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PERIODIC TRAVELING WAVES AND ASYMPTOTIC SPREADING OF A MONOSTABLE REACTION-DIFFUSION EQUATIONS WITH NONLOCAL EFFECTS

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ABSTRACT. This article concerns the dynamical behavior for a reaction-diffusion equation with integral term. First, by using bifurcation analysis and center manifold theorem, the existence of periodic steady-state solution are established for a special kernel function and a general kernel function respectively. Then, we prove the model admits periodic traveling wave solutions connecting this periodic steady state to the uniform steady state $u = 1$ by applying center manifold reduction and the analysis to phase diagram. By numerical simulations, we also show the change of the wave profile as the coefficient of aggregate term increases. Also, by introducing a truncation function, a shift function and some auxiliary functions, the asymptotic behavior for the Cauchy problem with initial function having compact support is investigated.

1. Introduction

In this article, we study the integro-differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\{1 + \alpha u - \beta u^2 - (1 + \alpha - \beta)(\phi * u)\}, \quad (t, x) \in (0, \infty) \times \mathbb{R}, \tag{1.1}$$

where $\alpha > 0$, $0 < \beta < 1 + \alpha$ and

$$(\phi * u)(t, x) := \int_{\mathbb{R}} \phi(x - y)u(t, y)dy, \quad t \in (0, \infty),\ x \in \mathbb{R}.$$

The kernel function $\phi(x) \in L^1(\mathbb{R})$ satisfies the conditions:

(A1) $\phi(0) > 0$, $\phi(x) = \phi(-x)$, $\phi(x) \geq 0$, $\int_{\mathbb{R}} \phi(x)dx = 1$ and $\int_{\mathbb{R}} x^2\phi(x)dx < +\infty$.

The equation with nonlocal term was first introduced by Britton [7, 8], when considering model simpler than (1.1),

$$u_t = u_{xx} + u(1 - \phi * u) \quad \text{in } \mathbb{R} \times (0, \infty). \tag{1.2}$$

By using linear stability analysis and bifurcation analysis, he obtained that the uniform steady state $u \equiv 1$ can be bifurcate to standing waves, periodic steady states or periodic traveling waves. Later, Gourley [17] gave the existence of traveling wave of equation (1.2) when the nonlocality is sufficiently weak. Recently, there have been some great progress on traveling wave solutions of equation (1.2). Particularly, Berestycki et al. [5] pointed out that, for all $c \geq c^* = 2$, equation

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(1.2) admits traveling fronts connecting 0 to some unknown positive state, while no such traveling wave solutions exists for $c < c^*$. Thereafter, through numerical simulation, this unknown steady state is showed just the equilibrium $u \equiv 1$ for some traveling front solutions in [31]. Alfaro and coville [2] gave a rigorously analysis proof (that is, they prove that equation (1.2) admits the rapid traveling front connecting 0 to 1). More recently, Hamel and Ryzhik [20] proved that (1.2) exists a periodic steady state due to the instability of the equilibrium. By using a center manifold reduction, Faye and Holzer [13] proved that (1.2) admits modulated traveling wave solutions. For more results about (1.2) (or similar nonlocal model), we can refer to [1, 14, 15, 16, 23, 24, 29, 33, 34] and the references therein.

However, the advantage of individuals from local aggregation and competition of individuals for space or resources still does not embody in model (1.2). Gourley [19] considered model (1.1) which would be more realistic (in fact, the model (1.1) was first mention in Britton [7, 8], however the research about (1.1) was first introduced in [19]). In equation (1.1), the term αu denotes the advantage to individuals from local aggregation, the term $-\beta u^2$ indicates the competition for space and the nonlocal term $-(1+\alpha-\beta)\phi * u$ represents the competition for food resources. For more biological interpretation can be seen in [10, 11, 32, 30]. The earlier researches about traveling wave solutions of equation (1.1) only consider the kernel function with special form, for example the kernel with special form $\phi(x) = \frac{\lambda}{2}e^{-\lambda|x|}$. Specially, Gourley et al. [19] studied the weakly nonlocal case (i.e. λ is sufficiently large). By using an asymptotic explanation, they showed that equation (1.1) exists traveling wave solutions connecting the two uniform steady states and found this wave have a 'hump', which is differences with the wave of the classical reaction-diffusion equation. Further, through the stability analysis and numerical simulation, they verified that equation (1.1) indeed exists this kind of wave. The case of the strongly nonlocal was researched by Billingham [6] (i.e. λ is sufficiently small). By using numerical and asymptotic method, they showed that in different, well-defined regions of parameter space, periodic traveling waves, unsteady traveling waves and steady traveling waves develop from localized initial conditions. And they also presented that equation (1.1) locally exists the traveling wave solutions with speed $c < 2$ when λ is sufficiently large.

Recently, we considered traveling wave solutions of (1.1) whose kernel is without limit (strong or weak), and the special kernel in [21]. We proved that (1.1) has traveling wave solutions connecting 0 to an unknown positive steady state. We also showed that this unknown steady state may be 1 or a periodic steady state. In this article, we continue to study the properties of the solutions for (1.1) and try to find some other types of traveling wave solutions. We will continue to study the existence of periodic steady state for equation (1.1). Then, we will prove that (1.1) has periodic traveling wave solution connecting this periodic steady state to the uniform steady $u = 1$ (which is different from the one in [21]). We will also present the change of the wave profile as α increased by numerical simulations. Note that the initial condition here is different from the condition in [21]. In fact, the solutions of (1.1) have a great relationship with the initial conditions. Also we will study the asymptotic behavior of the Cauchy problem corresponding to the (1.1), with a non-negative initial condition $u_0 \in L^\infty(\mathbb{R})$. It should be pointed out that the periodic steady state we are considering here is quite different from that in [21] (the causes are different and the amplitudes are different). In [21], it is considered that the periodic steady state is caused by the natural growth rate μ. And we mainly consider it caused by α and β. In addition, to clearly describe the effect of the initial conditions on solution of (1.1), and the asymptotic spreading speed, we present numerical simulations. Now we state our main results.

Theorem 1.1. (i) *For a special kernel function $\phi(x) = \frac{3a}{2}e^{-a|x|} - e^{-|x|}$, if $\alpha = \alpha_T$, then the Turing bifurcation will occur in system* (2.2) *around the unique positive equilibrium at the critical wave number σ_T, where σ_T and α_T are defined in* (2.6).

(ii) *For a general kernel function. Assume $\phi(x)$ satisfies (A1) and there exist $\alpha_c > 0, \beta_c > 0$ and $\sigma_c > 0$ such that $\alpha_c, \beta_c, \sigma_c$ satisfies*
 (a) $f(0, \sigma_c, \alpha_c, \beta_c) = 0$.
 (b) $\partial_\sigma f(0, \sigma_c, \alpha_c, \beta_c) = 0$.
 (c) $\partial_{\sigma\sigma} f(0, \sigma_c, \alpha_c, \beta_c) < 0$.
 (d) $-\sigma_c^2 + (\alpha_c - 2\beta_c) < 0$.

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and

$$\delta^2 < -\frac{\widehat{\phi}(\sigma_c)}{2 + (1 + \alpha_c - \beta_c)\widehat{\phi}''(\sigma_c)}\varepsilon^2,$$

equation (1.1) *has a periodic stationary solution of the form*

$$u_{\varepsilon,\delta}(x) = 1 + \sqrt{\frac{\frac{\widehat{\phi}(\sigma_c)}{2}\varepsilon^2 + (1 + \frac{1+\alpha_c-\beta_c}{2}\widehat{\phi}''(\sigma_c))\delta^2}{\varsigma}}\cos((\sigma_c + \delta)x) + O(|\varepsilon^2 - \delta^2|),$$

where

$$f(\lambda, \sigma, \alpha, \beta) = -\sigma^2 - (1 + \alpha - \beta)\widehat{\phi}(\sigma) + (\alpha - 2\beta) - \lambda, \tag{1.3}$$

and λ, σ are given in (2.4), *$\varsigma < 0$ is defined in* (2.10) *below.*

Obviously, there exist kernel functions, for example, $\phi(x) = \frac{3}{2}ae^{-a|x|} - e^{-|x|}$, where $a \in (2/3, \sqrt{2/3})$, and $\phi(x) = e^{-\lambda|x|}/(2\lambda)$ with $\lambda > 0$.

Theorem 1.2. *Assume the kernel $\phi(x) = 3ae^{-a|x|}/2 - e^{-|x|}$ with $a \in (2/3, \sqrt{2/3})$ and the conditions in Theorem 1.1(ii) are satisfied. Then for all $s > s^*$, there exists a $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and*

$$\delta^2 < -\frac{\widehat{\phi}(\sigma_c)}{2 + (1 + \alpha_c - \beta_c)\widehat{\phi}''(\sigma_c)}\varepsilon^2,$$

equation (1.1) *admits traveling wave solutions of the form*

$$u(t, x) = U(x - \varepsilon st, x) = \sum_{n\in\mathbb{Z}} U_n(x - \varepsilon st, x)e^{-in(\sigma_c+\delta)x},$$

and satisfies the the boundary conditions

$$\lim_{\xi\to-\infty} U(\xi, x) = u_{\varepsilon,\delta}(x), \quad \lim_{\xi\to+\infty} U(\xi, x) = 1,$$

where

$$s^* = \sqrt{-2\widehat{\phi}(\sigma_c)(1 + \frac{1 + \alpha_c - \beta_c}{2}\widehat{\phi}''(\sigma_c))}$$

and U_n is defined by (3.3).

Theorem 1.3. *Let u be the solution of the Cauchy problem*

$$\begin{gathered} u_t = u_{xx} + u\{1 + \alpha u - \beta u^2 - (1 + \alpha - \beta)(\phi * u)\}, \quad t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad \text{on } \mathbb{R}, \end{gathered} \tag{1.4}$$

where $u_0 \in L^\infty(\mathbb{R})$ and $u_0 \not\equiv 0$. Then u has the following properties:

(i) $\liminf_{t\to+\infty} \Big(\min_{|x|\le ct} u(t, x)\Big) > 0$ *for all* $0 \le c \le 2$.

(ii) *If u_0 is compactly supported, then*

$$\lim_{t\to+\infty} \Big(\max_{|x|\ge ct} u(t, x)\Big) = 0 \quad \text{for all } c > c^*,$$

where c^* is the minimal speed of equation (4.12).

This article is organized as follows. In Section 2, we show the existence of stationary periodic solution, and prove complete the proof of Theorem 1.1. In Section 3, we prove the existence of periodic traveling wave solutions connecting the stationary periodic state to the steady state $u = 1$ for (1.1); that is prove Theorem 1.2. In Section 4 we study the asymptotic spreading speed of the Cauchy problem (1.4); that is we prove Theorem 1.3.

2. Existence of a periodic steady state

In this section, we show the existence of stationary periodic solutions around the steady state $u = 1$. Specially, in subsection 2.1, for a special kernel, we study the bifurcation of the equation (1.1) could occur Turing bifurcation under some conditions (that is to say equation (1.1) exist a periodic steady state, thus we proved (i) of Theorem 1.1). In subsection 2.2, we prove that equation (1.1) has stationary periodic solutions for a general kernel; thus we complete the proof of Theorem 1.1 part (ii).

