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POSITIVE SOLUTIONS TO A DIRICHLET PROBLEM WITH NON-LIPSCHITZ NONLINEARITIES

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Abstract. Let Ω be a bounded smooth domain in $\mathbb{R}^N$. We study the existence of positive solutions to the Dirichlet problem

$$\begin{gathered}-\Delta u=(1-u)u^{s-1}-\lambda u^{r-1}, \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega,\end{gathered}$$

where $1<r<s\le 2$, and $\lambda>0$. In particular, we answer to some questions posed in the recent paper [3] where this problem was considered.

1. Introduction

In the recent paper [3], the authors investigate the existence and multiplicity of nonzero nonnegative solutions to the problem

$$\begin{gathered}-\Delta u=(1-u)u^{s-1}-\lambda u^{r-1}, \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N\ge 1$), $1<s\le 2$, $r>1$, and λ is a real parameter. The equation $-\Delta u=(1-u)u^{s-1}-\lambda u^{r-1}$, with $\lambda>0$, is the stationary version of a reaction diffusion equation employed to model certain isothermal autocatalytic chemical reactions (see [6]). In [3], various situations, which correspond to different values of the parameters s,r,λ, are analyzed and, for each of them, the authors prove some existence and multiplicity results of nonzero nonnegative solutions. Moreover, the cases in which a nonzero nonnegative solution is actually a positive solution (via the Strong Maximum Principle) are pointed out and, for each of these cases, the uniqueness of positive solution is also investigated.

In particular, one can observe that the Strong Maximum Principle applies in the following cases (see Lemma 2.2 of [3])

- $\lambda\le 0$;
- $\lambda>0$, $r\ge 2$;
- $\lambda>0$, $r>s$;
- $0<\lambda<1$, $r=s$.

It is worth noting to recall that positive solutions are often the only ones that have a physical meaning. If the function $t\in(0,+\infty)\to(1-t)t^{s-1}-\lambda t^{r-1}$ is negative and non-Lipschitz near 0 (for instance, when $1<r<s\le 2$), the standard conditions

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that allow to apply the Strong Maximum Principle are not satisfied any longer, and in this case the detection of positive solutions to problem (1.1) becomes a more delicate question.

In particular, the possible existence of positive solutions (as well as compact support solutions) in the case $\lambda > 0$ and $1 < r < s \leq 2$ is an open question posed in [3].

In this article, we will prove that a positive solution actually exists for $\lambda > 0$ small enough. Moreover, we will also show that the existence of a positive solution entails the existence of a second nonnegative solution with positive energy. This means that the technical condition involving the exponents r, s imposed in [3] to guarantee the existence of this second solution can be removed.

In what follows, by a nonnegative solution of problem (1.1) we mean a nonnegative function $u \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega [(1-u)u^{s-1} - u^{r-1}] v dx, \quad \text{for each } v \in W_0^{1,2}(\Omega).$$

By the standard regularity theory of elliptic equations, if Ω is of class C^2, any nonnegative solution u to problem (1.1) (in the sense given above) is classical (see [8, Appendix B]). More precisely, one has $u \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$, for some $\alpha \in (0,1)$, and u satisfies the equation and the boundary condition of (1.1) pointwise. Hereafter, we will always assume that Ω is of class C^2.

Throughout this article, we denote by

$$\Big(\| \cdot \| := \int_\Omega |\nabla (\cdot)|^2 dx \Big)^{1/2}$$

the Poincaré norm of $W_0^{1,2}(\Omega)$ and, for $p \in [1, \infty)$, we denote by

$$\| \cdot \|_p = \Big(\int_\Omega | \cdot |^p dx \Big)^{\frac{1}{p}}$$

the standard norm of $L^p(\Omega)$. Moreover, we put

$$c_p = \sup_{u \in W_0^{1,2}(\Omega),\, \|u\|=1} \|u\|_p \quad \text{and} \quad \lambda_1 = c_2^{-2}$$

By the Sobolev embeddings, we know that $c_p < +\infty$, if $p \leq \frac{2N}{N-2}$, when $N \geq 3$. The number λ_1 is the first eigenvalue of the Laplacian in Ω and it is well known that λ_1 is simple.

