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EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR FRACTIONAL LAPLACE PROBLEMS WITH CRITICAL GROWTH

YAJING ZHANG, QIAOQIN LI, LU PANG

Abstract. We prove the existence of multiple positive solutions of fractional Laplace problems with critical growth, we consider the concave power case or the convex power case. We establish the relationship between the number of the local maximum points of the coefficient function of the critical nonlinearity and the number of the positive solutions of the equation.

1. Introduction

Considerable attention has been devoted to fractional and non-local operators of elliptic type in recent years, both for their interesting theoretical structure and in view of concrete applications, like flame spropagation, chemical reactions of liquids, population dynamics, geophysical fluid dynamics, and American option, see [3, 12, 13, 17, 30, 31] and the references therein.

In this article we consider the critical problem involving the fractional Laplacian

$$\begin{gathered} (-\Delta)^s u = \lambda u^{q-1} + Q(x)u^{p-1} \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{1.1}$$

where $s \in (0,1)$ is fixed and $(-\Delta)^s$ is the fractional Laplace operator, $\Omega \subset \mathbb{R}^N (N > 2s)$ is a smooth bounded domain, $1 < q < p = 2_s^* := \frac{2N}{N-2s}$, $\lambda > 0$, and $Q \in C(\bar{\Omega})$ is a positive function.

The fractional Laplace operator $(-\Delta)^s$ (up to normalization factors) is defined by

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^N} \Big(u(x+y) + u(x-y) - 2u(x)\Big) K(y) dy, \quad x \in \mathbb{R}^N,$$

where $K(x) = |x|^{-(N+2s)}, x \in \mathbb{R}^N$. We will denote by $H^s(\mathbb{R}^N)$ the usual fractional Sobolev space endowed with the so-called Gagliardo norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \Big(\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y)\, dx\, dy\Big)^{1/2},$$

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while X_0 is the function space defined as

$$X_0 = \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

We refer to [18, 23, 24] for a general definition of X_0 and its properties. The embedding $X_0 \hookrightarrow L^q(\Omega)$ is continuous for any $q \in [1, 2_s^*]$ and compact for any $q \in [1, 2_s^*)$. The space X_0 is endowed with the norm

$$\|u\|_{X_0} = \Big(\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y)\,dx\,dy\Big)^{1/2}.$$

By [23, Lemma 5.1] we have $C_0^2(\Omega) \subset X_0$. Thus X_0 is non-empty. Note that $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space with scalar product

$$(u, v)_{X_0} = \int_{\mathbb{R}^{2N}} (u(x) - u(y))(v(x) - v(y))K(x-y)\,dx\,dy.$$

Problems similar to (1.1) have been also studied in the local setting by several authors. In particular, Brezis and Nirenberg[9] studied the equation

$$-\Delta u = |u|^{2^*-2}u + f(x, u),$$

where $f(x, u)$ is a lower order perturbation of $|u|^{2^*-2}u$ in the sense that $f(x, t)/t^{2^*} \to 0$ as $t \to +\infty$, and $2^* = \frac{2N}{N-2}$. A typical example to which their results apply is

$$\begin{gathered} -\Delta u = \lambda u^{q-1} + u^{2^*-1} \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $\lambda > 0$ is a parameter and $2 < q < 2^*$. When $N \geq 4$, problem (1.2) has a positive solution for every $\lambda > 0$. When $N = 3$ and $4 < q < 6$, problem (1.2) has a positive solution. When $N = 3$ and $2 < q \leq 4$, it is only for large values of λ that problem (1.2) has a positive solution. The case $q = 2$ in (1.2) is also studied by them. Ambrosetti et al. [1] investigated the following problem with concave-convex power nonlinearities,

$$\begin{gathered} -\Delta u = \lambda u^{q-1} + u^{p-1} \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where $1 < q < 2 < p \leq 2^*$. They proved that there exists $\lambda_0 > 0$ such that (1.3) admits at least two positive solutions for $\lambda \in (0, \lambda_0)$, one positive solution for $\lambda = \lambda_0$, and no positive solution for $\lambda > \lambda_0$. After the work[1], several papers have been devoted to problem (1.3), see for example [2, 7, 9, 10, 16].

Now, we focus our attention on critical nonlocal fractional problems. It is worth noting here that problem (1.1) with $\lambda = 0$ and $Q \equiv 1$ has no positive solution whenever Ω is a star-shaped domain, see [15, 21]. This fact motivates the perturbation terms λu^{q-1} since we are interested in the existence of positive solutions of (1.1). Servadei and Valdinoci[25, 26] studied problem (1.1), with $q = 2$ and $Q \equiv 1$, and obtained Brezis-Nirenberg type results. When $Q \equiv 1$, Barrios et al. [6] studied problem (1.1) and showed the existence and multiplicity of solutions to problem (1.1). Note that one can also define a fractional power of the Laplacian using spectral decomposition. The similar problem with (1.1) but for this spectral fractional Laplacian has been treated in [5, 11].

Taking into account that we are looking for positive solutions, we consider the energy functional associated with (1.1),

$$\begin{aligned} I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \frac{\lambda}{q}\int_\Omega (u^+)^q dx \\ - \frac{1}{p}\int_\Omega Q(x)(u^+)^p dx, \end{aligned} \tag{1.4}$$

where $u^+ = \max\{u,0\}$ denotes the positive part of u. By the Maximum Principle(Proposition 2.2.8 in [27]), it is easy to check that critical points of I are the positive solutions of (1.1).

We assume that Q satisfies the following hypotheses.

(H1) $Q \in C(\bar\Omega)$ is a positive function;

(H2) there exist m local maximum points $a^1, a^2, \dots, a^m \in \Omega$ of Q such that

$$Q(a^i) = \max_{x\in\bar\Omega} Q(x) = 1 \quad \text{for } 1 \le i \le m,$$
$$Q(x) - Q(a^i) = o(|x-a^i|^\sigma)$$

as $x \to a^i$ uniformly in i, where $\sigma := \frac{N-2s}{2}$;

(H2') there exist m local maximum points $a^1, a^2, \dots, a^m \in \Omega$ of Q such that

$$Q(a^i) = \max_{x\in\bar\Omega} Q(x) = 1 \quad \text{for } 1 \le i \le m,$$
$$Q(x) - Q(a^i) = o(|x-a^i|^\sigma)$$

as $x \to a^i$ uniformly in i, for some $\sigma := N - \frac{(N-2s)q}{2}$;

(H3) there exists $\rho_0 > 0$ such that

$$\overline{B_{\rho_0}(a^i)} \cap \overline{B_{\rho_0}(a^j)} = \emptyset \quad \text{for } i \ne j \text{ and } 1 \le i,j \le m,$$

and $\cup_{i=1}^m \overline{B_{\rho_0}(a^i)} \subset \Omega$, where $B_{\rho_0}(a^i) = \{x \in \mathbb{R}^N : |x-a^i| < \rho_0\}$.

We now summarize the main results of the paper. Note that we are facing two cases of $|u|^{q-2}u$ in problem (1.1), the concave case: $1 < q < 2$, and the convex case: $2 < q < 2_s^*$. Firstly, in Section 2 we look at the problem (1.1) in the concave case and prove the following result.

Theorem 1.1. *Assume that $1 < q < 2$ and Q satisfies* (H1)–(H3). *There exists a positive number Λ^* such that for $\lambda \in (0,\Lambda^*)$, problem* (1.1) *has at least $m+1$ positive solutions.*

The convex case is treated in Section 3. While the existence result for problem (1.1) is given in the next theorem.

Theorem 1.2. *Assume $2 < q < 2_s^*, N \ge 4$, and Q satisfies* (H1), (H2'), (H3). *Then there exists a positive number Λ^* such that for $\lambda \in (0,\Lambda^*)$, problem* (1.1) *has at least m positive solutions.*

We prove Theorem 1.1 and Theorem 1.2 by variational methods. We construct m compact Palais-Smale sequences which are localized in correspondence of m local maximum points of Q in Ω. Thus, we could prove multiplicity of positive solutions of (1.1). This paper is organized as follows. In Section 2 we study problem (1.1) in the case of the exponent $1 < q < 2$. In Section 3 we we study problem (1.1) in the case of the exponent $2 < q < 2_s^*$.

2. Critical and concave case $1 < q < 2$

This section is devoted to the study of problem (1.1) when the exponent satisfies $1 < q < 2$. Firstly, we prove the existence of a ground state solution of (1.1). By a ground state solution, we mean a solution $w \in X_0$ such that $I_\lambda(w) \le I_\lambda(v)$ for every nontrivial solution v of (1.1). Next, we establish the existence of $m+1$ positive solution of (1.1).

2.1. **Preliminaries and Nehari manifold.** Note that I_λ is unbounded below. We restrict I_λ to a suitable set in order to get rid of this problem. We define the Nehari manifold

$$\begin{aligned}\mathcal{N}_\lambda &= \{u \in X_0 \setminus \{0\} : \langle I'_\lambda(u), u\rangle = 0\}\\ &= \big\{u \in X_0 \setminus \{0\} : \|u\|_{X_0}^2 = \lambda \int_\Omega (u^+)^q dx + \int_\Omega Q(x)(u^+)^p dx\big\}.\end{aligned}$$

Obviously, the Nehari manifold contains all the nontrivial critical points of I_λ.

Lemma 2.1. *The functional I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Proof. For every $u \in \mathcal{N}_\lambda$, we have

$$\begin{aligned}I_\lambda(u) &= \big(\frac{1}{2} - \frac{1}{p}\big)\|u\|_{X_0}^2 - \lambda\big(\frac{1}{q} - \frac{1}{p}\big)\int_\Omega (u^+)^q dx\\ &\ge \frac{s}{N}\|u\|_{X_0}^2 - \lambda\big(\frac{1}{q} - \frac{1}{p}\big)|\Omega|^{\frac{p-q}{p}}|u|_p^q \qquad (2.1)\\ &\ge \frac{s}{N}\|u\|_{X_0}^2 - \lambda\big(\frac{1}{q} - \frac{1}{p}\big)|\Omega|^{\frac{p-q}{p}} S_s^{-q/2}\|u\|_{X_0}^q,\end{aligned}$$

consequently, I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$ since $1 < q < 2$. □

We define $\psi_\lambda(u) = \langle I'_\lambda(u), u\rangle$. Then for $u \in \mathcal{N}_\lambda$, we have

$$\langle \psi'_\lambda(u), u\rangle = 2\|u\|_{X_0}^2 - \lambda q\int_\Omega (u^+)^q dx - p\int_\Omega Q(x)(u^+)^p dx \qquad (2.2)$$

$$= (2-q)\|u\|_{X_0}^2 - (p-q)\int_\Omega Q(x)(u^+)^p dx \qquad (2.3)$$

$$= \lambda(p-q)\int_\Omega (u^+)^q dx - (p-2)\|u\|_{X_0}^2. \qquad (2.4)$$

Adopting a method similar to that used in [29], we split $\mathcal{N}_\lambda$ into three parts:

$$\begin{aligned}\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u\rangle > 0\};\\ \mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u\rangle = 0\};\\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u\rangle < 0\}.\end{aligned}$$

In our context, the Sobolev constant is

$$S_s = \inf_{u \in H^s(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}} (u(x)-u(y))^2 K(x-y)\,dx\,dy}{\big(\int_{\mathbb{R}^N} |u(x)|^p dx\big)^{2/p}}. \qquad (2.5)$$

Set

$$\Lambda := \frac{p-2}{p-q}\big(\frac{2-q}{p-q}\big)^{\frac{2-q}{p-2}}|\Omega|^{-\frac{p-q}{p}} S_s^{\frac{p-q}{p-2}}. \qquad (2.6)$$

Lemma 2.2. *If $\lambda \in (0, \Lambda)$, then $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof. Arguing by contradiction and assume that there exists $\lambda_0 \in (0,\Lambda)$ such that $\mathcal{N}_\lambda^0 \neq \emptyset$. For $u \in \mathcal{N}_\lambda^0$, by (2.3), we have

$$(2-q)\|u\|_{X_0}^2 = (p-q)\int_\Omega Q(x)(u^+)^p dx \le (p-q)S_s^{-p/2}\|u\|_{X_0}^p.$$

Consequently,

$$\|u\|_{X_0} \ge \Big(\frac{2-q}{p-q}S_s^{\frac{p}{2}}\Big)^{\frac{1}{p-2}}. \tag{2.7}$$

Similarly, by (2.4), we have

$$\|u\|_{X_0} \le \Big(\lambda_0\frac{p-q}{p-2}|\Omega|^{\frac{p-q}{p}}S_s^{-q/2}\Big)^{\frac{1}{2-q}}. \tag{2.8}$$

Combing (2.7) and (2.8), we have

$$\lambda_0 \ge \frac{p-2}{p-q}\big(\frac{2-q}{p-q}\big)^{\frac{2-q}{p-2}}|\Omega|^{-\frac{p-q}{p}}S_s^{\frac{p-q}{p-2}} = \Lambda.$$

We have a contradiction. □

Set

$$X_0^+ = X_0 \setminus \{u \in X_0 : u^+(x) = 0 \text{ a.e. in } \Omega\}.$$

Lemma 2.3. *For $\lambda \in (0,\Lambda)$ and $u \in X_0^+$, there exist unique positive numbers $t^+(u)$ and $t^-(u)$ such that $t^+(u)u \in \mathcal{N}_\lambda^+, t^-(u)u \in \mathcal{N}_\lambda^-$, and*

$$I_\lambda(t^+(u)u) = \inf_{t\in[0,t_{\max}]} I_\lambda(tu), \quad I_\lambda(t^-(u)u) = \sup_{t\in[t_{\max},+\infty)} I_\lambda(tu), \tag{2.9}$$

where

$$t_{\max} = \Big[\frac{(2-q)\|u\|_{X_0}^2}{(p-q)\int_\Omega Q(x)(u^+)^p dx}\Big]^{\frac{1}{p-2}}.$$

