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EXTINCTION IN FINITE TIME OF SOLUTIONS TO FRACTIONAL PARABOLIC POROUS MEDIUM EQUATIONS WITH STRONG ABSORPTION

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Dedicated to Prof. Jesus Ildefonso Díaz on his 70th birthday

Abstract. In this article we study the solutions of a general fractional parabolic porous medium equation with a non-Lipschitz absorption term. We obtain the existence of weak solutions, L^p-estimates, and decay estimates. Also, we show that weak solutions must vanish after a finite time, even for large initial data.

1. Introduction

In this article, we study the fractional parabolic porous medium equation with a non-Lipschitz absorption term,

$$\begin{gathered}\partial_t u - \operatorname{div}(|u|^{m_1}\nabla(-\Delta)^{-s}[|u|^{m_2-1}u]) + |u|^{\beta-1}u = 0 \quad \text{in } \mathbb{R}^N \times (0,T),\\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N,\end{gathered} \tag{1.1}$$

where $m_1, m_2 > 0$, $s \in (0,1)$, $\beta \in (0,1)$, and $N \geq 2$. Equations of type (1.1) with $s = 0$ and $m_2 = 1$, without the absorption term, correspond to the well-known porous medium equation $\partial_t u = \operatorname{div}(u^{m_1}\nabla u)$.

This equation appears in applications such as the standard model for gas flow through a porous medium (Darcy-Leibenzon-Muskat), Boussinesq's model of groundwater flow, and a model of population dynamics (Gurtin-McCamy) (see [19]). These applications have served as a motivation for many authors to study equation (1.1), see for example [2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 16, 17], and the references therein. Most of the known results concern the existence of weak solutions, decay estimates and finite speed of propagation. This is the main feature of porous media equations and gave rise to free boundary problems.

Now, we like to mention some recent results concerning equation (1.1). Biler et al. [1] studied (1.1) with $\alpha = 2(1-s)$, $m_1 = 1$, $m = m_2 + 1$, without the absorption term $|u|^{\beta-1}u$:

$$\partial_t u - \operatorname{div}(|u|\nabla^{\alpha-1}(|u|^{m-2}u)) = 0\,,$$

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They constructed nonnegative self-similar solutions of Barenblatt-Pattle-Zeldovich type, and obtained an existence of weak solutions u satisfying the decay estimate

$$\|u(t)\|_{L^p} \leq Ct^{-\frac{N(1-\frac{1}{p})}{N(m-1)+\alpha}} \|u_0\|_{L^1}^{\frac{N(m-1)/p+\alpha}{N(m-1)+\alpha}} . \tag{1.2}$$

Stan et al. [18] investigated (1.1) with $s \in (0,1)$, $m_2 = 1$, $m_1 = m-1 > 0$, without the absorption term. The authors studied the existence of nonnegative weak solutions for all integrable initial data u_0. They obtained the smoothing effect L^p-L^∞, for $p \geq 1$:

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{N}{N(m-1)+2p(1-s)}} \|u_0\|_{L^p}^{\frac{2p(1-s)}{N(m-1)+2p(1-s)}} , \tag{1.3}$$

with $C = C(N,s,m,p) > 0$. Moreover, the finite and infinite speed of propagation have been also studied by the same authors in [17]. Very recently, Dao-Díaz studied (1.1), and obtained the following result.

Theorem 1.1 ([12])**.** *Let $m_1, m_2 > 0$ and $s \in (0,1)$. Suppose that $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then, there exists a weak solution u of (1.1) satisfying the following properties:*

(i) *L^q-estimates: For any $1 \leq q \leq \infty$, we have*
$$\|u(t)\|_{L^q} \leq \|u_0\|_{L^q}, \quad \text{for a.e. } t \in (0,T) . \tag{1.4}$$
(ii) *Decay estimates: Let $p \geq 1$ be such that $m_1 + m_2 > 1 - \frac{2p(1-s)}{N}$. Then*
$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{1}{p(1-\alpha_0)+\sigma_0}} \|u_0\|_{L^p}^{\frac{p(1-\alpha_0)}{p(1-\alpha_0)+\sigma_0}}, \tag{1.5}$$
with $\alpha_0 = \big(N - 2(1-s)\big)/N$, and $\sigma_0 = m_1 + m_2 - 1$.