We denote by $\phi_1 \in C^1(\overline{\Omega})$ the unique (positive) eigenfunction associated with λ_1 and normalized with respect to sup-norm $\|u\|_\infty := \sup_\Omega |u|$. The function ϕ_1 satisfies

$$\begin{gathered} -\Delta \phi_1 = \lambda_1 \phi_1, \quad \text{in } \Omega, \\ 0 \leq \phi_1 \leq 1, \quad \text{in } \Omega, \\ \phi_1 = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

and, in particular,

$$\|\phi_1\|^2 = \lambda_1 \|\phi_1\|_2^2 \tag{1.2}$$

Finally, given any $u : \Omega \to \mathbb{R}$, we denote by $u_+, u_- : \Omega \to \mathbb{R}$ the functions defined by

$$u_+(x) = \max\{u(x), 0\}, \quad u_-(x) = \max\{-u(x), 0\}, \quad \text{all } x \in \Omega.$$

2. Main result

In what follows, we assume that $r, s \in]1, 2]$ are real numbers such that $1 < r < s$. Moreover, for each $\lambda \geq 0$, we consider the continuous function $f_\lambda : \mathbb{R} \to \mathbb{R}$ defined by

$$f_\lambda(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0), \\ (1-t)t^{s-1} - \lambda t^{r-1}, & \text{if } t \in [0, 1), \\ -\lambda t^{r-1}, & \text{if } t \in [1, \infty), \end{cases}$$

and the functional $I_\lambda : W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F_\lambda(u(x))dx, \quad \text{for each } u \in W_0^{1,2}(\Omega),$$

where

$$F_\lambda(\xi) = \int_0^\xi f_\lambda(t)dt, \quad \text{for each } \xi \in \mathbb{R}.$$

A routine argument shows that I_λ is sequentially weakly lower semicontinuous and of class C^1 in $W_0^{1,2}(\Omega)$, with

$$I'_\lambda(u)(v) = \int_\Omega \nabla u \nabla v dx - \int_\Omega f_\lambda(u) v dx, \quad \text{for each } u, v \in W_0^{1,2}(\Omega).$$

A further property of I_λ is given by the next lemma.

Lemma 2.1. *Let $u \in W_0^{1,2}(\Omega)$. Then*

(1) *u is a nonnegative solution of* (1.1) *if and only if $I'_\lambda(u) = 0$;*
(2) *$I'_\lambda(u) = 0$ implies $0 \leq u \leq 1$ in Ω.*

Proof. The proof is standard. We give it for completeness. Let $u \in W_0^{1,2}(\Omega)$ be a nonnegative solution to (1.1). Then

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega [(1-u)u^{s-1} - \lambda u^{r-1}] v dx, \quad \text{for each } v \in W_0^{1,2}(\Omega). \tag{2.1}$$

Testing this equation with $v(x) = \max\{u(x) - 1, 0\}$, $x \in \Omega$, we obtain

$$\int_{u \geq 1} |\nabla u(x)|^2 dx = \int_{u \geq 1} [(1-u)u^{s-1}(x) - \lambda u^{r-1}(x)](u(x) - 1)dx \leq 0.$$

This clearly implies $0 \leq u(x) \leq 1$, for all $x \in \Omega$, which in turn, recalling the definition of f_λ, implies the equivalence of (2.1) and $I_\lambda(u) = 0$. Suppose now that $u \in W_0^{1,2}(\Omega)$ satisfies $I'_\lambda(u) = 0$. Then, if we take again $v(x) = \max\{u(x) - 1, 0\}$, $x \in \Omega$, as a test function, we obtain

$$0 = I'_\lambda(u)(v) = \int_{u \geq 1} |\nabla u|^2 dx + \int_{u \geq 1} \lambda u^{r-1}(x) dx \geq \int_{u \geq 1} |\nabla u|^2 dx$$

which implies $u(x) \leq 1$, for all $x \in \Omega$. While, testing with $v = u_-$, we obtain

$$0 = I'_\lambda(u)(v) = \int_{u \leq 0} |\nabla u|^2 dx + \int_{u \leq 0} f(u) u dx = \int_{u \leq 0} |\nabla u|^2 dx$$