Proof. Set $\gamma(t) = t^{2-q}\|u\|_{X_0}^2 - t^{p-q}\int_\Omega Q(x)(u^+)^p dx$ and $\varphi(t) = I_\lambda(tu)$ for $t \ge 0$. Clearly, $tu \in \mathcal{N}_\lambda$ if and only if $\gamma(t) = \lambda\int_\Omega (u^+)^q dx$. Moreover,

$$\gamma'(t) = (2-q)t^{1-q}\|u\|_{X_0}^2 - (p-q)t^{p-q-1}\int_\Omega Q(u^+)^p dx, \tag{2.10}$$

and so it is easy to see that, if $tu \in \mathcal{N}_\lambda$, then $tu \in \mathcal{N}_\lambda^+$ (or $\mathcal{N}_\lambda^-$) if and only if $\gamma'(t) > 0$ (or < 0). By (2.10), $\gamma(t)$ has a unique critical point at $t = t_{\max}$, and γ is strictly increasing on $(0, t_{\max})$ and strictly decreasing on $(t_{\max}, +\infty)$ with $\lim_{t\to+\infty}\gamma(t) = -\infty$. By (2.5) and $\lambda \in (0,\Lambda)$, we have

$$\begin{aligned}
\gamma(t_{\max}) &= \frac{p-2}{p-q}\big(\frac{2-q}{p-q}\big)^{\frac{2-q}{p-2}}\frac{\|u\|_{X_0}^{\frac{2(p-q)}{p-2}}}{\big(\int_\Omega Q(x)(u^+)^p dx\big)^{\frac{2-q}{p-2}}} \\
&\ge \frac{p-2}{p-q}\Big(\frac{2-q}{p-q}Q_M^{-1}\Big)^{\frac{2-q}{p-2}}S_s^{\frac{p(2-q)}{2(p-2)}}\|u\|_{X_0}^q \\
&> \lambda S_s^{-q/2}|\Omega|^{\frac{p-q}{p}}\|u\|_{X_0}^q \\
&\ge \lambda\Big(\int_\Omega (u^+)^p dx\Big)^{\frac{q}{p}}|\Omega|^{\frac{p-q}{p}} \\
&\ge \lambda\int_\Omega (u^+)^q dx.
\end{aligned}$$

Thus, we have unique $t^+(u)$ with $0 < t^+(u) < t_{\max} < t^-(u)$ such that

$$\gamma(t^+(u)) = \int_\Omega (u^+)^q dx = \gamma(t^-(u))$$

and $\gamma'(t^+) > 0 > \gamma'(t^-)$. Equivalently, $t^+(u)u \in \mathcal{N}_\lambda^+$ and $t^-(u)u \in \mathcal{N}_\lambda^-$. Since

$$\varphi'(t) = t^{q-1}\big[\gamma(t) - \lambda \int_\Omega (u^+)^q dx\big],$$

we derive that φ is decreasing on the intervals $(0, t^+(u))$ and $(t^-(u), +\infty)$, and increasing on the interval $(t^+(u), t^-(u))$. Then we obtain (2.9). □

Applying Lemma 2.1 and Lemma 2.2, we write $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ and define

$$\alpha_\lambda = \inf_{u\in\mathcal{N}_\lambda} I_\lambda(u), \quad \alpha_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u), \quad \alpha_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} I_\lambda(u).$$

Lemma 2.4. (i) *If $\lambda \in (0, \Lambda)$, then $\alpha_\lambda \le \alpha_\lambda^+ < 0$;*
(ii) *if $\lambda \in (0, \frac{q}{2}\Lambda)$, then $\alpha_\lambda^- \ge d_0$, where*

$$d_0 = \big(\frac{2-q}{p-q}\big)^{\frac{q}{p-2}} S_s^{\frac{pq}{2(p-2)}} \big[\frac{s}{N}\big(\frac{2-q}{p-q}\big)^{\frac{2-q}{p-2}} S_s^{\frac{p(2-q)}{2(p-2)}} - \lambda\big(\frac{1}{q} - \frac{1}{p}\big)|\Omega|^{\frac{p-q}{p}} S_s^{-q/2}\big] > 0.$$

Proof. (i) By Lemma 2.2, $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$. Let $w_0 \in X_0^+$, by Lemma 2.7, there exists $t(w_0) > 0$ such that $t(w_0)w_0 \in \mathcal{N}_\lambda^+$. By (2.3), we have

$$\begin{aligned} I_\lambda(t(w_0)w_0) &= \big(\frac{1}{2} - \frac{1}{q}\big)t^2(w_0)\|w_0\|_{X_0}^2 + \big(\frac{1}{q} - \frac{1}{p}\big)t^p(w_0)\int_\Omega Q(x)(w_0^+)^p dx \\ &< -\frac{2-q}{q}\big(\frac{1}{2} - \frac{1}{p}\big)t^2(w_0)\|w_0\|_{X_0}^2 < 0. \end{aligned} \tag{2.11}$$

(ii) For $u \in \mathcal{N}_\lambda^-$, by (2.3) and (2.5), we have

$$\frac{2-q}{p-q}\|u\|_{X_0}^2 < \int_\Omega Q(x)(u^+)^p dx \le S_s^{-p/2}\|u\|_{X_0}^p,$$

which implies

$$\|u\|_{X_0} > \big(\frac{2-q}{p-q}\big)^{\frac{1}{p-2}} S_s^{\frac{p}{2(p-2)}}. \tag{2.12}$$

Consequently,

$$\begin{aligned} I_\lambda(u) &= \big(\frac{1}{2} - \frac{1}{p}\big)\|u\|_{X_0}^2 - \lambda\big(\frac{1}{q} - \frac{1}{p}\big)\int_\Omega (u^+)^q dx \\ &\ge \frac{s}{N}\|u\|_{X_0}^2 - \lambda\big(\frac{1}{q} - \frac{1}{p}\big)|\Omega|^{\frac{p-q}{p}} S_s^{-q/2}\|u\|_{X_0}^q \\ &= \|u\|_{X_0}^q\big[\frac{s}{N}\|u\|_{X_0}^{2-q} - \lambda\big(\frac{1}{q} - \frac{1}{p}\big)|\Omega|^{\frac{p-q}{p}} S_s^{-q/2}\big] > d_0 \end{aligned}$$

for $\lambda \in (0, \frac{q}{2}\Lambda)$. □

As a consequence of Lemma 2.2 we have the following result.

Lemma 2.5. *For each $u \in \mathcal{N}_\lambda$, there exist $\varepsilon > 0$ and a differentiable function $\xi : B_\varepsilon(0) \subset X_0 \to (0, +\infty)$ such that*

$$\xi(0) = 1, \quad \xi(w)(u - w) \in \mathcal{N}_\lambda \quad \text{for } w \in B_\varepsilon(0),$$

$$\langle \xi'(0), w\rangle = \frac{2(u, w)_{X_0} - \lambda q\int_\Omega (u^+)^{q-1} w dx - p\int_\Omega Q(u^+)^p dx}{(2-q)\|u\|_{X_0}^2 - (p-q)\int_\Omega Q(x)(u^+)^p dx} \tag{2.13}$$

for all $w \in X_0$.

Proof. We define $F : \mathbb{R} \times X_0 \to \mathbb{R}$ as

$$\begin{aligned} F(\xi, w) &= \langle I'_\lambda(\xi \cdot (u-w)), \xi \cdot (u-w)\rangle \\ &= \xi^2 \|u-w\|^2_{X_0} - \lambda \xi^q \int_\Omega [(u-w)^+]^q dx - \xi^p \int_\Omega Q(x)[(u-w)^+]^p dx. \end{aligned}$$

Then $F(1,0) = 0$, and by Lemma 2.2, we have

$$\frac{\partial F}{\partial \xi}\Big|_{(1,0)} = \langle \psi'_\lambda(u), u\rangle \neq 0$$

We can apply the implicit function theorem at the point $(1,0)$ and obtain the result. □

2.2. **Existence of a ground state solution.** We follow the idea in [29] to show the existence of a $(PS)_{\alpha_\lambda}$ sequence and a $(PS)_{\alpha_\lambda^-}$ sequence in X_0 for I_λ.

Lemma 2.6. (i) *For* $\lambda \in (0, \Lambda)$, *there exists a* $(PS)_{\alpha_\lambda}$ *sequence* $\{u_n\} \subset \mathcal{N}_\lambda$ *for* I_λ*;*

(ii) *For* $\lambda \in (0, \frac{q}{2}\Lambda)$, *there exists a* $(PS)_{\alpha_\lambda^-}$ *sequence* $\{u_n\} \subset \mathcal{N}^-$ *for* I_λ.

Proof. We only prove (i). (ii) has a similar proof. Applying Ekeland's variational principle[14] to the minimization problem $\alpha_\lambda = \inf_{u\in\mathcal{N}_\lambda} I_\lambda(u)$ we have a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ with the following properties:

$$I_\lambda(u_n) < \alpha_\lambda + \frac{1}{n}, \tag{2.14}$$

$$I_\lambda(w) \geq I_\lambda(u_n) - \frac{1}{n}\|w - u_n\|_{X_0}, \quad \forall w \in \mathcal{N}_\lambda. \tag{2.15}$$

By taking n large, from (2.14) and (2.11), we have

$$\begin{aligned} I_\lambda(u_n) &= \Big(\frac{1}{2} - \frac{1}{p}\Big)\|u_n\|^2_{X_0} - \lambda\Big(\frac{1}{q} - \frac{1}{p}\Big)\int_\Omega (u_n^+)^q dx \\ &< \alpha_\lambda + \frac{1}{n} \\ &< -\frac{2-q}{q}\Big(\frac{1}{2} - \frac{1}{p}\Big)t^2(w_0)\|w_0\|^2_{X_0} \end{aligned} \tag{2.16}$$

for some $w_0 \in X_0^+$, which implies that

$$|\Omega|^{\frac{p-q}{p}} S_s^{-q/2}\|u_n\|^q_{X_0} \geq \int_\Omega (u_n^+)^q dx > \frac{(2-q)(\frac{p}{2}-1)}{\lambda(p-q)} t^2(w_0)\|w_0\|^2_{X_0}. \tag{2.17}$$

By (2.16) and (2.17), we have

$$L_1 < \|u_n\|_{X_0} < L_2, \tag{2.18}$$

where

$$\begin{gathered} L_1 = \Big(|\Omega|^{-\frac{p-q}{p}} S_s^{q/2} \frac{(2-q)(\frac{p}{2}-1)}{\lambda(p-q)} t^2(w_0)\|w_0\|^2_{X_0}\Big)^{1/q}, \\ L_2 = \Big(\lambda \frac{p-q}{(\frac{p}{2}-1)q}|\Omega|^{\frac{p-q}{p}} S_s^{-q/2}\Big)^{\frac{1}{2-q}}. \end{gathered}$$

Now we show that $I'_\lambda(u_n) \to 0$ as $n \to \infty$.

Let $v \in X_0$ with $\|v\|_{X_0} = 1$. Applying Lemma 2.5 with $u = u_n$ and $w = \rho v$, $\rho > 0$ small, we obtain $\xi_n(\rho v)$ such that $w_\rho := \xi_n(\rho v)(u_n - \rho v) \in \mathcal{N}_\lambda$. By (2.15), we deduce

$$\langle I'_\lambda(u_n), w_\rho - u_n\rangle + o(\|w_\rho - u_n\|_{X_0}) = I_\lambda(w_\rho) - I_\lambda(u_n) \geq -\frac{1}{n}\|w_\rho - u_n\|_{X_0}.$$

Therefore,

$$\langle I'_\lambda(u_n), -\rho v\rangle + [\xi_n(\rho v) - 1]\langle I'_\lambda(u_n), u_n - \rho v\rangle \geq -\frac{1}{n}\|w_\rho - u_n\|_{X_0} + o(\|w_\rho - u_n\|_{X_0}).$$

Dividing by ρ we have

$$\begin{aligned} &\langle I'_\lambda(u_n), v\rangle \\ &\leq \frac{\xi_n(\rho v) - 1}{\rho}\langle I'_\lambda(u_n), u_n - \rho v\rangle + \frac{1}{n\rho}\|w_\rho - u_n\|_{X_0} + \frac{o(\|w_\rho - u_n\|_{X_0})}{\rho} \\ &= [1 - \xi_n(\rho v)]\langle I'_\lambda(u_n), v\rangle + \frac{1}{n\rho}\|w_\rho - u_n\|_{X_0} + \frac{o(\|w_\rho - u_n\|_{X_0})}{\rho}. \end{aligned} \tag{2.19}$$

Clearly,

$$\|w_\rho - u_n\|_{X_0} \leq |\xi_n(\rho v) - 1| \cdot \|u_n\|_{X_0} + \rho|\xi_n(\rho v)|,$$
$$\lim_{\rho \to 0} \frac{|\xi_n(\rho v) - 1|}{\rho} \leq \|\xi'_n(0)\|_{X_0^*}.$$

Consequently, passing to the limit as $\rho \to 0$ in (2.19), we find a constant $C > 0$ independent of ρ such that

$$\langle I'_\lambda(u_n), v\rangle \leq \frac{C}{n}(1 + \|\xi'_n(0)\|_{X_0^*}).$$

