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SOLUTIONS OF KIRCHHOFF PLATE EQUATIONS WITH INTERNAL DAMPING AND LOGARITHMIC NONLINEARITY

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Abstract. In this article we study the existence of weak solutions for the nonlinear initial boundary value problem of the Kirchhoff equation

$$\begin{gathered} u_{tt} + \Delta^2 u + M(\|\nabla u\|^2)(-\Delta u) + u_t = u \ln |u|^2, \text{ in } \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = \frac{\partial u}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0, \end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, $T > 0$ is a fixed but arbitrary real number, $M(s)$ is a continuous function on $[0, +\infty)$ and η is the unit outward normal on $\partial\Omega$. Our results are obtained using the Galerkin method, compactness approach, potential well corresponding to the logarithmic nonlinearity, and the energy estimates due to Nakao.

1. Introduction

In this article we study the existence and decay properties of global solutions for the nonlinear initial boundary value problem

$$u_{tt} + \Delta^2 u + M(\|\nabla u\|^2)(-\Delta u) + u_t = u \ln |u|^2, \text{in } \Omega \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = \frac{\partial u}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, $T > 0$ is a fixed but arbitrary real number, $M(s)$ is a continuous function on $[0, +\infty)$ and η is the unit outward normal on $\partial\Omega$. The boundary conditions (1.3) mean that boundary is clamped. We do not imposed a priori conditions on the function space, and it turns out that a weak solution automatically satisfies the boundary conditions.

The physical origin of this problem without logarithmic source term leads to the study of dynamic buckling of the hinged extensible beam which is either stretched or compressed by an axial force. The readers can see in Burgreen [10] and Eisley [16] for more physical justifications and the model background. From the mathematical point of view, we cite the pioneer works of Kirchhoff [21], Woinowsky-Krieger [33] and Berger [7]. For logarithmic source term, to the best of our knowledge, the first

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contribution in literature was given by Birula and Mycielski [9], where they studied the problem

$$\begin{gathered} u_{tt} - u_{xx} + u = \epsilon u \ln |u|^2, \quad (x,t) \in [a,b] \times (0,T), \\ u(x,0) = u^0(x), \quad u'(x,0) = u^1(x), \quad x \in [a,b], \\ u(a,t) = u(b,t) = 0, \quad t \in (0,T). \end{gathered} \tag{1.4}$$

Problem (1.1)-(1.3) is also called nonlocal because of the presence of the term

$$M\big(\|\nabla u(t)\|^2\big) = M\Big(\int_\Omega |\nabla u(x,t)|^2 dx\Big),$$

which implies that the equation is no longer has a pointwise dependence. The nonlocal term provokes some mathematical difficulties which make the study of such a problem particularly interesting. See the work of Arosio-Panizzi [3]. Nonlocal initial boundary value problems are important in the framework of their practical application to the modeling and investigation of various phenomena. In particular, the type of problems (1.1)-(1.3) has applications in nuclear physics, optics and geophysics, see for instance [6, 9, 18]. With logarithmic nonlinearity $u \ln |u|^2$ it appears naturally in inflation cosmology and supersymmetric field theories, quantum mechanics and nuclear physics, see [5, 17].

Now, we focus on a chronological literature overview. The one-dimensional nonlinear equation (1.5) of motion of an elastic string was proposed by Kirchhoff (1883) [21], in connection with some problems in nonlinear elasticity, and rediscovered by Carrier (1945) [11],

$$\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\tau_0}{m} + \frac{k}{2mL}\int_0^L \big(\frac{\partial u}{\partial x}\big)^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.5}$$

where τ_0 is the initial tension, m the mass of the string and k the Young's modulus of the material of the string. This model describes small vibrations of a stretched string of the length L when only the transverse component of the tension is considered. For mathematical aspects of (1.5) see Bernstein (1940) [8].

The model (1.5) is a generalization of the linearized problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\tau_0}{m}\frac{\partial^2 u}{\partial x^2} = 0,$$

obtained by Euler $(1707 - 1783)$ and d'Alembert $(1714 - 1793)$. A particular case of (1.5) can be written, in general, as

$$\frac{\partial^2 u}{\partial t^2} - M\Big(\int_\Omega |\nabla u(x,t)|^2 dx\Big)\Delta u = 0, \tag{1.6}$$

or

$$\frac{\partial^2 u}{\partial t^2} + M\big(\|u(t)\|^2\big)Au = 0, \tag{1.7}$$

in operator notation, where we consider the Hilbert spaces $V \hookrightarrow H \hookrightarrow V'$, where V' is the dual of V with the immersions continuous and dense. By $\|\cdot\|$ we denote the norm in V and $A : V \to V'$ a bounded linear operator.