and so $u(x) \geq 0$, for all $x \in \Omega$. □

The previous lemma says that the nonnegative solutions of (1.1) are exactly the critical points of I_λ. In particular, $u \in W_0^{1,2}(\Omega)$ is a positive solution to (1.1) if and only if u is a positive critical point of I_λ. We will see, via the Strong Maximum Principle, that for $\lambda = 0$ the non-zero critical points of I_λ are positive in Ω. To be more precise, let $\mathcal{P}$ be the interior of the positive cone of the space $C_0^1(\overline{\Omega}) := \{u \in C^1(\overline{\Omega}) : u(x) = 0, \text{ for each } x \in \partial\Omega\}$ equipped with its standard norm $\|u\|_{C^1(\overline{\Omega})} := \sup_\Omega |u| + \sum_{i=1}^N \sup_\Omega |\frac{\partial u}{\partial x_i}|$. As it is well known, the set $\mathcal{P}$ is given by

$$\mathcal{P} = \big\{u \in C_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega\big\},$$

where ν denotes the outer unit normal to $\partial\Omega$.

Lemma 2.2. *For $\lambda = 0$, any nonnegative and nonzero critical point of $I_\lambda = I_0$ belongs to $\mathcal{P}$.*

Proof. Assume $\lambda = 0$ and let $u \in W_0^{1,2}(\Omega)$ be a nonzero critical point of I_0. Then, in view of Lemma 2.1, u is a nonzero nonnegative solution of the problem

$$\begin{gathered} -\Delta u = f_0(u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$

where $f_\lambda(t) = f_0(t) = \max\{0, (1-t)\max\{0,t\}^{s-1}\}$, $t \in \mathbb{R}$, is a nonnegative function. Hence, conclusion follows by the Strong Maximum Principle (see [5, Lemma 3.4 and Theorem 3.5]). $\square$

For $\lambda > 0$, f_λ is negative and is not Lipschitz continuous in a right-neighborhood of 0, so we cannot use, as for the case $\lambda = 0$, the Strong Maximum Principle to deduce that any nonnegative and nonzero critical point of I_λ is positive in Ω. Nevertheless, using the regularity theory for elliptic equations, we will see that for λ small, positive critical points of I_λ actually exist. More precisely, we have the following lemma.

Lemma 2.3. *If $\lambda_1 < 1$ when $s = 2$, there exists $\Lambda > 0$ such that, for each $\lambda \in [0, \Lambda)$, I_λ admits a global nonzero minimum point u_λ in $W_0^{1,2}(\Omega)$, with $I_\lambda(u_\lambda) < 0$. Moreover, there also exists $\Lambda_0 \in (0, \Lambda)$ such that, for each $\lambda \in [0, \Lambda_0)$, any global minimum point of I_λ belongs to $\mathcal{P}$.*

Proof. Consider the positive eigenfunction ϕ_1 associated to λ_1 and normalized with respect to the sup-norm. For each $\tau \in (0, 1)$ and $\lambda \geq 0$, taking in mind (1.2) and that $0 \leq \phi_1 \leq 1$ in Ω, one has

$$\begin{aligned} I_\lambda(\tau_1\phi_1) &= \frac{\tau^2}{2}\|\phi_1\|^2 - \int_\Omega \Big(\int_0^{\tau\phi_1(x)} f_\lambda(t)dt\Big)dx \\ &= \frac{\tau^2}{2}\tau^2\|\phi_1\|^2 - \int_\Omega \Big(\int_0^{\tau\phi_1(x)} [(1-t)t^{s-1} - \lambda t^{r-1}]dt\Big)dx \\ &= \frac{\lambda_1\tau^2}{2}\|\phi_1\|_2^2 - \frac{\tau^s}{s}\|\phi_1\|_s^s + \frac{\tau^{s+1}}{s+1}\|\phi_1\|_{s+1}^{s+1} + \frac{\lambda\tau^r}{r}\|\phi_1\|_r^r. \end{aligned}$$