The will be complte once we show that $\|\xi'_n(0)\|_{X_0^*}$ is uniformly bounded in n. From (2.13) and (2.18) we obtain

$$\langle \xi'_n(0), v\rangle \leq \frac{C_1}{|(2-q)\|u_n\|^2_{X_0} - (p-q)\int_\Omega Q(x)(u_n^+)^p dx|}$$

for some suitable constant $C_1 > 0$. We only need to show that $|(2-q)\|u_n\|^2_{X_0} - (p-q)\int_\Omega Q(u_n^+)^p dx|$ is bounded away from zero. Arguing by contradiction, assume that for a subsequence, which we still call $\{u_n\}$, we have

$$(2-q)\|u_n\|^2_{X_0} - (p-q)\int_\Omega Q(x)(u_n^+)^p dx = o(1). \tag{2.20}$$

By (2.5), we have

$$(2-q)\|u_n\|^2_{X_0} = (p-q)\int_\Omega Q(x)(u_n^+)^p dx + o(1) \leq (p-q)S_s^{-p/2}\|u_0\|^p_{X_0} + o(1).$$

Since $\|u_n\|_{X_0}$ is bounded away from zero by (2.18), we obtain

$$\|u_n\|_{X_0} \geq \Big(\frac{2-q}{p-q}S_s^{\frac{p}{2}}\Big)^{\frac{1}{p-2}} + o(1). \tag{2.21}$$

In addition (2.20), and the fact that $u_n \in \mathcal{N}_\lambda$ give

$$\begin{gathered} \lambda\int_\Omega (u_n^+)^q dx = \|u_n\|^2_{X_0} - \int_\Omega Q(x)(u_n^+)^p dx = \frac{p-2}{p-q}\|u_n\|^2_{X_0} + o(1), \\ \|u_n\|_{X_0} \leq \big[\lambda\frac{p-q}{p-2}|\Omega|^{\frac{p-q}{p}}S_s^{-q/2}\big]^{\frac{1}{2-q}} + o(1). \end{gathered} \tag{2.22}$$

Combing (2.21) and (2.22), we have

$$\lambda \geq \Lambda + o(1),$$

which is clearly impossible. We obtain $\langle I'_\lambda(u_n), v\rangle \leq \frac{C}{n}$. This completes the proof of (i). □

Proposition 2.7. *Assume that* $1 < q < 2$ *and* Q *satisfies* (H1)–(H3). *Then* (1.1) *has at least one positive ground state solution if* $\lambda \in (0, \Lambda)$.

Proof. By Lemma 2.6 (i), there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ for I_λ such that

$$I_\lambda(u_n) \to \alpha_\lambda, \quad I'_\lambda(u_n) \to 0 \tag{2.23}$$

as $n \to \infty$. Since I_λ is coercive on $\mathcal{N}_\lambda$ by Lemma 2.1, we obtain that $\|u_n\|_{X_0}$ is bounded. Going if necessary to a subsequence, we can assume that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } X_0, \\ u_n \to u \quad \text{in } L^r(\Omega) \text{ for } r \in [1, p), \\ u_n \to u \quad \text{a.e. in } \Omega. \end{gathered}$$

From (2.23) We obtain that $\langle I'_\lambda(u), w\rangle = 0, \forall w \in X_0$, i.e. u is a solution of (1.1). In particular, $u \in \mathcal{N}_\lambda$. By the Maximum Principle [27, Proposition 2.2.8], u is strictly positive in Ω. By the definition of α_λ and weak lower semicontinuity of the norm, we have

$$\begin{aligned} \alpha_\lambda \leq I_\lambda(u) &= \big(\frac{1}{2} - \frac{1}{p}\big)\|u\|^2_{X_0} - \lambda\big(\frac{1}{q} - \frac{1}{p}\big)\int_\Omega (u^+)^q dx \\ &\leq \liminf_{n\to\infty} \big[\frac{s}{N}\|u_n\|^2_{X_0} - \lambda\big(\frac{1}{q} - \frac{1}{p}\big)\int_\Omega (u_n^+)^q dx\big] \\ &\leq \liminf_{n\to\infty} I_\lambda(u_n) = \alpha_\lambda. \end{aligned}$$

It follows that $I_\lambda(u) = \alpha_\lambda$ and $u_n \to u$ strongly in X_0.

We claim that $u \in \mathcal{N}_\lambda^+$. Assume by the contradiction that $u \in \mathcal{N}_\lambda^-$. By Lemma 2.3, there exist positive numbers $t^+(u) < t_{\max} < t^-(u) = 1$ such that $t^+(u)u \in \mathcal{N}_\lambda^+$ and $t_\lambda^-(u)u \in \mathcal{N}_\lambda^-$, and

$$I_\lambda(t^+(u)u) < I_\lambda(t^-(u)u) = I_\lambda(u) = \alpha_\lambda,$$

which is impossible. Hence, $u \in \mathcal{N}_\lambda^+$ and $I_\lambda(u) = \alpha_\lambda = \alpha_\lambda^+$. □

2.3. **Proof of Theorem 1.1.** In this section, we prove that (1.1) admits m positive solutions. First of all, we show that I_λ satisfies the $(PS)_\beta$ condition in X_0 for $\beta < \beta^*$, where

$$\beta^* := \frac{s}{N} S_s^{\frac{N}{2s}} + \alpha_\lambda.$$

Lemma 2.8. I_λ *satisfies the* $(PS)_\beta$ *condition in* X_0 *for* $\beta < \beta^*$.

Proof. Let $\{u_n\}$ be a $(PS)_\beta$ sequence for I_λ such that

$$I_\lambda(u_n) \to \beta \quad \text{and} \quad I'_\lambda(u_n) \to 0. \tag{2.24}$$

Then, for n large enough, we have

$$\beta + 1 + \|u_n\|_{X_0} \geq I_\lambda(u_n) - \frac{1}{p}\langle I'_\lambda(u_n), u_n\rangle$$

$$
\begin{aligned}
&= \frac{s}{N}\|u_n\|_{X_0}^2 - \lambda\big(\frac{1}{q}-\frac{1}{p}\big)\int_\Omega (u_n^+)^q dx\\
&\geq \frac{s}{N}\|u\|_{X_0}^2 - \lambda\big(\frac{1}{q}-\frac{1}{p}\big)|\Omega|^{\frac{p-q}{p}} S_s^{-q/2}\|u\|_{X_0}^q.
\end{aligned}
$$

It follows that $\|u_n\|_{X_0}$ is bounded. Going if necessary to a subsequence, we can assume that

$$
\begin{gathered}
u_n \rightharpoonup u_0 \quad \text{in } X_0,\\
u_n \to u_0 \quad \text{in } L^r(\Omega) \text{ for } r\in[1,p),\\
u_n \to u_0 \quad \text{a.e. in } \Omega.
\end{gathered} \tag{2.25}
$$

Set $v_n = u_n - u_0$. Since X_0 is a Hilbert space, we have

$$
\|u_n\|_{X_0}^2 = \|v_n\|_{X_0}^2 + \|u_0\|_{X_0}^2 + o(1). \tag{2.26}
$$

By Brezis-Lieb's Lemma, we have

$$
\int_\Omega Q(x)(u_n^+)^p dx = \int_\Omega Q(x)(v_n^+)^p dx + \int_\Omega Q(x)(u_0^+)^p dx + o(1). \tag{2.27}
$$

By (2.26) and (2.27), we have

$$
\beta - I_\lambda(u_0) = \frac{1}{2}\|v_n\|_{X_0}^2 - \frac{1}{p}\int_\Omega Q(x)(v_n^+)^p dx + o(1), \tag{2.28}
$$

and

$$
\|v_n\|_{X_0}^2 - \int_\Omega Q(x)(v_n^+)^p dx + \|u_0\|_{X_0}^2 - \lambda\int_\Omega (u_0^+)^q dx - \int_\Omega Q(x)(u_0^+)^p dx = o(1).
$$

By (2.24) and (2.25), we have

$$
0 = \lim_{n\to\infty}\langle I_\lambda'(u_n), u_0\rangle = \|u_0\|_{X_0}^2 - \lambda\int_\Omega (u_0^+)^q dx - \int_\Omega Q(x)(u_0^+)^p dx, \tag{2.29}
$$

consequently,

$$
\|v_n\|_{X_0}^2 - \int_\Omega Q(x)(v_n^+)^p dx = o(1). \tag{2.30}
$$

Now, we assume that

$$
\|v_n\|_{X_0}^2 \to b, \quad \int_\Omega Q(x)(v_n^+)^p dx \to b, \quad \text{as } n\to\infty. \tag{2.31}
$$

By (2.5) and (2.31), we obtain

$$
\|v_n\|_{X_0}^2 \geq S_s\Big(\int_{\mathbb{R}^N}|v_n|^p dx\Big)^{2/p} \geq S_s\Big(\int_{\mathbb{R}^N} Q(x)(v_n^+)^p dx\Big)^{2/p}.
$$

Passing to the limit, we have $b \geq S_s b^{2/p}$. This implies that $b=0$ or $b\geq S_s^{p/(p-2)} = S_s^{N/(2s)}$. If $b=0$, the proof is complete. Assume that $b \geq S_s^{N/(2s)}$. By (2.29) and (2.31), we have

$$
\beta = \big(\frac{1}{2}-\frac{1}{p}\big)b + I(u_0) \geq \frac{s}{N}S_s^{N/(2s)} + \alpha_\lambda.
$$

which implies a contradiction. Hence, $b=0$, that is $u_n \to u_0$ in X_0 as $n\to\infty$. $\square$

Recall that

$$
S_s := \inf_{v\in H^s(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}}|v(x)-v(y)|^2 K(x-y)\,dx\,dy}{\big(\int_{\mathbb{R}^N}|v|^p dx\big)^{2/p}}.
$$

It is well known from [26] that the infimum in formula above is attained at $\tilde{u}$, where

$$\tilde{u}(x) = \frac{\kappa}{(\mu^2 + |x - x_0|^2)^{\frac{N-2s}{2}}}, \quad x \in \mathbb{R}^N, \tag{2.32}$$

with $\kappa \in \mathbb{R} \setminus \{0\}, \mu > 0$ and $x_0 \in \mathbb{R}^N$ fixed constants. We suppose $\kappa > 0$ for our convenience. Equivalently, the function $\bar{u}$ is defined as

$$\bar{u} = \frac{\tilde{u}}{\|\tilde{u}\|_{L^p(\mathbb{R}^N)}}$$

is such that

$$S_s = \int_{\mathbb{R}^{2N}} |\bar{u}(x) - \bar{u}(y)|^2 K(x - y)\, dx\, dy.$$

The function

$$u^*(x) = \bar{u}\Big(\frac{x}{S_s^{1/(2s)}}\Big), \quad x \in \mathbb{R}^N,$$

is a solution of

$$(-\Delta)^s u = |u|^{p-2} u \quad \text{in } \mathbb{R}^N. \tag{2.33}$$

Now, we consider the family of function U_ε defined as

$$U_\varepsilon(x) = \varepsilon^{-(N-2s)/2} u^*(x/\varepsilon), \quad x \in \mathbb{R}^N,$$

for any $\varepsilon > 0$. The function U_ε is a solution of problem (2.33) and satisfies

$$\int_{\mathbb{R}^{2N}} |U_\varepsilon(x) - U_\varepsilon(y)|^2 K(x - y)\, dx\, dy = \int_{\mathbb{R}^N} |U_\varepsilon(x)|^p dx = S_s^{N/(2s)}. \tag{2.34}$$

Fix a maximum point a^i of Q, where $1 \le i \le m$. Let $\eta_i \in C^\infty$ be such that $0 \le \eta_i \le 1$ in $\mathbb{R}^N$, $\eta_i(x) = 1$ if $|x - a^i| < \rho_0/2$; $\eta_i(x) = 0$ if $|x - a^i| \ge \rho_0$. For every $\varepsilon > 0$ we define the function

$$u_{\varepsilon,i}(x) = \eta_i(x) U_\varepsilon(x - a^i), \quad x \in \mathbb{R}^N. \tag{2.35}$$

In what follows we suppose that up to a translation $x_0 = 0$ in (2.32). From [26] we have the following estimates

$$\int_{\mathbb{R}^{2N}} |u_{\varepsilon,i}(x) - u_{\varepsilon,i}(y)|^2 K(x - y)\, dx\, dy = S_s^{N/(2s)} + O(\varepsilon^{N-2s}), \tag{2.36}$$

$$\int_{\mathbb{R}^N} |u_{\varepsilon,i}|^p dx = S_s^{N/(2s)} + O(\varepsilon^N), \tag{2.37}$$

where C_s is a positive constant depending on s.