For $M : [0,\infty) \to \mathbb{R}$ real function, $M(\lambda) \geq m_0 > 0$, $M \in C^1(0,\infty)$, the global solution for operator given in (1.7) can be found in Lions (1969) [22]. Later, Pohozhaev (1974) [30] proved that the mixed problem for (1.6) has global solution in t when the initial data $u(x,0), u_t(x,0)$ are restricted the class of functions called Pohozhaev's Class. For the case $M(\lambda) \geq 0$ we cite the works of Hazoya-Yamada

(1991) [20], Arosio-Spagnolo (1996) [4] and Medeiros-Límaco-Menezes (2002) [23] with reference therein.

Considering Ω a bounded domain in $\mathbb{R}^2$, Cavalcanti et al. (2004) [13], studied the equation

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + g(u_t) + f(u) = 0 \tag{1.8}$$

with $g(s) = |s|^{\rho-1}s$ and $f(s) = |s|^{\gamma-1}s$ where ρ and γ are positive constants such that $1 < \rho, \gamma \leq n/(n-2)$ if $n \geq 3$; $\rho, \gamma > 1$ if $n = 1, 2$. The global existence and asymptotic stability were obtained using the fixed point theorem and continuity arguments.

The problem studied in (1.8) was investigated more generally by Zhijian (2013) [36] as follows

$$u_{tt} + \Delta^2 u + M(|\nabla u|^2)(-\Delta u) + g(u_t) + f(u) = h(x) \tag{1.9}$$

where the source terms $f, g \in C^1(\mathbb{R})$, $|f'(s)| \leq C(1 + |s|^{p-1})$ and

$$K_0|s|^{q-1} < g'(s) \leq C(1 + |s|^{q-1}), \quad K_0, C > 0$$

with $1 \leq p < \infty$, $1 \leq q < \infty$ if $n \leq 4$; $1 \leq p \leq p^* = (n+4)/(n-4)$ and $p \leq q$ if $N \geq 5$. By Galerkin approximation combined with the monotone arguments, the author proved the existence of a global solution.

Milla Miranda et al. (2017) [26], investigated the existence and uniqueness of local solutions of the initial value problem for the nonlinear mixed problem 1.10 of Kirchhoff type,

$$\begin{gathered} u'' - M\Big(t, \int_\Omega |\nabla u|^2 dx\Big)\Delta u + |u|^\rho = f, \quad \text{in } \Omega \times (0, T_0), \\ u(x,0) = u^0(x), \quad u'(x,0) = u^1(x), \quad x \in \Omega, \\ u = 0 \quad \text{on } \Gamma_0 \times (0, T_0), \quad \frac{\partial u}{\partial \nu} + \delta(x)h(u') = 0 \quad \text{on } \Gamma_1 \times (0, T_0), \end{gathered} \tag{1.10}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with its boundary consisting of two disjoint parts Γ_0 and Γ_1; $\rho > 1$ is a real number; $\nu(x)$ is the exterior unit normal vector at $x \in \Gamma_1$ and $\delta(x), h(s)$ are real functions defined in Γ_1 and $\mathbb{R}$, respectively. The authors used the Galerkin method with a special basis, a modification of the Tartar approach, compactness method and fixed-point theorem.

Mohammad et al. (2018) [1], by using the Galerkin method, established the existence of solutions for a plate equation with nonlinear damping and a logarithmic source term and proved an explicit and general decay rate result, by using the multiplier method and some properties of the convex functions for the problem

$$\begin{gathered} u'' + \Delta u^2 + u + h(u_t) = ku \ln |u|, \quad \text{in } \Omega \times (0, T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = \frac{\partial u}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega, \ t \geq 0, \end{gathered} \tag{1.11}$$

where Ω is a bounded domain of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$.

For the study of an extensible beam equation with internal damping and source terms, Pereira et al. (2019) [28], considered the nonlinear beam equation

$$\begin{gathered} u_{tt}+\Delta^2 u+M(|\nabla u|^2)(-\Delta u)+u_t=|u|^{r-1}u, \quad \text{in } \Omega\times(0,T), \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega, \\ u(x,t)=\frac{\partial u}{\partial \eta}(x,t)=0, \quad x\in\partial\Omega,\ t\geq 0, \end{gathered} \tag{1.12}$$

where $r>1$ is a real number, $M(s)$ is a continuous function on $[0,+\infty)$ and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The authors constructed the global solutions by using the Faedo-Galerkin approximations, taking the initial data in an appropriate set for the stability created from the Nehari manifold. The asymptotic behavior was obtained by the Nakao's method.

Meng et al. (2020) [24], used the Nehari manifold, Ekeland variational principle, and the theory of Lagrange multipliers, to prove that there are at least two positive solutions for the nonlocal biharmonic equation of Kirchhoff type involving concave-convex nonlinearities. They considered the system

$$\Delta^2 u-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)\Delta u+V(x)u=\lambda f_1(x)|u|^{q-2}u+f_2(x)|u|^{p-2}u.$$

Regarding system (1.12) Pereira et al. (2021) [29], taking initial data suitable for the stability created from the Nehari manifold, proved the existence of global solutions and energy decay estimate when the internal damping is $|u_t|^{p-1}u_t$.

