

## IMPROVED OSCILLATION CRITERIA FOR FIRST-ORDER DELAY DIFFERENTIAL EQUATIONS WITH VARIABLE DELAY

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ABSTRACT. This article concerns the oscillation of solutions to the delay differential equation  $x'(t) + p(t)x(\tau(t)) = 0$ . Conditions for oscillation have been stated as lower bounds for the limit superior and limit inferior of  $\int_{\tau}^t p$ . In this article we match the bound for the best case in [7], without using one of their hypotheses. Then assuming that hypothesis, we obtain a bound lower than the one in [12]. Then we apply our results to an equation with several delays. We employ iterated estimates of the solution.

### 1. INTRODUCTION

In this article we improve existing conditions for the oscillation of all solutions to the delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \quad (1.1)$$

where  $p, \tau \in C([t_0, \infty), [0, \infty))$ ,  $\tau$  is non-decreasing,  $\tau(t) \leq t$  for all  $t \in [t_0, \infty)$ , and  $\lim_{t \rightarrow \infty} \tau(t) = \infty$ .

Let  $T_0 = \inf\{\tau(t) : t \geq t_0\}$ . By a solution, we mean a function that is continuous for  $t \geq T_0$ , differentiable for  $t \geq t_0$ , and satisfies (1.1). Given an initial function  $\phi$  defined on  $[T_0, t_0]$ , we can obtain a unique solution by integrating (1.1) in successive intervals (a process known as the method of steps).

A solution is called oscillatory if it has arbitrarily large zeros; otherwise it is called non-oscillatory. A solution  $x$  is called eventually positive if  $x(t) > 0$  for all  $t$  sufficiently large.

Throughout this article we use the following notation:  $\tau^{n+1}(t) = \tau^n(\tau(t))$  with  $\tau^1(t) = \tau(t)$  and  $\tau^0(t) = t$ ,

$$\alpha = \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds, \quad \beta = \limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds. \quad (1.2)$$

Moreover, we consider the equation

$$\lambda = e^{\alpha\lambda},$$

where  $\lambda$  is a function of  $\alpha$ . If  $\alpha = 0$ , then  $\lambda = 1$  is the only solution. If  $0 < \alpha < 1/e$ , then there are two solutions,  $\lambda_1 < \lambda_2$ ; furthermore,  $\lambda_1$  is a continuous and increasing function of  $\alpha$ . If  $\alpha = 1/e$ , then  $\lambda = e$  is the only solution. If  $\alpha > 1/e$ , there is

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no solution. For  $0 \leq \alpha \leq 1/e$ . we denote the solutions to this equation by  $\lambda_1, \lambda_2$ , where  $\lambda_1 \leq \lambda_2$ .

It is well known [1, 3, 10] that if

$$\alpha > \frac{1}{e} \quad \text{or} \quad \beta > 1, \quad (1.3)$$

then every solution of (1.1) is oscillatory. On the other hand if  $\int_{\tau(t)}^t p(s) ds \leq 1/e$  holds sufficiently large  $t$ , then there is a non-oscillatory solution; see [3, Corollary 2.1.1] and [8]. From these statements, we see that if  $\lim_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds$  does not exist, then there is gap in the results. Many results have improved the above bounds, We just mention a few of them, and direct the reader to the references in this article.

**Lemma 1.1** ([7, Lemma 1]). *Let  $0 < \alpha$  and  $x$  be an eventually positive solution of (1.1). Then  $0 < \alpha \leq 1/e$  and*

$$\lambda_1 \leq \liminf_{t \rightarrow \infty} \frac{x(\tau(t))}{x(t)} \leq \lambda_2. \quad (1.4)$$

**Lemma 1.2** ([6, Corollary 1]). *Assume  $0 < \alpha \leq 1/e, \beta < 1$  and*

$$\beta > \frac{\ln(\lambda_1) + 1}{\lambda_1} - \frac{(1 - \alpha) - \sqrt{1 - 2\alpha - \alpha^2}}{2}. \quad (1.5)$$

*Then every solution of (1.1) is oscillatory.*

**Lemma 1.3** ([7, Theorem]). *Assume that  $0 < \alpha \leq 1/e, \beta < 1$ , that there exists  $\omega > 0$  such that*

$$\int_{\tau(u)}^{\tau(t)} p(s) ds \geq \omega \int_u^t p(s) ds \quad \text{for } \tau(t) \leq u \leq t. \quad (1.6)$$

*If*

$$\beta > \frac{\ln(\lambda_1) + 1}{\lambda_1} - \frac{(1 - \alpha) - \sqrt{(1 - \alpha)^2 - 4A}}{2}, \quad (1.7)$$

*where*

$$A = \frac{e^{\alpha\omega\lambda_1} - \alpha\omega\lambda_1 - 1}{(\omega\lambda_1)^2}, \quad (1.8)$$

*then every solution of (1.1) is oscillatory.*

**Lemma 1.4** ([12, Theorem]). *Under the assumptions of Lemma 1.3, if*

$$\beta > \frac{\ln(\lambda_1)}{\lambda_1} + \frac{-1 + \sqrt{1 + 2\omega - 2\omega\lambda_1 B}}{\omega\lambda_1}, \quad (1.9)$$

*where  $B = (1 - \alpha - \sqrt{(1 - \alpha)^2 - 4A})/2$  and  $A$  is given by (1.8), then every solution of (1.1) is oscillatory.*

Note that the bounds for  $\beta$  depend on the value of  $\alpha$ . In particular when  $\alpha = 0$  conditions (1.3), (1.5) and (1.7) become  $\beta > 1$ . A table of numerical values for these and other bounds can be found in [7, 12, 13]. Table 1 towards the end of this paper compares our results with the bounds from (1.7), (1.9).

