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EXACT FORMS OF ENTIRE SOLUTIONS FOR FERMAT TYPE PARTIAL DIFFERENTIAL EQUATIONS IN $\mathbb{C}^2$

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Abstract. This article studies the existence and the exact form of entire solutions of several Fermat type partial differential equations in $\mathbb{C}^2$, by utilizing the Nevanlinna theory of meromorphic functions in several complex variables. We obtain results about the existence and form of transcendental entire solutions with finite order for some variations of Fermat type functional equations. Our results are extensions and generalizations of the previous theorems by Xu and Cao [29, 30], Liu and Dong [19].

1. Introduction and statement of main results

In 1939, Iyer [10] studied solutions of the Fermat type functional equation
$$f^2(z) + g^2(z) = 1, \tag{1.1}$$
and proved the classical result that the entire solutions of equation (1.1) are $f = \cos a(z)$, $g = \sin a(z)$, where $a(z)$ is an entire function, no other solutions exist. After his work, many scholars had paid considerable attention to the existence and the form of entire and meromorphic solutions of some variations of (1.1); for details, we refer readers to [8, 25, 31, 32].

In 2004, Yang and Li [31] discussed the form of solutions of the equations, where $g(z)$ is replaced by $f'(z)$ in (1.1), that is,
$$f^2(z) + (f'(z))^2 = 1, \tag{1.2}$$
they proved that (1.2) has only transcendental entire solutions of the form
$$f(z) = \frac{1}{2}\Big(Pe^{\alpha z} + \frac{1}{P}e^{-\alpha z}\Big),$$
where P, α are nonzero constants. They also studied the existence of solutions of the equation when $f'(z)$ is replaced by a differential polynomial in f and obtained the following theorem.

Theorem 1.1 ([31, Theorem 2]). *Let b_n and b_{n+1} be nonzero constants. Then*
$$f^2(z) + [b_n f^{(n)}(z) + b_{n+1} f^{(n+1)}(z)]^2 = 1 \tag{1.3}$$
has no transcendental meromorphic solutions.

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In 2015, Liu and Dong [19] further investigated the existence of solutions of the Fermat type equation (1.1), when both $f(z)$ and $g(z)$ are replaced by differential polynomials in $f(z)$. they proved the following result.

Theorem 1.2 ([19, Theorem 1.7]). *The equation*

$$[f(z)+f'(z)]^2+[f(z)+f''(z)]^2=1 \tag{1.4}$$

has no transcendental meromorphic solutions.

It is always an interesting and quite difficult problem to prove the existence and the form of the entire or meromorphic solution of differential equation in the complex plane $\mathbb{C}$. In the past five or more decades, Nevanlinna theory of meromorphic functions has been used widely to deal with these problems and derive many interesting results of meromorphic solutions of differential equations in complex plane (see, e.g.,[1, 13, 17]). Especially, Yang [33], Yi and Yang [34], and Li and Yang [16] studied the existence and the form of the entire and meromorphic solutions of complex Fermat type differential equations in $\mathbb{C}$, by employing the Nevanlinna theory.

Very recently, with the development of the Nevanlinna theory with several complex variables (see [2, 3, 12]), Xu and Cao [29, 30], Xu and coauthors [27, 28] investigated the existence of solutions for some Fermat type partial differential equations with two complex variables by using the difference logarithmic derivative lemma of several complex variables, and extended the results of Yang and Li [31] from one complex variable to several complex variables.

Theorem 1.3 ([30, Corollary 1.4]). *Any transcendental entire solution with finite order of the partial differential equation of the Fermat type*

$$f^2(z_1,z_2)+\Big(\frac{\partial f(z_1,z_2)}{\partial z_1}\Big)^2=1 \tag{1.5}$$

has the form of $f(z_1,z_2)=\sin(z_1+g(z_2))$, where $g(z_2)$ is a polynomial in one variable z_2.

The study of complex partial differential equations has a long history, see for example [5, 7, 22], and for equations with several complex variables see [9, 11, 15, 20, 22]. Khavinson [11] pointed out that any entire solution of the partial differential equation

$$\Big(\frac{\partial f}{\partial z_1}\Big)^2+\Big(\frac{\partial f}{\partial z_2}\Big)^2=1 \tag{1.6}$$

in $\mathbb{C}^2$ is necessarily linear. This partial differential equations in the real variable case occur in the study of characteristic surfaces and in wave propagation theory, and it is the two dimensional eiconal equation, one of the main equations of geometric optics (see [6, 7]). In 1999, Saleeby [22] studied the entire solution of Fermat type partial differential equation (1.6) and obtain the following result.