(iii) *Finite time extinction: If $m_1 + m_2 < \alpha_0$ then, there is a finite time $T_0 > 0$ such that*
$$u(x,t) = 0, \quad \text{for } (x,t) \in \mathbb{R}^N \times (T_0, \infty) . \tag{1.6}$$

Inspired by the above results, we want to prove the existence of weak solutions to equation (1.1), which satisfies estimates (1.4), (1.5). After that, we show that such a weak solution must vanish after a finite time, even when beginning with a large initial data u_0. It is known that this phenomenon occurs because of the strong absorption term $|u|^{\beta-1}u$. see [11, 13, 14] for another strong absorption term $u^{-\beta}\chi_{\{u>0\}}$.

Let us define

$$\Theta(u) = |u|^{m_1}\nabla(-\Delta)^{-s}[|u|^{m_2-1}u], \quad Q_T = \mathbb{R}^N \times (0,T).$$

Definition 1.2. Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We say that u is a weak solution of (1.1) if $u \in L^1(0,T; L^\infty(\mathbb{R}^N)) \cap L^\infty(Q_T)$ satisfies $\operatorname{div}\Theta(u) \in L^2(0,T; Y(B_R))$, and

$$\int_0^T \int_{\mathbb{R}^N} (-u\varphi_t + \Theta(u)\cdot\nabla\varphi + |u|^{\beta-1}u\varphi)\,dx\,dt = 0, \quad \forall \varphi \in \mathcal{C}_c^\infty(Q_T),$$

where

$$Y(B_R) = \begin{cases} H^{-1}(B_R), & \text{if } s \in [1/2,1), \\ W^{-2,p}(B_R),\ p > 1 \text{ such that } \frac{m_2 p}{p-1} \geq 1, & \text{if } s \in (0,1/2). \end{cases}$$

Note that $H^{-1}(B_R)$ is the dual space of $H_0^1(B_R)$, and $W^{-2,p}(B_R)$ the dual space of $W_0^{2,p}(B_R)$. Here B_R is the ball in $\mathbb{R}^N$, with center at 0 and radius R.

Remark 1.3. It follows from Definition 1.2 that $u \in \mathcal{C}([0,T];Y(B_R))$, for any $R>0$. Thus, $u(t)$ possesses an initial trace u_0 in this sense. In particular, if either $s\in[1/2,1)$ or $m_2>m_1$, then $u\in\mathcal{C}([0,T];H^{-1}(B_R))$ for every $R>0$.

Our main results read as follows.

Theorem 1.4. *Let $s\in(0,1)$, $\beta\in(0,1)$, and $m_1,m_2>0$. Suppose that $u_0\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. Then, there exists a weak solution of* (1.1) *satisfying* (1.4), (1.5)*, and* (1.6) *in Theorem 1.1.*

Concerning the finite time extinction of solutions, it suffices to consider $m_1+m_2\ge\alpha_0$ in the following theorem since u vanishes after a finite time if provided $m_1+m_2<\alpha_0$.

Theorem 1.5. *Assume the hypotheses in Theorem 1.4. Suppose that $m_1+m_2\ge\alpha_0$. Then, there exists a finite time $T_0>0$ such that*

$$u(x,t)=0,\quad \textit{for } (x,t)\in\mathbb{R}^N\times(T_0,\infty)\,. \tag{1.7}$$

And T_0 can be estimated as follows

$$T_0\le C\|u_0\|_{L^p}^{p(1-\gamma_0)}\,, \tag{1.8}$$

for some constant $C>0$ (independent of u_0), with

$$\gamma_0=\frac{1}{1+\frac{2(1-s)(1-\beta)}{2(1-s)(p-1)+N(m_1+m_2)-\beta(N-2(1-s))}}\,.$$

Note that $\gamma_0\in(0,1)$ since $m_1+m_2\ge\alpha_0$.

Through this paper, the constant C may change value from step by step. Moreover, $C=C(\alpha,\beta,\gamma)$ means that the constant C merely depends on the parameters α,β,γ. We denote $\|\cdot\|_{X(\mathbb{R}^N)}=\|\cdot\|_X$, and $\int_{\mathbb{R}^N}f(x)dx=\int f(x)dx$. Finally, $A\lesssim B$ means that there exists a positive constant c, independent of the data, such that $A\le cB$.