In this article, without assuming (1.6), we establish the bound

$$\beta > 2\alpha + \frac{2}{\lambda_1} - 1$$

which matches the bound given in [7] using assumption (1.6). In fact when (1.6) is assumed, we obtain a bound slightly lower than the one in (1.9).

Our main tool is to integrate (1.1) which expresses  $x$  as an integral transform of itself. We then substitute the expression obtained into the integral. This substitution was done one time in the book by Erbe [3, Lemma 2.1.3]. Here we iterate this substitution several times as in [14, 15]. This process yields a multiple integral over an interval  $[\tau^n(t), t^*]$ . Then we partition each interval  $[\tau^k(t), \tau^{k-1}(t)]$  and use Riemann sums to estimate a multiple integral. The idea of partitioning the domain comes from [15]. However our partition is different from theirs, and as  $t \rightarrow \infty$  the resulting number of integrals we obtain can approach infinity. This is stark contrast to [15] where partitioning intervals depends of a positive parameter  $\delta$ . However as  $n$  increases,  $\delta$  becomes negative and the partitioning process stops.

**Equations with constant delay.** A function  $f$  is called slowly varying at infinity [11] if for every  $s \in \mathbb{R}$ ,

$$\lim_{t \rightarrow \infty} f(t+s) - f(t) = 0.$$

Garab et al [4, Theorem 4] showed the following result which is optimal for the constant delay case. Let  $\tau(t) = t - \tau_0$ ,  $p$  be a non-negative bounded and uniformly continuous function such that  $0 < \alpha$ ,  $1/e < \beta$ . Also let the mapping  $t \mapsto \int_{\tau(t)}^t p(s) ds$  be slowly varying at infinity. Then all solutions of (1.1) are oscillatory.

This article is organized as follows. In Section 2, we study oscillation of solutions without assuming condition (1.6). In Section 3, we assume condition (1.6) for obtaining an oscillation criterion. Also we compare the bounds that we obtain with some bounds in the literature. In Section 4, we extend our results to equations with multiple delays.

## 2. RESULTS WITHOUT ASSUMING (1.6)

**Lemma 2.1.** *For  $n \geq 1$  and  $t^* > t$ , we have*

$$\begin{aligned} & \int_t^{t^*} p(s_1) \int_t^{s_1} p(s_2) \int_t^{s_2} \cdots \int_t^{s_{n-1}} p(s_n) ds_n \dots ds_1 \\ &= \int_t^{t^*} p(s_1) \int_{s_1}^{t^*} p(s_2) \int_{s_2}^{t^*} \cdots \int_{s_{n-1}}^{t^*} p(s_n) ds_n \dots ds_1 \\ &= \frac{1}{n!} \left( \int_t^{t^*} p(s) ds \right)^n \end{aligned}$$

The above lemma can be proved by induction on the number of integrals, using integration by substitution.

**Lemma 2.2.** *Let  $0 < \hat{\alpha} < \alpha$  and  $n \geq 1$ . Then there exists  $t_2$ , and for each  $t \geq t_2$ , there exists  $t^*$  such that  $\int_t^{t^*} p(s) ds = \hat{\alpha}$ , with  $\tau(t^*) \leq t < t^*$ , and*

$$\begin{aligned} \rho_n(t) &:= \int_t^{t^*} p(s_1) \int_{\tau(s_1)}^t p(s_2) \int_{\tau(s_2)}^{\tau(t)} p(s_3) \cdots \int_{\tau(s_{n-1})}^{\tau^{n-2}(t)} p(s_n) ds_n \dots ds_1 \\ &\geq \frac{\hat{\alpha}^n}{n!} \quad \text{for all } t \geq t_2, \end{aligned} \tag{2.1}$$

*Proof.* From  $\hat{\alpha} < \alpha$ , we have  $\int_{\tau(t)}^t p \geq \hat{\alpha}$  for all  $t$  large enough. For each one of those sufficiently large values of  $t$ , the continuity of the map  $u \mapsto \int_t^u p(s) ds$  and

$\int_{\tau(t)}^t p \geq \hat{\alpha}$ , yield a  $t^*$  such that  $\int_t^{t^*} p(s) ds = \hat{\alpha}$ . The fact that  $t < t^*$  follows from  $\int_t^{t^*} p > 0$ , and the fact that  $\tau(t^*) \leq t$  follows from  $\int_{\tau(t^*)}^{t^*} p \geq \hat{\alpha}$ .