2.1. **Special kernel.** To discuss the bifurcation of the equation (1.1), we take the kernel with a special form,

$$\phi(x) = \frac{3a}{2} e^{-a|x|} - e^{-|x|}, \tag{2.1}$$

where $a \in (\frac{2}{3}, \sqrt{2/3})$. We define

$$v(t, x) := \big(\frac{3a}{2} e^{-a|x|} * u\big)(t, x) \quad \text{and} \quad w(t, x) := \big(- e^{-|x|} * u\big)(t, x)\,.$$

Then equation (1.1) can be replaced by

$$\begin{aligned} u_t &= u_{xx} + u(1 + \alpha u - \beta u^2 - (1 + \alpha - \beta)(v + w)), \\ 0 &= v_{xx} - a^2 v + 3a^2 u, \\ 0 &= w_{xx} - w - 2u, \end{aligned} \tag{2.2}$$

which has three equilibria $(0, 0, 0)$, $(-1/\beta, -3/\beta, 2/\beta)$ and $(1, 3, -2)$. We are mainly interested in the third equilibrium point from the biological point of view. Next, we analyze system (2.2) to obtain the behavior of equation (1.1).

Now, linearizing system (2.2) near the point $(1, 3, -2)$, then we obtain the linear system

$$\begin{aligned} \widetilde{u}_t &= \widetilde{u}_{xx} + (\alpha - 2\beta)\widetilde{u} - (1 + \alpha - \beta)(\widetilde{v} + \widetilde{w}), \\ 0 &= \widetilde{v}_{xx} - a^2\widetilde{v} + 3a^2\widetilde{u}, \\ 0 &= \widetilde{w}_{xx} - \widetilde{w} - 2\widetilde{u}. \end{aligned} \tag{2.3}$$

Taking the test function of the form

$$\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \\ \widetilde{w} \end{pmatrix} = \begin{pmatrix} C_\sigma^1 \\ C_\sigma^2 \\ C_\sigma^3 \end{pmatrix} e^{\lambda t + i\sigma x} \tag{2.4}$$

and substituting (2.4) into system (2.3), we obtain the characteristic equation for λ,

$$\begin{vmatrix} \alpha - 2\beta - \sigma^2 - \lambda & -(1 + \alpha - \beta) & -(1 + \alpha - \beta) \\ 3a^2 & -a^2 - \sigma^2 & 0 \\ -2 & 0 & -1 - \sigma^2 \end{vmatrix} = 0,$$

which is equivalent to

$$\lambda = \frac{(1 + \alpha - \beta)\left(-a^2 + (-3a^2 + 2)\sigma^2\right)}{(a^2 + \sigma^2)(1 + \sigma^2)} - \alpha + 2\beta + \sigma^2. \tag{2.5}$$

Following [9, 19, 22, 25, 32, 35], we search for the Hopf bifurcation and the Turing bifurcation of system (2.2). For the spatially homogeneous Hopf bifurcation, we know that it occurs when $\mathrm{Im}(\lambda) \neq 0$, $\mathrm{Re}(\lambda) = 0$ at $\sigma = 0$ of equation (2.5), while $\lambda \in \mathbb{R}$ for any α, β, σ, so the Hopf bifurcation cannot occur in system (2.2).

Next, we consider the spatially homogeneous Turing bifurcation, that is to prove (i) of Theorem 1.1.

Proof Theorem 1.1(1). It is known that when $\mathrm{Im}(\lambda) = 0$ and $\mathrm{Re}(\lambda) = 0$ at $\sigma = \sigma_T$, system (2.3) will occur Turing bifurcation. For this purpose, let $\lambda = 0$, which implies

$$\alpha = \frac{(2\beta + \sigma^2)(a^2 + \sigma^2)(1 + \sigma^2) - (1 - \beta)(-a^2 + (-3a^2 + 2)\sigma^2)}{(a^2 + \sigma^2)(1 + \sigma^2) + [-a^2 + (-3a^2 + 2)\sigma^2]}.$$

Since

$$\lim_{\sigma \to 0^+} \alpha(\sigma) = \lim_{\sigma \to +\infty} \alpha(\sigma) = +\infty,$$

then there exists a $\sigma_T \in (0, +\infty)$ such hat

$$\alpha_T := \alpha(\sigma_T) = \min_{\sigma \in (0, +\infty)} \alpha(\sigma). \tag{2.6}$$

Thus, when $\sigma = \sigma_T$ and $\alpha = \alpha_T$, the uniform steady state $(1, 3, -2)$ lose stability and has a new, non-uniformly steady state with the $\exp(i\sigma_c x)$-like spatial structure, i.e. system (2.2) has a Turing bifurcation at the critical wave number σ_T. The bifurcation here is caused by α; of course we can also consider other factors (such as β, σ or α and β working together, see Figure 1. This completes the proof. $\square$

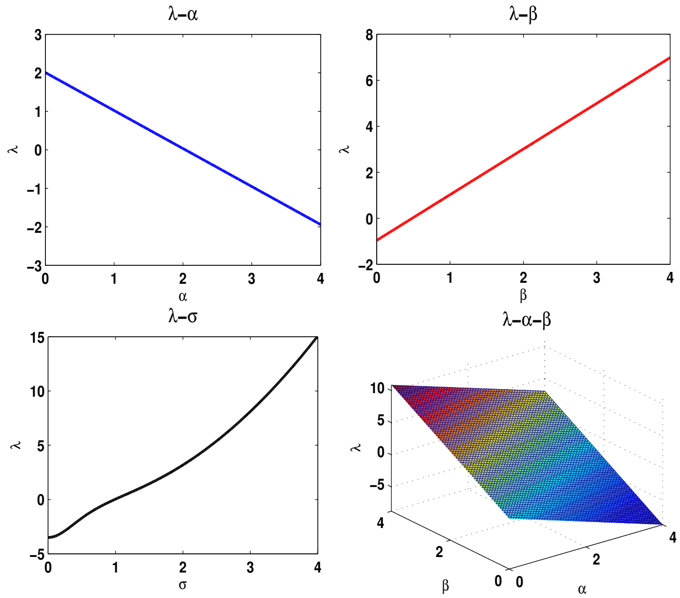

FIGURE 1. Bifurcation diagram for (2.5). The top left panel shows the relationship between λ and α for coefficients $\beta = 0.5$, $a = 0.7$, and $\sigma = 1$. The top right panel shows the relationship between λ and β for coefficients $\alpha = 0.2$, $a = 0.7$, $\sigma = 1$. The bottom left panel shows the relationship between λ and σ for coefficients $\alpha = 2$, $\beta = 0.5$, $a = 0.7$; The bottom right panel shows the relationship between λ and α, β, for coefficients $\sigma = 1$, $a = 0.7$.

2.2. **General kernel.** In subsection 2.1, for a special kernel function, it is shown that (1.1) can have Turing bifurcation under certain conditions. In this subsection, we prove the existence of periodic stationary solutions around the steady state $u = 1$ for the general kernel function. First set $u(t,x) = 1 + v(t, \sigma x)$, then (1.1) can be written as

$$v_t = \sigma^2 v_{xx} + (\alpha - 2\beta)v - (1+\alpha-\beta)(\phi_\sigma * v) + (\alpha - 3\beta)v^2 - (1+\alpha-\beta)v(\phi_\sigma * v) - \beta v^3,$$

where $\phi_\sigma(x) = \frac{1}{\sigma}\phi(\frac{x}{\sigma})$ and v is 2π-periodic in x. We define $\sigma := \sigma_c + \delta$, $\alpha := \alpha_c + \varepsilon^2$, $\beta := \beta_c + \frac{\varepsilon^2}{2}$ with $0 < \varepsilon \ll 1$, $0 < \delta \ll 1$ and

$$\begin{gathered}\mathcal{B}(\sigma,\alpha,\beta)v := \sigma^2 v_{xx} + (\alpha - 2\beta)v - (1+\alpha-\beta)(\phi_\sigma * v),\\ \mathcal{Q}(v,\sigma,\alpha,\beta) := (\alpha - 3\beta)v^2 - (1+\alpha-\beta)v(\phi_\sigma * v) - \beta v^3.\end{gathered}$$

Then we obtain

$$v_t = \mathcal{B}_c v + \mathcal{C}(\varepsilon,\delta)v + \mathcal{Q}\Big(v, \sigma_c + \delta, \alpha_c + \varepsilon^2, \beta_c + \frac{\varepsilon^2}{2}\Big), \tag{2.7}$$

where

$$\mathcal{B}_c = \mathcal{B}(\sigma_c,\alpha_c,\beta_c) \quad \text{and} \quad \mathcal{C}(\varepsilon,\delta) = \mathcal{B}(\sigma,\alpha,\beta) - \mathcal{B}_c.$$

We define

$$\mathcal{Y} := L^2_{per}[0,2\pi] = \{u \in L^2_{\rm loc}(\mathbb{R}) | u(x+2\pi) = u(x), \quad x \in \mathbb{R}\},$$

$$\mathcal{W} := \mathcal{D}(\mathcal{B}) = H^2_{per}[0,2\pi] = \{u \in H^2_{\rm loc}(\mathbb{R}) | u(x+2\pi) = u(x), \quad x \in \mathbb{R}\}.$$

Since the linear operator $\mathcal{B}_c : \mathcal{W} \to \mathcal{Y}$ is continuous and $\mathcal{W}$ is dense in $\mathcal{Y}$ and compactly embedded into $\mathcal{Y}$, the resolvent of $\mathcal{B}_c$ is compact, which implies the spectrum $\sigma(\mathcal{B}_c)$ only includes eigenvalues, λ. It follows from (1.3) that

$$\sigma(\mathcal{B}_c) = \big\{\lambda_l \in \mathbb{C} | \lambda_l = -\sigma_c^2 l^2 - (1+\alpha_c - \beta_c)\widehat{\phi}(l\sigma_c) + (\alpha_c - 2\beta_c),\ l \in \mathbb{Z}\big\}, \tag{2.8}$$

so $\sigma(\mathcal{B}_c) \cap i\mathbb{R} = \{0\}$, and the geometric multiplicity of $\lambda = 0$ is two, whose corresponding eigenvectors are $\mathbf{e}(x) = e^{ix}$ and $\bar{\mathbf{e}}(x) = e^{-ix}$. In addition, through calculation, we can find that the algebraic multiplicity of $\lambda = 0$ is also two. We define $\mathcal{Y}_c$ as the subspace expanded by $\mathbf{e}$ and $\bar{\mathbf{e}}$ (i.e. $\mathcal{Y}_c = \{\mathbf{e}, \bar{\mathbf{e}}\}$) and the spectral projection $\mathcal{G}_c : \mathcal{Y}_c \to \mathcal{Y}_c$ as

$$\mathcal{G}_c = \langle u, \mathbf{e}\rangle \mathbf{e} + \langle u, \bar{\mathbf{e}}\rangle \bar{\mathbf{e}},$$

where $\langle u, v\rangle = \frac{1}{2\pi}\int_0^{2\pi} u(x)\overline{v}(x)dx$. From (2.8), we deduce that

$$\|(i\nu - \mathcal{B}_c)^{-1}\|_{(id-\mathcal{G}_c)\mathcal{Y}} \le \frac{E}{1+|\nu|}, \quad \nu \in \mathbb{R},$$

with some positive constant $E > 0$. So, by using the center manifold theorem (see [26, 13]), we have the following conclusion.