This article is organized as follows. In section 2 we present some hypothesis needed in the proof of our results. In section 3 we construct global weak solutions by means of the Galerkin approximations. In section 4 we present the potential well corresponding to the logarithmic nonlinearity. In section 5 we apply the results due to Nakao [27] to prove the exponential decay of solutions.

2. Preliminaries

In this section, we present some material needed in the proof of our results. For simplicity of notation we denote by $\|\cdot\|$ the norm in the Lebesgue space $L^2(\Omega)$, and by $\|\cdot\|_2$ the norm in the Sobolev space $H_0^2(\Omega)$. We consider the hypothesis

(H1) $M\in C([0,\infty))$ with $M(\lambda)\geq-\beta$, for all $\lambda\geq 0$, $0<\beta<\lambda_1$, where λ_1 is the first eigenvalue of the problem $\Delta^2 u=\lambda(-\Delta u)$.

Remark 2.1 (see Miklin [25]). The first eigenvalue λ_1 of $\Delta^2 u=\lambda(-\Delta u)$ with the clamped boundary conditions

$$u\big|_{\partial\Omega}=0, \quad \frac{\partial u}{\partial\eta}\big|_{\partial\Omega}=0,$$

satisfies

$$\lambda_1=\inf_{u\in H_0^2(\Omega)}\frac{\|\Delta u\|^2}{\|\nabla u\|^2}>0, \quad \text{and} \quad \|\nabla u\|^2\leq\frac{1}{\lambda_1}\|\Delta u\|^2.$$

Now, we enunciate the preliminary results.

Lemma 2.2 (Logarithmic Sobolev inequality [14, 19]). *Let u be a function in $H_0^1(\Omega)$ and $a>0$. Then*

$$\int_\Omega u^2\ln|u|\,dx\leq\|u\|^2\ln\|u\|^2+\frac{a^2}{2\pi}\|\nabla u\|^2-(1+\ln a)\|u\|^2. \tag{2.1}$$

Corollary 2.3. *Let u be a function in $H_0^2(\Omega)$ and $a>0$. Then*

$$\int_\Omega u^2 \ln|u|\,dx \le \|u\|^2\ln\|u\|^2 + \frac{a^2}{2\lambda_1\pi}\|\Delta u\|^2 - (1+\ln a)\|u\|^2. \tag{2.2}$$

Lemma 2.4 (Logarithmic Gronwall inequality [12])**.** *Let $\gamma \in L^1(0,T;\mathbb{R}^+)$ and $c>0$. Also assume that the function $w:[0,T]\to[1,\infty)$ satisfies*

$$w(t) \le c\Big(1+\int_0^t \gamma(s)w(s)\ln w(s)\,ds\Big), \quad 0\le t\le T. \tag{2.3}$$

Then

$$w(t) \le c\exp\Big(c\int_0^t \gamma(s)\,ds\Big), \quad 0\le t\le T. \tag{2.4}$$

Lemma 2.5 (Nakao's Lemma [27])**.** *Suppose that $\phi(t)$ is a bounded nonnegative function on $\mathbb{R}^+$ satisfying*

$$\operatorname{ess\,sup}_{t\le s\le t+1}\phi(s) \le C_0[\phi(t)-\phi(t+1)], \quad \forall t\ge 0,$$

where C_0 is a positive constant. Then $\phi(t)\le Ce^{-\alpha t}$ for all $t\ge 0$, where C and α are positive constants.

3. Existence of global weak solutions

Theorem 3.1. *Let $u_0\in H_0^1(\Omega)$, $u_1\in L^2(\Omega)$, and assume* (H1) *holds. Then there exists a function $u:[0,T]\to L^2(\Omega)$ with*

$$u\in L^\infty(0,T;H_0^2(\Omega)), \quad u_t\in L^\infty(0,T;L^2(\Omega)), \tag{3.1}$$

such that for all $w\in H_0^2(\Omega)$,

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}(u_t(t),w) + \langle\Delta u(t),\Delta w\rangle + M(\|\nabla u(t)\|^2)(-\Delta u(t),w)\\ +(u_t(t),w) - \big(u(t)\ln|u(t)|^2,w\big) = 0 \quad \text{in } \mathcal{D}'(0,T),\end{gathered} \tag{3.2}$$

$$u(0)=u_0, \quad u_t(0)=u_1. \tag{3.3}$$

Proof. We use Faedo-Galerkin's method to prove the global existence of solutions.

