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OPTIMIZATION PROBLEMS AND MATHEMATICAL ANALYSIS OF OPTIMAL VALUES IN ORLICZ SPACES

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Abstract. This article concerns a minimization problem related to an elliptic equation in Orlicz-Sobolev spaces. We prove existence and uniqueness of optimal solutions and show that they are monotone and stable. Furthermore, by employing a characterization of the tangent cones in L^∞ spaces, we derive some qualitative properties of the optimal solutions. We also derive some results regarding the optimal values.

1. Introduction

1.1. **General overview.** This article addresses an optimization problem related to the boundary value problem

$$\begin{aligned} -\nabla\cdot(a(|\nabla u|)\nabla u) = f(x) \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

The conditions that we employ in (1.1) will be described in the next section. As we shall see the imposed restrictions on the function $a(\cdot)$ suggest considering Orlicz-Sobolev space as the underlying function space in which we seek the solution of (1.1). The existence and uniqueness of a solution to the boundary value problem is a straightforward task of implementing the direct method to prove the former and a typical strict convexity argument to guarantee the latter. Denoting the solution by u_f, to stress the dependence on the force function $f(\cdot)$, the goal function

$$\gamma(f) := \int_\Omega (f u_f - \Phi(|\nabla u_f|))\,\mathrm{d}x,$$

is minimized relative to

$$f \in A_\alpha = \big\{f \in L^\infty(\Omega) : 0 \le f \le 1, \int_\Omega f(x)\,\mathrm{d}x = \alpha\big\}.$$

The function Φ that appears in the definition of $\gamma(\cdot)$ is an appropriate N-function closely related to the function $a(\cdot)$. In order to appreciate the results reported in this paper a thorough understanding of the admissible set A_α is an advantage. This set can be decomposed as $A_\alpha = \mathcal{C}_+ \cap \mathcal{B}(0,1) \cap \Lambda^{-1}(\alpha)$. Here $\mathcal{C}_+$ denotes the positive cone of $L^\infty(\Omega)$, $\mathcal{B}(0,1)$ the closed unit ball in $L^\infty(\Omega)$, and $\Lambda(f) = \int_\Omega f\,\mathrm{d}x$ the continuous linear functional on $L^\infty(\Omega)$. Identifying $L^\infty(\Omega)$ with the

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dual of $L^1(\Omega)$, it is readily verified that A_α is convex and weak*-compact in $L^\infty(\Omega)$. Unfortunately this decomposition does not reveal more properties of A_α which happen to be core in what follows. To discover other properties of A_α we first recall the definition of a measure preserving transformation from a measure space into another measure space. The mapping $\xi : (X, \sigma_X, \mu_X) \to (Y, \sigma_Y, \mu_Y)$ is a measure preserving transformation if and only if

(i) ξ is measurable i.e. for every $S \in \sigma_Y$, $\xi^{-1}(S) \in \sigma_X$;
(ii) the equation $\mu_X(\xi^{-1}(S)) = \mu_Y(S)$ holds for every $S \in \sigma_Y$.

Here is an example of a measure preserving transformation when $X = Y = [0,1]$, σ_X and σ_Y are the Borel sets, and $\mu_X = \mu_Y = d\mathcal{L}$, the Lebesgue measure: Let $\xi(t) : [0,1] \to [0,1]$ be defined by $\xi(t) = kt \mod 1$, for some $k \in \mathbb{N}$. Whence

$$\xi(t) = \sum_{i=0}^{k-1} k\Big(t - \frac{i}{k}\Big)\chi_{[i/k,(i+1)/k)}(t).$$

Henceforth χ_E denotes the characteristic function supported on the set E. So $\chi(x)$ is equal to 1 when $x \in E$ and equal to 0 otherwise. Let us consider the open interval $(a,b) \subseteq [0,1]$. Observe that

$$\xi^{-1}(a,b) = \cup_{i=0}^{k-1}\Big(\frac{a+i}{k}, \frac{b+i}{k}\Big),$$

so $\mathcal{L}(\xi^{-1}(a,b)) = \mathcal{L}(a,b) = b - a$. Since the family of open intervals (a,b) generates the open sets of $[0,1]$, we infer that $\mathcal{L}(\xi^{-1}(O)) = \mathcal{L}(O)$ for every O, an open subset of $[0,1]$. Finally using a well-known extension theorem in Ergodic theory we deduce that $\mathcal{L}(\xi^{-1}(B)) = \mathcal{L}(B)$, for every $B \in \mathcal{B}$, the Borel sets of $[0,1]$. So ξ is a measure preserving transformation as desired.