2. Functional setting

Let $p\ge1$, and $s\in(0,1)$. For a given domain $\Omega\subset\mathbb{R}^N$, we define the fractional Sobolev space

$$W^{s,p}(\Omega)=\big\{u\in L^p(\Omega):\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dx\,dy<\infty\big\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)}=\Big(\|u\|_{L^p(\Omega)}^p+\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dx\,dy\Big)^{1/p}.$$

We also denote the homogeneous fractional Sobolev space by $\dot W^{s,p}(\Omega)$, endowed with the seminorm

$$\|u\|_{\dot W^{s,p}(\Omega)}=\Big(\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dx\,dy\Big)^{1/p}.$$

In particular, we denote $W^{s,2}(\mathbb{R}^N)$ by $H^s(\mathbb{R}^N)$, which turns out to be a Hilbert space. It is well-known that we have the equivalent characterization

$$H^s(\mathbb{R}^N)=\big\{u\in L^2(\mathbb{R}^N):\int(1+|\xi|^{2s})|\mathcal{F}\{u\}(\xi)|^2\,d\xi<\infty\big\}\,,$$

where $\mathcal{F}$ denotes the Fourier transform, and that we have

$$\|u\|_{H^s(\mathbb{R}^N)} = \Big(\int (1+|\xi|^{2s})|\mathcal{F}\{u\}(\xi)|^2 \, d\xi \Big)^{1/2}.$$

In addition, for $u \in H^s(\mathbb{R}^N)$, the fractional Laplacian is defined by

$$(-\Delta)^s u(x) = C(N,s)\text{p.v.} \int \frac{u(x)-u(y)}{|x-y|^{N+2s}} \, dy = \mathcal{F}^{-1}\{|\xi|^{2s}\mathcal{F}(u)(\xi)\} . \tag{2.1}$$

Then

$$\|u\|^2_{H^s(\mathbb{R}^N)} = \|u\|^2_{L^2} + C\|(-\Delta)^{s/2}u\|^2_{L^2} .$$

We emphasize that if $s > 0$, then $(-\Delta)^{-s} = \mathcal{I}_{2s}$ is the Riesz potential. Moreover, the fractional gradient ∇^s can be written as $\nabla \mathcal{I}_{1-s}$. And for any smooth bounded function $v : \mathbb{R}^N \to \mathbb{R}$, we have

$$\nabla^s v = C(N,s) \int_{\mathbb{R}^N} (v(x) - v(x+z)) \frac{z}{|z|^{N+1+s}} \, dz ,$$

with a suitable constant $C(N,s)$, see [1].

Approximation of the fractional Laplacian. For any $s \in (0,1)$, and for $\varepsilon > 0$, let us define the operator

$$\mathcal{L}^s_\varepsilon[f](x) := \int \frac{f(x)-f(y)}{(|x-y|^2+\varepsilon^2)^{\frac{N+2s}{2}}} \, dy, \tag{2.2}$$

for $x \in \mathbb{R}^N$, and $f \in \mathcal{S}(\mathbb{R}^N)$ (the Schwartz space). Note that $\mathcal{L}^s_\varepsilon$ can be considered as a regularization of the fractional Laplacian $(-\Delta)^s$ (see [7]).

• Square root: By symmetry, we observe that

$$\langle \mathcal{L}^s_\varepsilon[f], f\rangle_{L^2} = \frac{1}{2} \int\int \frac{|f(x)-f(y)|^2}{(|x-y|^2+\varepsilon^2)^{\frac{N+2s}{2}}} \, dx \, dy .$$

Then, we denote $\mathcal{L}^{s/2}_\varepsilon[f]$ as a square root of $\mathcal{L}^s_\varepsilon[f]$ in the Fourier transform sense, and

$$\|\mathcal{L}^{s/2}_\varepsilon[f]\|^2_{L^2} = \langle \mathcal{L}^s_\varepsilon[f], f\rangle_{L^2} .$$

The following lemmas will be useful in proving Theorems 1.4 and 1.5. Their proof can be found in [12].

Lemma 2.1. *Let $\{f_\varepsilon\}_{\varepsilon>0}$ be a sequence in $L^2(\mathbb{R}^N)$ such that $f_\varepsilon \to f$ in $L^2(\mathbb{R}^N)$ as $\varepsilon \to 0$. Then, for any $s \in (0,1)$, it holds*

$$\|(-\Delta)^{-s}\mathcal{L}^s_\varepsilon[f_\varepsilon] - f\|_{L^2} \to 0 . \tag{2.3}$$

Next, we recall a generalized version of Stroock-Varopoulos's inequality.