Theorem 1.4 ([22, Theorem 1]). *If f is an entire solution of* (1.6) *in $\mathbb{C}^2$, then $f=c_1z_1+c_2z_2+c$, where $c_1,c_2,c\in\mathbb{C}$ and $c_1^2+c_2^2=1$.*

Later, Li and his coauthors [4, 14, 15] discussed some variations of the partial differential equation (1.6), and obtained interesting and important results, of which we mention the following.

Theorem 1.5 ([4, Corollary 2.3]). *Let $P(z_1,z_2)$ and $Q(z_1,z_2)$ be arbitrary polynomials in $\mathbb{C}^2$. Then f is an entire solution of the equation*

$$\Big(P\frac{\partial f}{\partial z_1}\Big)^2+\Big(Q\frac{\partial f}{\partial z_2}\Big)^2=1$$

if and only if $f=c_1z_1+c_2z_2+c_3$ is a linear function, where c_j's are constants, and exactly one of the following holds:

(i) *$c_1=0$ and Q is a constant satisfying that $(c_2Q)^2=1$;*
(ii) *$c_2=0$ and P is a constant satisfying that $(c_1P)^2=1$;*
(iii) *$c_1c_2\neq 0$ and P,Q are both constants satisfying that $(c_1P)^2+(c_2Q)^2=1$.*

From Theorems 1.1–1.3, a question can be naturally raised:

> What will happen to the existence and the form of the solutions when equations (1.3) and (1.4) are turned from one complex variable to several complex variables?

Motivated by this question, this article considers the description of entire solutions for some variations of the partial differential equation (1.5) in more general form. The main tool in this paper is the Nevanlinna theory with several complex variables. Our main results generalize the previous theorems given by Xu and Cao, Liu and Dong [19, 30]. Throughout this article, for convenience, we assume that $z+w=(z_1+w_1,z_2+w_2)$ for any $z=(z_1,z_2),w=(w_1,w_2)$.

Firstly, we consider the transcendental entire solution with finite order of the first order partial differential equation of Fermat type,

$$\Big[a_1f(z)+a_2\frac{\partial f}{\partial z_1}\Big]^2+\Big[a_3f(z)+a_4\frac{\partial f}{\partial z_2}\Big]^2=1, \tag{1.7}$$

where $a_1,a_2,a_3,a_4\in\mathbb{C}$.

Theorem 1.6. *Let $a_1,a_2,a_3,a_4\in\mathbb{C}$ be four nonzero constants. Then the transcendental entire solution $f(z_1,z_2)$ with finite order of the partial differential equation (1.7) must be of the form*

$$f(z_1,z_2)=\pm\frac{1}{\sqrt{a_1^2+a_3^2}}+\eta e^{-(\frac{a_1}{a_2}z_1+\frac{a_3}{a_4}z_2)},$$

or

$$\begin{aligned}f(z_1,z_2)=&\frac{a_3+ia_1}{2(\alpha_1a_2a_3-\alpha_2a_1a_4)}e^{L(z)+B}\\&-\frac{a_3-ia_1}{2(\alpha_1a_2a_3-\alpha_2a_1a_4)}e^{-L(z)-B}+\eta e^{-(\frac{a_1}{a_2}z_1+\frac{a_3}{a_4}z_2)},\end{aligned}$$

where $L(z)=\alpha_1z_2+\alpha_2z_2$, $\alpha_1=\frac{a_3}{a_2}i$, $\alpha_2=-\frac{a_1}{a_4}i$, and $\eta,B\in\mathbb{C}$.

The following example shows that the forms of the solutions in Theorem 1.6 are precise.

Example 1.7. Let $\eta\in\mathbb{C}$ and $\eta\neq 0$, and

$$f(z_1,z_2)=\pm\frac{1}{\sqrt5}+\eta e^{-(2z_1+z_2)},$$

$$g(z_1,z_2)=\frac{1+2i}{10i}e^{i(z_1-2z_2)}-\frac{1-2i}{10i}e^{-i(z_1-2z_2)}+\eta e^{-(2z_1+z_2)}.$$

Then $\rho(f)=\rho(g)=1$ and $f(z_1,z_2),g(z_1,z_2)$ are the finite order transcendental entire solutions for (1.7) with $a_1=2,a_2=a_3=a_4=1$.