Proposition 2.1. *There exists $\mathcal{U} \subset \mathcal{Y}_c$, $\mathcal{V} \subset (id - \mathcal{G}_c)\mathcal{W}$, $\mathcal{S} \in \mathbb{R}^2$, such that for each $n < \infty$, the C^n-map $\Phi : \mathcal{U} \times \mathcal{S} \to \mathcal{V}$ satisfy the following properties:*

(i) *For each $(\varepsilon, \delta) \in \mathcal{S}$, each bounded solution v of (2.7) satisfies*
$$v(t) = \mathbf{B}(t)\mathbf{e} + \overline{\mathbf{B}}(t)\bar{\mathbf{e}} + \Phi(\mathbf{B}(t), \overline{\mathbf{B}}(t), \varepsilon, \delta), \quad \forall t \in \mathbb{R},$$
where $\mathbf{B}(t)$ and $\overline{\mathbf{B}}(t)$ is a vector and only depends on t.

(ii) $\|\Phi(\mathbf{B}(t), \overline{\mathbf{B}}(t), \varepsilon, \delta)\|_{\mathcal{W}} = O(|\varepsilon|^2|\mathbf{B}| + |\delta||\mathbf{B}| + |\mathbf{B}|^2)$.

(iii) $\frac{d\mathbf{B}}{dt} = g(\mathbf{B}(t),\ \overline{\mathbf{B}}(t), \varepsilon, \delta) = \mathbf{B}h(|\mathbf{B}|^2, \varepsilon, \delta)$, *where h is real-valued and a $C^{n-1}-$ map in $(\mathbf{B}(t), \overline{\mathbf{B}}(t), \varepsilon, \delta)$.*

Lemma 2.2. *The map h has the form*

$$\begin{aligned} h(|\mathbf{B}|^2, \varepsilon, \delta) = &-\frac{\widehat{\phi}(\sigma_c)}{2}\varepsilon^2 - \Big(1 + \frac{1+\alpha_c-\beta_c}{2}\widehat{\phi}''(\sigma_c)\Big)\delta^2 \\ &+ \varsigma|\mathbf{B}|^2 + O(|\delta|^3 + |\varepsilon|^2|\delta| + |\mathbf{B}|^4), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} \varsigma := &-\frac{2(1+\alpha_c-\beta_c)\widehat{\phi}(\sigma_c) - 2(\alpha_c - 3\beta_c)}{1+\beta_c}\Big(\alpha_c - 1 - 5\beta_c + \widehat{\phi}(\sigma_c)\Big) \\ &+ \frac{\alpha_c - 3\beta_c - (1+\alpha_c-\beta_c)\widehat{\phi}(\sigma_c)}{4\sigma_c^2 - (\alpha_c - 2\beta_c) + (1+\alpha_c-\beta_c)\widehat{\phi}(2\sigma_c)} \\ &\times \Big(2(\alpha_c - 3\beta_c) - (1+\alpha_c-\beta_c)\widehat{\phi}(2\sigma_c) + \widehat{\phi}(\sigma_c)\Big) - 3\beta_c < 0, \end{aligned} \tag{2.10}$$

and $\widehat{\phi}$ is the Fourier transform of ϕ.

The proof of the above lemma is similar to that of [13, Lemma 2.1], see Appendix A for details.

Proof of Theorem 1.1(ii). We define

$$\Lambda := \frac{\frac{\widehat{\phi}(\sigma_c)}{2}\varepsilon^2 + \left(1 + \frac{1+\alpha_c-\beta_c}{2}\widehat{\phi}''(\sigma_c)\right)\delta^2}{\varsigma} > 0.$$

Our goals is to find the nontrivial stationary solution $\mathbf{B}_0 \in \mathbb{C}$ that satisfies

$$h(|\mathbf{B}|^2, \varepsilon, \delta) = 0. \tag{2.11}$$

Up to a rescaling $\mathbf{B}_0 = \sqrt{\Lambda}\widetilde{\mathbf{B}}_0$, (2.11) can be written as

$$\Lambda \cdot \left(-\varsigma + \varsigma|\widetilde{\mathbf{B}}_0|^2 + O(\sqrt{\Lambda})\right), \quad \text{as } \Lambda \to 0.$$

It follows from the implicit function theorem that the solutions have the form

$$|\widetilde{\mathbf{B}}_0| = 1 + O(\sqrt{\Lambda}), \quad \text{as } \Lambda \to 0.$$

Thus

$$v_{\varepsilon,\delta}(x) = \sqrt{\Lambda}\cos((\sigma_c+\delta)x) + O\Big(\Big|\frac{\widehat{\phi}(\sigma_c)}{2}\varepsilon^2 + \Big(1 + \frac{1+\alpha_c-\beta_c}{2}\widehat{\phi}''(\sigma_c)\Big)\delta^2\Big|\Big),$$

for some $\varepsilon \in (0, \varepsilon_0]$ and δ satisfies

$$\Big(1 + \frac{1+\alpha_c-\beta_c}{2}\widehat{\phi}''(\sigma_c)\Big)\delta^2 < -\frac{\widehat{\phi}(\sigma_c)}{2}\varepsilon^2.$$

Further, we know that equation (1.1) has the periodic solution of the form

$$u_{\varepsilon,\delta}(x) = 1 + \sqrt{\Lambda}\cos((\sigma_c+\delta)x) + O\Big(\Big|\frac{\widehat{\phi}(\sigma_c)}{2}\varepsilon^2 + \Big(1 + \frac{1+\alpha_c-\beta_c}{2}\widehat{\phi}''(\sigma_c)\Big)\delta^2\Big|\Big)$$

for some $\varepsilon \in (0, \varepsilon_0]$ and $\big(1 + \frac{1+\alpha_c-\beta_c}{2}\widehat{\phi}''(\sigma_c)\big)\delta^2 < -\frac{\widehat{\phi}(\sigma_c)}{2}\varepsilon^2$. This completes the proof. $\square$

3. Traveling wave solutions connecting 1 to a periodic steady state

In the previous section, we show that equation (1.1) has periodic steady around $u = 1$, in this section, we prove that (1.1) has traveling wave solutions connecting this periodic steady to the uniform steady $u = 1$. Specifically, in subsection 3.1, by using center manifold reduction (similar to [13]), we prove the existence of this traveling wave solution. Then, we study the dynamic behavior of the solution for equation (1.1) through numerical Simulation in subsection 3.2.

3.1. **Existence of traveling wave solutions.** In this subsection, we consider the kernel function with the special form (2.1) and study the corresponding system (2.2). Substituting $u = 1 + \widetilde{u}$, $v = 3 + \widetilde{v}$, $w = -2 + \widetilde{w}$ into (2.2) (in the case of no confusion, we still write as u, v, w), we obtain

$$\begin{aligned} u_t = u_{xx} + (\alpha - 2\beta)u - (1+\alpha-\beta)(v+w) + (\alpha - 3\beta)u^2 \\ - (1+\alpha-\beta)u(v+w) - \beta u^3, \\ 0 = v_{xx} - a^2 v + 3a^2 u, \\ 0 = w_{xx} - w - 2u. \end{aligned} \tag{3.1}$$

Let $\varepsilon^2 = \alpha - \alpha_c = 2(\beta - \beta_c)$ and $U = (u, v, w)^T$. Then (2.7) can be written as

$$DU_t = U_{xx} + K_c U + \varepsilon^2 K_r U - \Big(1 + \alpha_c - \beta_c + \frac{\varepsilon^2}{2}\Big)L(U) - \Big(\beta_c + \frac{\varepsilon^2}{2}\Big)J(U), \tag{3.2}$$

where $D(1,1) = 1$ and the other items are zero, and

$$K_c = \begin{pmatrix} \alpha_c - 2\beta_c & -(1+\alpha_c-\beta_c) & -(1+\alpha_c-\beta_c) \\ 3a^2 & -a^2 & 0 \\ -2 & 0 & -1 \end{pmatrix},$$

$$K_r = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L(U) = \begin{pmatrix} u(v+w) \\ 0 \\ 0 \end{pmatrix}, \quad J(U) = \begin{pmatrix} u^3 \\ 0 \\ 0 \end{pmatrix}.$$

Next, we find a solution of (3.2) of the form

$$U(t,x) = V(x-\varepsilon st, x) = \Sigma_{n\in\mathbb{Z}} V_n(x-\varepsilon st) e^{-in\sigma_c x}. \tag{3.3}$$

Let $\zeta = x - \varepsilon st$, plugging this into (3.2), we obtain

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\varepsilon s \partial_\zeta V_n^u \\ -\varepsilon s \partial_\zeta V_n^v \\ -\varepsilon s \partial_\zeta V_n^w \end{pmatrix} = \begin{pmatrix} \partial_{\zeta\zeta} V_n^u - 2in\sigma_c \partial_\zeta V_n^u - n^2\sigma_c^2 V_n^u \\ \partial_{\zeta\zeta} V_n^v - 2in\sigma_c \partial_\zeta V_n^v - n^2\sigma_c^2 V_n^v \\ \partial_{\zeta\zeta} V_n^w - 2in\sigma_c \partial_\zeta V_n^w - n^2\sigma_c^2 V_n^w \end{pmatrix} \\ &+ \begin{pmatrix} \alpha_c - 2\beta_c & -\iota & -\iota \\ 3a^2 & -a^2 & 0 \\ -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} V_n^u \\ V_n^v \\ V_n^w \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_n^u \\ V_n^v \\ V_n^w \end{pmatrix} \\ &- \Big(\iota + \frac{\varepsilon^2}{2}\Big) \begin{pmatrix} \Sigma_{p+q=n} V_p^u V_q^v + V_p^u V_q^w \\ 0 \\ 0 \end{pmatrix} - \Big(\beta_c + \frac{\varepsilon^2}{2}\Big) \begin{pmatrix} \Sigma_{p+q+r=n} V_p^u V_q^u V_r^u \\ 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{3.4}$$

where $\iota = 1 + \alpha_c - \beta_c$. Further, let $X := (X_n)_{n\in\mathbb{Z}}$ and

$$X_n := (V_n^u, \partial_\zeta V_n^u, V_n^v, \partial_\zeta V_n^v, V_n^w, \partial_\zeta V_n^w)^T,$$

then (2.9) can be simplified to

$$\partial_\zeta X_n = N_n^\varepsilon X_n - \Big(1 + \alpha_c - \beta_c + \frac{\varepsilon^2}{2}\Big) L_n(X) - \Big(\beta_c + \frac{\varepsilon^2}{2}\Big) J_n(X), \quad n \in \mathbb{Z}, \tag{3.5}$$

where

$$N_n^\varepsilon = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ n^2\sigma_c^2 - (\alpha_c - 2\beta_c) & 2in\sigma_c - \varepsilon s & \iota + \frac{\varepsilon^2}{2} & 0 & \iota + \frac{\varepsilon^2}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -3a^2 & 0 & a^2 + n^2\sigma_c^2 & 2in\sigma_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 + n^2\sigma_c^2 & 2in\sigma_c \end{pmatrix}$$

and

$$L_n(X) = \begin{pmatrix} 0 \\ \Sigma_{p+q=n} V_p^u V_q^v + V_p^u V_q^w \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad J_n(X) = \begin{pmatrix} 0 \\ \Sigma_{p+q+r=n} V_p^u V_q^v V_r^w \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

By using a center manifold reduction (see Appendix C), we obtain

$$\partial_\zeta X = \mathcal{N}^\varepsilon X - \Big(1 + \alpha_c - \beta_c + \frac{\varepsilon^2}{2}\Big) \mathcal{L}(X) - \Big(\beta_c + \frac{\varepsilon^2}{2}\Big) \mathcal{J}(X), \quad X \in \mathcal{E}, \tag{3.6}$$

where $(\mathcal{N}^\varepsilon X)_n = (N_n^\varepsilon)_n$, $(\mathcal{L}(X))_n = L_n(X)$ and $(\mathcal{J}(X))_n = J_n(X)$ for $n \in \mathbb{N}$. It is easy to see that there are only two center directions corresponding to the eigenvalues $\lambda_\pm^\varepsilon$ ($\lambda_\pm^\varepsilon := \lambda_{-,1}^{\varepsilon,\pm}$) of N_1^ε (see Appendix B). Define $\varphi_\pm^\varepsilon$ as the corresponding eigenvectors of $\lambda_\pm^\varepsilon$, i.e.