Lemma 2.2. *Let $s \in (0,1)$, and let $\psi, \phi \in \mathcal{C}^1(\mathbb{R})$ be such that $\psi', \phi' \geq 0$. Then*

$$\int \psi(f)\mathcal{L}^s_\varepsilon[\phi(f)]dx \geq 0. \tag{2.4}$$

If we take $\psi(f) = f$, then we obtain

$$\int f\mathcal{L}^s_\varepsilon[\phi(f)] \, dx \geq \int |\mathcal{L}^{s/2}_\varepsilon \Phi(f)|^2 \, dx , \tag{2.5}$$

where $\phi' = (\Phi')^2$.

Finally, we have the following fundamental inequality.

Lemma 2.3. *Let $\alpha, \beta > 0$, and $\theta = \frac{\alpha+\beta}{2}$. Then, there is a constant $C > 0$ such that*

$$\left||a|^{\theta-1}a - |b|^{\theta-1}b\right|^2 \le C\left||a|^{\alpha-1}a - |b|^{\alpha-1}b\right|\left||a|^{\beta-1}a - |b|^{\beta-1}b\right|, \quad \forall a, b \in \mathbb{R}. \tag{2.6}$$

3. Existence of solutions

In this section, we prove Theorem 1.4 using the Lemmas and the Propositions below.

A regularized problem. We consider the following regularizing version of (1.1),

$$\begin{gathered}\partial_t u - \delta_1 \Delta u + \delta_2 \mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)] - \operatorname{div}\Theta_{\varepsilon,\nu}(u) + |u|^{\beta-1}u\Phi_\mu(u) = 0, \\ \text{in } \mathbb{R}^N \times (0,T), \\ u(0) = u_0, \quad \text{in } \mathbb{R}^N,\end{gathered} \tag{3.1}$$

where $s_0 = (1-2s)_+$, $\Theta_{\varepsilon,\nu}(u) = H_\nu(u)\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)]$, and

$$H_\nu(u) = \frac{|u|^{m_1+2}}{\nu^2+u^2}, \quad G_\nu(u) = \frac{|u|^{m_2+1}u}{\nu^2+u^2}, \quad J_\kappa(u) = \frac{|u|^{m_0+1}u}{u^2+\kappa^2},$$

with $m_0 = \frac{1}{2}\min\{m_1, \frac{m_2(N-2s_0)}{N}\}$, and $\Phi_\mu(u)$ is a cut-off function in a neighborhood of $u=0$, defined by $\Phi_\mu(u) = \Phi(\frac{u}{\mu})$, where $\Phi(s) \in \mathcal{C}^\infty(\mathbb{R})$, $0 \le \Phi(s) \le 1$, for all $s \in \mathbb{R}$, and

$$\Phi(s) = \begin{cases} 0, & \text{if } |s| \le 1, \\ 1, & \text{if } |s| \ge 2, \end{cases}$$

for $\delta_1, \delta_2, \varepsilon, \kappa, \mu, \nu \in (0,1)$.

We shall prove the existence of solutions of (3.1) in a suitable functional space by using the fixed-point theorem, and derive some energy estimates in order to pass to the limit as $\varepsilon, \kappa, \nu, \delta_1, \delta_2, \mu \to 0$ alternatively. The proof is most likely to the one in Section 3, [12]. Here, we just present some different points with the presence of the absorption.

Let us put

$$X = L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),$$

with the associated norm $\|\cdot\|_X = \|\cdot\|_{L^1(\mathbb{R}^N)} + \|\cdot\|_{L^\infty(\mathbb{R}^N)}$.

Lemma 3.1. *Let $u_0 \in X$ and $f \in L^1(Q_T) \cap L^\infty(Q_T)$. Then, there exists a weak solution $u \in \mathcal{C}([0,T];X)$ satisfying problem (3.1) in the weak sense, i.e.*

$$\int_0^T \int \left(-u\varphi_t + \delta_1 \nabla u \cdot \nabla\varphi + \delta_2 \mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)]\varphi - \Theta_{\varepsilon,\nu}(u)\cdot\nabla\varphi + |u|^{\beta-1}u\Phi_\mu(u)\varphi\right) dx\, dt = 0,$$

for all $\varphi \in \mathcal{C}_c^\infty(Q_T)$.