We partition the interval  $[t, t^*]$  using the  $m + 1$  points

$$t = u_{0,m} < u_{0,m-1} < \cdots < u_{0,0} = t^*,$$

so that

$$\int_{u_{0,k}}^{u_{0,0}} p(s) ds = \frac{\hat{\alpha}k}{m} \quad \text{for } k = 0, 1, \dots, m.$$

We partition the interval  $[\tau(t), t]$  using the  $m + 2$  points

$$\tau(t) = u_{1,m+1} \leq u_{1,m} < \cdots < u_{1,0} = t,$$

so that

$$\int_{u_{1,k}}^{u_{1,0}} p(s) ds = \int_{u_{0,k}}^{u_{0,0}} p(s) ds = \frac{\hat{\alpha}k}{m} \quad \text{for } k = 0, 1, \dots, m.$$

Then

$$\int_{\tau(t)}^t p(s) ds \geq \int_{u_{1,m}}^{u_{1,0}} p(s) ds = \int_{u_{0,m}}^{u_{0,0}} p(s) ds = \int_t^{t^*} p(s) ds.$$

Note that we can not guarantee this inequality without the assumption  $\int_t^{t^*} p = \hat{\alpha}$ . In a similarly way, we can partition the intervals  $[\tau^2(t), \tau(t)]$ ,  $\dots$ ,  $[\tau^{n-1}(t), \tau^{n-2}(t)]$ . Then

$$\rho_n(t) \geq \sum_{k_1=0}^{m-1} \int_{u_{0,k_1+1}}^{u_{0,k_1}} p(s_1) \sum_{k_2=0}^{k_1-1} \int_{u_{1,k_2+1}}^{u_{1,k_2}} p(s_2) \cdots \sum_{k_n=0}^{k_{n-1}-1} \int_{u_{n-1,k_n+1}}^{u_{n-1,k_n}} p(s_n) ds_n \dots ds_1.$$

Since  $\int_{u_{j,k}}^{u_{j,0}} p = \int_{u_{0,k}}^{u_{0,0}} p$  and  $p$  is continuous, the expression above is a Riemann sum that approximates

$$\int_t^{t^*} p(s_1) \int_{s_1}^{t^*} p(s_2) \int_{s_2}^{t^*} p(s_3) \cdots \int_{s_{n-1}}^{t^*} p(s_n) ds_n \dots ds_1.$$

Then by Lemma 2.1, this multiple multiple equals  $(\int_t^{t^*} p(s) ds)^n/n!$  Taking the limit as  $n \rightarrow \infty$  and using that  $\int_t^{t^*} p(s) ds = \hat{\alpha}$ , we have the desired result.  $\square$

**Lemma 2.3.** *Let  $0 < \hat{\alpha} < \alpha \leq 1/e$ , and  $x$  be an eventually positive solution of (1.1). Then there exists  $t_1 \geq t_0$ , so that for each  $t \geq t_1$  there exists  $n = n(t)$  with  $\lim_{t \rightarrow \infty} n(t) = \infty$ , and*

$$\frac{x(t)}{x(\tau(t))} \geq d(n, t, \hat{\alpha}) \quad \forall t \geq t_1, \quad (2.2)$$

where  $d(n, t, \hat{\alpha})$  is the smaller root of quadratic equation

$$d^2 - (1 - \hat{\alpha})d + f_n(t, \hat{\alpha}) = 0 \quad (2.3)$$

and

$$f_n(t, \hat{\alpha}) = \frac{\hat{\alpha}^2}{2!} + \frac{\hat{\alpha}^3 \hat{\lambda}}{3!} + \cdots + \frac{\hat{\alpha}^n \hat{\lambda}^{n-2}}{n!}, \quad (2.4)$$

where  $\hat{\lambda}$  is the smaller solution of  $\lambda = e^{\hat{\alpha}\lambda}$ .

*Proof.* Since  $0 < \hat{\alpha} < \alpha \leq 1/e$ , each one of the equations  $\lambda = e^{\hat{\alpha}\lambda}$  and  $\lambda = e^{\alpha\lambda}$  has two solutions. Then by Lemma 1.1,

$$\hat{\lambda} := \hat{\lambda}_1 < \lambda_1 \leq \liminf_{t \rightarrow \infty} \frac{x(\tau(t))}{x(t)} \leq \lambda_2 < \hat{\lambda}_2.$$

From the fact that  $x$  is eventually positive and  $\lim_{t \rightarrow \infty} \tau(t) = \infty$ , there is a  $t_1$  such that for all  $t \geq t_1$  the following 4 conditions hold:  $0 < x(t)$ ,  $0 < x(\tau(t))$ ,  $\hat{\alpha} \leq \int_{\tau(t)}^t p(s) ds$  (because  $\hat{\alpha} < \alpha$  which is a limit inferior), and  $\hat{\lambda} \leq x(\tau(t))/x(t)$  (because  $\hat{\lambda} < \lambda$  which is a limit inferior).

For each  $t > t_1$ , we select  $n = n(t)$  as the largest integer for which

$$\tau^n(t) \leq t_1 \leq \tau^{n-1}(t). \tag{2.5}$$

Then

$$\hat{\lambda} \leq \frac{x(\tau^{j+1}(t))}{x(\tau^j(t))} \quad \text{for } j = 0, 1, \dots, n-1. \tag{2.6}$$

Note that  $n$  is a non-decreasing function of  $t$  because  $\tau$  is non-decreasing. Since  $\tau$  is continuous and  $\lim_{t \rightarrow \infty} \tau(t) = \infty$ , we have, for each finite  $n$ , that

$$\lim_{t \rightarrow \infty} \tau^n(t) = \tau(\dots \tau(\lim_{t \rightarrow \infty} \tau(t)) \dots) = \infty.$$

Now we claim that  $n \rightarrow \infty$  as  $t \rightarrow \infty$ . To reach a contradiction, assume that  $n$  remains bounded as  $t \rightarrow \infty$ . Taking the limit in (2.5),

$$\infty = \lim_{t \rightarrow \infty} \tau^n(t) \leq t_1$$

which is a contradiction; therefore,  $n$  can not remain bounded as  $t \rightarrow \infty$ .