Lemma 2.9. *We have*

$$\Big(\int_\Omega Q(x)(u^+_{\varepsilon,i})^p dx\Big)^{2/p} = \Big(\int_\Omega u^p_{\varepsilon,i} dx\Big)^{2/p} + o(\varepsilon^\sigma). \tag{2.38}$$

Proof. It is easy to see that

$$\begin{aligned} \Big|\int_\Omega [Q(x) - 1](u^+_{\varepsilon,i})^p dx\Big| &\le \int_\Omega |Q(x) - Q(a^i)| u^p_{\varepsilon,i} dx \\ &= \int_{B_{\rho_0}(a^i)} |Q(x) - Q(a^i)| u^p_{\varepsilon,i} dx. \end{aligned}$$

By (H2), for any $\gamma > 0$ there exists $\delta \in (0, \rho)$ such that

$$|Q(x) - Q(a^i)| < \gamma |x - a^i|^\sigma \quad \text{for all } |x - a^i| < \delta.$$

Recall that

$$U_\varepsilon(x) = \tilde\kappa \varepsilon^{-(N-2s)/2}\Big(\mu^2 + \big|\frac{x}{\varepsilon S_s^{1/(2s)}}\big|^2\Big)^{-(N-2s)/2}.$$

By using the shorthand notation $\rho_\varepsilon := \rho_0/(\varepsilon S_s^{1/(2s)}\mu)$, $\delta_\varepsilon := \delta/(\varepsilon S_s^{1/(2s)}\mu)$ and the change of variable, we have

$$\begin{aligned}
&\int_{B_{\rho_0}(a^i)} |Q(x) - Q(a^i)| u_{\varepsilon,i}^p dx \\
&= \tilde\kappa S_s^{\frac{N}{2s}} \mu^{-N} \int_{B_{\rho_\varepsilon}(0)} |Q(\mu S_s^{\frac{1}{2s}} \varepsilon x + a^i) - Q(a^i)| \eta_i^p(\mu S_s^{\frac{1}{2s}} \varepsilon x + a^i)(1+|x|^2)^{-N} dx \\
&= \tilde\kappa S_s^{\frac{N}{2s}} \mu^{-N} \Big(\int_{B_{\delta_\varepsilon}(0)} + \int_{B_{\rho_\varepsilon}(0)\setminus B_{\delta_\varepsilon}(0)}\Big) \big|Q(\mu S_s^{\frac{1}{2s}} \varepsilon x + a^i) - Q(a^i)\big| \\
&\quad \times \eta_i^p(\mu S_s^{\frac{1}{2s}} \varepsilon x + a^i)(1+|x|^2)^{-N} dx \\
&\le C\gamma\varepsilon^\sigma \int_{B_{\delta_\varepsilon}(0)} \frac{|x|^\sigma}{(1+|x|^2)^N} dx + C\int_{B_{\rho_\varepsilon}(0)\setminus B_{\delta_\varepsilon}(0)} \frac{1}{(1+|x|^2)^N} dx \\
&= CN\omega_N\gamma\varepsilon^\sigma \int_0^{\delta_\varepsilon} \frac{r^{\sigma+N-1}}{(1+r^2)^N} dr + CN\omega_N \int_{\delta_\varepsilon}^{\rho_\varepsilon} \frac{r^{N-1}}{(1+r^2)^N} dr \\
&\le C'\gamma\varepsilon^\sigma + C'\varepsilon^N,
\end{aligned}$$

where $C, C' > 0$ are constants independent of ε, and ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. Consequently,

$$\limsup_{\varepsilon\to 0} \varepsilon^{-\sigma} \Big| \int_\Omega [Q(x) - 1](u_{\varepsilon,i}^+)^p dx \Big| \le C'\gamma.$$

The arbitrariness of γ implies (2.38). $\square$

The following lemma is a key for proving our main result.

Lemma 2.10. *There exist $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ and $\lambda \in (0, \Lambda)$,*

$$\sup_{t\ge 0} I_\lambda(u_\lambda + t u_{\varepsilon,i}) < \beta^* \quad \text{uniformly in } i, \tag{2.39}$$

where u_λ is the positive solution obtained in Proposition 2.7.

Proof. Since I_λ is continuous in X_0 and $u_{\varepsilon,i}$ is uniformly bounded in X_0, there exists $t_1 > 0$ such that for $t \in [0, t_1]$,

$$I_\lambda(u_\lambda + t u_{\varepsilon,i}) < \alpha_\lambda + \frac{s}{N} S_s^{\frac{N}{2s}}.$$

Direct computations show that

$$\begin{aligned}
I_\lambda(u_\lambda + t u_{\varepsilon,i}) &= \frac{1}{2}\|u_\lambda\|_{X_0}^2 + t(u_\lambda, u_{\varepsilon,i})_{X_0} + \frac{t^2}{2}\|u_{\varepsilon,i}\|_{X_0}^2 \\
&\quad - \frac{\lambda}{q}\int_\Omega (u_\lambda + t u_{\varepsilon,i})^q dx - \frac{1}{p}\int_\Omega Q(x)(u_\lambda + t u_{\varepsilon,i})^p dx.
\end{aligned} \tag{2.40}$$

From (2.37), we have

$$\int_\Omega u_{\varepsilon,i}^p dx \ge \frac{1}{2} S_s^{\frac{N}{2s}}$$

for ε small enough. Note that the last term in (2.40) satisfies

$$\begin{aligned}\frac{1}{p}\int_\Omega Q(x)(u_\lambda+tu_{\varepsilon,i})^p dx &\geq \frac{t^p}{p}\int_\Omega Q(x)\left(\frac{u_\lambda}{t}+u_{\varepsilon,i}\right)^p dx\\ &\geq \min_{x\in\bar\Omega} Q(x)\frac{t^p}{p}\int_\Omega u_{\varepsilon,i}^p dx\\ &\geq \min_{x\in\bar\Omega} Q(x)\frac{S_s^{\frac{N}{2s}}}{2p}t^p.\end{aligned}$$

Thus, $I(u_\lambda+tu_{\varepsilon,i})\to-\infty$ as $t\to\infty$ uniformly in ε and i. Consequently, there exists $t_2>t_1$ such that $I_\lambda(u_\lambda+tu_{\varepsilon,i})<\alpha_\lambda+\frac{s}{N}S_s^{\frac{N}{2s}}$ for $t\geq t_2$. Then, we only need to verify that inequality

$$\sup_{t_1\leq t\leq t_2} I_\lambda(u_\lambda+tu_{\varepsilon,i})<\beta^* \quad \text{uniformly in } i,$$

for ε small enough.

From now on, we assume that $t\in[t_1,t_2]$. Then there exists a constant $C>0$ such that

$$\begin{aligned}&\int_\Omega Q(x)(u_\lambda+tu_{\varepsilon,i})^p dx\\ &\geq \int_\Omega Q(x)u_\lambda^p dx+t^p\int_\Omega Q(x)u_{\varepsilon,i}^p dx+pt\int_\Omega Q(x)u_\lambda^{p-1}u_{\varepsilon,i}dx\\ &\quad +p\ t^{p-1}\int_\Omega Q(x)u_{\varepsilon,i}^{p-1}u_\lambda dx-Ct^{p/2}\int_\Omega Q(x)u_\lambda^{p/2}u_{\varepsilon,i}^{p/2}dx.\end{aligned}\tag{2.41}$$

We have used the following inequality (see [4]) for $r>2$, there exists a constant C_r (depending on r) such that

$$(\alpha+\beta)^r\geq\alpha^r+\beta^r+r\left(\alpha^{r-1}\beta+\alpha\beta^{r-1}\right)-C_r\alpha^{r/2}\beta^{r/2}\quad\forall\alpha,\beta>0.$$

Using that u_λ is a positive solution of (1.1), and (2.41), (2.36), and by Lemma 2.9, we have

$$\begin{aligned}&I_\lambda(u_\lambda+tu_{\varepsilon,i})\\ &\leq \frac{1}{2}\|u_\lambda\|_{X_0}^2+t(u_\lambda,u_{\varepsilon,i})_{X_0}+\frac{t^2}{2}\|u_{\varepsilon,i}\|_{X_0}^2-\frac{\lambda}{q}\int_\Omega(u_\lambda+tu_{\varepsilon,i})^q dx\\ &\quad -\frac{1}{p}\int_\Omega Q(x)u_\lambda^p dx-\frac{1}{p}t^p\int_\Omega Q(x)u_{\varepsilon,i}^p dx-t\int_\Omega Q(x)u_\lambda^{p-1}u_{\varepsilon,i}dx\\ &\quad -t^{p-1}\int_\Omega Q(x)u_{\varepsilon,i}^{p-1}u_\lambda dx+C_pt^{p/2}\int_\Omega Q(x)u_\lambda^{p/2}u_{\varepsilon,i}^{p/2}dx\\ &=\frac{1}{2}\|u_\lambda\|_{X_0}^2+\lambda t\int_\Omega u_\lambda^{q-1}u_{\varepsilon,i}dx+\frac{t^2}{2}\|u_{\varepsilon,i}\|_{X_0}^2-\frac{\lambda}{q}\int_\Omega(u_\lambda+tu_{\varepsilon,i})^q dx\\ &\quad -\frac{1}{p}\int_\Omega Q(x)u_\lambda^p dx-\frac{1}{p}t^p\int_\Omega Q(x)u_{\varepsilon,i}^p dx-t^{p-1}\int_\Omega Q(x)u_{\varepsilon,i}^{p-1}u_\lambda dx\\ &\quad +C_pt^{p/2}\int_\Omega Q(x)u_\lambda^{p/2}u_{\varepsilon,i}^{p/2}dx\\ &=I_\lambda(u_\lambda)+\lambda t\int_\Omega u_\lambda^{q-1}u_{\varepsilon,i}dx+\frac{t^2}{2}\|u_{\varepsilon,i}\|_{X_0}^2-\frac{\lambda}{q}\int_\Omega(u_\lambda+tu_{\varepsilon,i})^q dx\end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda}{q}\int_\Omega u_\lambda^q dx-\frac{1}{p}t^p\int_\Omega Q(x)u_{\varepsilon,i}^p dx-t^{p-1}\int_\Omega Q(x)u_{\varepsilon,i}^{p-1}u_\lambda dx\\
&+C_pt^{p/2}\int_\Omega Q(x)u_\lambda^{p/2}u_{\varepsilon,i}^{p/2}dx\\
&=I_\lambda(u_\lambda)+\frac{t^2}{2}\|u_{\varepsilon,i}\|_{X_0}^2-\frac{\lambda}{q}\int_\Omega\big[(u_\lambda+tu_{\varepsilon,i})^q-u_\lambda^q-qtu_\lambda^{q-1}u_{\varepsilon,i}\big]dx\\
&-\frac{1}{p}t^p\int_\Omega Q(x)u_{\varepsilon,i}^p dx-t^{p-1}\int_\Omega Q(x)u_{\varepsilon,i}^{p-1}u_\lambda dx+C_pt^{p/2}\int_\Omega Q(x)u_\lambda^{p/2}u_{\varepsilon,i}^{p/2}dx\\
&\le I_\lambda(u_\lambda)+\frac{t^2}{2}\|u_{\varepsilon,i}\|_{X_0}^2-\frac{t^p}{p}\int_\Omega Q(x)u_{\varepsilon,i}^p dx-t^{p/2}\Big(t^{\frac{p-2}{2}}\int_\Omega Q(x)u_{\varepsilon,i}^{p-1}u_\lambda dx\\
&-C_p\int_\Omega Q(x)u_\lambda^{p/2}u_{\varepsilon,i}^{p/2}dx\Big)\\
&\le I_\lambda(u_\lambda)+S_s^{\frac{N}{2s}}\big(\frac{t^2}{2}-\frac{t^p}{p}\big)-t^{p/2}\Big(t^{\frac{p-2}{2}}\int_\Omega u_{\varepsilon,i}^{p-1}Q(x)u_\lambda dx\\
&-C_p\int_\Omega Q(x)u_\lambda^{p/2}u_{\varepsilon,i}^{p/2}dx\Big)+O(\varepsilon^{N-2s})+o(\varepsilon^\sigma)\\
&\le I_\lambda(u_\lambda)+\frac{s}{N}S_s^{\frac{N}{2s}}-t^{p/2}\Big(t^{\frac{p-2}{2}}\int_\Omega Q(x)u_{\varepsilon,i}^{p-1}u_\lambda dx-C_p\int_\Omega Q(x)u_\lambda^{p/2}u_{\varepsilon,i}^{p/2}dx\Big)\\
&+O(\varepsilon^{N-2s})+o(\varepsilon^\sigma).
\end{aligned}\tag{2.42}
$$

Here we have used the elementary inequality: $(\alpha+\beta)^q\ge\alpha^q+q\alpha^{q-1}\beta$ for $\alpha,\beta>0$.

Now, we estimate the third term in (2.42). There exists a constant $C_1>0$ independent of i such that $Q(x)u_\lambda(x)\ge C_1$ for all $x\in B_{\rho_0/2}(a^i)$. Then

$$
\int_\Omega Q(x)u_{\varepsilon,i}^{p-1}u_\lambda dx\ge C_1\int_{B_{\rho_0/2}(a^i)}U_\varepsilon^{p-1}(x-a^i)dx\ge C_1\varepsilon^{\frac{N-2s}{2}}. \tag{2.43}
$$

Direct computations show that there exists a constant $C_2>0$ independent of i such that

$$
\int_\Omega Q(x)u_\lambda^{p/2}u_{\varepsilon,i}^{p/2}dx\le C_2\int_{B_{\rho_0}(a^i)}U_\varepsilon^{p/2}(x-a^i)dx\le C_2\varepsilon^{\frac{N}{2}}|\ln\varepsilon|. \tag{2.44}
$$

By (2.42), (2.43) and (2.44), we have

$$
\sup_{t_1\le t\le t_2}I(u_\lambda+tu_{\varepsilon,i})<I(u_\lambda)+\frac{s}{N}S_s^{\frac{N}{2s}}
$$

for ε small enough. □

We define $X^+:=\{u\in X_0:u^+\not\equiv0\}$, and

$$
\mathcal{A}_1:=\big\{u\in X_0^+:\frac{1}{\|u\|_{X_0}}t^-\big(\frac{u}{\|u\|_{X_0}}\big)>1\big\},
$$
$$
\mathcal{A}_2:=\big\{u\in X_0^+:\frac{1}{\|u\|_{X_0}}t^-\big(\frac{u}{\|u\|_{X_0}}\big)<1\big\}.
$$

Following the idea in [29], we have the following results.