$$N_1^\varepsilon \varphi_\pm^\varepsilon = \lambda_\pm^\varepsilon \varphi_\pm^\varepsilon.$$

Similarly, $\psi_\pm^\varepsilon \in \mathbb{C}^6$ are denoted as the corresponding eigenvectors of the eigenvalue $\overline{\lambda}_\pm^\varepsilon$ (which are the eigenvectors of the adjoint matrix $(N_1^\varepsilon)^*$, i.e.

$$(N_1^\varepsilon)^* \psi_\pm^\varepsilon = \overline{\lambda}_\pm^\varepsilon \psi_\pm^\varepsilon.$$

By calculations, it is easy to check that

$$\langle \psi_\pm^\varepsilon, \varphi_\pm^\varepsilon \rangle = \pm \frac{\sqrt{\vartheta_1^2 - 4\vartheta_0\vartheta_2}}{\vartheta_2}\varepsilon \quad \text{and} \quad \langle \varphi_\pm^\varepsilon, \psi_\pm^\varepsilon \rangle = 0.$$

We define $\Phi_\pm^\varepsilon, \Psi_\pm^\varepsilon \in \mathcal{E}$ as

$$\begin{aligned} &(\Phi_\pm^\varepsilon)_1 = \phi_\pm^\varepsilon, \quad (\Phi_\pm^\varepsilon)_n = 0_{\mathbb{C}^6}, \quad n \neq 1; \\ &(\Psi_\pm^\varepsilon)_1 = \psi_\pm^\varepsilon, \quad (\Psi_\pm^\varepsilon)_n = 0_{\mathbb{C}^6}, \quad n \neq 1. \end{aligned}$$

Now, we define the spectral projection

$$\mathcal{G}_c^\varepsilon = c_+^\varepsilon \langle \Psi_+^\varepsilon, X \rangle_l \Phi_+^\varepsilon + c_-^\varepsilon \langle \Psi_-^\varepsilon, X \rangle_l \Phi_-^\varepsilon,$$

where

$$c_\pm^\varepsilon = \frac{\pm\vartheta_2}{2^l \varepsilon \sqrt{\vartheta_1^2 - 4\vartheta_0\vartheta_2}} - O(1), \quad \text{as } \varepsilon \to 0.$$

Obviously, there exists a constant $d_1 > 0$, such that

$$\sigma(N^\varepsilon|_{(id - \mathcal{G}_c^\varepsilon)\mathcal{E}}) \subset \left\{ \lambda \in \mathbb{C} || \mathcal{R}(\lambda)| \geq d_1 \sqrt{\varepsilon} \right\}.$$

Then the center manifold theorem [27, 12] can be used to obtain the following result.

Proposition 3.1. *For each $0 < r < 1/3$ and sufficiently small $\varepsilon > 0$, there exist $\mathcal{U}^\varepsilon \subset \mathcal{E}_c$, $\mathcal{V}^\varepsilon \subset (id - \mathcal{G}_c^\varepsilon)\mathcal{E}$ and a C^m-map $\Upsilon^\varepsilon : \mathcal{U}^\varepsilon \to \mathcal{V}^\varepsilon$ (for any $m < \infty$), such that the following properties hold:*

(i) *All bounded solutions of* (3.6) *satisfy $X = X_c + \Upsilon^\varepsilon(X_c)$;*
(ii) *$\|\Upsilon^\varepsilon(X_c)\|_l = O_\varepsilon(\|X_c\|_l^2)$;*
(iii) *The neighborhood $\mathcal{U}^\varepsilon$ is of size $O\big(\varepsilon^{\frac{2}{3}+r}\big)$.*

Next, by projecting equation (3.6) with $\mathcal{G}_c^\varepsilon$, we obtain

$$\begin{aligned} \frac{dX_c}{d\zeta} =& \mathcal{N}^\varepsilon X_c - \Big(1 + \alpha_c - \beta_c + \frac{\varepsilon^2}{2}\Big) \mathcal{G}_c^\varepsilon(\mathcal{L}(X_c + \Upsilon^\varepsilon(X_c))) \\ &- \Big(\beta_c + \frac{\varepsilon^2}{2}\Big) \mathcal{G}_c^\varepsilon(\mathcal{J}(X_c + \Upsilon^\varepsilon(X_c))). \end{aligned}$$

Let $X_c = x_+ \Phi_+^\varepsilon + x_- \Phi_-^\varepsilon$ on the center manifold, then

$$X = x_+ \Phi_+^\varepsilon + x_- \Phi_-^\varepsilon + \Upsilon^\varepsilon(x_+, x_-),$$

where

$$\Upsilon^\varepsilon(x_+, x_-) = \Sigma_{|\mathbf{n}|=2} x_+^{n_1} x_-^{n_2} \overline{x}_+^{n_3} \overline{x}_-^{n_4} \Phi_{\mathbf{n}}^\varepsilon + O_\varepsilon(|x_+ + x_-|^3).$$

Furthermore, we obtain

$$
\begin{aligned}
\frac{dx_+}{d\zeta} &= \lambda_+^\varepsilon x_+ - (1+\alpha_c-\beta_c+\frac{\varepsilon^2}{2})c_+^\varepsilon\langle\psi_+^\varepsilon,\mathcal{L}(x_+\Phi_+^\varepsilon+x_-\Phi_-^\varepsilon+\Upsilon^\varepsilon(x_+,x_-))\rangle_l\\
&\quad -(\beta_c+\frac{\varepsilon^2}{2})c_+^\varepsilon\langle\psi_+^\varepsilon,\mathcal{J}(x_+\Phi_+^\varepsilon+x_-\Phi_-^\varepsilon+\Upsilon^\varepsilon(x_+,x_-))\rangle_l,\\
\frac{dx_-}{d\zeta} &= \lambda_-^\varepsilon x_- - (1+\alpha_c-\beta_c+\frac{\varepsilon^2}{2})c_-^\varepsilon\langle\psi_-^\varepsilon,\mathcal{L}(x_+\Phi_+^\varepsilon+x_-\Phi_-^\varepsilon+\Upsilon^\varepsilon(x_+,x_-))\rangle_l\\
&\quad -(\beta_c+\frac{\varepsilon^2}{2})c_-^\varepsilon\langle\psi_-^\varepsilon,\mathcal{J}(x_+\Phi_+^\varepsilon+x_-\Phi_-^\varepsilon+\Upsilon^\varepsilon(x_+,x_-))\rangle_l,
\end{aligned}
\tag{3.7}
$$

In addition, we have

$$
\begin{aligned}
D_X\Upsilon^\varepsilon(X_c)\frac{dX_c}{d\zeta} =&\mathcal{N}^\varepsilon\Upsilon^\varepsilon(X_c)-\Big(1+\alpha_c-\beta_c+\frac{\varepsilon^2}{2}\Big)(\mathcal{G}_c^\varepsilon)^\perp(\mathcal{L}(X_c+\Upsilon^\varepsilon(X_c)))\\
&-\Big(\beta_c+\frac{\varepsilon^2}{2}\Big)(\mathcal{G}_c^\varepsilon)^\perp(\mathcal{J}(X_c+\Upsilon^\varepsilon(X_c))),
\end{aligned}
$$

where $(\mathcal{G}_c^\varepsilon)^\perp := id-\mathcal{G}_c^\varepsilon$. It follows from the definition of $\Psi_\pm^\varepsilon$, $\mathcal{N}$ and $\mathcal{J}$, that

$\langle\Psi_\pm^\varepsilon,\mathcal{L}(x_+\Phi_+^\varepsilon+x_-\Phi_-^\varepsilon+\Upsilon^\varepsilon(x_+,x_-))\rangle_l = 2^l\langle\psi_\pm^\varepsilon,\mathcal{L}_1(x_+\Phi_+^\varepsilon+x_-\Phi_-^\varepsilon+\Upsilon^\varepsilon(x_+,x_-))\rangle$

and

$\langle\Psi_\pm^\varepsilon,\mathcal{J}(x_+\Phi_+^\varepsilon+x_-\Phi_-^\varepsilon+\Upsilon^\varepsilon(x_+,x_-))\rangle_l = 2^l\langle\psi_\pm^\varepsilon,\mathcal{J}_1(x_+\Phi_+^\varepsilon+x_-\Phi_-^\varepsilon+\Upsilon^\varepsilon(x_+,x_-))\rangle.$

From the form of $\mathcal{L}_1(X)$, $\mathcal{J}_1(X)$ and Υ^ε, we obtain

$$
\begin{aligned}
\mathcal{L}_1(x_+\Phi_+^\varepsilon+x_-\Phi_-^\varepsilon+\Upsilon^\varepsilon(x_+,x_-)) = O_\varepsilon(|x_++x_-|^3),\\
\mathcal{J}_1(x_+\Phi_+^\varepsilon+x_-\Phi_-^\varepsilon+\Upsilon^\varepsilon(x_+,x_-)) = O_\varepsilon(|x_++x_-|^3).
\end{aligned}
$$

Because for each sub-system, namely the ones left after linearization is invariant, we can work on each mode. Let Υ_n^ε be the n-th mode of Υ^ε and $x_c := x_+\phi_+^\varepsilon+x_-\phi_-^\varepsilon$, then (3.7) can be written as

$$
\begin{aligned}
D_x\Upsilon_n^\varepsilon(X_c)\frac{dx_c}{d\zeta} =&\mathcal{N}^\varepsilon\Upsilon^\varepsilon(x_c)-\Big(1+\alpha_c-\beta_c+\frac{\varepsilon^2}{2}\Big)(\mathcal{G}_c^\varepsilon)^\perp(\mathcal{L}(X_c+\Upsilon^\varepsilon(X_c)))\\
&-\Big(\beta_c+\frac{\varepsilon^2}{2}\Big)(\mathcal{G}_c^\varepsilon)^\perp(\mathcal{J}(X_c+\Upsilon^\varepsilon(X_c))).
\end{aligned}
\tag{3.8}
$$

Next we compute the value of $\Upsilon_n^\varepsilon(X_c)(n=0,1,2)$. When $n=0$, we assume that

$$
\Upsilon_0^\varepsilon(x_+,x_-) = \Sigma_{|\mathbf{n}|=2}x_+^{n_1}x_-^{n_2}\overline{x}_+^{n_3}\overline{x}_-^{n_4}\gamma_{\mathbf{n}}^0(\varepsilon)+O_\varepsilon(|x_++x_-|^3),\quad \gamma_{\mathbf{n}}^0(\varepsilon)\in\mathbb{C}^6. \tag{3.9}
$$

Substituting (3.9) into (3.8), we obtain

$$
\Xi_n\gamma_{\mathbf{n}}^0(\varepsilon) = \mathcal{N}_0^\varepsilon\gamma_{\mathbf{n}}^0(\varepsilon)-(1+\alpha_c-\beta_c+\frac{\varepsilon^2}{2})\mathcal{L}_{0,n}^\varepsilon-(\beta_c+\frac{\varepsilon^2}{2})\mathcal{J}_{0,n}^\varepsilon,
$$

where

$$
\begin{gathered}
\Xi_n = \Sigma_{j=1}^4\lambda_jn_j,\quad \lambda_{1,2}=\lambda_\pm^\varepsilon,\quad \lambda_{3,4}=\overline{\lambda}_\pm^\varepsilon,\\
L_{0,n}^\varepsilon = (0,\mathbf{L}_{0,n},0,0,0,0)^T,\quad J_{0,n}^\varepsilon = (0,\mathbf{J}_{0,n},0,0,0,0)^T.
\end{gathered}
$$