From Theorem 1.6, one can easily obtain the following corollary.

Corollary 1.8. *Let $f(z_1, z_2)$ be a transcendental entire solution with finite order of the partial differential equation*

$$\Big[f(z) + \frac{\partial f}{\partial z_1}\Big]^2 + \Big[f(z) + \frac{\partial f}{\partial z_2}\Big]^2 = 1. \tag{1.8}$$

Then $f(z_1, z_2)$ is of the form

$$f(z_1, z_2) = \pm\frac{\sqrt{2}}{2} + \eta e^{-(z_1+z_2)},$$

or

$$f(z_1, z_2) = \sin(z_2 - z_1 + \eta_1) - \cos(z_2 - z_1 + \eta_1) + \eta_2 e^{-(z_1+z_2)},$$

where $\eta, \eta_1, \eta_2 \in \mathbb{C}$.

Secondly, we study the existence and the form of transcendental entire solutions of several second order partial differential equations of Fermat type,

$$\Big[a_1 f(z) + a_2\frac{\partial f}{\partial z_1}\Big]^2 + \Big[a_3 f(z) + a_4\frac{\partial^2 f}{\partial z_1^2}\Big]^2 = 1, \tag{1.9}$$

and

$$\Big(a_2\frac{\partial f}{\partial z_1}\Big)^2 + \Big[a_3 f(z) + a_4\frac{\partial^2 f}{\partial z_1^2}\Big]^2 = 1, \tag{1.10}$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}$.

Theorem 1.9. *Let $a_1, a_2, a_3, a_4 \in \mathbb{C}$ be four nonzero constants such that $D := -(a_1^2 a_4 + a_2^2 a_3) \neq 0$. Then the partial differential equation* (1.9) *does not admit any transcendental entire solution with finite order.*

From Theorem 1.9, we have the following corollary.

Corollary 1.10. *The partial differential equation*

$$\Big[f(z) + \frac{\partial f}{\partial z_1}\Big]^2 + \Big[f(z) + \frac{\partial^2 f}{\partial z_1^2}\Big]^2 = 1$$

does not admit any transcendental entire solution with finite order.

For $a_1 = 0$ in (1.9), we have the following result.

Theorem 1.11. *Let $a_2, a_3, a_4 \in \mathbb{C}$ be three nonzero constants. Then* (1.10) *admits any transcendental entire solution $f(z_1, z_2)$ with finite order, and $f(z_1, z_2)$ must be of the form*

$$f(z_1, z_2) = -\frac{\alpha_1 a_4 + i a_2}{a_2 a_3} sh(\alpha_1 z_1 + \varphi(z_2)),$$

where $\varphi(z_2)$ is a polynomial in z_2, and

$$\alpha_1 = \frac{(-a_2 \pm \sqrt{a_2^2 + 4a_3 a_4})i}{2a_4}.$$

Similar to the above argument, we discuss the transcendental entire solutions of some second mix partial differential equations. We obtain the following theorem.

Theorem 1.12. *Let a_2, a_3, a_4 be three nonzero constants and $a_1 \in \mathbb{C}$, and*

$$\Big[a_1 f(z) + a_2\frac{\partial f}{\partial z_1}\Big]^2 + \Big[a_3 f(z) + a_4\frac{\partial^2 f}{\partial z_1 \partial z_2}\Big]^2 = 1. \tag{1.11}$$

Then

(i) *if* $a_1 \neq 0$, *then equation* (1.11) *has no any transcendental entire solution with finite order;*
(ii) *if* $a_1 = 0$, *then the finite order transcendental entire solution* $f(z_1, z_2)$ *of equation* (1.11) *must be of the form*

$$f(z_1, z_2) = -\frac{\alpha_1 a_4 + i a_2}{a_2 a_3} sh(\alpha_1 z_1 + \alpha_2 z_2 + B),$$

where α_1, α_2, B *are constants and satisfy* $\alpha_1 = -\frac{a_3}{\alpha_2 a_4 + a_2 i}$.

The following example shows the existence of a transcendental entire solution of equation (1.11).