Lemma 2.11. *Assume that $\lambda\in(0,\Lambda)$. We have*

(i) $X_0^+=\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{N}_\lambda^-$,
(ii) $\mathcal{N}^+\subset\mathcal{A}_1$,
(iii) *there exists $t_{\varepsilon,i}>1$ such that $u_\lambda+t_{\varepsilon,i}u_{\varepsilon,i}\in\mathcal{A}_2$ for each $1\le i\le m$,*

(iv) *there exists* $s_{\varepsilon,i} \in (0,1)$ *such that* $u_\lambda + s_{\varepsilon,i} t_{\varepsilon,i} u_{\varepsilon,i} \in \mathcal{N}_\lambda^-$ *for each* $1 \le i \le m$,

(v) $\alpha_\lambda^- < \alpha_\lambda^+ + \frac{s}{N} S_s^{\frac{N}{2s}}$.

Proof. (i) Let

$$\mathcal{S} := \{u \in X_0^+ : \frac{1}{\|u\|_{X_0}} t^-\big(\frac{u}{\|u\|_{X_0}}\big) = 1\}.$$

It suffices to prove that $\mathcal{N}_\lambda^- = \mathcal{S}$. Let $v = u/\|u\|_{X_0}$ for $u \in \mathcal{N}_\lambda^-$. By Lemma 2.3, there exists $t^-(v) > 0$ such that $t^-(v)v \in \mathcal{N}_\lambda^-$, that is $\frac{t^-(v)}{\|u\|_{X_0}} u \in \mathcal{N}_\lambda^-$. Since $u \in \mathcal{N}_\lambda^-$, we have $t^-(v) = \|u\|_{X_0}$. Hence, we obtain $\mathcal{N}_\lambda^- \subset \mathcal{S}$. On the other hand, let $u \in \mathcal{S}$. Then,

$$u = t^-\Big(\frac{u}{\|u\|_{X_0}}\Big)\frac{u}{\|u\|_{X_0}} \in \mathcal{N}^-.$$

Thus, $\mathcal{S} \subset \mathcal{N}_\lambda^-$. Consequently, $\mathcal{N}_\lambda^- = \mathcal{S}$.

(ii) For any $u \in \mathcal{N}_\lambda^+$, let $v = \frac{u}{\|u\|_{X_0}}$. By Lemma 2.3, there exists $t^-(v) > 0$ such that $t^-(v)v \in \mathcal{N}^-$, that is

$$\frac{1}{\|u\|_{X_0}} t^-\Big(\frac{u}{\|u\|_{X_0}}\Big) u \in \mathcal{N}^-.$$

Hence,

$$t^-(u) = \frac{1}{\|u\|_{X_0}} t^-\Big(\frac{u}{\|u\|_{X_0}}\Big).$$

By Lemma 2.3, we have $1 = t^+(u) < t_{\max}(u) < t^-(u)$. Therefore, $\mathcal{N}^+ \subset \mathcal{A}_1$.

(iii) Firstly, we claim that there exists a positive constant C independent of i such that

$$\sup_{t \ge 0} t^-\Big(\frac{u_\lambda + t u_{\varepsilon,i}}{\|u_\lambda + t u_{\varepsilon,i}\|_{X_0}}\Big) < C.$$

Assume by contradiction that there exists s sequence $\{t_{n,i}\}$ such that $t_{n,i} \to +\infty$ and $t^-(v_{n,i}) \to +\infty$ as $n \to \infty$, where

$$v_{n,i} := \frac{u_\lambda + t_n u_{\varepsilon,i}}{\|u_\lambda + t_n u_{\varepsilon,i}\|_{X_0}}.$$

Since $t^-(v_{n,i})v_{n,i} \in \mathcal{N}_\lambda^-$, by Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}
\int_\Omega (v_{n,i}^+)^p dx &= \frac{1}{\|t_{n,i}^{-1} u_\lambda + u_{\varepsilon,i}\|_{X_0}^p} \int_\Omega (t_{n,i}^{-1} u_\lambda + u_{\varepsilon,i})^p dx \\
&\to \frac{\int_\Omega u_{\varepsilon,i}^p dx}{\|u_{\varepsilon,i}\|_{X_0}^p} \\
&= \frac{S_s^{\frac{N}{2s}} + O(\varepsilon^N)}{[S_s^{\frac{N}{2s}} + O(\varepsilon^{N-2s})]^{p/2}}
\end{aligned}$$

as $n \to \infty$. Thus

$$\begin{aligned}
& I_\lambda(t^-(v_{n,i})v_{n,i}) \\
&= \frac{1}{2}(t^-(v_{n,i}))^2 - \frac{\lambda}{q}(t^-(v_{n,i}))^q \int_\Omega (v_{n,i}^+)^q dx - \frac{(t^-(v_{n,i}))^p}{p} \int_\Omega (v_{n,i}^+)^p dx \\
&\to -\infty
\end{aligned}$$

as $n \to \infty$, which is impossible since I is bounded from below on $\mathcal{N}_\lambda$ by Lemma 2.1. Set

$$t_{\varepsilon,i} = \frac{\|u_\lambda\|_{X_0} + \big(\|u_\lambda\|^2_{X_0} + |C^2 - \|u_\lambda\|^2_{X_0}|\big)^{1/2}}{\|u_{\varepsilon,i}\|_{X_0}} + 1.$$

Then

$$\begin{aligned}\|u_\lambda + t_{\varepsilon,i}u_{\varepsilon,i}\|^2_{X_0} &= \|u_\lambda\|^2_{X_0} + t^2_{\varepsilon,i}\|u_{\varepsilon,i}\|^2_{X_0} + 2t_{\varepsilon,i}\,(u_\lambda, u_{\varepsilon,i})_{X_0}\\ &> \|u_\lambda\|^2_{X_0} + \big|C^2 - \|u_\lambda\|^2_{X_0}\big|\\ &\geq C^2 > \big[t^-\big(\frac{u_\lambda + tu_{\varepsilon,i}}{\|u_\lambda + tu_{\varepsilon,i}\|_{X_0}}\big)\big]^2.\end{aligned}$$

Hence, we obtain $u_\lambda + t_{\varepsilon,i}u_{\varepsilon,i} \in \mathcal{A}_2$.

(iv) Define $\gamma_i : [0,1] \to \mathbb{R}$ as

$$\gamma_i(s) := \frac{1}{\|u_\lambda + st_{\varepsilon,i}u_{\varepsilon,i}\|_{X_0}} t^-\Big(\frac{u_\lambda + st_{\varepsilon,i}u_{\varepsilon,i}}{\|u_\lambda + st_{\varepsilon,i}u_{\varepsilon,i}\|_{X_0}}\Big) \quad \text{for all } s \in [0,1].$$

Note that $\gamma(s)$ is a continuous function of s. Since $\gamma(0) > 1$ and $\gamma(1) < 1$ there exists $s_{\varepsilon,i} \in (0,1)$ such that $\gamma(s_{\varepsilon,i}) = 1$, that is $u_\lambda + s_{\varepsilon,i}t_{\varepsilon,i}u_{\varepsilon,i} \in \mathcal{N}^-_\lambda$.

(v) By Lemma 2.10 and (iv), we have $\alpha^-_\lambda < \alpha^+_\lambda + \frac{s}{N}S_s^{\frac{N}{2s}}$. $\square$

Let

$$\mathcal{P} = \{a^i : 1 \leq i \leq m\} \quad \text{and} \quad \mathcal{P}_{\rho_0} = \cup_{i=1}^m B_{\rho_0}(a^i).$$

Let $r_0 = \max_{1\leq i\leq m}|a^i| + \rho_0$. We minimize the energy functional I_λ on some submanifolds of $\mathcal{N}_\lambda$. To this end, we define a barycenter map (cf. [8]) $\mathbf{K} : X_0\backslash\{0\} \to \mathbb{R}^N$ as

$$\mathbf{K}(u) = \frac{\int_\Omega \chi(x)|u|^p dx}{\int_\Omega |u|^p dx},$$

where

$$\chi(x) = \begin{cases} x, & |x| \leq r_0,\\ r_0x/|x|, & |x| > r_0.\end{cases}$$

Lemma 2.12. $\mathbf{K}(t^-(u_{\varepsilon,i})u_{\varepsilon,i}) \to a^i$ *as* $\varepsilon \to 0$*. In particular, there exists* $\varepsilon_1 \in (0, \varepsilon_0)$ *such that if* $\varepsilon \in (0, \varepsilon_1)$*, then* $\mathbf{K}(t^-(u_{\varepsilon,i})u_{\varepsilon,i}) \in \mathcal{P}_{\rho_0}$ *for each* $1 \leq i \leq m$*.*

Proof. Direct computations imply that

$$\begin{aligned}\mathbf{K}(t^-(u_{\varepsilon,i})u_{\varepsilon,i}) &= \frac{\int_\Omega \chi(x)\eta_i^p(x)U_\varepsilon^p(x-a^i)dx}{\int_\Omega \eta_i^p(x)U_\varepsilon^p(x-a^i)dx}\\ &= \frac{\int_{B_{\rho_0}(a^i)} \chi(x)\eta_i^p(x)U_\varepsilon^p(x-a^i)dx}{\int_{B_{\rho_0}(a^i)} \eta_i^p(x)U_\varepsilon^p(x-a^i)dx}\\ &= a^i + \frac{\varepsilon\int_{B_{\rho_0/\varepsilon}(0)} x\eta_i^p(\varepsilon x + a^i)\big(\mu^2 + |\frac{x}{S_s^{1/(2s)}}|^2\big)^{-N}dx}{\int_{B_{\rho_0/\varepsilon}(0)} \eta_i^p(\varepsilon x + a^i)\big(\mu^2 + |\frac{x}{S_s^{1/(2s)}}|^2\big)^{-N}dx}.\end{aligned}$$

Since $\int_{B_{\rho_0/\varepsilon}(0)} x\eta_i^p(\varepsilon x + a^i)(\mu^2 + |\frac{x}{S_s^{1/(2s)}}|^2)^{-N}dx$ is bounded and

$$\int_{B_{\rho_0/\varepsilon}(0)} \eta_i^p(\varepsilon x + a^i)(\mu^2 + |\frac{x}{S_s^{1/(2s)}}|^2)^{-N}dx$$

is bounded away from zero, we have

$$\mathbf{K}(t^-(u_{\varepsilon,i})u_{\varepsilon,i}) \to a^i$$

as $\varepsilon \to 0$. Consequently, there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that $\mathbf{K}(t^-(u_{\varepsilon,i})u_{\varepsilon,i}) \in \mathcal{P}_{\rho_0}$ for any $\varepsilon \in (0, \varepsilon_1)$ and each $1 \le i \le m$. □

For each $1 \le i \le m$, we define

$$\begin{gathered}
\mathcal{O}_{\lambda,i} = \{u \in \mathcal{N}_\lambda^- : |\mathbf{K}(u) - a^i| < \rho_0\}, \\
\partial\mathcal{O}_{\lambda,i} = \{u \in \mathcal{N}_\lambda^- : |\mathbf{K}(u) - a^i| = \rho_0\}, \\
\beta_{\lambda,i} = \inf_{u \in \mathcal{O}_{\lambda,i}} I_\lambda(u), \quad \tilde{\beta}_{\lambda,i} = \inf_{u \in \partial\mathcal{O}_{\lambda,i}} I_\lambda(u).
\end{gathered}$$

Consider the critical problem

$$\begin{gathered}
(-\Delta)^s u = |u|^{p-2}u \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{gathered} \tag{2.45}$$

We define the energy functional $J : X_0 \to \mathbb{R}$ associated with the critical problem (2.45) as

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^{2N}} (u(x) - u(y))^2 K(x-y)\,dx\,dy - \frac{1}{p}\int_\Omega |u|^p dx.$$

Set

$$\mathcal{M}(\Omega) = \{u \in X_0 \setminus \{0\} : \langle J'(u), u\rangle = 0\}, \quad \gamma(\Omega) = \inf_{u \in \mathcal{M}(\Omega)} J(u).$$

Similarly, we define $J_\infty : \dot{H}^s(\mathbb{R}^N) \to \mathbb{R}$ as

$$J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^{2N}} (u(x) - u(y))^2 K(x-y)\,dx\,dy - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx,$$

where $\dot{H}^s(\mathbb{R}^N)$ denotes the space of functions $u \in L^p(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^{2N}}(u(x) - u(y))^2 K(x-y)\,dx\,dy < \infty$. Set

$$\mathcal{M}(\mathbb{R}^N) = \{u \in \dot{H}^s(\mathbb{R}^N) : \langle J_\infty(u), u\rangle = 0\}, \quad \gamma(\mathbb{R}^N) = \inf_{u \in \mathcal{M}(\mathbb{R}^N)} J_\infty(u).$$

It is easy to see that $\gamma(\mathbb{R}^N) = \frac{s}{N} S_s^{\frac{N}{2s}}$. The following results corresponds to the classical results of [9, 28].

Lemma 2.13. *(i) $\gamma(\Omega) = \gamma(\mathbb{R}^N)$ and $\gamma(\Omega)$ is never achieved except when $\Omega = \mathbb{R}^N$;*
(ii) $\gamma(\Omega) = \alpha_0$.