3.1. **Approximated problem.** Let $(w_\nu)_{\nu\in\mathbb{N}}$ be a basis of $H_0^2(\Omega)$ consisting of eigenvectors of the operator $-\Delta$ and $V_m=\operatorname{span}\{w_1,w_2,\dots,w_m\}$. For $w\in V_m$, let

$$u^m(t)=\sum_{j=1}^m k_{jm}(t)w_j$$

be a solution of the approximated problem

$$\begin{gathered}(u_{tt}^m(t),w) + (\Delta u^m(t),\Delta w) + M(\|\nabla u^m(t)\|^2)(-\Delta u^m(t),w)\\ +(u_t^m(t),w) - \big(u^m(t)\ln|u^m(t)|^2,w\big) = 0,\end{gathered} \tag{3.4}$$

$$u^m(0)=u_{0m}\to u_0 \quad \text{strongly in } H_0^2(\Omega, \tag{3.5}$$

$$u_t^m(0)=u_{1m}\to u_1 \quad \text{strongly in } L^2(\Omega). \tag{3.6}$$

System (3.4)-(3.6) has a local solution in $[0,t_m)$, $0<t_m\le T$, by Carathéodory's theorem [15]. The extension of the solution to the whole interval $[0,T]$ is a consequence of a priori estimates.

3.2. **A priori estimates.** Replacing $w = u_t^m(t)$ in (3.4), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u_t^m(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\Delta u^m(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\hat{M}(\|\nabla u^m(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u^m(t)\|^2 \\ &- \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega (u^m(t))^2 \ln|u^m(t)|^2\,dx = -\|u_t^m(t)\|^2, \end{aligned} \tag{3.7}$$

where $\hat{M}(s) = \int_0^s M(\xi)d\xi$. Integrating (3.7) from 0 to t, $0 \le t \le t_m$, we obtain

$$\begin{aligned} &\frac{1}{2}\|u_t^m(t)\|^2 + \frac{1}{2}\|\Delta u^m(t)\|^2 + \frac{1}{2}\hat{M}(\|\nabla u^m(t)\|^2) + \frac{1}{2}\|u^m(t)\|^2 + \int_0^t \|u_t^m(s)\|^2\mathrm{d}s \\ &= \frac{1}{2}\|u_{1m}\|^2 + \frac{1}{2}\|\Delta u_{0m}\|^2 + \frac{1}{2}\hat{M}(\|\nabla u_{0m}\|^2) - \frac{1}{2}\int_\Omega (u_{0m})^2\ln|u_{0m}|^2\,dx \\ &\quad + \frac{1}{2}\int_\Omega (u^m(t))^2\ln|u^m(t)|^2\,dx. \end{aligned} \tag{3.8}$$

Now, by hypothesis (H1), we have

$$\hat{M}(\|\nabla u^m(t)\|^2) \ge -\frac{\beta}{\lambda_1}\|\Delta u^m(t)\|^2, \tag{3.9}$$

$$\hat{M}(\|\nabla u_{0m}\|^2) \le m_0\|\nabla u_{0m}\|^2 \le \frac{m_0}{\lambda_1}\|\Delta u_0 m\|^2 \tag{3.10}$$

where

$$m_0 = \max_{0\le s\le \|\nabla u_{0m}\|^2 \le C_0} M(s),$$

with C_0 a positive constant. Replacing (3.9) and (3.10) in (3.8), and using logarithmic Sobolev inequality 2.1 we obtain

$$\begin{aligned} &\frac{1}{2}\|u_t^m(t)\|^2 + \frac{1}{2}\big(1 - \frac{\beta}{\lambda_1} - \frac{a^2}{2\lambda_1\pi}\big)\|\Delta u^m(t)\|^2 + \big(\frac{3}{2} + \ln a\big)\|u^m(t)\|^2 \\ &\le C + \|u^m(t)\|^2\ln\|u^m(t)\|^2, \end{aligned} \tag{3.11}$$

where C is a positive constant, independent of m and t by (3.5) and (3.6).

Choosing $e^{-3/2} < a < \sqrt{2\pi(\lambda_1 - \beta)}$ we have

$$1 - \frac{\beta}{\lambda_1} - \frac{a^2}{2\pi\lambda_1} > 0, \quad \frac{3}{2} + \ln a > 0.$$

Taking

$$C_1 = \min\Big\{\frac{1}{2}, \frac{1}{2}\Big(1 - \frac{\beta}{\lambda_1} - \frac{a^2}{2\lambda_1\pi}\Big), \big(\frac{3}{2} + \ln a\big)\Big\}$$

we have the estimate

$$\|u_t^m(t)\|^2 + \|\Delta u^m(t)\|^2 + \|u^m(t)\|^2 \le C_2\Big(1 + \|u^m(t)\|^2\ln\|u^m(t)\|^2\Big). \tag{3.12}$$

Now, observe that

$$u^m(\cdot,t) = u^m(\cdot,0) + \int_0^t \frac{\partial u^m}{\partial s}(\cdot,s)\,ds,$$

so, using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned} \|u^m(t)\|^2 &\le 2\|u_{0m}\|^2 + 2\Big|\Big|\int_0^t \frac{\partial u^m}{\partial s}(s)\,ds\Big|\Big|^2 \\ &\le 2\|u_{0m}\|^2 + 2T\int_0^t \|\frac{\partial u^m}{\partial s}(s)\|^2\,ds. \end{aligned} \tag{3.13}$$