Example 1.13. Let

$$f(z_1, z_2) = -\frac{(1+\sqrt{2})i}{4}\Big(e^{(\sqrt{2}-1)iz_1 + z_2^n} - e^{-[(\sqrt{2}-1)iz_1 + z_2^n]}\Big), \quad n \in \mathbb{N}_+.$$

Then $\rho(f) = n$ and $f(z_1, z_2)$ is a finite order transcendental entire solution for (1.11) with $a_2 = 2$, $a_3 = a_4 = 1$ and $\alpha_1 = (\sqrt{2} - 1)i$.

From Theorem 1.9, we can easily obtain the following corollary.

Corollary 1.14. *The partial differential equation*

$$\Big[f(z) + \frac{\partial f}{\partial z_1}\Big]^2 + \Big[f(z) + \frac{\partial^2 f}{\partial z_1 \partial z_2}\Big]^2 = 1$$

does not admit any transcendental entire solution with finite order.

Finally, we can obtain the following results by using the same arguments as in Theorem 1.8.

Theorem 1.15. *Let* b_1 *and* b_2 *be two nonzero constants in* $\mathbb{C}$. *Then*

$$f^2(z) + \Big[b_1 \frac{\partial f}{\partial z_1} + b_2 \frac{\partial^2 f}{\partial z_1^2}\Big]^2 = 1 \tag{1.12}$$

has no finite order transcendental entire solutions.

Theorem 1.16. *Let* b_1 *and* b_2 *be two nonzero constants in* $\mathbb{C}$. *Then the finite order transcendental entire solution* $f(z_1, z_2)$ *of equation*

$$f^2(z) + \Big[b_1 \frac{\partial f}{\partial z_1} + b_2 \frac{\partial^2 f}{\partial z_1 \partial z_2}\Big]^2 = 1 \tag{1.13}$$

must be of the form $f = \sin\big(\frac{1}{b_1} z_1 + \eta\big)$, *where* $\eta \in \mathbb{C}$.

Theorems 1.15 and 1.16 are extensions of Theorem 1.1 from one complex variable to two complex variables.

2. Proof of Theorem 1.6

The following lemmas play the key roles in proving our results.

Lemma 2.1 ([23, 24])**.** *For an entire function* F *on* $\mathbb{C}^n$, *with* $F(0) \neq 0$ *and* $\rho(n_F) = \rho < \infty$. *Then there exists a canonical function* f_F *and a function* $g_F \in \mathbb{C}^n$ *such that* $F(z) = f_F(z) e^{g_F(z)}$. *For the special case* $n = 1$, f_F *is the canonical product of Weierstrass.*

Here, $\rho(n_F)$ denotes the order of the counting function of zeros of F.

Lemma 2.2 ([21]). *. If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases: either*

(a) *the internal function h is a polynomial and the external function g is of finite order; or*
(b) *the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.*

Prof of Theorem 1.6. Suppose that $f(z)$ is a transcendental entire solution with finite order of (1.7). Two cases will be discussed below.

Case 1: $f(z) + \frac{\partial f}{\partial z_1}$ is a constant. We set

$$a_1 f(z) + a_2 \frac{\partial f}{\partial z_1} = K_1, \quad K_1 \in \mathbb{C}. \tag{2.1}$$

In view of (1.7), it follows that $a_3 f(z) + a_4 \frac{\partial f}{\partial z_2}$ is a constant, let

$$a_3 f(z) + a_4 \frac{\partial f}{\partial z_2} = K_2,\ K_2 \in \mathbb{C}. \tag{2.2}$$

This leads to $K_1^2 + K_2^2 = 1$. In view of (2.1) and (2.2), it follows that

$$a_2 a_3 \frac{\partial f}{\partial z_1} - a_1 a_4 \frac{\partial f}{\partial z_2} = a_3 K_1 - a_1 K_2. \tag{2.3}$$

The characteristic equations of (2.3) are

$$\frac{dz_1}{dt} = a_2 a_3, \quad \frac{dz_2}{dt} = -a_1 a_4, \quad \frac{df}{dt} = a_3 K_1 - a_1 K_2.$$

Using the initial conditions: $z_1 = 0, z_2 = s$, and $f = f(0, s) := \phi(s)$ with a parameter s. Thus, we obtain the following parametric representation for the solutions of the characteristic equations: $z_1 = a_2 a_3 t$, $z_2 = -a_1 a_4 t + s$,

$$f(t, s) = \int_0^t a_3 K_1 - a_1 K_2 dt + \phi(s) = (a_3 K_1 - a_1 K_2)t + \phi(s),$$

where $\phi(s)$ is a transcendental entire function with finite order in s. Noting that $t = \frac{z_1}{a_2 a_3}$ and $s = z_2 + \frac{a_1 a_4}{a_2 a_3} z_1$, then the solution of (2.3) is of the form

$$f(z_1, z_2) = (a_3 K_1 - a_1 K_2) \frac{z_1}{a_2 a_3} + \phi(z_2 + \frac{a_1 a_4}{a_2 a_3} z_1). \tag{2.4}$$

