda}{\mu}\int_\Omega \phi_2(u_n) u_n \, dx - \lambda \int_\Omega \Phi_2(u_n)\, dx \\ &= J_\lambda(u_n) - \frac{1}{\mu}\langle J'_\lambda(u_n), u_n\rangle - \Psi(u_n) + \frac{1}{\mu}\int_{\Omega\times\Omega} \phi_1(D^s u_n) D^s u_n \, d\mu \\ &\quad + \int_\Omega F(x, u_n)\, dx - \frac{1}{\mu} f(x, u_n) u_n \, dx, \\ &\le \|u_n\| + 1 + c + \Big(\frac{\phi_1^+}{\mu} - 1\Big)\Psi(u_n) \\ &\le \|u_n\| + 1 + c. \end{aligned}$$

So,

$$\int_\Omega \Phi_2(u_n)\, dx \le c(\|u_n\| + 1).$$

Then, by (2.6) and (3.6) we have

$$\begin{aligned} \Psi(u_n) &= J_\lambda(u_n) + \lambda \int_\Omega \Phi_2(u_n)\, dx + \int_\Omega F(x, u_n)\, dx \\ &\le J_\lambda(u_n) + \lambda \int_\Omega \Phi_2(u_n)\, dx + \varepsilon \int_\Omega \Phi_1(u_n)\, dx + c_\varepsilon \int_\Omega \Phi_2(u_n)\, dx \\ &\le c + o_n(1) + (\lambda + c_\varepsilon)\int_\Omega \Phi_2(u_n)\, dx + \varepsilon\lambda_1 \Psi(u_n). \end{aligned}$$

This implies

$$(1 - \varepsilon\lambda_1)\Psi(u_n) \le c(1 + \|u_n\|) + o_n(1).$$

Since ε is arbitrary, then for ε sufficiency small and for n sufficiently large, we have

$$\Psi(u_n) \le c(1 + \|u_n\|).$$

If $\|u_n\| > 1$, by proposition 2.7, it following that

$$\|u_n\|^{\phi_1^-} \le c(1 + \|u_n\|).$$

Using that $\phi_1^- > 1$, the above inequality shows that $\{u_n\}$ is bounded in $W_0^s L_{\Phi_1}(\Omega)$. From Lemma 3.6, we obtain the desired assertion. $\square$

Proof Theorem 1.2. By (A3) f is odd, then J_λ is an even functional. From Lemmas 3.8 and 3.9, the functional J_λ satisfies all the conditions of the Fountain theorem 2.16 Hence, J_λ has an unbounded sequence of critical values, that is there exists a sequence $\{u_n\} \subset W_0^s L_{\Phi_1}(\Omega)$ such that $J'_\lambda(u_k) = 0$ and $J_\lambda(u_k) \to \infty$ as $k \to \infty$. This completes the proof. $\square$

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Elhoussine Azroul
Departamento de Matemáticas, Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz, Laboratory of Mathematical Analysis and Applications, Fez, Morocco
Email address: elhoussine.azroul@gmail.com

Abdelmoujib Benkirane
Departamento de Matemáticas, Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz, Laboratory of Mathematical Analysis and Applications, Fez, Morocco
Email address: abd.benkirane@gmail.com

Mohammed Srati
Departamento de Matemáticas, Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz, Laboratory of Mathematical Analysis and Applications, Fez, Morocco
Email address: srati93@gmail.com

César E. Torres Ledesma
Departamento de Matemáticas, Universidad Nacional de Trujillo, Av. Juan Pablo II s/n. Trujillo, Perú
Email address: ctl_576@yahoo.es