Using estimates (3.12) and (3.13) we have

$$\|u^m(t)\|^2 \le \|u_{0m}\|^2 + 2TC_2 \int_0^t \Big(1 + \|u^m(t)\|^2 \ln \|u^m(t)\|^2\Big) ds$$

and choosing $C_3 = \max\{\|u_{0m}\|^2, TC_2\}$, we obtain

$$\|u^m(t)\|^2 \le 2C_3\Big(1 + \int_0^t \|u^m(t)\|^2 \ln \|u^m(t)\|^2 ds\Big).$$

Without loss of generality, we take $C_3 \ge 1$ which gives

$$\|u^m(t)\|^2 \le 2C_3\Big(1 + \int_0^t \Big(C_3 + \|u^m(t)\|^2\Big) \ln \big(C_3 + \|u^m(t)\|^2\big) ds\Big)$$

and then by Lemma 2.4, we obtain

$$\|u^m(t)\|^2 \le 2C_3\, e^{2C_3 T} \le C_4.$$

Hence, from inequality (3.12) it follows that

$$\|u_t^m(t)\|^2 + \|\Delta u^m(t)\|^2 + \|u^m(t)\|^2 \le C_5, \tag{3.14}$$

with C_4, C_5 positive constants independent of m and t.

Therefore, we can extend the approximate solutions $u_t^m(t)$ to the whole interval $[0,T]$. Then by (3.14) we have that

$$(u^m) \text{ is bounded in } L^\infty(0,T;H_0^2(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \tag{3.15}$$

$$(u_t^m) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)). \tag{3.16}$$

3.3. **Passage to the limit.** From the estimates (3.15)-(3.16), there exists a subsequence of (u^m), still denoted by (u^m), such that

$$u^m \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;H_0^2(\Omega)), \tag{3.17}$$

$$u^m \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{3.18}$$

$$u_t^m \overset{*}{\rightharpoonup} u_t \quad \text{in } L^\infty(0,T;L^2(\Omega)). \tag{3.19}$$

Applying the Lions-Aubin compactness lemma [22], we have from (3.17)-(3.18) that

$$u^m \to u \quad \text{strongly in } L^2(0,T;H_0^1(\Omega)), \tag{3.20}$$

$$u^m \to u \quad \text{a.e. in } \Omega \times (0,T). \tag{3.21}$$

Taking into account that M is continuous and the convergences (3.20), (3.21), we have that

$$M(\|\nabla u^m\|^2) \to M(\|\nabla u\|^2) \quad \text{strongly in } L^2(0,T).$$

Therefore,

$$M(\|\nabla u^m\|^2)(-\Delta u^m) \rightharpoonup M(\|\nabla u\|^2)(-\Delta u) \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{3.22}$$

Since the map $s \to\ s \ln |s|^2$ is continuous, we have assured the convergence

$$u^m \ln |u^m|^2 \to u \ln |u|^2 \quad \text{a.e. in } (\Omega) \times (0,T).$$

By using the immersion of $H_0^1(\Omega)$ in $L^\infty(\Omega)$ because $(\Omega \subset \mathbb{R}^2)$, it is clear that $u^m \ln |u^m|^2$ is bounded in $L^\infty(\Omega \times (0,T))$. So, by the Lebesgue dominated convergence theorem,

$$u^m \ln |u^m|^2 \to u \ln |u|^2 \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{3.23}$$

By the convergence (3.17), (3.18) and (3.23), we can pass to the limit in the approximate problem (3.4) and obtain the equation

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(u_t(t), w) + \langle \Delta u(t), \Delta w\rangle + M(\|\nabla u\|^2)(-\Delta u, w) + (u_t(t), w) \\ &- \big(u^m(t)\ln|u^m(t)|^2, w\big) = 0, \end{aligned} \tag{3.24}$$

for all $w \in V_m$, in $\mathcal{D}'(0,T)$. Since the V_m is dense in $H_0^2(\Omega)$ it follows that (3.24) is valid for all $w \in H_0^2(\Omega)$. The verification of the initial data can be obtained in a standard way. □

4. Potential well

In this section, we present the potential well corresponding to the logarithmic nonlinearity. It is well known that the energy of a PDE system is, in some sense, split into kinetic and potential energy. Following the idea by Ye [35] and [31], we are able to construct a set of stability as follows. We will prove that there is a valley or a "well" of depth d created in the potential energy. If this height d is strictly positive, we find that for solutions with initial data in the "good part" of the well, the potential energy of the solution can never escape the well. In general, it is possible for the energy from the source term to cause the blow-up in finite time. However, in the good part of the well, it remains bounded. As a result, the total energy of the solution remains finite on any time interval $[0,T)$, which provides the global existence of the solution.