By Lemma 2.2 there exists  $t_2 \geq t_1$ , such that for each  $t \geq t_2$  there exists  $t^*$  such that  $\int_t^{t^*} p = \hat{\alpha}$ . Integrating (1.1), we have inequalities of the form

$$x(t) = x(t^*) + \int_t^{t^*} p(s_1)x(\tau(s_1)) ds_1, \tag{2.7}$$

$$x(\tau(s_1)) = x(t) + \int_{\tau(s_1)}^t p(s_2)x(\tau(s_2)) ds_2, \quad \dots, \tag{2.8}$$

$$x(\tau(s_{n-1})) = x(\tau^{n-2}(t)) + \int_{\tau(s_{n-1})}^{\tau^{n-2}(t)} p(s_n)x(\tau(s_n)) ds_n. \tag{2.9}$$

Since  $x(t) > 0$ , by (1.1),  $x'(t) \leq 0$  and  $x$  is non-increasing, and

$$\int_{\tau(s_{n-1})}^{\tau^{n-2}(t)} p(s_n)x(\tau(s_n)) ds_n \geq x(\tau^{n-1}(t)) \int_{\tau(s_{n-1})}^{\tau^{n-2}(t)} p(s_n) ds_n.$$

Substituting (2.8)–(2.9) into (2.7), and using the definition of  $\rho$ , it follows by the above inequality, that

$$x(t) \geq x(t^*) + \hat{\alpha}x(t) + [\rho_2(t)x(\tau(t)) + \rho_3(t)x(\tau^2(t)) + \dots + \rho_n(t)x(\tau^{n-1}(t))].$$

From (2.6), we have  $x(\tau^{j+1}(t)) \geq \hat{\lambda}^j x(\tau(t))$  for  $j = 1, 2, \dots, n-1$ . Then from Lemma 2.2,  $\int_t^{t^*} p = \hat{\alpha}$ ,  $\int_{\tau(t)}^t p \geq \hat{\alpha}$ , and the result in (2.1), we have

$$x(t) \geq x(t^*) + \hat{\alpha}x(t) + f_n(t, \hat{\alpha})x(\tau(t)), \quad \forall t \geq t_2,$$

where  $f_n$  is defined by (2.4). Then

$$(1 - \hat{\alpha})x(t) \geq x(t^*) + f_n(t, \hat{\alpha})x(\tau(t)). \tag{2.10}$$

Since  $x$  and  $f_n$  are positive,  $(1 - \hat{\alpha}) > 0$  which agrees with assumption  $0 < \hat{\alpha} < 1/e$ . Ignoring the term  $x(t^*)$ , we have

$$\frac{x(t)}{x(\tau(t))} \geq \frac{f_n(t, \hat{\alpha})}{1 - \hat{\alpha}} := d_1$$

which is positive. Recalling that  $x$  is non-increasing and  $\tau(t^*) \leq t$ , we have

$$x(t^*) \geq d_1 x(\tau(t^*)) \geq d_1 x(t).$$

Using this inequality in (2.10) yields

$$(1 - \hat{\alpha} - d_1)x(t) \geq x(t^*) + f_n(t, \hat{\alpha})x(\tau(t)).$$

Since  $x$  and  $f_n$  are positive,  $(1 - \hat{\alpha} - d_1) > 0$ , which implies  $d_1 < 1 - \hat{\alpha}$ . Then

$$\frac{x(t)}{x(\tau(t))} \geq \frac{f_n(t, \hat{\alpha})}{1 - \hat{\alpha} - d_1} := d_2.$$

Proceeding as above,  $(1 - \hat{\alpha} - d_2) > 0$ , which implies  $d_2 < 1 - \hat{\alpha}$ . Also because  $d_1 > 0$ , we have

$$d_1 = \frac{f_n(t, \hat{\alpha})}{1 - \hat{\alpha}} < \frac{f_n(t, \hat{\alpha})}{1 - \hat{\alpha} - d_1} = d_2.$$

As in [3, Lemma 2.1.3], repeating the above process, we have an increasing sequence  $\{d_k\}$  that is bounded above by  $1 - \hat{\alpha}$ ; therefore the sequence converges to the smaller solution of the quadratic equation  $d^2 - (1 - \hat{\alpha})d + f_n(t, \hat{\alpha}) = 0$ . By (2.10), we have  $(1 - \hat{\alpha} - d)x(t) \geq x(t^*) + f_n(t, \hat{\alpha})x(\tau(t))$ , and

$$\frac{x(t)}{x(\tau(t))} \geq \frac{f_n(t, \hat{\alpha})}{1 - \hat{\alpha} - d} = d.$$

This completes the proof.  $\square$

**Lemma 2.4.** *Let  $0 < \alpha \leq 1/e$ , and  $x$  be an eventually positive solution of (1.1). Then*

$$\liminf_{t \rightarrow \infty} \frac{x(t)}{x(\tau(t))} \geq 1 - \alpha - \frac{1}{\lambda_1}.$$