Since $1 < s \leq 2$, and $\lambda_1 < 1$ when $s = 2$, we can find $\tau_0 > 0$ such that

$$C_0 := \frac{\lambda_1\tau_0^2}{2}\|\phi_1\|_2^2 - \frac{\tau_0^s}{s}\|\phi_1\|_s^s + \frac{\tau_0^{s+1}}{s+1}\|\phi_1\|_{s+1}^{s+1} < 0.$$

Consequently, if $\Lambda = -\frac{rC_0}{\tau_0^r \|\phi_1\|_r^r}$, then

$$\inf_{W_0^{1,2}(\Omega)} I_\lambda \le I_\lambda(\tau_0\phi_1) < 0, \quad \text{for each } \lambda \in [0,\Lambda) \tag{2.3}$$

In addiction, since $f_\lambda(t) \le 1$ for each $t \in [0,\infty)$, and $f_\lambda(t) = 0$ for each $t \in (-\infty, 0)$, one has

$$I_\lambda(u) \ge \frac{1}{2}\|u\|^2 - \|u\|_1 \ge \frac{1}{2}\|u\|^2 - c_1\|u\|, \quad \text{for each } u \in W_0^{1,2}(\Omega).$$

From this inequality, it follows that

$$\lim_{\|u\|\to+\infty} I_\lambda(u) = +\infty. \tag{2.4}$$

Therefore, since I_λ is sequentially lower semicontinuous, it admits at least a global minimum point in $W_0^{1,2}(\Omega)$. Moreover, if $\lambda \in [0,\Lambda)$ and u_λ is any global minimum point of I_λ, in view of (2.3) one has $I_\lambda(u_\lambda) < 0$. In particular, u_λ is nonzero.

Now, let us to show that the global minimum points of I_λ belong to $\mathcal{P}$, provided that $\lambda > 0$ is small. Arguing by contradiction, assume that there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ in $(0,\Lambda)$, with $\lambda_n \to 0$, and a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,2}(\Omega)$ such that, for each $n \in \mathbb{N}$, u_n is a global minimum point of I_{λ_n} and $u_n \notin \mathcal{P}$.

For each $n \in \mathbb{N}$, one has $I'_{\lambda_n}(u_n) = 0$. Therefore, by Lemma 2.1, it turns out $0 \le u_n(x) \le 1$, for each $x \in \Omega$. Moreover, if we fix $q > N/2$, by a classical regularity result (see [1, Theorem 8.2']), we have $u \in W^{2,q}(\Omega)$ and there exists a constant $C > 0$, independent of n, such that

$$\|u_n\|_{W^{2,q}(\Omega)} \le C(\|f_{\lambda_n}(u_n)\|_q + \|u_n\|_q) \le C(2+\lambda_n)|\Omega|^{1/q}, \tag{2.5}$$

where the last inequality follows by $0 \le u_n(x) \le 1$ and $0 \le |f(u_n(x))| \le (1+\lambda_n)$, for each $x \in \Omega$. In particular, in view of the embedding $W^{2,q}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, from (2.5) we infer that

$$\|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \le C_1(2+\lambda_n) \tag{2.6}$$

for a suitable constant $C_1 > 0$ independent of $n \in \mathbb{N}$. As a consequence, we obtain the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $C^{1,\alpha}(\overline{\Omega})$. Thus, by the Ascoli-Arzelá Theorem, we can assume that, up to a subsequence, $\{u_n\}_{n\in\mathbb{N}}$ converges in $C^1(\overline{\Omega})$ to a function $u_0 \in C^1(\overline{\Omega})$. We claim that u_0 is a global minimum point of I_0. Indeed, consider the function $g : [0,\Lambda) \to \mathbb{R}$ defined by

$$\begin{aligned} g(\lambda) &= \inf_{u\in W_0^{1,2}(\Omega)} I_\lambda(u) \\ &= \inf_{u\in W_0^{1,2}(\Omega)} \Big(\frac{1}{2}\|u\|^2 - \int_{0\le u\le 1}\int_0^{u(x)} (1-t)t^{s-1}dx + \frac{\lambda}{r}\int_{u\ge 0} u(x)^r dx\Big), \end{aligned}$$