We started by introducing the functionals $J, I : H_0^2(\Omega) \to \mathbb{R}$ by

$$\begin{gathered} J(u) := \frac{1}{2}\big(1 - \frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 - \frac{1}{2}\int_\Omega |u|^2 \ln|u|^2\,dx + \frac{1}{2}\|u\|^2, \\ I(u) := \big(1 - \frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 - \int_\Omega |u|^2\ln|u|^2\,dx. \end{gathered}$$

From the above definitions, it is clear that

$$J(u) = \frac{1}{2}I(u) + \frac{1}{2}\|u\|^2.$$

For $u \in H_0^2(\Omega)$ we define the functional

$$J(\lambda u) = \frac{1}{2}I(\lambda u) + \frac{\lambda^2}{2}\|u\|^2,$$

Associated with the J we have the Nehari Manifold,

$$\mathcal{N} := \{u \in H_0^2(\Omega) : I(u) = 0,\ \|\Delta u\| \neq 0\}.$$

Lemma 4.1. *For $u \in H_0^2(\Omega)$ with $\|u\| \neq 0$, let $g(\lambda) = J(\lambda u)$. Then we have*

$$I(\lambda u) = \lambda g'(\lambda), \quad \text{where } \lambda g'(\lambda) \begin{cases} > 0, & 0 < \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & 0 \le \lambda^* < \lambda < +\infty. \end{cases}$$

with

$$\lambda^* = \exp\Big(\frac{(1 - \frac{\beta}{\lambda_1})\|\Delta u\|^2 - \int_\Omega |u|^2\ln|u|^2\,dx}{2\|u\|^2}\Big).$$

Proof. Note that

$$\begin{aligned}
g(\lambda) &= J(\lambda u)\\
&= \frac{\lambda^2}{2}\big(1-\frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 - \frac{\lambda^2}{2}\int_\Omega |u|^2 \ln \lambda^2|u|^2\,dx + \frac{\lambda^2}{2}\|u\|^2\\
&= \frac{\lambda^2}{2}\big(1-\frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 - \frac{\lambda^2}{2}\int_\Omega |u|^2 \ln \lambda^2\,dx - \frac{\lambda^2}{2}\int_\Omega |u|^2 \ln |u|^2\,dx + \frac{\lambda^2}{2}\|u\|^2\\
&= \frac{\lambda^2}{2}\Big[\big(1-\frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 + (1-2\ln\lambda)\,\|u\|^2 - \int_\Omega |u|^2 \ln |u|^2\,dx\Big]
\end{aligned}$$

and that

$$\begin{aligned}
g'(\lambda) &= \lambda\Big[\big(1-\frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 + (1-2\ln\lambda)\,\|u\|^2 - \int_\Omega |u|^2 \ln |u|^2\,dx\Big] + \frac{\lambda^2}{2}\frac{-2\|u\|^2}{\lambda}\\
&= \lambda\Big[\big(1-\frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 + (1-2\ln\lambda)\,\|u\|^2 - \int_\Omega |u|^2 \ln |u|^2\,dx\Big] - \lambda\|u\|^2\\
&= \lambda\Big[\big(1-\frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 - 2\|u\|^2\ln\lambda - \int_\Omega |u|^2 \ln |u|^2\,dx\Big].
\end{aligned}$$

Then

$$\lambda g'(\lambda) = \lambda^2\Big[\big(1-\frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 - 2\|u\|^2\ln\lambda - \int_\Omega |u|^2 \ln |u|^2\,dx\Big].$$

So that $I(\lambda u)=0$ implies

$$\lambda^2\Big[\big(1-\frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 - 2\|u\|^2\ln\lambda - \int_\Omega |u|^2 \ln |u|^2\,dx\Big] = 0, \tag{4.1}$$

$$\lambda = \exp\Big(\frac{\big(1-\frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 - \int_\Omega |u|^2\ln|u|^2\,dx}{2\|u\|^2}\Big) = \lambda^*. \tag{4.2}$$

Now, observe that, for $0<\lambda<\lambda^*$ we have $-\ln\lambda > -\ln\lambda^*$, and then

$$\begin{aligned}
I(\lambda u) &= \lambda g'(\lambda)\\
&= \lambda^2\Big[\big(1-\frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 - 2\|u\|^2\ln\lambda - \int_\Omega |u|^2 \ln |u|^2\,dx\Big]\\
&> \lambda^2\Big[\big(1-\frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 - 2\|u\|^2\ln\lambda^* - \int_\Omega |u|^2 \ln |u|^2\,dx\Big] = 0.
\end{aligned} \tag{4.3}$$

In the same way, for $\lambda^* < \lambda$, we obtain

$$I(\lambda u) = \lambda g'(\lambda) < 0. \tag{4.4}$$

Finally, from (4.2), (4.3) and (4.4) the proof is complete. □

The potential well depth is defined as

$$d := \inf\{\sup_{\lambda\ge 0} J(\lambda u); u\in H_0^2(\Omega),\ \|\Delta u\|\neq 0\}. \tag{4.5}$$