Proof. (i) Since $\mathcal{M}(\Omega) \subset \mathcal{M}(\mathbb{R}^N)$, we have $\gamma(\mathbb{R}^N) \le \gamma(\Omega)$. Conversely, let $\{u_n\} \subset \dot{H}^s(\mathbb{R}^N)$ be a minimizing sequence for $\gamma(\mathbb{R}^N)$. By density of $C_0^\infty(\mathbb{R}^N)$ in $\dot{H}^s(\mathbb{R}^N)$ we may assume that $u_n \in C_0^\infty(\mathbb{R}^N)$. We can choose $y_n \in \mathbb{R}^N$ and $\lambda_n > 0$ such that

$$u_n^{y_n,\lambda_n}(\cdot) := \lambda_n^{\frac{N-2s}{2}} u_n(\lambda_n \cdot + y_n) \in C_0^\infty(\Omega).$$

Since

$$\|u_n^{y_n,\lambda_n}\|_{X_0} = \|u_n\|_{\dot{H}(\mathbb{R}^N)}, \quad \int_\Omega |u_n^{y_n,\lambda_n}|^p dx = \int_{\mathbb{R}^N} |u_n|^p dx,$$

we obtain $\gamma(\Omega) \le \gamma(\mathbb{R}^N)$. Thus, $\gamma(\Omega) = \gamma(\mathbb{R}^N)$.

Assume by contradiction that $\Omega \neq \mathbb{R}^N$ and $u \in X_0$ is a minimizer for $\gamma(\Omega)$. Let $t > 0$ such that $t|u| \in \mathcal{M}(\Omega)$. Then

$$t = \Big(\frac{\big\||u|\big\|_{X_0}^2}{\int_\Omega |u|^p dx} \Big)^{\frac{1}{p-2}} \leq \Big(\frac{\|u\|_{X_0}^2}{\int_\Omega |u|^p dx} \Big)^{\frac{1}{p-2}} = 1.$$

Consequently,

$$\gamma(\Omega) \leq J(t|u|) = t^p \big(\frac{1}{2} - \frac{1}{p} \big) \int_\Omega |u|^p dx \leq \gamma(\Omega).$$

Thus, $t = 1$ and $|u| \in \mathcal{M}(\Omega)$ is another minimizer for $\gamma(\Omega)$. For this reason we assume straight away that $u \geq 0$. Clearly, $u \in \mathbb{R}^{\mathcal{N}}$ is a minimizer for J_∞. Therefore, we obtain that $J'_\infty(u) = 0$. So that u is a solution of

$$(-\Delta)^s u = u^p \quad \text{in } \mathbb{R}^N.$$

By maximum principle [27, Proposition 2.2.8], $u > 0$ in $\mathbb{R}^N$. This is a contradiction.

(ii) For every $u \in \mathcal{N}_0$, one sees immediately that $tu \in \mathcal{M}(\Omega)$ for some $t > 0$. Indeed, $tu \in \mathcal{M}(\Omega)$ is equivalent to

$$\|tu\|_{X_0}^2 = \int_\Omega |tu|^p dx,$$

which has solution

$$t = \Big(\frac{\|u\|_{X_0}^2}{\int_\Omega |u|^p dx} \Big)^{\frac{1}{p-2}} > 0.$$

Since $u \in \mathcal{N}_0$ and $\max_{x \in \bar{\Omega}} Q(x) = 1$, we have

$$\|u\|_{X_0}^2 = \int_\Omega Q(x)(u^+)^p dx \leq \int_\Omega |u|^p dx,$$

which implies $t \leq 1$. Therefore,

$$\gamma(\Omega) \leq J(tu) = \big(\frac{1}{2} - \frac{1}{p} \big) \|tu\|_{X_0}^2 \leq \big(\frac{1}{2} - \frac{1}{p} \big) \|u\|_{X_0}^2.$$

By the arbitrariness of $u \in \mathcal{N}_0$, we have $\gamma(\Omega) \leq \alpha_0$.

By (2.36) and (2.38), we have

$$\frac{\|u_{\varepsilon,i}\|_{X_0}^2}{\big(\int_\Omega Q(x)(u_{\varepsilon,i}^+)^p dx \big)^{2/p}} = S_s + O(\varepsilon^{N-2s}) + o(\varepsilon^\sigma). \tag{2.46}$$

Direct computations show that

$$\sup_{t \geq 0} \Big(\frac{a}{2} t^2 - \frac{b}{p} t^p \Big) = \frac{s}{N} \Big(\frac{a}{b^{2/p}} \Big)^{N/(2s)}$$

for any $a, b > 0$. By (2.46), we obtain that

$$\begin{aligned} \sup_{t \geq 0} I_0(tu_{\varepsilon,i}) &= \frac{s}{N} \Big(\frac{\|u_{\varepsilon,i}\|_{X_0}^2}{\big(\int_\Omega Q(x)(u_{\varepsilon,i}^+)^p dx \big)^{2/p}} \Big)^{N/(2s)} \\ &\leq \frac{s}{N} S_s^{N/(2s)} + O(\varepsilon^{N-2s}) + o(\varepsilon^\sigma). \end{aligned} \tag{2.47}$$

Let $t_{\varepsilon,i} > 0$ be such that $t_{\varepsilon,i} u_{\varepsilon,i} \in \mathcal{N}_0$. Then, by (2.47), we have

$$\alpha_0 \leq I_0(t_{\varepsilon,i} u_{\varepsilon,i}) \leq \sup_{t \geq 0} I_0(tu_{\varepsilon,i}) \leq \frac{s}{N} S_s^{N/(2s)} + O(\varepsilon^{N-2s}) + o(\varepsilon^\sigma).$$

Passing to the limit, we obtain that $\alpha_0 \leq \gamma(\Omega)$. Thus $\gamma(\Omega) = \alpha_0$. □

To show that $\beta_{\lambda,i}$ are (PS) values, we need the following Palais-Smale decomposition theorem, see [20, Theorem 1.1] or [19, Theorem 4].

Theorem 2.14. *Assume that $\{u_n\}$ is a $(PS)_c$ sequence in X_0 for J. Then there exists a (possibly trivial) solution $u^0 \in X_0$ to problem (2.45) such that, up to a subsequence, $u_n \rightharpoonup u^0$ in X_0. Moreover, either the convergence is strong, or there exist $\ell \in \mathbb{N}$, nontrivial solutions $u^1, \dots, u^\ell \in \dot{H}^s(\mathbb{R}^N)$ to the equations*

$$(-\Delta)^s u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{2.48}$$

or

$$(-\Delta)^s u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N_+, \quad u = 0 \quad \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+, \tag{2.49}$$

sequences of points $x_n^1, \dots, x_n^\ell \subset \Omega$, and finitely many sequences of numbers $r_n^1, \dots, r_n^\ell \subset (0, +\infty)$ converging to zero such that, up to a subsequence,

$$u_n^j := (r_n^j)^{\frac{N-2s}{2}} u_n(x_n^j + r_n^j x) \rightharpoonup u^j \quad \text{in } \dot{H}^s(\mathbb{R}^N),$$

for $j = 1, \dots, \ell$, and

$$\lim_{n\to\infty} \big\| u_n - u^0 - \sum_{j=1}^{\ell} (r_n^j)^{\frac{2s-N}{2}} u^j\Big(\frac{x - x_n^j}{r_n^j}\Big) \big\|_{\dot{H}(\mathbb{R}^N)} = 0,$$

$$\lim_{n\to\infty} \|u_n\|_{X_0}^2 = \sum_{j=0}^{\ell} \|u^j\|^2_{\dot{H}^s(\mathbb{R}^N)},$$

$$\lim_{n\to\infty} J(u_n) = J(u^0) + \sum_{j=1}^{\ell} J_\infty(u^j),$$

$$\big|\ln \frac{r_n^i}{r_n^j}\big| + \big|\frac{x_n^i - x_n^j}{r_n^i}\big| \to \infty \quad \text{as } n \to \infty, \quad \text{for } i \neq j,\ i,j = 1, \dots, \ell.$$

Lemma 2.15. *There exists $\delta_0 > 0$ such that if $u \in \mathcal{N}_0$ and $I_0(u) \le \alpha_0 + \delta_0$, then $\mathbf{K}(u) \in \mathcal{P}_{\rho_0/2}$.*

Proof. Assume by contradiction that there exists a sequence $\{u_n\} \subset \mathcal{N}_0$ such that

$$I_0(u_n) = \alpha_0 + o(1) \quad \text{and} \quad \mathbf{K}(u_n) \notin \mathcal{P}_{\rho_0/2},$$

for all $n \in \mathbb{N}$. Let $s_n > 0$ be such that $s_n u_n \in \mathcal{M}(\Omega)$. By Lemma 2.13, We obtain

$$\gamma(\Omega) \le J(s_n u_n) \le I_0(s_n u_n) \le \sup_{s\ge 0} I_0(s u_n) = I_0(u_n) = \gamma(\Omega) + o(1).$$

Then, $s_n = 1 + o(1)$ and $J(s_n u_n) = \gamma(\Omega) + o(1)$. By Ekeland's variational principle[14], there exists a sequence $\{v_n\} \subset \mathcal{M}(\Omega)$ such that

$$J'(v_n) \to 0, \quad J(v_n) \to \gamma(\Omega), \quad \|v_n - s_n u_n\|_{X_0} \to 0.$$

By Lemma 2.13 and Theorem 2.14, there exists a (possibly trivial) solution $v^0 \in X_0$ to problem (2.45) such that $v_n \rightharpoonup v^0$ in X_0, and there exist $\ell \in \mathbb{N}$, nontrivial solutions $v^1, \dots, v^\ell \in \dot{H}^s(\mathbb{R}^N)$ to (2.48) or (2.49), sequences of points $x_n^1, \dots, x_n^\ell \subset \Omega$ and finitely many sequences of numbers $r_n^1, \dots, r_n^\ell \subset (0, +\infty)$ converging to zero such that, up to a subsequence,

$$v_n = v^0 + \sum_{j=1}^{\ell} (r_n^j)^{\frac{2s-N}{2}} v^j\Big(\frac{x - x_n^j}{r_n^j}\Big) + o(1) \quad \text{in } \dot{H}^s(\mathbb{R}^N), \tag{2.50}$$

and

$$J(v_n) = J(v^0) + \sum_{j=1}^{\ell} J_\infty(v^j) + o(1). \tag{2.51}$$

If $v^0 \neq 0$ or $\ell > 1$, then by (2.51), we have $J(v_n) \to J(v^0) + \sum_{j=1}^{\ell} J_\infty(v^j) > \gamma(\Omega)$, which is a contradiction. Thus, by (2.50),

$$v_n = (r_n^1)^{\frac{2s-N}{2}} v^1\Big(\frac{x - x_n^1}{r_n^1}\Big) + o(1) \quad \text{in } \dot{H}^s(\mathbb{R}^N). \tag{2.52}$$

By following the argument in [20, Theorem 1.1], we have $\operatorname{dist}(x_n^1, \partial\Omega)/r_n^1 \to \infty$ as $n \to \infty$. We may assume $x_n^1 \to x_0^1 \in \bar{\Omega}$ since Ω is bounded. By Lebesgue dominated convergence theorem, we obtain that

$$\begin{aligned}
S_s^{N/(2s)} &\leq \int_\Omega Q(x)|v_n|^p dx + o(1) \\
&= (r_n^1)^{-N} \int_\Omega Q(x)\big|v^1\big(\frac{x - x_n^1}{r_n^1}\big)\big|^p dx + o(1) \\
&= \int_{\Omega_n} Q(xr_n^1 + x_n^1)|v^1(x)|^p dx + o(1) \\
&= Q(x_0^1) \int_{\mathbb{R}^N} |v^1(x)|^p \ \mathbf{1}_{\Omega_n} dx + \int_{\mathbb{R}^N} \big[Q(xr_n^1 + x_n^1) \\
&\quad - Q(x_0^1)\big] |v^1(x)|^p \ \mathbf{1}_{\Omega_n} dx + o(1) \\
&= Q(x_0^1) \int_{\mathbb{R}^N} |v^1(x)|^p dx + o(1),
\end{aligned}$$

where $\mathbf{1}_{\Omega_n}$ is the indicator function,

$$\mathbf{1}_{\Omega_n} := \begin{cases} 1, & \text{if } x \in \Omega_n, \\ 0, & \text{if } x \notin \Omega_n, \end{cases}$$

$\Omega_n := \{x \in \mathbb{R}^N : xr_n^1 + x_n^1 \in \Omega\} \to \mathbb{R}^N$ as $n \to \infty$. Thus, we obtain that $x_0^1 \in \mathcal{P}$. Consequently,

$$\begin{aligned}
\mathbf{K}(u_n) &= \frac{\int_\Omega \chi(x)\big|v^1\big(\frac{x - x_n^1}{r_n^1}\big)\big|^p dx}{\int_\Omega \big|v^1\big(\frac{x - x_n^1}{r_n^1}\big)\big|^p dx} + o(1) \\
&= \frac{\int_{\Omega_n} \chi(xr_n^1 + x_n^1)|v^1(x)|^p dx}{\int_{\Omega_n} |v^1(x)|^p dx} + o(1) \\
&\to x_0^1 \in \mathcal{P}_{\rho_0/2}
\end{aligned}$$

as $n \to \infty$. We get a contradiction. □

Lemma 2.16. *There exists $\Lambda^* \in \big(0, \frac{q}{2}\Lambda\big)$ such that if $\lambda \in (0, \Lambda^*)$ and $u \in \mathcal{N}_\lambda^-$ with $I_\lambda(u) \leq \frac{s}{N} S_s^{N/(2s)} + \frac{\delta_0}{2}$ (δ_0 is the constant from Lemma 2.15), then $\mathbf{K}(u) \in \mathcal{P}_{\rho_0/2}$.*