Let $\mathcal{M}_{\Omega\to[0,1]} = \{\xi : \Omega \to [0,1] : \xi \text{ is a measure preserving transformation}\}$, and $f^\Delta : [0, |\Omega|] \to [0,1]$ defined by $f^\Delta(t) = \chi_{[0,\alpha)}(t)$. Define

$$\mathcal{R} = \{f^\Delta \circ \xi : \xi \in \mathcal{M}_{\Omega\to[0,1]}\}.$$

The fact that $A_\alpha = \overline{\mathcal{R}}^{\sigma(L^\infty,L^1)}$, the w^* closure of $\mathcal{R}$ in $L^\infty(\Omega)$, and $\mathcal{R} = \operatorname{ext} A_\alpha$, the set of extreme points of A_α in $L^\infty(\Omega)$, belong to the folklore, see for example [4, 5, 16]. Note that functions in $\mathcal{R}$ belong to $\{0,1\}^\Omega$ i.e. they are $\{0,1\}$-valued whereas clearly those in A_α belong to $[0,1]^\Omega$. The existence of optimal solutions for the minimization

$$\inf_{f\in A_\alpha} \gamma(f)$$

shall be shown using the w^* continuity of $\gamma(\cdot)$ in conjunction with the w^* compactness of A_α, in $L^\infty(\Omega)$. However, similar to many other optimization problems, particularly from the numerical point of view, it would be significantly more efficient to know that the optimal solutions belong to a smaller set than A_α. Indeed, we shall prove that they belong to the extreme points of A_α i.e. $\mathcal{R}$. This milestone will be achieved using a very friendly characterization of the tangent cones of subsets of $L^\infty(\Omega)$. The uniqueness of the optimal solution is another achievement which is an immediate consequence of the strict convexity of $\gamma(\cdot)$. Since the optimal solutions are of type $\chi_{\hat{\Omega}} \in \mathcal{R}$ we can identify them with the shape $\hat{\Omega}$. One then could explore the qualitative properties of $\hat{\Omega}$. We shall see, for example, that $\hat{\Omega}$ behave monotonically with respect to the parameter α in the sense that for $\beta \le \alpha$, $\hat{\Omega}_\beta \subseteq \hat{\Omega}_\alpha$, where $\hat{\Omega}_\beta$ and $\hat{\Omega}_\alpha$ denote the optimal shapes relative to A_β and A_α, respectively. It will also be shown that $\hat{\Omega}$, an essentially open set, is connected, thanks to the fact that Ω is simply connected, and also that $\hat{\Omega}$ forms a layer around $\partial\Omega$, the boundary of Ω.

In the final part of this note we shall derive some mathematical analysis results about the optimal value:

$$\ell(\alpha) = \inf_{f\in A_\alpha} \gamma(f).$$

In particular, we shall prove that $\ell(\cdot)$ is Lipschitz continuous, strictly convex and differentiable. We shall also apply a Lagrange multiplier argument to show the estimate $\ell(\alpha) \le C\alpha$ for some positive constant C.

1.2. **Description of the minimization problem and preliminaries.** Let Ω be a bounded smooth domain in $\mathbb{R}^N (N \ge 2)$ and $a : (0,\infty) \to (0,\infty)$ a function such that the map

$$\varphi(t) = \begin{cases} a(|t|)t & t \neq 0,\\ 0 & t = 0, \end{cases}$$

is an odd strictly increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Thus, the function $\Phi(t) = \int_0^t \varphi(s)\,\mathrm{d}s$, $t \in \mathbb{R}$, is an N-function, see for example [1] for the definition. The conjugate of Φ, denoted Φ^*, is defined by $\Phi^*(t) = \int_0^t \varphi^{-1}(s)\,\mathrm{d}s$, for all $t \in \mathbb{R}$. It is known that Φ^* is also an N-function, and can be reformulated as

$$\Phi^*(t) = \sup_{s\ge 0}(st - \Phi(s)).$$

The set

$$K_\Phi(\Omega) = \big\{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega \Phi(|u(x)|)\,\mathrm{d}x < \infty\big\},$$

is called the *generalized Orlicz class* while the *generalized Orlicz space* is defined by

$$L^\Phi(\Omega) = \big\{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \lim_{\tau\to 0^+} \int_\Omega \Phi(\tau|u(x)|)\,\mathrm{d}x = 0\big\}.$$