On the other hand, differentiating both two sides of the equations (2.1), (2.2) for the variables z_2, z_1, respectively, and noting the fact that $\frac{\partial^2 f}{\partial z_1 \partial z_2} = \frac{\partial^2 f}{\partial z_2 \partial z_1}$, it follows that

$$a_2 a_3 \frac{\partial f}{\partial z_1} = a_1 a_4 \frac{\partial f}{\partial z_2},$$

which implies that $a_3 K_1 = a_1 K_2$. Thus, it follows that

$$K_1 = \pm \frac{a_1}{\sqrt{a_1^2 + a_3^2}}, \quad K_2 = \pm \frac{a_3}{\sqrt{a_1^2 + a_3^2}}, \quad f(z_1, z_2) = \phi(z_2 + \frac{a_1 a_4}{a_2 a_3} z_1).$$

Substituting these into (2.2) and (2.3), we obtain

$$\phi(z_2 + \frac{a_1 a_4}{a_2 a_3} z_1) + \frac{a_4}{a_3} \phi'(z_2 + \frac{a_1 a_4}{a_2 a_3} z_1) = \pm \frac{1}{\sqrt{a_1^2 + a_3^2}}.$$

This means that

$$f(z_1,z_2)=\phi(z_2+\frac{a_1a_4}{a_2a_3}z_1)=\pm\frac{1}{\sqrt{a_1^2+a_3^2}}+\eta e^{-(\frac{a_1}{a_2}z_1+\frac{a_3}{a_4}z_2)}.$$

Case 2: $a_1f(z)+a_2\frac{\partial f}{\partial z_1}$ is not a constant. From the fact that the entire solutions of equation $f^2+g^2=1$ are $f=\cos a(z)$, $g=\sin a(z)$, we can deduce that $a_1f(z)+a_2\frac{\partial f}{\partial z_1}$ is transcendental, where $a(z)$ is an entire function. Thus, we rewrite (1.7) in the form

$$\begin{aligned}&\Big[a_1f(z)+a_2\frac{\partial f}{\partial z_1}+i\Big(a_3f(z)+a_4\frac{\partial f}{\partial z_2}\Big)\Big]\\&\times\Big[a_1f(z)+a_2\frac{\partial f}{\partial z_1}-i\Big(a_3f(z)+a_4\frac{\partial f}{\partial z_2}\Big)\Big]=1,\end{aligned}\tag{2.5}$$

which implies that both $a_1f+a_2\frac{\partial f}{\partial z_1}+i\big(a_3f+a_4\frac{\partial f}{\partial z_2}\big)$ and $a_1f+a_2\frac{\partial f}{\partial z_1}-i\big(a_3f+a_4\frac{\partial f}{\partial z_2}\big)$ have no poles and zeros. Thus, by Lemmas 2.1 and 2.2, there thus exists a polynomial $p(z)$ such that

$$\begin{gathered}a_1f(z)+a_2\frac{\partial f}{\partial z_1}+i\Big(a_3f(z)+a_4\frac{\partial f}{\partial z_2}\Big)=e^{p(z)},\\a_1f(z)+a_2\frac{\partial f}{\partial z_1}-i\Big(a_3f(z)+a_4\frac{\partial f}{\partial z_2}\Big)=e^{-p(z)},\end{gathered}$$

which leads to

$$a_1f(z)+a_2\frac{\partial f(z)}{\partial z_1}=\frac{e^{p(z)}+e^{-p(z)}}{2},\tag{2.6}$$

$$a_3f(z)+a_4\frac{\partial f(z)}{\partial z_2}=\frac{e^{p(z)}-e^{-p(z)}}{2i}.\tag{2.7}$$

This means that

$$a_2a_3\frac{\partial f(z)}{\partial z_1}-a_1a_4\frac{\partial f(z)}{\partial z_2}=\frac{a_3+ia_1}{2}e^{p(z)}+\frac{a_3-ia_1}{2}e^{-p(z)}.\tag{2.8}$$