Therefore,

$$
\gamma_{\mathbf{n}}^0(\varepsilon) =\Big(1+\alpha_c-\beta_c+\frac{\varepsilon^2}{2}\Big)\left(\mathcal{N}_0^\varepsilon-\Xi_nid\right)^{-1}\mathcal{L}_{0,n}^\varepsilon+\Big(\beta_c+\frac{\varepsilon^2}{2}\Big)\mathcal{J}_{0,n}^\varepsilon
$$

$$= (1+\alpha_c-\beta_c)\,(\mathcal{N}_0^0)^{-1}\mathcal{L}_{0,n}^0 + \beta_c(N_0^0)^{-1}\mathcal{J}_{0,n}^0 + O(\varepsilon).$$

When $n=1$, by using the same methods, we obtain

$$\begin{aligned}\gamma_{\mathbf{n}}^1(\varepsilon) =&\Big(1+\alpha_c-\beta_c+\frac{\varepsilon^2}{2}\Big)(\mathcal{N}_1^\varepsilon-\Xi_n id)^{-1}\mathcal{L}_{1,n}^\varepsilon + \Big(\beta_c+\frac{\varepsilon^2}{2}\Big)\mathcal{J}_{1,n}^\varepsilon\\ =&(1+\alpha_c-\beta_c)(\mathcal{N}_1^0)^{-1}\mathcal{L}_{1,n}^0 + \beta_c(\mathcal{N}_1^0)^{-1}\mathcal{J}_{1,n}^0 + O(\varepsilon).\end{aligned}$$

Similarly, when $n=2$, we have

$$\begin{aligned}\gamma_{\mathbf{n}}^2(\varepsilon) =&\Big(1+\alpha_c-\beta_c+\frac{\varepsilon^2}{2}\Big)(\mathcal{N}_2^\varepsilon-\Xi_n id)^{-1}\mathcal{L}_{2,n}^\varepsilon + \left(\beta_c+\frac{\varepsilon^2}{2}\right)\mathcal{J}_{2,n}^\varepsilon\\ =&(1+\alpha_c-\beta_c)(\mathcal{N}_2^0)^{-1}\mathcal{L}_{2,n}^0 + \beta_c(\mathcal{N}_1^0)^{-1}\mathcal{J}_{2,n}^0 + O(\varepsilon).\end{aligned}$$

It is easy to see that, in the inner product

$$\langle\psi_\pm^\varepsilon, \mathcal{L}_1(x_+\Phi_+^\varepsilon + x_-\Phi_-^\varepsilon + \Upsilon^\varepsilon(x_+,x_-))\rangle, \text{ as } \varepsilon\to 0.$$

About the cubic order, we see that

$$\begin{aligned}&\Big(1+\alpha_c-\beta_c+\frac{\varepsilon^2}{2}\Big)c_+^\varepsilon\langle\psi_+^\varepsilon, \mathcal{L}(x_+\Phi_+^\varepsilon + x_-\Phi_-^\varepsilon + \Upsilon^\varepsilon(x_+,x_-))\rangle_l\\ &+\Big(\beta_c+\frac{\varepsilon^2}{2}\Big)c_+^\varepsilon\langle\psi_+^\varepsilon, \mathcal{J}(x_+\Phi_+^\varepsilon + x_-\Phi_-^\varepsilon + \Upsilon^\varepsilon(x_+,x_-))\rangle_l\\ =&-\frac{\varsigma c_+^\varepsilon 2^l}{(1+\alpha_c-\beta_c)\varrho_0}(x_++x_-)|x_++x_-|^2 + O_\varepsilon(|x_++x_-|^4),\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}&\Big(1+\alpha_c-\beta_c+\frac{\varepsilon^2}{2}\Big)c_-^\varepsilon\langle\psi_-^\varepsilon, \mathcal{L}(x_+\Phi_+^\varepsilon + x_-\Phi_-^\varepsilon + \Upsilon^\varepsilon(x_+,x_-))\rangle_l\\ &+\Big(\beta_c+\frac{\varepsilon^2}{2}\Big)\langle\psi_-^\varepsilon, \mathcal{J}(x_+\Phi_+^\varepsilon + x_-\Phi_-^\varepsilon + \Upsilon^\varepsilon(x_+,x_-))\rangle_l\\ =&-\frac{\varsigma c_-^\varepsilon 2^l}{(1+\alpha_c-\beta_c)\varrho_0}(x_++x_-)|x_++x_-|^2 + O_\varepsilon(|x_++x_-|^4),\end{aligned}\tag{3.11}$$

where

$$\varrho_0 = -4\sigma_c^2(1+\sigma_c^2)\Big(\frac{3a^2}{(a^2+\sigma_c^2)^3} - \frac{2}{(1+\sigma_c^2)^3}\Big)$$

and ς is defined in (2.10). From (3.7), (3.10) and (3.11), we have

$$\begin{aligned}\frac{dx_+}{d\zeta} &= \lambda_+^\varepsilon x_+ + \frac{\varsigma c_+^\varepsilon 2^l(1+\sigma_c^2)}{(1+\alpha_c-\beta_c)\varrho_0}(x_++x_-)|x_++x_-|^2 + O_\varepsilon(|x_++x_-|^4),\\ \frac{dx_-}{d\zeta} &= \lambda_-^\varepsilon x_+ + \frac{\varsigma c_-^\varepsilon 2^l(1+\sigma_c^2)}{(1+\alpha_c-\beta_c)\varrho_0}(x_++x_-)|x_++x_-|^2 + O_\varepsilon(|x_++x_-|^4).\end{aligned}$$

Through the three variable substitutions (similar to [13]),

$$\begin{gathered}Y = x_+ + x_-, \quad Z = x_+ - x_-,\\ Y(t) = \varepsilon u(\varepsilon t), \quad Z(t) = \varepsilon v(\varepsilon t),\\ u = q, \quad v = \frac{\vartheta_1}{\sqrt{\vartheta_1^2-4\vartheta_0\vartheta_2}}q + \frac{2\vartheta_2}{\sqrt{\vartheta_1^2-4\vartheta_0\vartheta_2}}p,\end{gathered}$$

we obtain the system

$$
\begin{aligned}
q' &= p + O(\varepsilon),\\
p' &= \frac{1}{\vartheta_2}\Big(-\vartheta_0 q - \vartheta_1 p + \frac{\varsigma(1+\sigma_c^2)\vartheta_2}{(1+\alpha_c-\beta_c)\sigma_c} q|q|^2\Big) + O(\varepsilon),
\end{aligned}
\tag{3.12}
$$

which is equivalent to

$$
q'' + \frac{\vartheta_1}{\vartheta_2}q' + \frac{\vartheta_0}{\vartheta_2}q - \frac{\varsigma(1+\sigma_c^2)}{(1+\alpha_c-\beta_c)\vartheta_0} q|q|^2 = 0. \tag{3.13}
$$

To compare (3.13) and (3.2), let

$$
-\frac{\widehat{\phi}(\sigma_c)}{2\gamma} = \frac{\vartheta_0}{\vartheta_2}, \quad \frac{s}{\gamma} = \frac{\vartheta_1}{\vartheta_2}, \quad \frac{\varsigma}{\gamma} = -\frac{\varsigma(1+\sigma_c^2)}{(1+\alpha_c-\beta_c)\varrho_0},
$$

then system (3.12) is equivalent to (3.2), namely

$$
\begin{aligned}
q' &= p + O(\varepsilon),\\
p' &= \frac{1}{\gamma}\Big(\frac{1}{2}\widehat{\phi}(\sigma_c)q - sp - \varsigma q|q|^2\Big) + O(\varepsilon).
\end{aligned}
\tag{3.14}
$$

Next, analyzing the properties of (3.14), we prove Theorem 1.2, i.e. prove that system (3.14) has heteroclinic orbits corresponding to the modulated traveling wave solution.

Proof of Theorem 1.2. When $\varepsilon = 0$, system (3.14) can be written as

$$
\begin{aligned}
q' &= p,\\
p' &= \frac{1}{\gamma}\Big(\frac{1}{2}\widehat{\phi}(\sigma_c)q - sp - \varsigma q|q|^2\Big).
\end{aligned}
\tag{3.15}
$$

Obviously, for every q on the circle $|q| = \frac{\widehat{\phi}(\sigma_c)}{2\varsigma}$, system (3.15) has a saddle connection C_0, which is tangent to the unstable direction at that point and connects the point to the origin $(q,p) = (0,0)$, this property are similar to [4] and the proof is also similar, we will not repeat it.

When $\varepsilon > 0$, $(p,q) = (0,0)$ has two eigenvalues $\rho_{1,2} = \frac{-s\pm\sqrt{s^2+2\widehat{\phi}(\sigma_c)\gamma}}{2\gamma}$, in addition $\rho_{1,2} < 0$ if $s > \sqrt{-2\widehat{\phi}(\sigma_c)(1+\frac{1+\alpha_c-\beta_c}{2}\widehat{\phi}''(\sigma_c))}$. Thus, $(0,0)$ is stable node for sufficiently small ε. To complete the proof of Theorem 1.2, we need another two facts. One is that (3.14) has a circle of normally hyperbolic fixed points approaching $|q| = \sqrt{\frac{\widehat{\phi}(\sigma_c)}{2\varsigma}}$, $p = 0$ as $\varepsilon \to 0$, which is similar to [13, Lemma 4.3]. The other is that (3.14) has a family of heteroclinic connections C_0 (related to one another via $q \to e^{i\theta}q$ and $p \to e^{i\theta}p$) between the circle of fixed points and the origin, which is given in [12, Lemma 4.2]. Thus, when $s > \sqrt{-2\widehat{\phi}(\sigma_c)(1+\frac{1+\alpha_c-\beta_c}{2}\widehat{\phi}''(\sigma_c))}$, there exists a $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ equation (1.1) has the modulated traveling wave solutions of frequency σ_c, of the form

$$
u(t,x) = u(x-\varepsilon st, x) = \Sigma_{n\in\mathbb{Z}} V_n^u(x-\varepsilon st)e^{-in\sigma_c x}
$$

and having the boundary conditions at infinity

$$
\lim_{\zeta\to-\infty} u(\zeta,x) = u_\varepsilon(x) \approx 1 + \varepsilon\sqrt{\frac{\widehat{\phi}(\sigma_c)}{2\varsigma}}\cos(\sigma_c x), \quad \lim_{\zeta\to+\infty} u(\zeta,x) = 1.
$$

All our results are still valid if σ_c is replaced by any σ, which satisfy $\sigma = \sigma_c + \delta$ and $\left(1 + \frac{1+\alpha_c-\beta_c}{2}\widehat{\phi}''(\sigma_c)\right)\delta^2 < -\frac{\widehat{\phi}(\sigma_c)}{2}\varepsilon^2$. This completes the proof of Theorem 1.2. $\square$

3.2. **Numerical simulations.** In section 2, we showed that equation (1.1) has a periodic steady state under some conditions. Further, the existence of traveling wave solutions connecting this periodic steady to the uniform steady $u = 1$ was proved in subsection 3.1. In this subsection, we show a process of forming a steady state around the uniform steady $u = 1$, by numerical simulation. We show that equation (1.1) has traveling wave solutions connecting this periodic steady state to the uniform steady $u = 1$. Here we only consider the special kernel function $\phi(x)$ which defined by (2.1).