*Proof.* First in Lemma 2.3, for each value of  $t$  we select the largest possible  $n$ , and observe that  $n \rightarrow \infty$  as  $t \rightarrow \infty$ . With the notation in Lemma 2.3, we have

$$\lim_{t \rightarrow \infty} f_n(t, \hat{\alpha}) = \frac{1}{(\hat{\lambda})^2} [e^{\hat{\alpha}\hat{\lambda}} - \hat{\alpha}\hat{\lambda} - 1] = \frac{1}{(\hat{\lambda})^2} [\hat{\lambda} - \hat{\alpha}\hat{\lambda} - 1].$$

Recall that the roots of a quadratic equation depend continuously on their coefficients. Then, as  $n \rightarrow \infty$ , the roots of (2.3) approach the roots of

$$d^2 - (1 - \hat{\alpha})d + \frac{1}{(\hat{\lambda})^2} [\hat{\lambda} - \hat{\alpha}\hat{\lambda} - 1] = 0.$$

Then as  $\hat{\lambda} \rightarrow \lambda_1$  and  $\hat{\alpha} \rightarrow \alpha$ , the roots of the above equation approach the roots of

$$d^2 - (1 - \alpha)d + \frac{1}{\lambda_1^2} [\lambda_1 - \alpha\lambda_1 - 1] = 0,$$

which are  $d = 1 - \alpha - 1/\lambda_1$  and  $d = 1/\lambda_1$ . For  $\alpha \in [0, 1/e]$  and the corresponding lambda with  $\lambda = e^{\alpha\lambda}$ , the first root is smaller than the second. To complete the proof we compute the limits in (2.2) first as  $t \rightarrow \infty$ , and then as  $\hat{\alpha} \rightarrow \alpha$ .  $\square$

**Theorem 2.5.** *Let  $0 < \alpha \leq 1/e$ , and*

$$\beta > 2\alpha + \frac{2}{\lambda_1} - 1. \quad (2.11)$$

*Then every solution of (1.1) is oscillatory.*

*Proof.* To obtain a contradiction, assume that  $x$  is an eventually positive solution of (1.1). Then by [6, Theorem 1],

$$\beta \leq \frac{\ln(\lambda_1) + 1}{\lambda_1} - \liminf_{t \rightarrow \infty} \frac{x(t)}{x(\tau(t))}.$$

Then by Lemma 2.4

$$\beta \leq \frac{\ln(\lambda_1) + 1}{\lambda_1} - \left(1 - \alpha - \frac{1}{\lambda_1}\right) = 2\alpha + \frac{2}{\lambda_1} - 1,$$

which contradicts the assumption and completes the proof for eventually positive solutions. If a solution  $y$  is an eventually negative solution, we consider  $x = -y$  which is an eventually positive solution.  $\square$

Note that the above theorem does not assume (1.6), and matches the best possible case of (1.7), i.e. when  $\omega = 1$ . Based on the example in [7], we build an example that satisfies the hypotheses in Theorem 2.5, but does not satisfy (1.6). Let  $\tau(t) = t - 2\sin^2(t) - 4/(e-2)$  and  $p(t) = (e-2)/(4e)$ . Then

$$\int_{\tau(t)}^t p(s) ds = \frac{e-2}{4e} \left(2\sin^2(t) + \frac{4}{e-2}\right),$$

so that  $\liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p = 1/e$  and  $\limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p = 1/2$ . Thus the assumptions in Theorem 2.5 are satisfied. Condition (1.6) becomes

$$\frac{e-2}{4e} [t - u - 2\sin^2(t) + 2\sin^2(u)] \geq \omega \frac{e-2}{4e} (t - u).$$

which is equivalent to

$$\frac{\sin^2(t) - \sin^2(u)}{t - u} \geq \frac{1 - \omega}{2}.$$

Because  $0 < \omega \leq 1$ , we have  $(1 - \omega)/2 \leq 1/2$ . Meanwhile a linear approximation on the numerator of the left-hand side gives a term of the form  $\sin(2u)$ , so we can select  $t$  and  $u$  close to each other for which the above inequality is not satisfied. Therefore (1.6) does not hold in this example.

### 3. BOUNDS USING CONDITION (1.6)

**Lemma 3.1.** *If  $\omega > 1$  in (1.6), then (1.1) has a non-oscillatory solution.*

*Proof.* Let  $\hat{\alpha} < \alpha$  and  $\hat{\beta} > \beta$ . Then from the definition of the limit inferior and the limit superior, there exists  $t_1$  such that

$$\hat{\alpha} \leq \int_{\tau(t)}^t p(s) ds \quad \text{and} \quad \int_{\tau(t)}^t p(s) ds \leq \hat{\beta} \quad \forall t \geq t_1.$$

For each  $t \geq t_1$ , let  $n = n(t)$  be the largest integer for which  $\tau^n(t) \geq t_1$ . Then  $n \rightarrow \infty$  as  $t \rightarrow \infty$ . By applying (1.6) repeatedly, we see that

$$\hat{\beta} \geq \int_{\tau^n(t)}^{\tau^{n-1}(t)} p(s) ds \geq \omega^{n-1} \int_{\tau(t)}^t p(s) ds.$$

As  $t \rightarrow \infty$ , we see that  $n \rightarrow \infty$  and  $\omega^{n-1} \rightarrow \infty$ . Thus  $\lim_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds = 0$ . Then  $\int_{\tau(t)}^t p(s) ds \leq 1/e$  holds eventually, so (1.1) has a non-oscillatory solution [3, Corollary 2.1.1].  $\square$

**Remark 3.2.** In view of Lemma 3.1, we restrict our attention to  $0 < \omega \leq 1$ . Using the series expansion of  $A$  in (1.8), we can show that  $A$  is an increasing function of  $\omega$ . Therefore,  $A$  possesses its maximal value and the bound in (1.7) has its minimal value when  $\omega = 1$ . Also the constant  $B$  has its maximal value, and the bound in (1.9) has its minimal value when  $\omega = 1$ .