Proof. We look for a mild solution $u \in \mathcal{C}([0,T];X)$ as a fixed point of the map

$$\begin{aligned}\mathcal{T}(u) = e^{t\delta_1\Delta}u_0 &+ \int_0^t \nabla e^{(t-\tau)\delta_1\Delta}\Theta_{\varepsilon,\nu}(u)\, d\tau \\ &- \int_0^t e^{(t-\tau)\delta_1\Delta}(\delta_2\mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)] + |u|^{\beta-1}u\Phi_\mu(u))\, d\tau,\end{aligned}$$

where $e^{t\Delta}$ is the semigroup corresponding to the heat kernel $(4\pi t)^{-N/2}\exp(\frac{-|x|^2}{4t})$.

We note that $|u|^{\beta-1}u\Phi_\mu(u)$ is a locally Lipschitz function, then we can mimic the proof of [12, Theorem 4] to obtain that $\mathcal{T}$ maps $\mathcal{C}([0,T];X)$ into itself. Moreover, there is a real number $\gamma \in (0,1)$ such that

$$\|\mathcal{T}(u)-\mathcal{T}(v)\|_{\mathcal{C}([0,T];X)} \le C(R)T^\gamma \|u-v\|_{\mathcal{C}([0,T];X)}\,,$$

for all $u, v \in \overline{B(0,R)} \subset \mathcal{C}([0,T];X)$. This implies that

$$\|\mathcal{T}(u)-\mathcal{T}(v)\|_{\mathcal{C}([0,T];X)} \le \frac{1}{2}\|u-v\|_{\mathcal{C}([0,T];X)}\,,$$

if $T>0$ is chosen small enough. Thanks to the contraction mapping theorem, we obtain a unique mild solution u to equation $\mathcal{T}(u)=u$.

Finally, since the terms in (3.1) are regular, then it follows from the standard regularity theory that u is smooth in $\mathbb{R}^N \times (0,T)$. The proof is complete. □

Now, we prove an L^q-estimate of u.

Proposition 3.2. *Let u be a solution of (3.1) in Q_T. Then, for every $q \in [1,\infty]$, we have*

$$\|u(t)\|_{L^q(\mathbb{R}^N)} \le \|u_0\|_{L^q(\mathbb{R}^N)}, \quad \forall t \in (0,T)\,. \tag{3.2}$$

Proof. For every $q>1$, by testing $|u|^{q-2}u$ to (3.1), we obtain

$$\begin{aligned} &\frac{1}{q}\frac{d}{dt}\int |u(t)|^q\,dx + (q-1)\int |u|^{q-2}H_\nu(u)\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)]\cdot\nabla u\,dx \\ &+ \delta_1(q-1)\int |u|^{q-2}|\nabla u|^2\,dx + \delta_2\int \mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)]|u|^{q-2}u\,dx \\ &+ \int |u|^{\beta+q-1}\Phi_\mu(u)\,dx = 0\,. \end{aligned} \tag{3.3}$$

By applying Lemma 2.2 to $\psi(u)=|u|^{q-2}u$, and $\phi(u)=J_\kappa(u)$, we obtain

$$\int |u|^{q-2}u\mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)]\,dx \ge 0\,. \tag{3.4}$$

Next, we observe that

$$\begin{aligned} &\int |u|^{q-2}H_\nu(u)\nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)]\cdot\nabla u\,dx \\ &= \int \nabla(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)]\cdot\nabla\tilde{H}_\nu(u)\,dx \\ &= \int \tilde{H}_\nu(u)(-\Delta)(-\Delta)^{-1}\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)]\,dx \\ &= \int \tilde{H}_\nu(u)\mathcal{L}_\varepsilon^{1-s}[G_\nu(u)]\,dx \ge 0\,, \end{aligned} \tag{3.5}$$

with

$$\tilde{H}_\nu(u) = \int_0^u |s|^{q-2}H_\nu(s)\,ds\,.$$

Note that the inequality in (3.5) was also obtained by Lemma 2.2. Thus,

$$\frac{d}{dt}\int |u(t)|^q\,dx \le 0\,.$$

This implies (3.2) for any $q\in[1,\infty)$. Finally, passing to the limit as $q\to\infty$, we also obtain (3.2) for the L^∞-estimate.