From the Mountain Pass theorem due to Ambrosetti and Rabinowitz [2], it is well-known that the depth of the well d is a strictly positive constant, see [34, Theorem 4.2], and that

$$d = \inf_{u\in\mathcal{N}} J(u). \tag{4.6}$$

With this approach, we introduce the potential well

$$W = \{u\in H_0^2(\Omega) : I(u)\neq 0,\ J(u)<d\}\cup\{0\} \tag{4.7}$$

and from lemma 4.1 we can partition W into two sets

$$\mathcal{W} = \{u \in H_0^2(\Omega) : I(u) > 0,\ J(u) < d\} \cup \{0\},$$
$$U = \{u \in H_0^2(\Omega) : I(u) < 0,\ J(u) < d\}.$$

We will refer to $\mathcal{W}$ as the "good" part of the well. Then we define the stability set for problem (1.1)-(1.3) by

$$\mathcal{W} = \big\{u \in H_0^2(\Omega) : \big(1 - \frac{\beta}{\lambda_1}\big)\|\Delta u\|^2 > \int_\Omega u^2 \ln |u|^2\,dx,\ J(u) < d\big\} \cup \{0\}.$$

The following lemma establishes a criterion for the solution u to remain in the stability set W.

Lemma 4.2. *Let $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ such that*

$$0 < E(0) < d \quad \text{and} \quad I(u_0) > 0. \tag{4.8}$$

Then every solution of (1.1)-(1.3) *belongs to $\mathcal{W}$.*

Proof. Let T be maximal existence time of a weak solution u. From (3.7), we defined the energy

$$E(t) = \frac{1}{2}\Big(\|u_t\|^2 + \|\Delta u\|^2 + \hat{M}(\|\nabla u\|^2) + \|u\|^2 - \int_\Omega u^2 \ln |u|^2 dx\Big), \tag{4.9}$$

where $\hat{M}(s) = \int_0^s M(\xi)\,d\xi$. Differentiating (4.9), and using (1.1)-(1.3), lead to

$$\frac{d}{dt}E(t) = -\int_\Omega \|u_t\|^2\,dx \le 0, \tag{4.10}$$

so

$$\frac{1}{2}\|u_t(t)\|^2 + J(u) \le \frac{1}{2}\|u_1\|^2 + J(u_0) < d, \tag{4.11}$$

for all $t \in [0, T]$.

We claim that $u(t) \in \mathcal{W}$ for all $t \in [0.T]$. If not, then there exists a $t_0 \in (0, T)$ such that $u(t_0) \in \partial\mathcal{W}$, so either $I(u(t_0)) = 0$ and $\|\Delta u(t_0)\| \neq 0$, or $J(u(t_0)) = d$. By (4.11), $J(u(t_0)) < 0$, thus we have $I(u_0) = 0$ and $\|\Delta u_0(t_0)\| \neq 0$. However, (4.6) implies $J(u(t_0)) > d$, which contradicts (4.11). So, we conclude that $u(t) \in \mathcal{W}$. $\square$

5. Exponential decay

We prove the exponential decay of the problem (1.1)-(1.3), using the Nakao's lemma.

Theorem 5.1. *Let $u_0 \in \mathcal{W}$ and $u_1 \in L^2(\Omega)$, and $0 < E(0) < d$. If* (H1) *holds, then the energy associated with* (1.1)-(1.3) *satisfies*

$$E(t) \le Ce^{-\alpha t}, \quad \forall t \ge 0,$$

where C and α are positive constants.

Proof. Let $w = u_t(t)$ in equation (3.24). Then

$$\begin{aligned}&\frac{d}{dt}\|u_t(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\Delta u(t)\|^2 + \frac{1}{2}\frac{d}{dt}\hat{M}(\|\nabla u(t)\|^2) + \frac{1}{2}\frac{d}{dt}\|u(t)\|^2\\ &- \frac{1}{2}\frac{d}{dt}\int_\Omega u^2(t) \ln |u(t)|^2\,dx + \|u_t(t)\|^2 = 0;\end{aligned}$$

that is,

$$\frac{d}{dt}E(t) + \|u_t(t)\|^2 < 0$$

where, $E(t)$ is define by (4.9). Integrating from t to $t+1$, we obtain

$$\int_t^{t+1} \|u_t(s)\|^2\,ds \le E(t) - E(t+1) := F^2(t). \tag{5.1}$$

Then there exists $t_1 \in [t, t+\frac{1}{2}]$ and $t_2 \in [t+\frac{3}{2}, t+1]$ such that

$$\|u_t(t_i)\| \le 2F(t_i), \quad i = 1, 2. \tag{5.2}$$

Let $w = u(t)$ in equation (3.24). Then we have

$$\begin{aligned} &\|\Delta u(t)\|^2 + M(\|\nabla u(t)\|^2)\|\nabla u(t)\|^2 - \int_\Omega u^2(t)\ln|u(t)|^2\,dx \\ &= -\frac{d}{dt}\big(u_t(t), u(t)\big) - \big(u_t(t), u(t)\big). \end{aligned} \tag{5.3}$$