**Lemma 3.3.** *Assume (1.6) holds and  $\tau(t) \leq t_* \leq t$ . Then*

$$\begin{aligned} \hat{\rho}_n(t) &:= \int_{t_*}^t p(s_1) \int_{\tau(s_1)}^{\tau(t)} p(s_2) \int_{\tau(s_2)}^{\tau^2(t)} p(s_3) \cdots \int_{\tau(s_{n-1})}^{\tau^{n-1}(t)} p(s_n) ds_n \cdots ds_1 \\ &\geq \frac{\omega^{1+\cdots+n-1}}{n!} \left( \int_{t_*}^t p(s) ds \right)^n. \end{aligned}$$

*Proof.* This is achieved by induction on the number of integrals. For the basic step  $n = 2$ , we have

$$\int_{t_*}^t p(s_1) \int_{\tau(s_1)}^{\tau(t)} p(s_2) ds_2 ds_1 \geq \omega \int_{t_*}^t p(s_1) \int_{s_1}^t p(s_2) ds_2 ds_1 = \frac{\omega}{2!} \left( \int_{t_*}^t p(s) ds \right)^2,$$

where the equality follows from Lemma 2.1.

For the induction step, we assume the inequality holds for  $n - 1$  integrals, and show it holds for  $n$  integrals. Under this assumption and using (1.6), we see that

$$\begin{aligned} \hat{\rho}_n(t) &\geq \int_{t_*}^t p(s_1) \frac{\omega^{1+\cdots+n-2}}{(n-1)!} \left( \int_{\tau(s_1)}^{\tau(t)} p(s) ds \right)^{n-1} \\ &\geq \int_{t_*}^{\tau(t)} p(s_1) \frac{\omega^{1+\cdots+n-1}}{(n-1)!} \left( \int_{s_1}^t p(s) ds \right)^{n-1}. \end{aligned}$$

By the substitution method with  $u = \int_{s_1}^t p(s) ds$ , we have  $du = -p(s_1) ds_1$ , and  $\int u^{n-1} du = \frac{1}{n} u^n$  which yields the desired result.  $\square$

**Lemma 3.4.** *Assume  $0 < \alpha \leq 1/e$ , (1.6), and that  $x$  is an eventually positive solution of (1.1). Then*

$$\beta \leq \alpha + \frac{1}{\omega \lambda_1} \ln(1 + 2\omega - \omega \lambda_1 + \alpha \omega \lambda_1).$$

*Proof.* Since  $0 < \alpha \leq 1/e$ , we have  $1 < \lambda_1$ , so there exists  $\hat{\lambda} \in (1, \lambda_1)$ . Then the conditions in Lemma 1.1 are satisfied, therefore (2.6) holds. As in [12, Lemma 3] we consider the function  $g(t) := x(\tau(t))/x(t)$  which is continuous,  $g(\tau(t)) = 1 < \hat{\lambda}$ , and  $g(t) > \hat{\lambda}$ . Then there exists  $t_* \in (\tau(t), t)$  such that

$$\frac{x(\tau(t))}{x(t_*)} = \hat{\lambda}. \quad (3.1)$$

From Lemma 2.4 for each  $\hat{d} < 1 - \alpha - \frac{1}{\lambda_1}$  there exists  $t_2$  such that

$$\frac{x(t)}{x(\tau(t))} \geq \hat{d} \quad \text{for all } t \geq t_2; \quad (3.2)$$



If necessary we may increase  $t_2$  to make it greater than the  $t_1$  in (2.6). Dividing (1.1) by  $x(t)$  and then integrating, by (2.6), we have

$$\int_{\tau(t)}^{t_*} \frac{x'(s)}{x(s)} ds = - \int_{\tau(t)}^{t_*} p(s) \frac{x(\tau(s))}{x(s)} ds \leq -\hat{\lambda} \int_{\tau(t)}^{t_*} p(s) ds.$$

From this inequality and (3.1),

$$\int_{\tau(t)}^{t_*} p(s) ds \leq \frac{\ln(\hat{\lambda})}{\hat{\lambda}}. \quad (3.3)$$

Now we estimate  $\Delta := \int_{t_*}^t p(s) ds$ . Integrating (1.1) from  $t_*$  to  $t$ , and proceeding as in (2.7), (2.9), we have

$$x(t_*) - x(t) \geq \Delta x(\tau(t)) + [\hat{\rho}_2(t)x(\tau^2(t)) + \hat{\rho}_3(t)x(\tau^3(t)) + \cdots + \hat{\rho}_n(t)x(\tau^n(t))].$$

By (2.6),

$$x(t_*) - x(t) \geq \Delta x(\tau(t)) + [\hat{\rho}_2(t)\hat{\lambda} + \hat{\rho}_3(t)\hat{\lambda}^2 + \cdots + \hat{\rho}_n(t)\hat{\lambda}^{n-1}]x(\tau(t)).$$