Proof. Fix any $u \in \mathcal{N}_\lambda^-$ with $I_\lambda(u) \leq \frac{s}{N} S_s^{N/(2s)} + \frac{\delta_0}{2}$, and let

$$t(u) = \Big(\frac{\|u\|_{X_0}^2}{\int_\Omega Q(x)(u^+)^p dx}\Big)^{1/(p-2)}.$$

Clearly,$t(u)u \in \mathcal{N}_0$. For $\lambda \in (0, \Lambda_0)$, by (2.1), we have

$$\begin{aligned}\big(\frac{1}{2}-\frac{1}{p}\big)\|u\|_{X_0}^2 &\le I_\lambda(u)+\lambda\big(\frac{1}{q}-\frac{1}{p}\big)|\Omega|^{\frac{p-q}{p}}S_s^{-q/2}\|u\|_{X_0}^q\\ &\le \frac{s}{N}S_s^{N/(2s)}+\frac{\delta_0}{2}+\Lambda_0\big(\frac{1}{q}-\frac{1}{p}\big)|\Omega|^{\frac{p-q}{p}}S_s^{-q/2}\|u\|_{X_0}^q.\end{aligned}$$

Thus, there exists a constant C_1 independent of λ and u such that $\|u\|_{X_0} \le C_1$. By (2.3), we obtain

$$\frac{2-q}{p-q} \le \frac{\int_\Omega Q(x)(u^+)^p dx}{\|u\|_{X_0}^2}.$$

Consequently,

$$t(u) \le \Big(\frac{p-q}{2-q}\Big)^{1/(p-2)}.$$

Since $t^-(u) = 1$ and $t_{\max} < t(u)$ ($t_{\max}$ is defined in Lemma 2.3), by Lemma 2.3, we have

$$\begin{aligned}\frac{s}{N}S_s^{N/(2s)}+\frac{\delta_0}{2} \ge I_\lambda(u) &= \sup_{t\ge t_{\max}} I_\lambda(tu)\\ &\ge I_\lambda(t(u)u)\\ &\ge I_0(t(u)u)-\frac{\lambda}{q}\int_\Omega (t(u)u^+)^q dx.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}I_0(t(u)u) &\le \frac{s}{N}S_s^{N/(2s)}+\frac{\delta_0}{2}+\frac{\lambda}{q}\int_\Omega (t(u)u^+)^q dx\\ &\le \frac{s}{N}S_s^{N/(2s)}+\frac{\delta_0}{2}+\frac{\lambda}{q}|\Omega|^{\frac{p-q}{p}}S_s^{-q/2}t^q(u)\|u\|_{X_0}^q\\ &\le \frac{s}{N}S_s^{N/(2s)}+\frac{\delta_0}{2}+\frac{\lambda}{q}|\Omega|^{\frac{p-q}{p}}S_s^{-q/2}C_1^q\big(\frac{p-q}{2-q}\big)^{q/(p-2)}.\end{aligned}$$

Consequently, there exists $\Lambda^* \in (0, q\Lambda/2)$ such that

$$I_0(t(u)u) \le \frac{s}{N}S_s^{N/(2s)}+\delta_0$$

for $\lambda \in (0, \Lambda^*)$. By Lemma 2.15, we have

$$\mathbf{K}(t(u)u) = \frac{\int_{\mathbb{R}^N}\chi(x)|t(u)u|^p dx}{\int_{\mathbb{R}^N}|t(u)u|^p dx} \in \mathcal{P}_{\rho_0/2}$$

for $\lambda \in (0, \Lambda^*)$. Meanwhile, $\mathbf{K}(u) \in \mathcal{P}_{\rho_0/2}$ for $\lambda \in (0, \Lambda^*)$. □

Lemma 2.17. *For each $u \in \mathcal{N}_\lambda$, there exist $\varepsilon > 0$ and a differential function $\eta : B_\varepsilon(0) \subset X_0 \to (0, +\infty)$ such that*

$$\begin{gathered}\eta(0)=1, \quad \eta(w)(u-w)\in\mathcal{N}_\lambda \quad \text{for } w\in B_\varepsilon(0),\\ \langle \eta'(0), w\rangle = \frac{2(u,w)_{X_0}-\lambda q\int_\Omega (u^+)^{q-1}w dx - p\int_\Omega Q(u^+)^p dx}{(2-q)\|u\|_{X_0}^2-(p-q)\int_\Omega Q(u^+)^p dx}\end{gathered} \tag{2.53}$$

for all $w \in X_0$.

Since the proof of the above Lemma is similar to that of Lemma 2.5, we omit it.

Lemma 2.18. *For each $1 \le i \le m$, there exists a $(PS)_{\beta_{\lambda,i}}$ sequence $\{u_n^i\} \subset \mathcal{O}_{\lambda,i}$ for I_λ.*

Proof. By Lemma 2.16, we have

$$\tilde{\beta}_{\lambda,i} \geq \frac{s}{N} S_s^{N/(2s)} + \frac{\delta_0}{2} \tag{2.54}$$

for all $\lambda \in (0, \Lambda^*)$. By Lemma 2.10, we have

$$\beta_{\lambda,i} \leq \alpha_\lambda^- < \beta^* \tag{2.55}$$

for all $\lambda \in (0, \Lambda^*)$. For each $1 \leq i \leq m$, by (2.54) and (2.55), we have

$$\beta_{\lambda,i} < \tilde{\beta}_{\lambda,i} \tag{2.56}$$

for all $\lambda \in (0, \Lambda^*)$. Then

$$\beta_{\lambda,i} = \inf_{u \in \mathcal{O}_{\lambda,i} \cup \partial \mathcal{O}_{\lambda,i}} I_\lambda(u)$$

for all $\lambda \in (0, \Lambda^*)$. By Lemma 2.17 and Ekeland's variational principle, we can prove Lemma 2.18. The rest of proof is similar to that of Lemma 2.6, we omit it. □

Proof of Theorem 1.1. For each $1 \leq i \leq m$, by Lemma 2.18, there exists a $(PS)_{\beta_{\lambda,i}}$ sequence $\{u_n^i\} \subset \mathcal{O}_{\lambda,i}$ for I_λ. Since I_λ satisfies the $(PS)_\beta$ condition for $\beta < \beta^*$, by (2.55), I_λ has at least m critical points in $\mathcal{N}_\lambda^-$ for $\lambda \in (0, \Lambda^*)$. Consequently, problem (1.1) has m positive solutions. Furthermore, since $u \in \mathcal{N}_\lambda^+$ is a solution of (1.1), as shown in Theorem 1.1, problem (1.1) has $m+1$ positive solutions. □

3. Critical and convex case $q > 2$

This section is devoted to the study of problem (1.1) when the exponent satisfies $2 < q < p$. As the energy functional I_λ is not bounded below on X_0, it is useful to consider the functional on the Nehari manifold

$$\begin{aligned}
\mathcal{N}_\lambda &= \{u \in X_0 \setminus \{0\} : \langle I_\lambda'(u), u \rangle = 0\} \\
&= \big\{u \in X_0 \setminus \{0\} : \|u\|_{X_0}^2 = \lambda \int_\Omega (u^+)^q dx + \int_\Omega Q(x)(u^+)^p dx \big\}.
\end{aligned}$$

Now, we give some properties of $\mathcal{N}_\lambda$.

Lemma 3.1. *The functional I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Proof. For every $u \in \mathcal{N}_\lambda$, we have

$$I_\lambda(u) = \big(\frac{1}{2} - \frac{1}{q}\big) \|u\|_{X_0}^2 + \big(\frac{1}{q} - \frac{1}{p}\big) \int_\Omega Q(x)(u^+)^p dx > 0, \tag{3.1}$$

since $2 < q < p$. Thus, I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$. □

Lemma 3.2. *For each $u \in X_0^+$, there exists $t(u) > 0$ such that $t(u)u \in \mathcal{N}_\lambda$ and $I_\lambda(t(u)u) = \sup_{t \geq 0} I_\lambda(tu)$.*

Proof. We define $\gamma(t) = I_\lambda(tu)$ for $t \geq 0$. It is easy to see that there exists $t(u) > 0$ such that $\gamma'(t) > 0$ for $t \in (0, t(u))$ and $\gamma'(t) < 0$ for $t \in (t(u), +\infty)$. Then $\sup_{t \geq 0} \gamma(t)$ is attained at some $t(u) > 0$. This implies that $\gamma'(t(u)) = 0$. Consequently, $t(u)u \in \mathcal{N}_\lambda$. □

To prove the existence of positive solutions, we claim that I_λ satisfies the $(PS)_\beta$ condition in X_0 for $\beta < \frac{s}{N} S_s^{\frac{N}{2s}}$.

Lemma 3.3. *I_λ satisfies the $(PS)_\beta$ condition in X_0 for $\beta < \frac{s}{N} S_s^{\frac{N}{2s}}$.*

The proof of the above lemma is similar to that of Lemma 2.8, we omit it. Direct computation yields the following estimates.

Lemma 3.4. *There exists a positive constant C_r such that*

$$\int_\Omega |u_{\varepsilon,i}|^r dx \geq \begin{cases} C_r\varepsilon^{N-\frac{(N-2s)r}{2}}, & \text{if } r > \frac{N}{N-2s}, \\ C_r\varepsilon^{\frac{N}{2}}|\ln\varepsilon|, & \text{if } r = \frac{N}{N-2s}, \\ C_r\varepsilon^{\frac{(N-2s)r}{2}}, & \text{if } r < \frac{N}{N-2s}. \end{cases} \tag{3.2}$$

Next, we want to obtain an estimate of $\sup_{t\geq 0} I_\lambda(tu_{\varepsilon,i})$.

Lemma 3.5. *There exists $\varepsilon_0 \in (0, \rho_0/2)$ such that if $\varepsilon \in (0, \varepsilon_0)$, then*

$$\sup_{t\geq 0} I_\lambda(tu_{\varepsilon,i}) < \frac{s}{N} S_s^{\frac{N}{2s}} \quad \text{uniformly in } i. \tag{3.3}$$

Proof. Since I_λ is continuous in X_0 and $u_{\varepsilon,i}$ is uniformly bounded in X_0, there exists $t_1 > 0$ such that for $t \in [0, t_1]$,

$$I_\lambda(tu_{\varepsilon,i}) < \frac{s}{N} S_s^{\frac{N}{2s}}.$$

By (2.37), we have

$$\int_\Omega u_{\varepsilon,i}^p dx \geq \frac{1}{2} S_s^{\frac{N}{2s}}$$

for ε small enough. Thus, $I(tu_{\varepsilon,i}) \to -\infty$ as $t \to \infty$ uniformly in ε and i. Consequently, there exists $t_2 > t_1$ such that $I_\lambda(tu_{\varepsilon,i}) < \frac{s}{N} S_s^{\frac{N}{2s}}$ for $t \geq t_2$. Then, we only need to verify that inequality

$$\sup_{t_1\leq t\leq t_2} I_\lambda(tu_{\varepsilon,i}) < \frac{s}{N} S_s^{\frac{N}{2x}} \quad \text{uniformly in } i,$$

for ε small enough.

From now on, we assume that $t \in [t_1, t_2]$. Since $N \geq 4$, we obtain that $q > 2 > \frac{N}{N-2s}$. Consequently, by Lemma 3.4,

$$\int_\Omega |u_{\varepsilon,i}|^q dx \geq C_q \varepsilon^{N-\frac{(N-2s)q}{2}}. \tag{3.4}$$

By (2.36), (2.38) and (3.4), we have

$$\begin{aligned} I_\lambda(tu_{\varepsilon,i}) &= \frac{1}{2}t^2\|u_{\varepsilon,i}\|_{X_0}^2 - \frac{\lambda}{q}t^q\int_\Omega |u_{\varepsilon,i}|^q dx - \frac{1}{p}t^p\int_\Omega Q(x)|u_{\varepsilon,i}|^p dx \\ &= S_s^{\frac{N}{2s}}\Big(\frac{t^2}{2} - \frac{t^p}{p}\Big) - \frac{\lambda}{q}t^q\int_\Omega |u_{\varepsilon,i}|^q dx + O(\varepsilon^{N-2s}) + o(\varepsilon^\sigma) \\ &\leq \frac{s}{N}S_s^{\frac{N}{2s}} - C_q t_1^q \frac{\lambda}{q}\varepsilon^{N-\frac{(N-2s)q}{2}} + O(\varepsilon^{N-2s}) + o(\varepsilon^\sigma). \end{aligned}$$

Since $N \geq 4$ and $\sigma = N - \frac{(N-2s)q}{2}$, we can choose $\varepsilon_0 > 0$ small enough such that

$$-C_q t_1^q \frac{\lambda}{q}\varepsilon^{N-\frac{(N-2s)q}{2}} + O(\varepsilon^{N-2s}) + o(\varepsilon^\sigma) < 0$$

for all $\varepsilon \in (0, \varepsilon_0)$. Thus, we obtain (3.3). □

By Lemma 3.2, there exists $t(u_{\varepsilon,i}) > 0$ such that $t(u_{\varepsilon,i})u_{\varepsilon,i} \in \mathcal{N}_\lambda$. Then we have the following lemma.

Lemma 3.6. $\mathbf{K}(t(u_{\varepsilon,i})u_{\varepsilon,i}) \to a^i$ *as* $\varepsilon \to 0$. *In particular, there exists* $\varepsilon_1 \in (0, \varepsilon_0)$ *such that if* $\varepsilon \in (0, \varepsilon_1)$, *then* $\mathbf{K}(t(u_{\varepsilon,i})u_{\varepsilon,i}) \in \mathcal{P}_{\rho_0}$ *for each* $1 \le i \le m$.