for each $\lambda \in [0,\Lambda)$. The function g is convex and non-decreasing in $[0,\Lambda)$ (as a lower envelope of a family of affine non-decreasing functions). In particular, g is continuous in $[0,\Lambda)$. Consequently, recalling (2.3) and taking in mind that $u_n \to u_0$ in $C^1(\overline{\Omega})$ and $\lambda_n \to 0$, we infer that

$$\begin{aligned} 0 &> \inf_{u\in W_0^{1,2}(\Omega)} I_0(u) = g(0) \\ &= \lim_{n\to+\infty} g(\lambda_n) = \lim_{n\to+\infty} \inf_{u\in W_0^{1,2}(\Omega)} I_{\lambda_n}(u) \\ &= \lim_{n\to+\infty} I_{\lambda_n}(u_n) = I_0(u_0) \end{aligned}$$

which proves our claim.

Therefore, the function u_0 is in particular a nonzero critical point of I_0. So, by Lemma 2.2, one has $u_0 \in \mathcal{P}$. Finally, being $\mathcal{P}$ an open set of $C^1(\overline{\Omega})$, from $u_n \to u_0$ in $C^1(\overline{\Omega})$ we infer that $u_n \in \mathcal{P}$, for $n \in \mathbb{N}$ large. This is a contradiction with $u_n \notin \mathcal{P}$, for each $n \in \mathbb{N}$. $\square$

Lemma 2.3 ensures that the set

$$\mathcal{S} := \{\lambda > 0 : (1.1) \text{ admits a solution } u \in \mathcal{P}\} \tag{2.7}$$

is nonempty since it contains an interval of the type $[0, \Lambda_0)$. In the next Lemma we show that the set $\mathcal{S}$ is an interval. This means that we can take $\Lambda_0 = \sup \mathcal{S}$.

Lemma 2.4. *Let $\mathcal{S}$ be as in (2.7). Then, for each $\lambda \in (0, \sup \mathcal{S})$, there exists a solution $u_\lambda \in \mathcal{P}$ of problem (1.1) which is also a local minimum point of I_λ.*

Proof. Let $\lambda \in (0, \sup \mathcal{S})$ and fix $\lambda' \in (\lambda, \sup \mathcal{S}] \cap \mathcal{S}$. Moreover, fix a solution $u_{\lambda'}$ of $(P_{\lambda'})$ belonging to $\mathcal{P}$. Consider the function $\tilde{f}_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{f}_\lambda(x,t) = \begin{cases} f_\lambda(u_{\lambda'}(x)), & \text{if } (x,t) \in \Omega \times (-\infty, u_{\lambda'}(x)), \\ f_\lambda(t) & \text{if } (x,t) \in \Omega \times [u_{\lambda'}(x), \infty). \end{cases}$$

and the functional $\tilde{I}_\lambda : W_0^{1,2}(\Omega) \to \mathbb{R}$, defined by

$$\tilde{I}_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \Big(\int_0^{u(x)} \tilde{f}_\lambda(x,t) dt \Big) dx, \quad \text{for each } u \in W_0^{1,2}(\Omega).$$

Clearly, as for I_λ, we have that $\tilde{I}_\lambda$ is sequentially weakly lower continuous and of class C^1 in $W_0^{1,2}(\Omega)$. Moreover, since $\tilde{f}_\lambda(x,t) \le 1$, for each $x \in \Omega$ and $t \in \mathbb{R}$, with $\tilde{f}_\lambda(x,t) = 0$ if $t \le 0$, one has

$$\tilde{I}_\lambda(u) \ge \frac{1}{2}\|u\|^2 - \|u\|_1 \ge \frac{1}{2}\|u\|^2 - c_1\|u\|, \quad \text{for each } u \in W_0^{1,2}(\Omega),$$

which implies $\lim_{\|u\| \to +\infty} \tilde{I}_\lambda(u) = +\infty$. Therefore, $\tilde{I}_\lambda$ admits a global minimum point u_λ in $W_0^{1,2}(\Omega)$. We claim that $u_\lambda \in \mathcal{P}$ and that u_λ is also a local minimum point of I_λ. First of all, note that u_λ is a solution of the problem

$$\begin{gathered} -\Delta u = \tilde{f}_\lambda(x,u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.8}$$