Similar to subsection 2.1, equation (1.1) can be replaced by (2.2), that is

$$\begin{gathered}
u_t = u_{xx} + u(1 + \alpha u - \beta u^2 - (1 + \alpha - \beta)(v + w)), \\
0 = v_{xx} - a^2 v + 3a^2 u, \\
0 = w_{xx} - w - 2u,
\end{gathered}$$

Before our numerical simulation, the initial value problem needs to be stated. We define the initial value of $u(t, x)$ as

$$u(0, x) = \begin{cases} 1 - \tau \sin bx, & x < L_0, \\ 1, & x > L_0, \end{cases} \tag{3.16}$$

where τ, b, L_0 are some constants. Since $v(0, x) = \int_{\mathbb{R}} \frac{3a}{2} e^{-a|x-y|} u(y)dy$, it follows that

$$v(0, x) = \begin{cases} 3 - \frac{3a}{2(a^2+b^2)}\tau \left[2a \sin bx - (a \sin bL_0 + b \cos bL_0)e^{ax-aL_0}\right], & x < L_0, \\ 3 - \frac{3a}{2(a^2+b^2)}\tau e^{(L_0-x)a}(a \sin bL_0 - b \cos bL_0), & x > L_0. \end{cases} \tag{3.17}$$

Since $w(0, x) = \int_{\mathbb{R}} -e^{-|x-y|} u(y)dy$, we have

$$w(0, x) = \begin{cases} -2 + \frac{1}{1+b^2}\tau \left[2 \sin bx - (\sin bL_0 + b \cos bL_0)e^{x-L_0}\right], & x < L_0, \\ -2 + \frac{1}{1+b^2}\tau e^{L_0-x}(\sin bL_0 - b \cos bL_0), & x > L_0. \end{cases} \tag{3.18}$$

In addition, the zero-flux boundary condition is considered here. Along with (3.16), (3.17) and (3.18), the system (2.2) can be simulated through the pdepe package in Matlab (see Figure 1).

Now, we explain our numerical results. Firstly, we see that the uniform steady $u = 1$ will lose its stability as the value of α increasing. And then a periodic steady state will occur (the theoretical analysis of this part is given in section 2.1, we do not repeat the narrative). Secondly, Figure 2 shows a specific traveling wave which connecting the uniform steady state $u = 1$ to a periodic steady state (it made a perfect complement to the previous section). Lastly, we know that equation (1.1) exists traveling wave solutions connecting 0 to 1 (or a periodic steady state), see [21, Figure 3] when the kernel $\phi(x) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}$, $\sigma > 0$ and the initial condition is

$$u(x, 0) = \begin{cases} 1, & \text{for } x < L_0, \\ 0, & \text{for } x \geq L_0. \end{cases}$$

However, here we know that equation (1.1) has traveling wave solutions connecting 1 to a periodic steady state when $\phi(x) = \frac{3a}{2}e^{-a|x|} - e^{-|x|}$ and $u(x, 0)$ id defined in

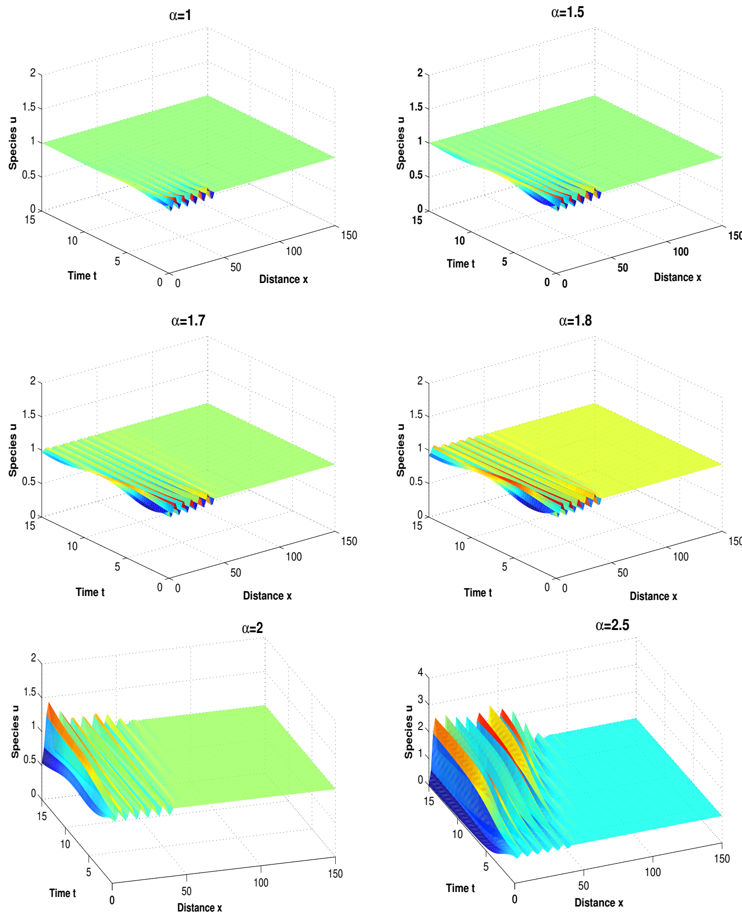

FIGURE 2. Time and space evolution for nonlocal equation (1.1) with kernel $\phi(x) = \frac{3a}{2}e^{-a|x|} - e^{-|x|}$. The computational domain is $x \in [0, 150]$ and $t \in [0, 15]$. The corresponding parameters are $L_0 = 40$, $\beta = 0.5$, $\tau = 0.1$, $a = 0.7$, $b = 5$, and α takes the values of $1, 1.5, 1.7, 1.8, 2, 2.5$.

(3.16). That is to say, the solution of equation (1.1) has a great relationship with the form of the kernel function and the initial condition. Next, we consider the influence of the initial conditions on the solution of equation (1.1).

4. Asymptotic rate of the Cauchy problem (1.4)

In this section, we study the asymptotic spreading speed for the solutions of the Cauchy problem (1.4). Also we complete the proof of the Theorem 1.3. First, we give a uniformly bounded of the solution u.

Lemma 4.1 ([10, Theorem 4.1]). *There exists a positive constant C such that the solution $u(x,t)$ of the Cauchy problem* (1.4) *satisfies*

$$u(x,t) \le C \quad \textit{for } (x,t) \in \mathbb{R} \times (0,\infty).$$

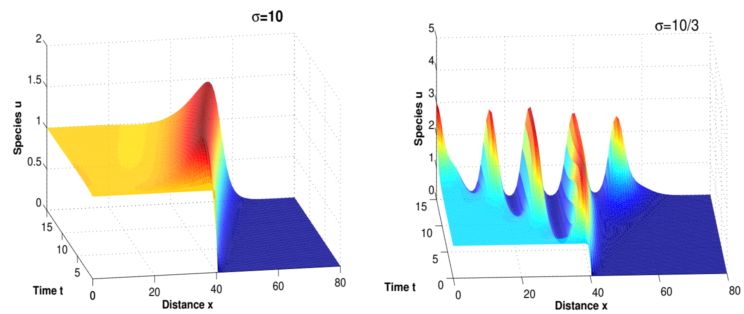

FIGURE 3. Time and space evolution for nonlocal equation (1.1) with the kernel $\phi(x) = \frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}$. The computational domain is $x \in [0, 80]$ and $t \in [0, 15]$. The corresponding parameters are $\alpha = 0.9$, $\beta = 0.5$ in the left figure, and $\alpha = 2$, $\beta = 0.4$ in the right figure.

Proof of Theorem 1.3(i). By contradictions, we assume the result is not true. Then, for $0 \le c < 2$, there exist two sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $(t_n)_{n\in\mathbb{N}}$ in $(0, +\infty)$ such that

$$|x_n| \le ct_n, \quad \text{for all } n \in \mathbb{N},$$

and

$$t_n \to +\infty, \quad u(t_n, x_n) \to 0 \quad \text{as } n \to +\infty. \tag{4.1}$$

We define the shifting functions

$$u_n(t, x) = u(t + t_n, x + x_n), \quad \text{for all } (t, x) \in (-t_n, +\infty) \times \mathbb{R}, n \in \mathbb{N}.$$

It follows from Lemma 4.1 that $(\|u_n\|_{L^\infty(-t_n,+\infty)\times\mathbb{R}})_{n\in\mathbb{N}}$ is bounded. Further, by the standard parabolic estimating, we know that u_n converges in $C^{1,2}_{\rm loc}(\mathbb{R} \times \mathbb{R})$, extracting a subsequence and letting $n \to +\infty$, we obtain u_∞ satisfying

$$(u_\infty)_t = (u_\infty)_{xx} + u_\infty\{1 + \alpha u_\infty - \beta(u_\infty)^2 - (1 + \alpha - \beta)(\phi * u_\infty)\} \quad \text{in } \mathbb{R} \times \mathbb{R},$$

and $u_\infty \ge 0$ in $\mathbb{R} \times \mathbb{R}$, $u_\infty(0, 0) = 0$. By regarding $1 + \alpha u_\infty - \beta(u_\infty)^2 - (1 + \alpha - \beta)(\phi * u_\infty)$ as a coefficient in $L^\infty(\mathbb{R} \times \mathbb{R})$ and using the strong maximum principle and the uniqueness of the solutions of the Cauchy problem (1.4), we know that $u_\infty(t, x) = 0$ for all $(x, t) \in \mathbb{R} \times \mathbb{R}$.

Further, we define

$$c_n = \frac{x_n}{t_n} \in [-c, c], \tag{4.2}$$

and

$$v_n(t, x) = u_n(t, x + c_n t) = u(t + t_n, x + c_n(t + t_n)) \quad \text{in } (-t_n, +\infty) \times \mathbb{R},$$

then $v_n(t, x)$ locally uniformly converge to 0 in $\mathbb{R} \times \mathbb{R}$. Thus, combining the boundedness of $(\|v_n\|_{L^\infty((-t_n,+\infty)\times\mathbb{R})})_{n\in\mathbb{N}}$, we know that $\phi * v_n$ also converges to 0 locally uniformly in $\mathbb{R} \times \mathbb{R}$.