It remains to prove the L^1-estimate of u. For every $\eta > 0$, let us put

$$\chi_\eta(r) = \begin{cases} \operatorname{sign}(r), & \text{if } |r| > \eta,\\ r/\eta, & \text{if } |r| \le \eta, \end{cases}$$

Testing (3.1) with $\chi_\eta(u)$ yields

$$\begin{aligned} &\int \big(u_t\chi_\eta(u) + \delta_1\nabla u\cdot\nabla\chi_\eta(u) + \delta_2\mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)]\chi_\eta(u) + \Theta(u)\cdot\nabla\chi_\eta(u)\big)\,dx \\ &+ \int |u|^{\beta-1}u\Phi_\mu(u)\chi_\eta(u)\,dx = 0\,. \end{aligned} \tag{3.6}$$

Since $\chi_\eta'(u) \ge 0$, it is clear that

$$\int \nabla u\cdot\nabla\chi_\eta(u)\,dx = \int |\nabla u|^2\chi_\eta'(u)\,dx \ge 0\,,$$

and by Lemma 2.2, we have

$$\int \mathcal{L}_\varepsilon^{s_0}[J_\kappa(u)]\chi_\eta(u)\,dx \ge 0, \quad \int \Theta(u)\cdot\nabla\chi_\eta(u)\,dx \ge 0\,.$$

From (3.6) after integrating on $(0,t)$ it follows that

$$\int S_\eta(u(t))\,dx \le \int S_\eta(u_0)\,dx\,,$$

with

$$S_\eta(u) = \int_0^u \chi_\eta(r)\,dr = \frac{u^2}{2\eta}\chi_{\{|u|<\eta\}} + (|u| - \frac{\eta}{2})\chi_{\{|u|\ge\eta\}}\,,$$

where χ_A denotes the characteristic function of the set A. Note that

$$\lim_{\eta\to0}\int S_\eta(u(t))\,dx = \int |u(t)|\,dx\,.$$

So, (3.2) follows with $q = 1$. This completes the proof. □

The following results are similar to the ones in [12], so we omit their proofs.

Proposition 3.3. *Let u be as in Proposition 3.2. Then, there is a constant $C = C(m_0, u_0) > 0$ such that for every $\kappa, \varepsilon, \mu, \nu > 0$,*

$$\delta_2\|\mathcal{L}_\varepsilon^{\frac{s_0}{2}}[J_\kappa(u_\varepsilon)]\|_{L^2(Q_T)} \le C\,. \tag{3.7}$$

Limit as $\varepsilon \to 0$.

Proposition 3.4. *Let u_ε be the solution of problem (3.1). Then, there exists a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ (still denoted as $\{u_\varepsilon\}_{\varepsilon>0}$) such that for any $R > 0$,*

$$u_\varepsilon \to u, \quad \text{in } L^2(B_R\times(0,T)).$$

Moreover, $u \in L^\infty(0,T;L^1(\mathbb{R}^N)) \cap L^\infty(Q_T) \cap L^2(0,T;H^1(\mathbb{R}^N))$ is a solution of the problem

$$\begin{aligned} &u_t - \delta_1\Delta u - \operatorname{div}(H_\nu(u)\nabla(-\Delta)^{-s}[G_\nu(u)]) + \delta_2(-\Delta)^{s_0}J_\kappa(u) + |u|^{\beta-1}u\Phi_\mu(u) \\ &= 0, \quad \text{in } Q_T\,. \end{aligned} \tag{3.8}$$

3.1. **Limit as $\kappa \to 0$.**

Proposition 3.5. *Let u_κ be the solution of problem* (3.8)*. Then, for any $R > 0$ it holds*

$$u_\kappa \to u, \quad \text{in } L^2(B_R \times (0,T))$$

up to a subsequence. Moreover, $u \in L^\infty(0,T; L^1(\mathbb{R}^N)) \cap L^\infty(Q_T) \cap L^2(0,T; H^1(\mathbb{R}^N))$ is a solution of the problem

$$u_t - \delta_1 \Delta u - \operatorname{div} \Theta_\nu(u) + \delta_2(-\Delta)^{s_0}(|u|^{m_0-1}u) + |u|^{\beta-1} u \Phi_\mu(u) = 0, \quad \text{in } Q_T\,, \tag{3.9}$$

where we denote $\Theta_\nu(u) = H_\nu(u)\nabla(-\Delta)^{-s}[G_\nu(u)]$.