Next, let us consider the open set $A = \{x \in \Omega : u_\lambda(x) < u_{\lambda'}(x)\}$ and assume A non-empty. Then, the function $u_\lambda - u_{\lambda'}$, which is negative in A, satisfies

$$\begin{gathered} \begin{aligned} -\Delta(u_\lambda - u_{\lambda'})(x) &= \tilde{f}_\lambda(x, u_\lambda(x)) - f_{\lambda'}(u_{\lambda'}(x)) \\ &= (\lambda' - \lambda) u_{\lambda'}(x)^{r-1} > 0, \quad \text{if } x \in A, \end{aligned} \\ (u_\lambda - u_{\lambda'})(x) = 0, \quad \text{if } x \in \partial A. \end{gathered}$$

As a consequence, by the Maximum Principle, we should also have that $u_\lambda - u_{\lambda'}$ is positive in A, a contradiction. This means that A is empty and thus

$$u_\lambda(x) \ge u_{\lambda'}(x), \quad \text{for each } x \in \Omega.$$

In addiction, note that there can be no point $x_0 \in \Omega$ such that $u_\lambda(x_0) = u_{\lambda'}(x_0)$ for, otherwise, x_0 should be a global minimum point for $u_\lambda - u_{\lambda'}$, and then

$$0 \ge -\Delta(u_\lambda - u_{\lambda'})(x_0) = (\lambda' - \lambda) u_{\lambda'}(x_0)^{r-1} > 0,$$

a contradiction. Thus, the function $u_\lambda - u_{\lambda'}$ is actually positive in Ω. Moreover, it satisfies

$$
\begin{aligned}
&-\Delta(u_\lambda - u_{\lambda'})(x)\\
&= u_\lambda(x)^{s-1} - u_{\lambda'}(x)^{s-1} - [u_\lambda(x)^s - u^s_{\lambda'}(x) + \lambda u_\lambda(x)^{r-1} - \lambda' u^{r-1}_{\lambda'}(x)]\\
&\geq -\Big[u_\lambda(x)^{s-1}\frac{\big(\frac{u_\lambda(x)}{u_\lambda(x)}\big)^s - 1}{\frac{u_\lambda(x)}{u_\lambda(x)} - 1} + \lambda u_{\lambda'}(x)^{r-2}\frac{\big(\frac{u_\lambda(x)}{u_{\lambda'}(x)}\big)^{r-1} - 1}{\frac{u_\lambda(x)}{u_{\lambda'}(x)} - 1}\Big](u_\lambda - u_{\lambda'})(x)\\
&\geq -c(x)(u_\lambda - u_{\lambda'})(x)
\end{aligned}
\tag{2.9}
$$

for all $x \in \Omega$, where

$$
c(x) = u_\lambda(x)^{s-1}\sup_{t>1}\frac{t^s - 1}{t - 1} + \lambda u_{\lambda'}(x)^{r-2}\sup_{0<t<1}\frac{t^{r-1} - 1}{t - 1}.
$$

Now, observe that, since $u'_\lambda \in \mathcal{P}$, one has

$$
0 < c(x) \leq k d(x, \partial\Omega)^{r-2}, \quad \text{for all } x \in \Omega,
$$

for some constant $k > 0$, where $d(\cdot, \partial\Omega)$ denotes the distance from $\partial\Omega$. Thus, we can apply Lemma 1 of [2] and obtain $u_\lambda - u_{\lambda'} \in \mathcal{P}$ or, equivalently, $u_\lambda \in U := u_{\lambda'} + \mathcal{P}$, where $U \subset \mathcal{P}$ is an open set in $C^1(\overline{\Omega})$.