Dividing by  $x(\tau(t))$ , using Lemma 3.3 and (3.2), we have

$$\begin{aligned} \Delta + \frac{\Delta^2}{2!}\omega\hat{\lambda} + \frac{\Delta^3}{3!}\omega^{1+2}\hat{\lambda}^2 + \cdots + \frac{\Delta^n}{n!}\omega^{1+\cdots+n-1}\hat{\lambda}^{n-1} &\leq \frac{x(t_*)}{x(\tau(t))} - \frac{x(t)}{x(\tau(t))} \\ &\leq \frac{1}{\hat{\lambda}} - \hat{d}. \end{aligned} \quad (3.4)$$

To solve the above inequality we define the polynomial

$$Q_n(\Delta) = \Delta + \frac{\Delta^2}{2!}\omega\hat{\lambda} + \frac{\Delta^3}{3!}\omega^{1+2}\hat{\lambda}^2 + \cdots + \frac{\Delta^n}{n!}\omega^{1+\cdots+n-1}\hat{\lambda}^{n-1} - \left(\frac{1}{\hat{\lambda}} - \hat{d}\right).$$

Note that all the coefficients of  $\Delta$  are positive and the independent term is negative, so by the Descartes' rule of signs,  $Q_n$  has at most one positive root. Since  $Q_n(0) < 0$  and  $\lim_{\Delta \rightarrow \infty} Q_n(\Delta) = \infty$ , it follows that  $Q_n$  has exactly one positive root. To satisfy (3.4),  $\Delta$  must be less than or equal to the positive root of  $Q_n$ . When  $t \rightarrow \infty$ , it follows that by definition  $t_* \rightarrow \infty$  and  $n \rightarrow \infty$ . Therefore we can increase  $n$ , which provides more accurate estimates for  $\Delta$ . There are formulas for obtaining the roots when  $n = 1, 2, 3, 4$ , but not for  $n \geq 5$ . Sficas et al [12] solved this equation when  $n = 2$ , by using its positive root as an estimate for the solution of (3.4). Our approach is to use the solution of

$$\lim_{n \rightarrow \infty} Q_n(\Delta) = 0$$

as an estimate for the solution of (3.4).

When  $\omega = 1$  we need to find the positive solution of

$$\Delta + \frac{e^{\Delta\hat{\lambda}} - \Delta\hat{\lambda} - 1}{\hat{\lambda}} - \left(\frac{1}{\hat{\lambda}} - \hat{d}\right) = 0.$$

In this case (3.4) is satisfied if

$$\Delta \leq \frac{1}{\hat{\lambda}} \ln(2 - \hat{\lambda}\hat{d}), \quad (3.5)$$

which corresponds to (3.7), below, with  $\omega = 1$ .

For  $0 < \omega < 1$ , we define a new polynomial

$$\hat{Q}_n(\Delta) = \Delta + \frac{\Delta^2}{2!}\omega\hat{\lambda} + \frac{\Delta^3}{3!}\omega^2\hat{\lambda}^2 + \cdots + \frac{\Delta^n}{n!}\omega^{n-1}\hat{\lambda}^{n-1} - \left(\frac{1}{\hat{\lambda}} - \hat{d}\right). \quad (3.6)$$

Note that  $\hat{Q}_2 \equiv Q_2$ , and  $Q_2(\Delta) < Q_n(\Delta) < \hat{Q}_n(\Delta)$  for  $n > 2$ , because  $0 < \omega < 1$ . Therefore the positive root of  $\hat{Q}_n$  is less than the positive root of  $Q_n$ . The equation  $\lim_{n \rightarrow \infty} \hat{Q}_n(\Delta) = 0$  is

$$\Delta + \frac{e^{\Delta\omega\hat{\lambda}} - \Delta\omega\hat{\lambda} - 1}{\omega\hat{\lambda}} - \left(\frac{1}{\hat{\lambda}} - \hat{d}\right) = 0.$$

Therefore, (3.4) is satisfied if

$$\Delta \leq \frac{1}{\omega\hat{\lambda}} \ln(1 + \omega - \omega\hat{\lambda}\hat{d}). \tag{3.7}$$

Note that the right-hand side of this inequality is less than the right-hand side of (3.5), because they correspond to the roots of  $\hat{Q}_n$  and of  $Q_n$ , respectively.

Adding (3.3) and (3.7), and then computing the limit as  $\hat{\lambda} \rightarrow \lambda_1$  and  $\hat{d} \rightarrow 1 - \alpha - \frac{1}{\lambda_1}$  yields

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p(s) ds &\leq \frac{\ln(\lambda_1)}{\lambda_1} + \frac{1}{\omega\lambda_1} \ln\left(1 + \omega - \omega\lambda_1\left(1 - \alpha - \frac{1}{\lambda_1}\right)\right) \\ &= \alpha + \frac{1}{\omega\lambda_1} \ln(1 + 2\omega - \omega\lambda_1 + \alpha\omega\lambda_1). \end{aligned}$$

This completes the proof. □

**Theorem 3.5.** *Assume (1.6),  $0 < \alpha \leq 1/e$ , and*

$$\beta > \alpha + \frac{1}{\omega\lambda_1} \ln(1 + 2\omega - \lambda_1\omega + \alpha\lambda_1\omega). \tag{3.8}$$

*Then every solution of (1.1) is oscillatory.*

*Proof.* For the sake of contradiction, assume that there is an eventually positive solution. Then by Lemma 3.4 we have a contradiction to (3.8). On the other hand if  $y$  is an eventually negative solution of (1.1), we may consider  $x = -y$  which is an eventually positive solution. □

**Remark 3.6.** For  $\omega = 1$ , the bound in Theorem 3.5 is slightly lower than the one in [12], see Table 1. For an example of an equation that satisfies the assumptions in Theorem 3.5, we refer the reader to [12].