Differentiating on z_2,z_1 for both two sides of equations (2.6), (2.7), respectively, and noting the fact that $\frac{\partial^2f}{\partial z_1\partial z_2}=\frac{\partial^2f}{\partial z_2\partial z_1}$, we can conclude that

$$a_2a_3\frac{\partial f}{\partial z_1}-a_1a_4\frac{\partial f}{\partial z_2}=-\frac{1}{2}(ia_2\frac{\partial p}{\partial z_1}+a_4\frac{\partial p}{\partial z_2})e^{p(z)}+\frac{1}{2}(-ia_2\frac{\partial p}{\partial z_1}+a_4\frac{\partial p}{\partial z_2})e^{-p(z)}.\tag{2.9}$$

Thus, it follows from (2.8) and (2.9) that

$$e^{2p}\Big(a_2i\frac{\partial p}{\partial z_1}+a_4\frac{\partial p}{\partial z_2}+a_1i+a_3\Big)=-a_2i\frac{\partial p}{\partial z_1}+a_4\frac{\partial p}{\partial z_2}+a_1i-a_3.\tag{2.10}$$

Suppose that $a_2i\frac{\partial p}{\partial z_1}+a_4\frac{\partial p}{\partial z_2}+a_1i+a_3\neq0$ and $-a_2i\frac{\partial p}{\partial z_1}+a_4\frac{\partial p}{\partial z_2}+a_1i-a_3\neq0$. Since $f(z)$ is a finite order transcendental entire solution of equation (1.7), by Lemma 2.1, 2.2 and (2.10), we conclude that $p(z)$ is a nonconstant polynomial in $\mathbb{C}^2$. Thus, a contradiction can be obtained from (2.10) using Nevanlinna theory. In fact, if $T(r,F)$ denotes the Nevanlinna characteristic function of a meromorphic function F in $\mathbb{C}^2$, then by (2.10) we deduce that $T(r,e^{2p})=O\{T(r,p)+\log r\}$, outside possibly a set of finite Lebesgue measure, using the results (see e.g. [26, p.99], [24]) that $T(r,F_{z_j})=O\{T(r,F)\}$ for any meromorphic function F outside a set of finite Lebesgue measure and that $T(r,P)=O\{\log r\}$ for any polynomial P.

But, $\lim_{r\to\infty}\frac{T(r,e^{2p})}{T(r,p)+\log r}=+\infty$ when p is a nonconstant polynomial. Therefore, p must be constant, a contradiction. Thus, equation (2.10) implies that

$$a_2 i\frac{\partial p}{\partial z_1}+a_4\frac{\partial p}{\partial z_2}+a_1 i+a_3=0,\quad -a_2 i\frac{\partial p}{\partial z_1}+a_4\frac{\partial p}{\partial z_2}+a_1 i-a_3=0.$$

Hence, it follows that $\alpha_1:=\frac{\partial p}{\partial z_1}=\frac{a_3}{a_2}i$ and $\alpha_2:=\frac{\partial p}{\partial z_2}=-\frac{a_1}{a_4}i$, which means that $p(z_1,z_2)=\alpha_1 z_1+\alpha_2 z_2+\eta_1=\frac{a_3}{a_2}iz_1-\frac{a_1}{a_4}iz_2+B$ where $B\in\mathbb{C}$.

On the other hand, it follows from (2.8) that

$$a_2a_3\frac{\partial f(z)}{\partial z_1}-a_1a_4\frac{\partial f(z)}{\partial z_2}=\frac{a_3+ia_1}{2}e^{\alpha_1 z_1+\alpha_2 z_2+B}+\frac{a_3-ia_1}{2}e^{-(\alpha_1 z_1+\alpha_2 z_2+B)}.$$

Then the characteristic equations for this differential equation are

$$\frac{dz_1}{dt}=a_2a_3,\quad \frac{dz_2}{dt}=-a_1a_4,$$
$$\frac{df}{dt}=\frac{a_3+ia_1}{2}e^{\alpha_1 z_1+\alpha_2 z_2+B}+\frac{a_3-ia_1}{2}e^{-(\alpha_1 z_1+\alpha_2 z_2+B)}.$$