3.2. **Limit as $\nu \to 0$.**

Proposition 3.6. *Let u_ν be the solution, obtained in Proposition 3.5. Then, there exists a subsequence of $\{u_\nu\}_{\nu>0}$ converging to a function u in $L^2(B_R \times (0,T))$ for any $R > 0$. Moreover, $u \in L^\infty(0,T; L^1(\mathbb{R}^N)) \cap L^\infty(Q_T) \cap L^2(0,T; H^1(\mathbb{R}^N))$ is a solution of the equation*

$$u_t - \delta_1 \Delta u - \operatorname{div} \Theta(u) + \delta_2(-\Delta)^{s_0}(|u|^{m_0-1}u) + |u|^{\beta-1} u \Phi_\mu(u) = 0, \quad \text{in } Q_T\,. \tag{3.10}$$

Recall that $\Theta(u) = H(u)\nabla(-\Delta)^{-s}[G(u)]$, with $H(u) = |u|^{m_1}$ and $G(u) = |u|^{m_2-1}u$.

3.3. **Limit as $\delta_1, \delta_2 \to 0$.**

Proposition 3.7. *Let u_{δ_2} be a solution of* (3.10) *above. Then, there exists a subsequence of $\{u_{\delta_2}\}_{\delta_2>0}$, converging to a function u in $L^2(B_R \times (0,T))$ for any $R > 0$. Moreover, $u \in L^\infty(0,T; L^1(\mathbb{R}^N)) \cap L^\infty(Q_T) \cap L^2(0,T; H^1(\mathbb{R}^N))$ is a weak solution of the problem*

$$u_t - \delta_1 \Delta u - \operatorname{div} \Theta(u) + |u|^{\beta-1} u \Phi_\mu(u) = 0, \quad \text{in } Q_T\,. \tag{3.11}$$

We emphasize that the estimates in the proof of Proposition 3.7 are also independent of δ_1.

Proposition 3.8. *Let u_{δ_1} be a solution of* (3.11)*. Then there exists a subsequence of $\{u_{\delta_1}\}_{\delta_1>0}$, converging to a function u in $L^2(B_R \times (0,T))$ for any $R > 0$. Furthermore, $u \in L^\infty(0,T; L^1(\mathbb{R}^N)) \cap L^\infty(Q_T)$, which is a weak solution of the equation*

$$u_t - \operatorname{div} \Theta(u) + |u|^{\beta-1} u \Phi_\mu(u) = 0, \quad \text{in } Q_T\,. \tag{3.12}$$

In addition, $\operatorname{div}(\Theta(u))$ *satisfies the following regularity:*

- *If $s \in [1/2, 1)$, then*

$$\operatorname{div}(\Theta(u)) \in L^2(0,T; H^{-1}(B_R))\,. \tag{3.13}$$

- *If $s \in (0, 1/2)$, then*

$$\operatorname{div}(\Theta(u)) \in L^p(0,T; W^{-2,p}(\mathbb{R}^N))\,, \tag{3.14}$$

  for $p > 1$ such that $\frac{m_2 p}{p-1} \geq 1$, and $W^{-2,p}(\mathbb{R}^N)$ is the dual space of $W^{2,p}(\mathbb{R}^N)$.

Limit $\mu \to 0$.

Proposition 3.9. *Let u_μ be a solution of* (3.12)*. Then, there exists a subsequence of $\{u_\mu\}_{\mu>0}$, converging to a function u in $L^2(B_R \times (0,T))$ for any $R > 0$. Furthermore, $u \in L^\infty(0,T; L^1(\mathbb{R}^N)) \cap L^\infty(Q_T)$, which is a weak solution of equation* (1.1)*. Also* $\operatorname{div}(\Theta(u))$ *satisfies either* (3.13) *if $s \in [1/2, 1)$, or* (3.14) *if $s \in (0, 1/2)$.*

Then, it is clear that solution u, obtained from Proposition 3.9 is a weak solution of (1.1). Moreover, u also satisfies the energy inequality

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int |u(x,t)|^p\,dx \\
&+ (p-1)\int\int \frac{(G(u(x)) - G(u(y)))(|u|^{m_1+p-2}u(x) - |u|^{m_1+p-2}u(y))}{|x-y|^{N+2(1-s)}}\,dx\,dy \le 0\,,
\end{aligned}
$$

see (4.1) below. Thus, we can mimic the proof of [12, Theorem 2] to obtain decay estimate (1.5). This completes the proof of Theorem 1.4.