Next we fix some parameters. Let $\delta > 0$ satisfy

$$1 - (1 + \alpha - \beta)\delta \ge \frac{c^2}{4} + \delta, \tag{4.3}$$

and $R > 0$ satisfy

$$\frac{\pi^2}{4R^2} \le \delta. \tag{4.4}$$

Without loss of generality, one can assume that $t_n > 1$ for every $n \in \mathbb{N}$. Because $\phi * v_n \to 0$ uniformly locally in $\mathbb{R} \times \mathbb{R}$ as $n \to +\infty$, for sufficiently large $n (n \ge N)$, we can define

$$t_n^* = \inf \{ t \in [-t_n + 1, 0] | \phi * v_n \le \delta \quad \text{for all } (x,t) \in [-R, R] \times [t, 0] \}, \quad n \ge N,$$

where δ and R are respectively as in (4.3) and (4.4). From the definition of t_n^*, we can assume that $t_n^* < 0$. By the continuity of $\phi * v_n$ in $(-t_n, +\infty) \times \mathbb{R}$ and the definition of t_n^*, we know that for every $n \ge N$,

$$0 \le \phi * v_n \le \delta \quad \text{for all } (x,t) \in [-R, R] \times [t, 0]. \tag{4.5}$$

On the other hand, $u(1, \cdot)$ is continuous and combining with the strong maximum principle, we know that $u(1, \cdot)$ is positive. Then, there exist a $\zeta > 0$ such that

$$u(1, x) \ge \zeta > 0 \quad \text{for all } |x| \le R + c. \tag{4.6}$$

From (4.2) and (4.6), we know that, for $n \ge N$,

$$v_n(-t_n + 1, x) = u(1, x + c_n) \ge \zeta \quad \text{for all } |x| \le R.$$

Therefore, we have the following dichotomy: either

$$t_n^* > -t_n + 1 \quad \text{and} \quad \max_{[-R,R]} (\phi * v_n)(t_n^*, .) = \delta, \tag{4.7}$$

or

$$t_n^* = -t_n + 1 \quad \text{and} \quad \min_{[-R,R]} v_n(t_n^*, .) > \zeta. \tag{4.8}$$

Next, we claim that there exists $\gamma > 0$ such that

$$\min_{[-R,R]} v_n(t_n^*, .) > \gamma > 0 \quad \text{for all } n \ge N, \tag{4.9}$$

which is true when $t_n^* = -t_n + 1$ from (4.8). Assume (4.9) does not hold when $t_n^* > -t_n + 1$, then, up to extraction a subsequence, there exist a sequence $(z_n)_{n \ge \mathbb{N}}$ in $[-R, R]$ such that

$$v_n(t_n^*, z_n) \to 0 \quad \text{and} \quad z_n \to z_\infty \in [-R, R] \text{ as } n \to +\infty.$$

We define

$$w_n(t, x) = v_n(t + t_n^*, x) \quad \text{for all } \ (t, x) \in (-t_n - t_n^*, +\infty) \times \mathbb{R},$$

for any $n \ge N$. It is easy to see that v_n satisfies

$$(v_n)_t = (v_n)_{xx} + c_n (v_n)_x + v_n \{1 + \alpha v_n - \beta (v_n)^2 - (1 + \alpha - \beta)(\phi * v_n)\} \quad \text{in } (-t_n, +\infty) \times \mathbb{R}. \tag{4.10}$$

Similarly, w_n satisfies the same equation for $(x, t) \in \mathbb{R} \times (-t_n, +\infty)$. Since

$$-t_n - t_n^* \le 1 \ for \ all \ n \ge N, c_n \to c_\infty \ as \ n \to +\infty, w_n \ge 0$$

and $(\|w_n\|_{L^\infty(-t_n - t_n^*, +\infty) \times \mathbb{R}})_{n \ge N}$ is bounded, combining the standard parabolic estimates, we know that w_n converges in $C^{1,2}_{\rm loc}((-1, +\infty) \times \mathbb{R})$. Then, up to extraction of a subsequence and let $n \to +\infty$, we have w_∞ satisfying

$$\begin{aligned}(w_\infty)_t =& (w_\infty)_{xx} + c_\infty (w_\infty)_x + w_\infty \{1 + \alpha w_\infty \\ &- \beta (w_\infty)^2 - (1 + \alpha - \beta)(\phi * w_\infty)\} \text{ in } (-1, +\infty) \times \mathbb{R},\end{aligned}$$

and

$$w_\infty(t,x) \geq 0 \quad \text{for all } (t,x) \in (-1,+\infty) \times \mathbb{R} \quad \text{and} \quad w_\infty(0,z_\infty) = 0.$$

The uniqueness of the solutions for the Cauchy problem and the maximum principle imply that

$$w_\infty(t,x) = 0 \quad \text{for all } (t,x) \in (-1,+\infty) \times \mathbb{R}.$$

On the other hand, from $w_n \to 0$ as $n \to +\infty$ locally uniformly in $(-1,\infty) \times \mathbb{R}$ and the boundedness of $(\|w_n\|_{L^\infty((-1,\infty)\times\mathbb{R})})_{n\geq N}$ it follows that $\phi * w_n \to 0$ as $n \to +\infty$ locally uniformly in $(-1,\infty) \times \mathbb{R}$. This implies

$$v_n(t_n^*,\cdot) \to 0 \quad \text{and} \quad (\phi * v_n)(t_n^*,\cdot) \to 0 \quad \text{locally uniformly in } \mathbb{R} \text{ as } n \to +\infty.$$

This is a contradiction to (4.7), so (4.9) is proved.

Now, from (4.5), (4.9) and (4.10), it is known that for every $n \geq N$, $-t_n + 1 \leq t_n^* < 0$ and $v_n \geq 0$, v_n satisfies

$$\begin{aligned} (v_n)_t &= (v_n)_{xx} + c_n(v_n)_x + v_n\{1 + \alpha v_n - \beta(v_n)^2 - (1+\alpha-\beta)(\phi * v_n)\} \\ &\geq (v_n)_{xx} + c_n(v_n)_x + (1 - (1+\alpha-\beta)\delta v_n) \\ &\quad + (v_n)^2(\alpha - \beta v_n) \text{ in } [t_n^*, 0] \times [-R,R], \\ &\qquad v_n(t,\pm R) \geq 0 \quad \text{for all } t \in [t_n^*, 0], \\ &\qquad v_n(t_n^*, x) \geq \gamma \quad \text{for all } x \in [-R,R]. \end{aligned} \tag{4.11}$$

For $n \geq N$, we define φ_n in $[-R,R]$ as

$$\varphi_n(x) = \nu\gamma e^{-\frac{c_n x}{2} - \frac{cR}{2}} \cos\big(\frac{\pi x}{2R}\big),$$

where ν is sufficiently small and satisfying $\nu\gamma < \frac{\alpha}{\beta}$ and $\nu < 1$. From (4.2)-(4.4), it follows that $\varphi_n(x)$ satisfies $0 \leq \varphi_n(x) \leq \gamma$ in $[-R,R]$, $\varphi_n(\pm R) = 0$ and

$$\begin{aligned} &\varphi_n'' + c_n\varphi_n' + (1 - (1+\alpha-\beta)\delta)\varphi_n + (\varphi_n)^2(\alpha - \beta\varphi_n) \\ &\geq \Big(1 - (1+\alpha-\beta)\delta - \frac{c_n^2}{4} - \frac{\pi^2}{4R^2}\Big)\varphi_n \geq 0 \quad \text{in } [-R,R]. \end{aligned}$$

That is to say, φ_n is a subsolution of (4.11). According to the maximum principle, for all $n \geq N$, we have

$$v_n(x,t) \geq \varphi_n(x) \quad \text{for all } (t,x) \in [t_n^*, 0] \times [-R,R].$$

In particular,

$$u(t_n, x_n) = v_n(0,0) \geq \varphi_n(0) = \nu\gamma e^{-cR/2} \quad \text{for all } n \geq N.$$

However, (4.1) implies that $u(t_n,x_n) \to 0$ as $n \to +\infty$ and $\nu\gamma e^{-cR/2}$ is a fixed constant, which is a contradiction. Therefore, (i) holds. □

To complete the proof of Theorem 1.3, we need another Lemma.

Lemma 4.2. *Assume c^* is the minimal speed of*

$$u_t = u_{xx} + u(1 + \alpha u - \beta u^2), \tag{4.12}$$

then

$$\lim_{t\to+\infty}\Big(\max_{|x|\geq c^* t} u(t,x) = 0\Big).$$

Since $f(u) = u(1 + \alpha u - \beta u^2)$ satisfies the conditions [28, A1–A4]. By [28, Theorems 2.17 and 4.3], it is easy to see that the Lemma 4.2 is true; we omit its proof.

Proof of Theorem 1.3(ii). Since u_0 is compactly supported, there exists $R > 0$ such that $u_0(x) = 0$ for a.e. $|x| \geq R$. Combining $u(x,t) \geq 0$ for all $t > 0$ with $x \in \mathbb{R}$, we have

$$u(t,x)\{1+\alpha u(t,x)-\beta u^2(t,x)-(1+\alpha-\beta)(\phi * u)(t,x)\} \leq u(t,x)(1+\alpha u(t,x)-\beta u^2(t,x)),$$

for all $t > 0$ and $x \in \mathbb{R}$. Let $v(t,x)$ denote the solution of the Cauchy problem

$$\begin{gathered} v_t = v_{xx} + v(t,x)(1 + \alpha v(t,x) - \beta v^2(t,x)), \\ v(x,0) = u_0 \quad \text{for all } x \in \mathbb{R}. \end{gathered}$$

By the maximum principle, we have

$$0 \leq u(t,x) \leq v(t,x) \quad \text{for all } t > 0, x \in \mathbb{R}.$$

On the other hand, [21, Lemma 2.1, Lemma 2.2] suggested that c^* is the minimal speed of (4.12) satisfying $2 \leq c^* \leq 2\sqrt{1 + \frac{\alpha^2}{4\beta}}$, especially, $c^* = 2$ when $\alpha \leq \sqrt{\beta/2}$. Combining this with Lemma 4.2, we complete the proof. $\square$

5. Appendix A

Proof of Lemma 2.2. Substituting $v(t) = \mathbf{B}(t)\mathbf{e} + \overline{\mathbf{B}}(t)\bar{\mathbf{e}} + \Phi(\mathbf{B}(t), \overline{\mathbf{B}}(t), \varepsilon, \delta)$ in equation (2.7) and comparing the coefficients of $\mathbf{e}$, we obtain

$$\begin{aligned} \frac{d\mathbf{B}}{dt} &= \Big(-\sigma^2 + (\alpha - 2\beta) - (1 + \alpha - \beta)\widehat{\phi}(\sigma) \Big) \mathbf{B} + O_{\varepsilon,\delta}(\mathbf{B}|\mathbf{B}|^2) \\ &= \Big(-(\sigma_c + \delta)^2 + (\alpha_c - 2\beta_c) - \Big(1 + \alpha_c - \beta_c + \frac{\varepsilon^2}{2}\Big)\widehat{\phi}(\sigma_c + \delta) \Big) \mathbf{B} + O_{\varepsilon,\delta}(\mathbf{B}|\mathbf{B}|^2). \end{aligned}$$

From the assumption on $f(\lambda, \sigma, \alpha, \beta)$ (i.e. the assumption (A1)(2)), we know that

$$\begin{gathered} -\sigma_c^2 - (1 + \alpha_c - \beta_c)\widehat{\phi}(\sigma_c) + (\alpha_c - 2\beta_c) = 0, \\ -2\sigma_c - (1 + \alpha_c - \beta_c)\widehat{\phi}'(\sigma_c) = 0. \end{gathered}$$

Thus

$$\begin{aligned} \lambda(\varepsilon, \delta) :=& -(\sigma_c + \delta)^2 + (\alpha_c - 2\beta_c) - (1 + \alpha_c - \beta_c)\widehat{\phi}(\sigma_c + \delta) - \frac{\varepsilon^2}{2}\widehat{\phi}(\sigma_c + \delta) \\ =& -\Big((\sigma_c + \delta)^2 - \sigma_c^2 - 2\delta\sigma_c\Big) - (1 + \alpha_c - \beta_c) \\ & \times \Big(\widehat{\phi}(\sigma_c + \delta) - \widehat{\phi}(\sigma_c) - \delta\widehat{\phi}'(\sigma_c)\Big) - \frac{\varepsilon^2}{2}\widehat{\phi}(\sigma_c + \delta) \\ =& \Big(-1 - \frac{1 + \alpha_c - \beta_c}{2}\widehat{\phi}''(\sigma_c) \Big)\delta^2 - \frac{\varepsilon^2}{2}\widehat{\phi}(\sigma_c) + O(|\varepsilon|^2|\delta| + |\delta|^3), \end{aligned}$$

as $(\varepsilon, \delta) \to (0,0)$.