TABLE 1. Oscillation criteria when  $\omega = 1$

cond.	$\alpha = 1/e, \lambda_1 = e$	$\alpha = 2 \ln(e/2)/e, \lambda_1 = e/2$	$\alpha = 0, \lambda_1 = 1$
(1.7)	$\beta > 0.471518$	$\beta > 0.923057$	$\beta > 1$
(1.9)	$\beta > 0.459987$	$\beta > 0.741974$	$\beta > \sqrt{3} - 1 \approx 0.732750$
(3.8)	$\beta > 0.459188$	$\beta > 0.716267$	$\beta > \ln(2) \approx 0.693147$

## 4. EQUATIONS WITH MULTIPLE DELAYS

As an application of the above results, we present a condition for the oscillation of solutions to the equation

$$x'(t) + \sum_{i=1}^m p_i(t)x(\tau_i(t)) = 0, \quad (4.1)$$

where  $p_i, \tau_i \in C([t_0, \infty), [0, \infty))$ ,  $\tau_i(t) \leq t$ , and  $\lim_{t \rightarrow \infty} \tau_i(t) = \infty$ . In this section we do not require  $\tau_i$  to be monotonic, instead we redefine  $\tau$  as the non-decreasing function

$$\tau(t) = \max_{t_0 \leq s \leq t} \left\{ \max_{1 \leq i \leq m} \{\tau_i(s)\} \right\}.$$

If  $x$  is an eventually positive solution of (4.1), then  $x'(t) = -\sum_{i=1}^m p_i(t)x(\tau_i(t)) \leq 0$  so  $x$  is non-increasing, and

$$x'(t) + \left( \sum_{i=1}^m p_i(t) \right) x(\tau(t)) \leq 0. \quad (4.2)$$

In this section the summation  $\sum_{i=1}^m p_i$  plays the role of  $p$  in the previous sections, while  $\tau$  plays the same role as before. We redefine the constants

$$\alpha = \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t \sum_{i=1}^m p_i(s) ds, \quad \beta = \limsup_{t \rightarrow \infty} \int_{\tau(t)}^t \sum_{i=1}^m p_i(s) ds;$$

while  $\lambda_1 \leq \lambda_2$  remain as the roots of  $\lambda = e^{\alpha\lambda}$ . In Sections 2 and 3 we replace  $p(\cdot)$  by  $\sum_{i=1}^m p_i(\cdot)$ , and replace the sign  $=$  by  $\leq$ , in (2.7)–(2.9). The rest of the inequalities remain valid, so we only restate the main results.

**Theorem 4.1.** *Let  $0 < \alpha \leq 1/e$ , and*

$$\beta > 2\alpha + \frac{2}{\lambda_1} - 1. \quad (4.3)$$

*Then every solution of (4.1) is oscillatory.*

**Theorem 4.2.** *Assume (1.6),  $0 < \alpha \leq 1/e$ , and*

$$\beta > \alpha + \frac{1}{\omega\lambda_1} \ln(1 + 2\omega - \lambda_1\omega + \alpha\lambda_1\omega). \quad (4.4)$$

*Then every solution of (4.1) is oscillatory.*

Our conditions for the oscillation of solutions of equations with multiple delays are rather basic. For alternative oscillation criteria, we refer the reader to [10, sec. 2.6], [2, 9].

We conclude this article by stating that the optimal bound  $\beta > 1/e$  has not been reached yet; so there is room for improvement.

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## 5. ADDENDUM POSTED OCTOBER 13, 2022

The proof of Lemma 2.2 is incorrect. It uses that

$$\int_{u_{0,k}}^{u_{0,k}} p = \int_{u_{1,k}}^{u_{1,k}} p$$

implies  $\tau(u_{10,k}) = \tau(u_{10,k})$ , which is not necessarily true. Therefore Lemma 2.2 needs a new proof. One possible solution is to assume that  $\int_{\tau(u)}^{\tau(v)} p \geq \int_u^v p$ , i.e. (1.6) holds for  $\omega = 1$ . However this assumption causes some difficulties. If  $\omega = 1$  and  $u \leq v$ , then

$$\int_{\tau(v)}^v p = \int_{\tau(u)}^u p + \int_u^v p - \int_{\tau(u)}^{\tau(v)} p \geq \int_{\tau(u)}^u p.$$

Therefore the mapping  $t \mapsto \int_{\tau(t)}^t p$  is non-decreasing which implies

$$\alpha = \liminf_{t \rightarrow \infty} \int_{\tau(t)}^t p = \limsup_{t \rightarrow \infty} \int_{\tau(t)}^t p = \beta.$$

The equality  $\alpha = \beta$  does not allow condition (4.3) to be satisfied in the typical case  $\alpha = 1/e$  and  $\lambda_1 = e$ .

Another consequence of the above equality is that in Remark 3.6 and Table 1 we cannot use  $\omega = 1$ , we can only use approximations such as  $\omega = 0.9999$ .

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