Finally, observe that, for each $u \in U$, one has $u > u_{\lambda'}$ in Ω, and

$$
\begin{aligned}
&\tilde{I}_\lambda(u)\\
&= \frac{1}{2}\|u\|^2 - \int_\Omega\Big(\int_0^{u_{\lambda'}(x)}\tilde{f}_\lambda(x,t)dt + \int_0^{u(x)} f_\lambda(x,t)dt - \int_0^{u_{\lambda'}(x)} f_\lambda(x,t)dt\Big)dx\\
&= I_\lambda(u) - M
\end{aligned}
$$

where

$$
M = \int_\Omega\Big(\int_0^{u_{\lambda'}(x)}(\tilde{f}_\lambda(x,t)dt - f_\lambda(x,t))dt\Big)dx.
$$

Recalling that u_λ is a global minimum point of $\tilde{I}_\lambda$, from the previous identity we infer that u_λ is a local minimum point of I_λ with respect to the $C^1(\overline{\Omega})$ topology. Then, by a classical result (see [4]), u_λ turns out to be a local minimum point of I_λ with respect to the $W^{1,2}_0(\Omega)$-topology as well. □

We are now in position to prove our main result.

Theorem 2.5. *For each $\lambda \in (0, \sup\mathcal{S})$, problem* (1.1) *admits at least two nonzero nonnegative solutions, one of which belongs to $\mathcal{P}$ and is a local minimum point of I_λ.*

Proof. Let $\lambda \in (0, \sup\mathcal{S})$ and fix $q \in (2, \infty)$, with $q < \frac{2N}{N-2}$, if $N \geq 3$. Moreover, we put

$$
M = \sup_{t>0}\frac{F_\lambda(t)}{t^q}.
$$

It is easy to check that $M \in (0, +\infty)$ and

$$
I_\lambda(u) \geq \frac{1}{2}\|u\|^2 - M\|u\|^q_q \geq \frac{1}{2}\|u\|^2 - Mc^q_q\|u\|^q > 0,
$$

for each $u \in W^{1,2}_0(\Omega)$, with $0 < \|u\| < \big(2Mc^q_q\big)^{\frac{1}{2-q}}$. In particular, 0 is a (strict) local minimum point for I_λ. By Lemma 2.4 we know that there exists another local minimum point $u_\lambda \in \mathcal{P}$. In addiction, in view of (2.4) one has that any Palais-Smale sequence for I_λ (that is any sequence $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,2}(\Omega)$ such that $\{I_\lambda(u_n)\}_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ and $I'_\lambda(u_n) \to 0$ in $W_0^{1,2}(\Omega)$) is bounded. Therefore, by a standard result (see Proposition 2.2 of [8], for instance), we see that functional I_λ satisfies the Palais-Smale condition. At this point, by applying a well known Mountain Pass Theorem (see [7]), we can deduce the existence of a second critical point $v_\lambda \in W_0^{1,2}(\Omega)$ (besides u_λ) for I_λ, with $I_\lambda(v_\lambda) > 0$. This completes the proof. $\square$

Remark 2.6. Theorem 2.5 gives a positive answer to [3, Conjectures 3.5 and 7.4] about the existence of positive solutions to problem (1.1). However, the existence of possible compact-support solutions remains an open problem. Moreover, Theorem 2.5 confirms that, as anticipated in [3], condition (3.3) in [3] is technical and can be removed.

Remark 2.7. Consider the set

$$\tilde{\mathcal{S}} = \{\lambda > 0 : (1.1) \text{ admits a nonnegative solution } u \in W_0^{1,2}(\Omega) \setminus \{0\}\}.$$

Clearly, $\tilde{\mathcal{S}} \supseteq \mathcal{S}$ and it is easy to see that $\tilde{S}$ is an interval. Indeed, if $\lambda \in (0, \sup \tilde{S})$, choosing $\lambda' \in (\lambda, \sup \tilde{\mathcal{S}}) \cap \tilde{\mathcal{S}}$, and fixing a nonzero nonnegative solution $u_{\lambda'}$ to (1.1) with λ' instead of λ, we can see that u'_λ is a sub-solution of (1.1). Moreover, if $\bar{u}$ is the unique positive solution of $-\Delta u = 1$ in Ω, $u = 0$ on $\partial\Omega$, choosing $k > 0$ large enough, $k\bar{u}$ turns out to be a super-solution of (1.1), with $k\bar{u} > u_{\lambda'}$ in Ω. Then, by the sub-supersolution method, we easily derive the existence of a nonzero nonnegative solution to (1.1). Again in view of [3, Conjectures 3.5 and 7.4], it would be interesting to give an answer to the following questions:

(1) Does Problem (1.1) admit a nonzero nonnegative solution for $\lambda = \sup \tilde{\mathcal{S}}$?
(2) Is it true that $\sup \tilde{\mathcal{S}} = \sup \mathcal{S}$? In other words, is it true that problem (1.1) admits no nonzero solution for $\lambda > \sup \mathcal{S}$?