Using the initial conditions: $z_1=0, z_2=s$, and $f=f(0,s):=\varphi_0(s)$ with a parameter s. Thus, we obtain the following parametric representation for the solutions of the characteristic equations: $z_1=a_2a_3t$, $z_2=-a_1a_4t+s$,

$$\begin{aligned}
f(t,s)&=\int_0^t\Big(\frac{1+i}{2}e^{(\alpha_1a_2a_3-\alpha_2a_1a_4)t+\alpha_2 s+B}\\
&\quad+\frac{1-i}{2}e^{-[(\alpha_1a_2a_3-\alpha_2a_1a_4)t+\alpha_2 s+B]}\Big)dt+\varphi_0(s)\\
&=\frac{a_3+ia_1}{2(\alpha_1a_2a_3-\alpha_2a_1a_4)}e^{(\alpha_1a_2a_3-\alpha_2a_1a_4)t+\alpha_2 s+B}\\
&\quad-\frac{a_3-ia_1}{2(\alpha_1a_2a_3-\alpha_2a_1a_4)}e^{-[(\alpha_1a_2a_3-\alpha_2a_1a_4)t+\alpha_2 s+B]}+\varphi(s),
\end{aligned}$$

where $\varphi(s)$ is an entire function with finite order in s such that

$$\varphi(s)=\varphi_0(s)-\frac{a_3+ia_1}{2(\alpha_1a_2a_3-\alpha_2a_1a_4)}e^{\alpha_2 s+B}+\frac{a_3-ia_1}{2(\alpha_1a_2a_3-\alpha_2a_1a_4)}e^{-(\alpha_2 s+B)}.$$

Thus, it follows that

$$f(z_1,z_2)=\frac{a_3+ia_1}{2(\alpha_1a_2a_3-\alpha_2a_1a_4)}e^{L(z)+B}-\frac{a_3-ia_1}{2(\alpha_1a_2a_3-\alpha_2a_1a_4)}e^{-L(z)-B}+\varphi(s).$$

Substituting this expression into (2.6), we can deduce that $\varphi(s)$ satisfies

$$\frac{a_4}{a_3}\varphi'(s)+\varphi(s)=0, \tag{2.11}$$

which implies that $\phi(s)=\eta e^{-(\frac{a_1}{a_2}z_1+\frac{a_3}{a_4}z_2)}$.

Therefore, from Case 1 and Case 2, the proof of Theorem 1.6 is complete. $\square$

3. Proofs of Theorems 1.9–1.12

Proof of Theorem 1.9. Suppose that $f(z)$ is a transcendental entire solution with finite order of (1.9). By using the same argument as in Case 2 of Theorem 1.8, we can easily get that there exists a polynomial $p(z)$ in $\mathbb{C}^2$ such that

$$a_1f(z)+a_2\frac{\partial f}{\partial z_1}=\frac{e^{p(z)}+e^{-p(z)}}{2}, \tag{3.1}$$

$$a_3 f(z) + a_4 \frac{\partial^2 f}{\partial z_1^2} = \frac{e^{p(z)} - e^{-p(z)}}{2i}. \tag{3.2}$$

Thus, the partial derivative of (3.1) for z_1 is

$$a_1 \frac{\partial f}{\partial z_1} + a_2 \frac{\partial^2 f}{\partial z_1^2} = \frac{\partial p}{\partial z_1} \frac{e^{p(z)} - e^{-p(z)}}{2}. \tag{3.3}$$

By combining (3.2) with (3.3), it follows that

$$a_2 a_3 f(z) - a_1 a_4 \frac{\partial f}{\partial z_1} = \frac{e^{p(z)} - e^{-p(z)}}{2} (-a_2^2 i - \frac{\partial p}{\partial z_1} a_2 a_4). \tag{3.4}$$

In view of $D := -(a_1^2 a_4 + a_2^2 a_3) \neq 0$, and by combining with (3.1) and (3.4), we have

$$f(z) = \frac{a_2^2 i + a_2 a_4 \frac{\partial p}{\partial z_1} - a_1 a_4}{2D} e^{p(z)} - \frac{a_2^2 i + a_2 a_4 \frac{\partial p}{\partial z_1} + a_1 a_4}{2D} e^{-p(z)}, \tag{3.5}$$

$$\frac{\partial f}{\partial z_1} = -\frac{a_1 a_2 i + a_1 a_4 \frac{\partial p}{\partial z_1} + a_2 a_3}{2D} e^{p(z)} + \frac{a_1 a_2 i + a_1 a_4 \frac{\partial p}{\partial z_1} - a_2 a_3}{2D} e^{-p(z)}. \tag{3.6}$$