Now by (H1) we obtain

$$M(\|\nabla u(t)\|^2)\|\nabla u(t)\|^2 \ge -\frac{\beta}{\lambda_1}\|\Delta u(t)\|^2.$$

Integrating (5.3) from t_1 to t_2 we obtain

$$\begin{aligned} &\int_{t_1}^{t_2} \Big[\Big(1-\frac{\beta}{\lambda_1}\Big)\|\Delta u(s)\|^2 - \int_\Omega u^2(s)\ln|u(s)|^2\,dx\Big]\,ds \\ &\le (u_t(t_1), u(t_1)) - (u_t(t_2), u(t_2)) - \int_{t_1}^{t_2} (u_t(s), u(s))\,ds \\ &\le C_1 \operatorname{ess\,sup}_{t\le s\le t+1} \|\Delta u(s)\|[\|u_t(t_1)\| + \|u_t(t_2)\|] \\ &\quad + \frac{C_1^2}{\delta}\int_{t_1}^{t_2} \|u_t(s)\|^2\,ds + \delta\int_{t_1}^{t_2} \|\Delta u_t(s)\|^2\,ds, \end{aligned}$$

where $0 < \delta < 1 - \frac{\beta}{\gamma_1}$ and $C_1 > 0$ is a constant such that $\|u_t(s)\| \le C_1\|\Delta u_t(s)\|$.

Then, by (5.1) and (5.2), we obtain

$$\begin{aligned} &\int_{t_1}^{t_2} \Big[\Big(1-\frac{\beta}{\lambda_1}-\delta\Big)\|\Delta u(s)\|^2 - \int_\Omega u^2(s)\ln|u(s)|^2\,dx\Big]\,ds \\ &\le 4C_1\, F(t)\operatorname{ess\,sup}_{t\le s\le t+1}\|\Delta u(s)\| + \frac{C_1^2}{\delta}F^2(t). \end{aligned}$$

Whence

$$\begin{aligned} &\int_{t_1}^{t_2} \Big[\Big(1-\frac{\beta}{\lambda_1}-\delta\Big)\|\Delta u(s)\|^2 - \int_\Omega u^2(s)\ln|u(s)|^2\,dx\Big]\,ds \\ &\le C_2\big[F(t)\operatorname{ess\,sup}_{t\le s\le t+1}\|\Delta u(s)\| + F^2(t)\big] =: G^2(t). \end{aligned} \tag{5.4}$$

Thanks to (5.1), we have

$$\int_{t_1}^{t_2} \Big[\|u_t(s)\|^2 + \Big(1-\frac{\beta}{\lambda_1}-\delta\Big)\|\Delta u(s)\|^2 - \int_\Omega u^2(s)\ln|u(s)|^2dx\Big]ds \le F^2(t) + G^2(t).$$

Hence, there exists $t^* \in [t_1, t_2]$ such that

$$\|u_t(t^*)\|^2 + \Big(1-\frac{\beta}{\lambda_1}-\delta\Big)\|\Delta u(t^*)\|^2 - \int_\Omega u^2(t^*)\ln|u(t^*)|^2dx \le 2[F^2(t) + G^2(t)];$$

that is,

$$\|u_t(t^*)\|^2 + \|\Delta u(t^*)\|^2 - \int_\Omega u^2(t^*) \ln |u(t^*)|^2 dx \le C_3[F^2(t) + G^2(t)]. \tag{5.5}$$

By (H1), we obtain

$$\begin{aligned} \|u_t(t^*)\|^2 + \hat{M}\|\Delta u(t^*)\|^2 &\le C_1^2\|\Delta u(t^*)\|^2 + m_0\|\nabla u(t^*)\|^2 \\ &\le \Big(C_1^2 + \frac{m_0}{\lambda_1}\Big)\|\Delta u(t^*)\|^2 \\ &\le C_4[F^2(t) \ + \ G^2(t)]. \end{aligned} \tag{5.6}$$

From (4.9), (5.5) and (5.6), it follows that

$$E(t^*) \le C_5[F^2(t) \ + \ G^2(t)]. \tag{5.7}$$

Now, by (5.1),(5.4) and (5.7), we have

$$\operatorname{ess\,sup}_{t\le s\le t+1} \ E(s) \le \ E(t^*) + \int_t^{t+1} \|u_t(s)\|^2 \, ds \le C_6 F^2(t) + \frac{1}{2} \operatorname{ess\,sup}_{t\le s\le t+1} E(s).$$

Therefore,

$$\operatorname{ess\,sup}_{t\le s\le t+1} \ E(s) \le C_7[E(t) - E(t+1)],$$

where $C_i = 1, 2, \dots, 7$ are positive constant. Finally, from lemma (2.5), we have $E(t) \le Ce^{-\alpha t}$ for all $t \ge 0$, where C and α are positive constants. □

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