The proof of the above lemma is similar to that of Lemma 2.12, we omit it. For each $1 \le i \le m$, we define

$$\begin{gathered}\mathcal{G}_{\lambda,i} = \{u \in \mathcal{N}_\lambda : |\mathbf{K}(u) - a^i| < \rho_0\},\\ \partial\mathcal{G}_{\lambda,i} = \{u \in \mathcal{N}_\lambda : |\mathbf{K}(u) - a^i| = \rho_0\},\\ \delta_{\lambda,i} = \inf_{u\in\mathcal{G}_{\lambda,i}} I_\lambda(u), \quad \tilde{\delta}_{\lambda,i} = \inf_{u\in\partial\mathcal{G}_{\lambda,i}} I_\lambda(u).\end{gathered}$$

Lemma 3.7. *There exists* $\delta_0 > 0$ *such that if* $u \in \mathcal{N}_0$ *and* $I_0(u) \le \alpha_0 + \delta_0$, *then* $\mathbf{K}(u) \in \mathcal{P}_{\rho_0/2}$.

The proofof the above lemma is similar to that of Lemma 2.15, we omit it.

Lemma 3.8. *There exists* $\Lambda^* > 0$ *such that if* $\lambda \in (0, \Lambda^*)$ *and* $u \in \mathcal{N}_\lambda$ *with* $I_\lambda(u) \le \frac{s}{N} S_s^{N/(2s)} + \delta_0$ *(*δ_0 *is the constant from Lemma 3.7), then* $\mathbf{K}(u) \in \mathcal{P}_{\rho_0/2}$.

Proof. Fix any $u \in \mathcal{N}_\lambda$ with $I_\lambda(u) \le \frac{s}{N} S_s^{N/(2s)} + \frac{\delta_0}{2}$, and let

$$t(u) = \Big(\frac{\|u\|_{X_0}^2}{\int_\Omega Q(x)(u^+)^p dx}\Big)^{1/(p-2)}.$$

Clearly, $t(u)u \in \mathcal{N}_0$. Since

$$I_\lambda(v) \ge \frac{1}{2}\|v\|_{X_0}^2 - \frac{\lambda}{q}|\Omega|^{\frac{p-q}{p}} S_s^{-q/2}\|v\|_{X_0}^q - \frac{1}{p} S_s^{-p/2}\|v\|_{X_0}^p, \quad \forall v \in X_0,$$

there exist positive numbers d_1 and d_2 such that $I_\lambda(v) \ge d_2$ if $\|v\|_{X_0} = d_1$.

Obviously, there exists $t_0 > 0$ such that $\|t_0 u\| = d_1$. By Lemma 3.2, we have

$$\begin{aligned}0 < d_2 &\le I_\lambda(t_0 u)\\ &\le \sup_{t\ge0} I_\lambda(tu)\\ &= I_\lambda(u)\\ &= \big(\frac{1}{2} - \frac{1}{p}\big)\|u\|_{X_0}^2 - \lambda\big(\frac{1}{q} - \frac{1}{p}\big)\int_\Omega (u^+)^q dx\\ &\le \big(\frac{1}{2} - \frac{1}{p}\big)\|u\|_{X_0}^2.\end{aligned}$$

Consequently, there exists a constant C_1 independent of λ and u such that $\|u\|_{X_0} \ge C_1$. On the other side, we have

$$\begin{aligned}\frac{s}{N} S_s^{N/(2s)} + \frac{\delta_0}{2} &\ge I_\lambda(u)\\ &= \big(\frac{1}{2} - \frac{1}{q}\big)\|u\|_{X_0}^2 + \big(\frac{1}{q} - \frac{1}{p}\big)\int_\Omega Q(x)(u^+)^p dx\\ &\ge \big(\frac{1}{2} - \frac{1}{q}\big)\|u\|_{X_0}^2.\end{aligned}$$

Thus, there exists a constant C_2 independent of λ and u such that $\|u\|_{X_0} \le C_2$. Moreover,

$$\int_\Omega Q(x)(u^+)^p dx = \|u\|_{X_0}^2 - \lambda\int_\Omega (u^+)^q dx$$

$$
\begin{aligned}
&\geq C_1 - \lambda|\Omega|^{\frac{p-q}{p}} S_s^{-q/2}\|u\|_{X_0}^q \\
&\geq C_1 - \lambda|\Omega|^{\frac{p-q}{p}} S_s^{-q/2} C_2.
\end{aligned}
$$

It follows that there exists $\Lambda > 0$ such that for $\lambda \in (0, \Lambda)$,

$$
\int_\Omega Q(x)(u^+)^p dx \geq C_1 - \Lambda|\Omega|^{\frac{p-q}{p}} S_s^{-q/2} C_2 > 0.
$$

Hence, there exists a constant $C_3 > 0$ independent of u such that $t(u) \leq C_3$ for $\lambda \in (0, \Lambda)$.

By Lemma 3.2, we have

$$
\begin{aligned}
\frac{s}{N} S_s^{N/(2s)} + \frac{\delta_0}{2} \geq I_\lambda(u) &= \sup_{t\geq 0} I_\lambda(tu) \\
&\geq I_\lambda(t(u)u) \\
&\geq I_0(t(u)u) - \frac{\lambda}{q}\int_\Omega (t(u)u^+)^q dx.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
I_0(t(u)u) &\leq \frac{s}{N} S_s^{N/(2s)} + \frac{\delta_0}{2} + \frac{\lambda}{q}\int_\Omega (t(u)u^+)^q dx \\
&\leq \frac{s}{N} S_s^{N/(2s)} + \frac{\delta_0}{2} + \frac{\lambda}{q}|\Omega|^{\frac{p-q}{p}} S_s^{-q/2} t^q(u)\|u\|_{X_0}^q \\
&\leq \frac{s}{N} S_s^{N/(2s)} + \frac{\delta_0}{2} + \frac{\lambda}{q}|\Omega|^{\frac{p-q}{p}} S_s^{-q/2} C_1^q \big(\frac{p-q}{2-q}\big)^{q/(p-2)}.
\end{aligned}
$$

Consequently, there exists $\Lambda^* \in (0, \Lambda)$ such that

$$
I_0(t(u)u) \leq \frac{s}{N} S_s^{N/(2s)} + \delta_0
$$

for $\lambda \in (0, \Lambda^*)$. By Lemma 3.7, we have

$$
\mathbf{K}(t(u)u) = \frac{\int_{\mathbb{R}^N} \chi(x)|t(u)u|^p dx}{\int_{\mathbb{R}^N} |t(u)u|^p dx} \in \mathcal{P}_{\rho_0/2}
$$

for $\lambda \in (0, \Lambda^*)$. Meanwhile, $\mathbf{K}(u) \in \mathcal{P}_{\rho_0/2}$ for $\lambda \in (0, \Lambda^*)$. ◻

According to Lemma 3.5 and 3.6, there exists $\varepsilon_1 > 0$ such that

$$
\delta_{\lambda,i} \leq I_\lambda(t(u_{\varepsilon,i})u_{\varepsilon,i}) < \frac{s}{N} S_s^{\frac{N}{2s}} \tag{3.5}
$$

for all $\varepsilon \in (0, \varepsilon_1)$. By Lemma 3.8, we obtain

$$
\tilde{\delta}_{\lambda,i} \geq \frac{s}{N} S_s^{\frac{N}{2s}} + \frac{\delta_0}{2} \tag{3.6}
$$

for $\lambda \in (0, \Lambda^*)$. By (3.5) and (3.6), we obtain $\tilde{\delta}_{\lambda,i} > \delta_{\lambda,i}$ for $\lambda \in (0, \Lambda^*)$. Thus,

$$
\delta_{\lambda,i} = \inf_{u\in\overline{\mathcal{G}}_{\lambda,i}} I_\lambda(u).
$$

Consequently, similar to that of Lemma 2.18, we obtain the following lemma.

Lemma 3.9. *For each $1 \leq i \leq m$, there exists a $(PS)_{\delta_{\lambda,i}}$ sequence $\{u_n^i\} \subset \mathcal{G}_{\lambda,i}$ for I_λ.*

Proof of Theorem 1.2. For each $1 \le i \le m$, by Lemma 3.9, there exists a $(PS)_{\delta_{\lambda,i}}$ sequence $\{u_n^i\} \subset \mathcal{G}_{\lambda,i}$ for I_λ. Since I_λ satisfies the $(PS)_\beta$ condition for $\beta < \frac{s}{N} S_s^{\frac{N}{2s}}$, by (3.5), I_λ has at least m critical points in $\mathcal{N}_\lambda$ for $\lambda \in (0, \Lambda^*)$. Consequently, problem (1.1) has m positive solutions. □

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References

[1] A. Ambrosetti, H. Brezis, G. Cerami; *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal., 122 (1994), 519-543.

[2] A. Ambrosetti, J. Garcia, I. Peral; *Multiplicity reults for some nonlinear elliptic equations*, J. Func. Anal., 137 (1996), 219-242.

[3] J. H. Bae, Y. H. Kim; *Multiple solutions for discontinuous elliptic problems involving the fractional Laplacian*, Electron. J. Differential Equations, 2019 (18) (2019), 1-16.

[4] A. Bahri, Y. Li; *On a min-max procedure for the existence of a positive solution for certain scalar field equations in* $\mathbb{R}^N$, Rev. Mat. Iberoam., 6 (1990), 1-15.

[5] B. Barrios, E. Colorado, A. de Pablo, U. Sanchez; *On some critical problems for the fractional Laplacian operator*, J. Differ. Equ., 252 (2012), 6133-6162.

[6] B. Barriosa, E. Colorado, R. Servadeid, F. Soria; *A critical fractional equation with concave-convex power nonlinearities*, Ann. I. H. Poincaré-AN, 32 (2015), 875-900.

[7] T. Bartsch, M. Willem; *On an elliptic equation with concave and convex nonlinearities*, Proc. Amer. Math. Soc., 123 (1995), 3555-3561.

[8] V. Benci, G. Cerami; *Positive solutions of semilinear elliptic equation in exterior domins*, Arch. Ration. Mech. Anal., 99 (1987), 283-300.

[9] H. Brézis, L. Nirenberg; *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., 36 (1983), 437-477.

[10] K. J. Brown, T. Wu; *A fibering map approach to a semilinear elliptic boundary value problem*, Elec. J. Differ. Equ., 2007 (2007), 1-9.

[11] X. Cabré, J. Tan; *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math., 224 (2010), 2052-2093.

[12] M. Chhetri, P. Girg, E. Hollifield; *Existence of positive solutions for fractional Laplacian equations: theory and numerical experiments*, Electron. J. Differential Equations, 2020 (81) (2020), 1-31.

[13] R. Cont, P. Tankov; *Financial Modeling with Jump Processes*, Chapman & Hall/CRC Financ. Math. Ser., Chapman and Hall/CRC, BocaRaton, Fl, 2004.

[14] I. Ekeland; *On the variational principle*, J. Math. Anal. Appl., 17 (1974), 324-353.

[15] A. Fiscella, R. Servadei, E. Valdinoci; *Density properties for fractional Sobolev spaces*, Ann. Acad. Sci. Fenn. Math., 40 (2015), 235-253.

[16] J. Garcia, I. peral; *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term*, Trans. Amer. math. Soc., 323 (1991), 877-895.

[17] A. Majda, E. Tabak; *A two-dimensional model for quasigeostrophic flow: comparison with the two-dimensional Euler flow*, in: Nonlinear Phenomena in Ocean Dynamics, Los Alamos, NM, 1995, Physica D, 98 (1996), 515-522.

[18] E. D. Nezza, G. Palatucci, E. Valdinoci; *Hitchhikers guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136 (2012), 521-573.

[19] G. Palatucci, A. Pisante; *Improved Sobolev embeddings, profile decomposition, and concentration compactness for fractional Sobolev spaces*, Calc. Var. Partial Differ. Equ., 50 (2014), 799-829.

[20] G. Palatucci, A. Pisante; *A global compactness type result for Palais-Smale seuqences in fractional spaces*, Nonlinear Anal., 117 (2015), 1-7.

[21] X. Ros-Oton, J. Serra; *The Pohozaev identity for the fractional Laplacian*, Arch. Ration. Mech. Anal., 213 (2014), 587-628.

[22] S. Secchi, N. Shioji, M. Squassina; *Coron problem for fractional equations*, Differ. Integral Equ., 28 (2015), 103-118.
[23] R. Servadei, E. Valdinoci; *Lewy-Stampacchia type estimates for variational inequalities driven by (non)local operators*, Rev. Mat. Iberoam., 29 (2013), 1091-1126.
[24] R. Servadei, E. Valdinoci; *Variational methods for the non-local operators of elliptic type*, Discrete Contin. Dyn. Syst., 33 (2013), 2105-2137.
[25] R. Servadei, E. Valdinoci; *A Brezis-Nirenberg result for non-local critical equations in low dimension*, Comm. Pure Appl. Anal., 12 (2013), 2445-2464.
[26] R. Servadei, E. Valdinoci; *The Brezis-Nirenberg result for the fractional Laplacian*, Tran. Amer. Math. Soc., 367 (2015), 67-102.
[27] L. Silvestre; *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math., 60 (2007), 67-112.
[28] M. Struwe; *Variational methods, applications to nonlinear partial differential equations and Hamilton systems*, Fourth Edition, Springer-Verlag, Berlin, 2008.
[29] G. Tarantello; *On nonhomogeneous elliptic equations invoving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Lineairé, 9 (1992), 281-304.
[30] E. Valdinoci; *From the long jump random walk to the fractional Laplacian*, Bol. Soc. Esp. Mat. Apl. Se MA, 49 (2009), 33-44.
[31] L. Vlahos, H. Isliker, Y. Kominis, K. Hizonidis; *Normal and a nomalous diffusion: atutorial*, in: T. Bountis(Ed.), Order and Chaos, 10th Volume, Patras University Press, 2008.

Yajing Zhang (corresponding author)
School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, China
Email address: zhangyj@sxu.edu.cn

Qiaoqin Li
School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, China
Email address: 1525760982@qq.com

Lu Pang
School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, China
Email address: 757005378@qq.com