The next result answers positively to the first question.

Theorem 2.8. *Let $\tilde{\mathcal{S}}$ be as in Remark 2.7 and put $\lambda^* = \sup \tilde{\mathcal{S}}$. Then Problem* (1.1)*, with λ^* instead of λ, admits a nonzero and nonnegative solution.*

Proof. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, \lambda^*)$ such that $\lambda_n \uparrow \lambda^*$. Moreover, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,2}(\Omega)$ such that u_n is a nonzero and nonnegative solution to Problem (1.1), with λ_n instead of λ, for each $n \in \mathbb{N}$. Since $I'_{\lambda_n}(u_n) = 0$, arguing as in Lemma 2.3, we obtain an inequality as (2.6). Therefore, up to a subsequence, $u_n \to u^*$ in the $C^1(\overline{\Omega})$-topology, for some $u^* \in C^1(\overline{\Omega})$. Consequently,

$$I'_{\lambda^*}(u^*) = \lim_{n\to+\infty} I'_{\lambda_n}(u_n) = 0.$$

To conclude, it remains to show that u^* is nonzero. To this end, fix $q \in (2, \infty)$, with $q \le \frac{2N}{N-2}$ if $N \ge 3$, and put

$$M = \sup_{t>0} \frac{f_{\lambda_1}(t)}{t^{q-1}}.$$

Then, $M \in (0, \infty)$ and, for each $n \in \mathbb{N}$, one has

$$0 = I'_{\lambda_n}(u_n)(u_n) = \|u_n\|^2 - \int_\Omega f_{\lambda_n}(u_n(x))u_n(x)dx$$

$$
\begin{aligned}
&\geq \|u_n\|^2 - \int_\Omega f_{\lambda_1}(u_n(x))u_n(x)dx \\
&\geq \|u_n\|^2 - M\|u_n\|_q^q \geq \|u_n\|^2 - Mc_q^q\|u_n\|^q,
\end{aligned}
$$

which implies

$$
\|u_n\| \geq (Mc_q^q)^{\frac{1}{2-q}} > 0.
$$

Since $u_n \to u^*$ in $C^1(\overline{\Omega})$, passing to the limit in the above inequality, we finally obtain $\|u^*\| \geq (Mc_q^q)^{\frac{1}{2-q}}$, that is $u^* \neq 0$. $\square$

References

[1] S. Agmon; The L^p approach to the Dirichlet problem, *Ann. Scuola Norm. Sup. Pisa*, **13** (1959), 405–448.

[2] G. Anello; Existence and multiplicity of nonnegative solutions for semilinear elliptic problems involving nonlinearities indefinite in sign, *Nonlinear Anal.*, **75** (2012), 6103–6107.

[3] V. Bobkov, P. Drábek, J. Hernández; Existence and multiplicity results for a class of semilinear elliptic equations, *Nonlinear Analysis*, **200**, (2020), Article ID 112017, 24 p.

[4] H. Brezis, L. Nirenberg; H^1 versus C^1 local minimizers, *C. R. Acad. Sci. Paris Sér. I Math.*, **317**(5) (1993), 465-472.

[5] D. Gilbarg, N. S. Trudinger; *Elliptic partial differential equations of second order*, Springer, Berlin Heidelberg, 1998.

[6] J. Leach & D. Needham; *Matched asymptotic expansions in reaction-diffusion theory*, Springer, London, 2004.

[7] P. Pucci, J. Serrin; A mountain pass theorem, *J. Differ. Equations*, **60** (1985), 14–149.

[8] M. Struwe; *Variational Methods*, Springer Verlag, Berlin, 1996.

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