4. Finite time extinction of solutions

Proof of Theorem 1.5. For every $p > 1$, it follows from (3.3) that

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int |u(x,t)|^p\,dx + \int |u(x,t)|^{p-1+\beta}\,dx \\
&+ (p-1)\int\int \frac{(G(u(x)) - G(u(y)))(|u|^{m_1+p-2}u(x) - |u|^{m_1+p-2}u(y))}{|x-y|^{N+2(1-s)}}\,dx\,dy \le 0\,.
\end{aligned}
\tag{4.1}
$$

Thanks to Lemma 2.3, we obtain

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int |u(x,t)|^p\,dx + \int |u(x,t)|^{p-1+\beta}\,dx \\
&+ C(p-1)\int\int \frac{\big||u|^{\theta_0-1}u(x) - |u|^{\theta_0-1}u(y)\big|^2}{|x-y|^{N+2(1-s)}}\,dx\,dy\,dt \le 0\,,
\end{aligned}
$$

with $\theta_0 = (m_1 + m_2 + p - 1)/2$. Next, applying the Sobolev embedding yields

$$
\||u(t)|^{\theta_0}\|_{L^{2^\star}} \le C\||u(t)|^{\theta_0}\|_{\dot H^{1-s}}\,,
$$

with $C = C(N,s)$, and $2^\star = \frac{2N}{N-2(1-s)} = \frac{2}{\alpha_0}$. Combining these inequalities yields

$$
\frac{1}{p}\frac{d}{dt}\|u(t)\|^p_{L^p} + C(\|u(t)\|^{p-1+\beta}_{L^{p-1+\beta}} + \|u(t)\|^{2\theta_0}_{L^{2^\star\theta_0}}) \le 0\,. \tag{4.2}
$$

Thanks to the interpolation inequality, we obtain

$$
\begin{aligned}
\|u(t)\|_{L^p} &\le \|u(t)\|^{\gamma}_{L^{p-1+\beta}}\|u(t)\|^{1-\gamma}_{L^{2^\star\theta_0}} \\
&= (\|u(t)\|^{p-1+\beta}_{L^{p-1+\beta}})^{\frac{\gamma}{p-1+\beta}}(\|u(t)\|^{2\theta_0}_{L^{2^\star\theta_0}})^{\frac{1-\gamma}{2\theta_0}} \\
&\le (\|u(t)\|^{p-1+\beta}_{L^{p-1+\beta}} + \|u(t)\|^{2\theta_0}_{L^{2^\star\theta_0}})^{\frac{\gamma}{p-1+\beta}+\frac{1-\gamma}{2\theta_0}}\,,
\end{aligned}
\tag{4.3}
$$

where $\frac{1}{p} = \frac{\gamma}{p-1+\beta} + \frac{1-\gamma}{2^\star\theta_0}$. Note that $2^\star\theta_0 > p$ since $m_1 + m_2 \ge \alpha_0$. By (4.2) and (4.3), we obtain

$$
\frac{d}{dt}\|u(t)\|^p_{L^p} + C\|u(t)\|^{p\lambda_0}_{L^p} \le 0\,,
$$

with

$$\lambda_0 = \frac{1}{1 + \frac{p(1-\gamma)}{\theta_0}\left(\frac{1}{2} - \frac{1}{2^\star}\right)} \in (0,1)\,.$$

Thus, $y(t) = \|u(t)\|_{L^p}^p$ satisfies

$$y'(t) + Cy^{\lambda_0}(t) \leq 0\,. \tag{4.4}$$

This implies that there exists a finite time $T_0 > 0$ such that $y(t) = 0$ for $t > T_0$. Thus, we obtain (1.7).

Finally, to estimate T_0, we solve directly (4.4) and obtain

$$y^{1-\lambda_0}(t) + Ct \leq y^{1-\lambda_0}(0) = \|u_0\|_{L^p}^{(1-\lambda_0)p}\,.$$

Thus, (1.8) follows. This completes the proof. □

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