Obviously, $p(z)$ is a nonconstant polynomial. Otherwise, $f(z)$ is a constant, this is a contradiction with the assumption. And in view of (3.5) and (3.6), it follows that

$$(\beta + \gamma) e^{2p(z)} = \beta - \gamma, \tag{3.7}$$

where

$$\beta = a_1 a_2 i + a_2 a_4 \frac{\partial^2 p}{\partial z_1^2}, \quad \gamma = a_2 a_4 \Big(\frac{\partial p}{\partial z_1}\Big)^2 + a_2^2 i \frac{\partial p}{\partial z_1} + a_2 a_3.$$

Similar to the argument as in the proof of Theorem 1.6, it follows that $\beta + \gamma = 0$ and $\beta - \gamma = 0$, which implies that $\beta = 0$ and $\gamma = 0$. In view of $\gamma = 0$ and $a_2 \neq 0$, it follows that $a_4 \big(\frac{\partial p}{\partial z_1}\big)^2 + a_2 i \frac{\partial p}{\partial z_1} + a_3 = 0$, which leads to $\frac{\partial^2 p}{\partial z_1^2} = 0$ or $\frac{\partial p}{\partial z_1} = -2i \frac{a_4}{a_2}$. Combining this with $\beta = 0$, we have $a_1 = 0$, this is a contradiction with $a_1 \neq 0$. This completes the proof. □

Proof of Theorem 1.11. Suppose that $f(z)$ is a transcendental entire solution with finite order of (1.10). By using the same argument as in the proof of Theorem 1.9, we can easily obtain that there exists a nonconstant polynomial $p(z)$ in $\mathbb{C}^2$ such that

$$f(z) = -\frac{a_2 i + a_4 \frac{\partial p}{\partial z_1}}{2 a_2 a_3} (e^{p(z)} - e^{-p(z)}), \quad \frac{\partial f}{\partial z_1} = \frac{e^{p(z)} + e^{-p(z)}}{2 a_2}, \tag{3.8}$$

and

$$\Big[a_4 \frac{\partial^2 p}{\partial z_1^2} + a_4 \Big(\frac{\partial p}{\partial z_1}\Big)^2 + a_2 i \frac{\partial p}{\partial z_1} + a_3\Big] e^{2p(z)} = a_4 \frac{\partial^2 p}{\partial z_1^2} - a_4 \Big(\frac{\partial p}{\partial z_1}\Big)^2 - a_2 i \frac{\partial p}{\partial z_1} - a_3. \tag{3.9}$$

Thus, it follows that

$$a_4 \frac{\partial^2 p}{\partial z_1^2} + a_4 \Big(\frac{\partial p}{\partial z_1}\Big)^2 + a_2 i \frac{\partial p}{\partial z_1} + a_3 = 0, \quad a_4 \frac{\partial^2 p}{\partial z_1^2} - a_4 \Big(\frac{\partial p}{\partial z_1}\Big)^2 - a_2 i \frac{\partial p}{\partial z_1} - a_3 = 0. \tag{3.10}$$

Hence, it means that $a_4 \frac{\partial^2 p}{\partial z_1^2} = 0$ and $a_4 \big(\frac{\partial p}{\partial z_1}\big)^2 + a_2 i \frac{\partial p}{\partial z_1} + a_3 = 0$. Since a_2, a_3, a_4 are nonzero constants, it follows that $\frac{\partial^2 p}{\partial z_1^2} = 0$ and $\frac{\partial p}{\partial z_1}$ is a constant and a root of the equation $a_4 \omega^2 + a_2 i \omega + a_3 = 0$. Set $\alpha_1 = \frac{\partial p}{\partial z_1}$, then $\alpha_1 = \frac{(-a_2 \pm \sqrt{a_2^2 + 4 a_3 a_4}) i}{2 a_4}$ and

$p(z) = \alpha_1 z_1 + \varphi(z_2)$, where $\varphi(z_2)$ is a polynomial in z_2. Substituting these into (3.8), we have

$$f(z_1, z_2) = -\frac{\alpha_1 a_4 + i a_2}{a_2 a_3} sh(\alpha_1 z_1 + \varphi(z_2)).$$

This completes the proof. □

The proof of Theorem 1.12 follows the same argument as that of Theorems 1.9 and 1.11; we omit it.

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