To obtain the coefficient ς in (2.9), let $(\varepsilon, \delta) = (0,0)$ in (2.7) and we expect the solutions have the Taylor expansion

$$v = \mathbf{B}(t)\mathbf{e} + \overline{\mathbf{B}}(t)\bar{\mathbf{e}} + \mathbf{B}^2(t)\mathbf{e}_{2,0} + \mathbf{B}(t)\overline{\mathbf{B}}(t)\mathbf{e}_{1,1} + \overline{\mathbf{B}}^2(t)\mathbf{e}_{0,2} + O(|\mathbf{B}(t)|^3). \quad (5.1)$$

Substituting (5.1) into (2.7) and comparing the coefficient of $\mathbf{B}^2 e^{i2x}$ and $\mathbf{B}\overline{\mathbf{B}}$, we have

$$\mathbf{e}_{2,0} = \frac{\alpha_c - 3\beta_c - (1+\alpha_c-\beta_c)\widehat{\phi}(\sigma_c)}{4\sigma_c^2 - (\alpha_c - 2\beta_c) + (1+\alpha_c-\beta_c)\widehat{\phi}(2\sigma_c)} e^{i2x} + \operatorname{span}(\mathbf{e}, \bar{\mathbf{e}}),$$

and

$$\mathbf{e}_{1,1} = \frac{2(1+\alpha_c-\beta_c)\widehat{\phi}(\sigma_c) - 2(\alpha_c - 3\beta_c)}{\alpha_c - 2\beta_c - (1+\alpha_c-\beta_c)\widehat{\phi}(0)} + \operatorname{span}(\mathbf{e}, \bar{\mathbf{e}}).$$

It is easy to see that ς is the coefficient of the term $\mathbf{B}|\mathbf{B}\overline{\mathbf{B}}|$ and only occurs in the terms of $(\alpha-3\beta)v^2, -(1+\alpha-\beta)v\phi * v, -\beta v^3$. Thus

$$\begin{aligned}\varsigma = &- \langle (1+\alpha_c-\beta_c)(\mathbf{e}\phi * \mathbf{e}_{1,1} + \bar{\mathbf{e}}\phi * \mathbf{e}_{2,0} + \mathbf{e}_{1,1}\phi * \mathbf{e} + \mathbf{e}_{2,0}\phi * \bar{\mathbf{e}}), \mathbf{e}\rangle \\ &+ \langle 2(\alpha_c - 3\beta_c)(\mathbf{e}\cdot \mathbf{e}_{1,1} + \bar{\mathbf{e}}\cdot \mathbf{e}_{2,0}, \mathbf{e})\rangle - \langle 3\beta_c \mathbf{e}\cdot\mathbf{e}\cdot\bar{\mathbf{e}}, \mathbf{e}\rangle\cdot\end{aligned} \tag{5.2}$$

Substituting $\mathbf{e}_{1,1}$ and $\bar{\mathbf{e}}_{2,0}$ in (5.2), we obtain the value of ς. This completes the proof. □

6. Appendix B

Let $\varepsilon = 0$, then N_n^ε can be rewrite as

$$N_n^0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ n^2\sigma_c^2 - (\alpha_c - 2\beta_c) & 2in\sigma_c & \iota & 0 & \iota & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -3a^2 & 0 & a^2 + n^2\sigma_c^2 & 2in\sigma_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1+n^2\sigma_c^2 & 2in\sigma_c \end{pmatrix}$$

It is easy to see that the characteristic polynomial of N_n^0 is

$$\begin{aligned}\mathcal{G}_n(\lambda) &= \Big(\lambda^2 - 2in\sigma_c\lambda - n^2\sigma_c^2 - \sigma_c^2 - 1 - a^2 \\ &+ \frac{(a^2+\sigma_c^2)(1+\sigma_c^2)(a^2+(3a^2-2)\sigma_c^2)}{3a^2(1+\sigma_c^2)^2 - 2(a^2+\sigma_c^2)^2}\Big) \\ &\times (\lambda - i(n-1)\sigma_c)^2 (\lambda - i(n+1)\sigma_c)^2,\end{aligned} \tag{6.1}$$

which implies

$$\sigma_n^0 = \Big\{ i(n\pm 1)\sigma_c,\ in\sigma_c \pm \sqrt{1 + a^2 + \sigma_c^2 - \frac{(a^2+\sigma_c^2)(1+\sigma_c^2)(a^2+(3a^2-2)\sigma_c^2)}{3a^2(1+\sigma_c^2)^2 - 2(a^2+\sigma_c^2)^2}} \Big\}.$$

One can easily check that the algebraic multiplicity of each eigenvalue $\lambda_{\pm,n} = i(n\pm 1)\sigma_c$ is two and the geometric multiplicity of $\lambda_{\pm,n}$ is one. Now, we study how these eigenvalues are perturbed away when $\varepsilon > 0$. Here, we only consider $\lambda_{-,1}$. First, we write the characteristic polynomial of N_1^ε as

$$\mathcal{G}_1^\varepsilon(\lambda) = \mathcal{G}_1(\lambda) + \varepsilon D_1^\varepsilon(\lambda),$$

where $\mathcal{G}(\lambda)$ is given in (6.1) and

$$D_1^\varepsilon(\lambda) = \varepsilon\vartheta_0 + \vartheta_1\lambda + O(\varepsilon|\lambda| + |\lambda|^2), \quad \text{as } \varepsilon \to 0,\ |\lambda| \to 0.$$

Here, we have

$$\vartheta_0 = \big(1 - \frac{2}{3}a^2\big)\sigma_c^2 - \frac{1}{2}a^2, \quad \vartheta_1 = (a^2+\sigma_c^2)(1+\sigma_c^2)s.$$

Thus, we can seek $\lambda^{\varepsilon}_{-,1}$ of the form

$$\lambda^{\varepsilon}_{-,1} = \varepsilon\chi + O(\varepsilon^2),$$

where $\chi \in \mathbb{C}$. Since $\mathcal{G}^{\varepsilon}_1(\lambda^{\varepsilon}_{-,1}) = 0$, it follows that χ satisfies the quadratic equation

$$\vartheta_2\chi^2 + \vartheta_1\chi + \vartheta_0 = 0, \tag{6.2}$$

where

$$\vartheta_2 = 4\sigma_c^2\Big(2\sigma_c^2 + 1 + a^2 - \frac{(a^2+\sigma_c^2)(1+\sigma_c^2)(a^2+(3a^2-2)\sigma_c^2)}{3a^2(1+\sigma_c^2)^2 - 2(a^2+\sigma_c^2)^2}\Big).$$

Then equation (6.2) has the roots

$$\chi_{\pm} = \frac{-\vartheta_1 \pm \sqrt{\vartheta_1^2 - 4\vartheta_0\vartheta_2}}{2\vartheta_2}.$$

As a conclusion, the eigenvalue $\lambda_{-,1}$ perturbs into two eigenvalues with asymptotic:

$$\lambda^{\varepsilon,\pm}_{-,1} = \varepsilon\chi_{\pm} + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0.$$

Thus, there are only two center directions corresponding to the eigenvalues $\lambda^{\varepsilon}_{\pm}$ of N^{ε}_1.

7. Appendix C

Here the center manifold reduction is mainly to transform the infinite dimensional system into a finite dimensional system. From Appendix B, we know that N^{ε}_n has eigenvalues $\lambda^{\varepsilon}_{\pm}$ and

$$\varpi_{\pm n} := in\sigma_c \pm \sqrt{1 + a^2 + \sigma_c^2 - \frac{(a^2+\sigma_c^2)(1+\sigma_c^2)(a^2+(3a^2-2)\sigma_c^2)}{3a^2(1+\sigma_c^2)^2 - 2(a^2+\sigma_c^2)^2}}.$$

We define $\mathcal{E}_0 := \oplus_{n=0}^{\infty}\mathbb{C}^6$ and let $X = (X_{n,1}, X_{n,2}, X_{n,3}, X_{n,4}, X_{n,5}, X_{n,6}) \in \mathcal{E}_0, n \geq 0$. Let $\mathcal{E}$ denote the subset of $\mathcal{E}_0$, if $X_{0,j}, j = 1, 2, \dots, 6$ are real. In addition, for the functions of the form

$$V(\zeta, x) = \Sigma_{n\in\mathbb{Z}} V_n(\zeta)e^{-in\sigma_c x}, \quad V_n(\zeta) = (V^u_n(\zeta), V^v_n(\zeta), V^w_n(\zeta))^T,$$

we define a one-to-one map

$$I_\zeta : \big\{ (V^u_n, \partial_z V^u_n, V^v_n, \partial_z V^v_n, V^w_n, \partial_z V^w_n)^T \mid \text{for all } n \in \mathbb{Z}\big\} \to \mathcal{E}$$

for all $\zeta \in \mathbb{R}$ as

$$I_\zeta(V)_n = (V^u_n, \partial_z V^u_n|_{z=\zeta}, V^v_n, \partial_z V^v_n|_{z=\zeta}, V^w_n, \partial_z V^w_n|_{z=\zeta})^T \leftrightarrow X_n.$$

We note that $V_{-n} = \overline{V}_n$, so $I_\zeta(V)_n \in \mathbb{C}^6$, $n > 0$ and $I_\zeta(V)_0 \in \mathbb{R}$ uniquely determine by V. Following, we define an inner product about $\mathcal{E}_0$ as

$$\langle X, Y\rangle_l = \sum_{n=0}^{\infty}(1+n^2)^l\langle X_n, Y_n\rangle_{\mathbb{C}^6},$$

and define the Hilbert space $H^l_{\mathbb{C}} = \{X \in \mathcal{E}_0 | \langle X, Y\rangle_l < \infty\}$. The nonlinearity $L : \mathcal{E}_0 \to \mathcal{E}_0$ is defined as

$$L_n(X) = \Big(0, \sum_{p+q=n} (X_{p,1}X_{q,3} + X_{p,1}X_{q,5}), 0, 0, 0, 0\Big)^T, \quad n \ge 0,$$

$$J_n(X) = \Big(0, \sum_{p+q+r=n} X_{p,1}X_{q,1}X_{r,1}, 0, 0, 0, 0\Big)^T, \quad n \ge 0.$$

Obviously, $\mathcal{L}_n(X) \in C^1(H^l_{\mathbb{C}}(\mathcal{E}_0), H^l_{\mathbb{C}}(\mathcal{E}_0))$ as long as $l > 1/2$. Furthermore, since $H^l_{\mathbb{C}}(\mathcal{E}_0)$ is a Banach algebra, we have the estimate

$$\|L(X)\|_l \ge C\|X\|^2_l, \quad X \in H^l_{\mathbb{C}}(\mathcal{E}_0),$$

where $C > 0$ is a constant which is depends on l and the nonlinearity. Finally, we also need a bilinear map $\overline{L} : \mathcal{E}_0 \times \mathcal{E}_0 \to \mathcal{E}_0$ defined as

$$\overline{L}_n(X, Y) = \frac{1}{2}\Big(0, \sum_{p+q=n} (X_{p,1}Y_{q,3} + X_{p,1}Y_{q,5} + X_{q,3}Y_{p,1} + X_{q,5}Y_{p,3})\Big)^T, \quad n \ge 0,$$

for any $(X, Y) \in \mathcal{E}_0 \times \mathcal{E}_0$. Thus, by using a center manifold reduction, we can transform the system (2.10) into a finite dimensional system (3.6).

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