$L^\Phi(\Omega)$ is a Banach space endowed with the Luxemburg norm

$$|u|_\Phi = \inf\big\{\tau > 0 : \int_\Omega \Phi\big(\frac{|u(x)|}{\tau}\big)\,\mathrm{d}x \le 1\big\},$$

or the equivalent Orlicz norm

$$|u|_{L^\Phi} = \sup\big\{\big|\int_\Omega uv\,\mathrm{d}x\big| : v \in L^{\Phi^*}(\Omega), \int_\Omega \Phi^*(|v(x)|)\,\mathrm{d}x \le 1\big\}.$$

Moreover, the following Hölder type inequality holds, [1],

$$\big|\int_\Omega uv\,\mathrm{d}x\big| \le 2|u|_\Phi|v|_{\Phi^*}, \quad \forall u \in L^\Phi(\Omega),\ v \in L^{\Phi^*}(\Omega). \tag{1.2}$$

Henceforth we assume that there exist two positive constants λ and μ such that

$$1 < \lambda \le \frac{t\varphi(t)}{\Phi(t)} \le \mu < \infty, \quad \forall t > 0. \tag{1.3}$$

The relation (1.3) ensures that the differential equation in (1.1) is uniformly elliptic, see [13], and that Φ satisfies the Δ_2-condition:

$$\Phi(2t) \le C\Phi(t), \quad \forall t \ge 0, \tag{1.4}$$

where C is a positive constant, [15, Proposition 2.3]. In turn, the Δ_2-condition implies that $L^\Phi(\Omega)$ and $K_\Phi(\Omega)$ are identical, and the dual of $L^\Phi(\Omega)$ coincides with $L^{\Phi^*}(\Omega)$, see for example [1]. Furthermore, we assume that the function

$$[0,\infty) \ni t \to \Phi(\sqrt{t}), \tag{1.5}$$

is convex. Condition (1.5) guarantees $L^\Phi(\Omega)$ is uniformly convex and hence reflexive, see [15, Proposition 2.2]. The *generalized Orlicz-Sobolev space* is defined by

$$W^{1,\Phi}(\Omega) = \big\{u \in L^\Phi(\Omega) : \frac{\partial u}{\partial x_i} \in L^\Phi(\Omega), i = 1, \dots, N\big\}.$$

It is well known that $W^{1,\Phi}(\Omega)$ endowed with the norm $\|u\|_{1,\Phi} = |\ |\nabla u|\ |_\Phi + |u|_\Phi$ is a reflexive Banach space. The space $W_0^{1,\Phi}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to $\|u\|_{1,\Phi}$-norm. Using the Poincaré inequality in Orlicz-Sobolev spaces, it follows that $\|u\| := \|\nabla u\|_\Phi$ is equivalent to $\|u\|_{1,\Phi}$. The Orlicz-Sobolev space $W_0^{1,\Phi}(\Omega)$ is also a reflexive Banach space, [15].

In [15] it is shown that for $u \in L^\Phi(\Omega)$ the following holds

$$|u|_\Phi > 1 \ \Rightarrow\ |u|_\Phi^\lambda \le \int_\Omega \Phi(|u(x)|)\,\mathrm{d}x \le |u|_\Phi^\mu. \tag{1.6}$$

Also, from (1.3), one can prove that the following embeddings are continuous,

$$L^\mu(\Omega) \hookrightarrow L^\Phi(\Omega) \hookrightarrow L^\lambda(\Omega), \quad \text{and} \quad L^{\lambda'}(\Omega) \hookrightarrow L^{\Phi^*}(\Omega) \hookrightarrow L^{\mu'}(\Omega), \tag{1.7}$$

where λ' and μ' denote the conjugate component of λ and μ respectively, see for example [1].

Definition 1.1. Let $f \in L^{\Phi^*}(\Omega)$. We say that $u \in X := W_0^{1,\Phi}(\Omega)$ is a weak solution of (1.1) if

$$\int_\Omega a(|\nabla u|)\nabla u \cdot \nabla v\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x, \tag{1.8}$$

for all $v \in X$.

Using the direct method followed with a strict convexity argument one can prove the following basic result.

Theorem 1.2. *The boundary value problem* (1.1) *has a unique solution* $u_f \in W_0^{1,\Phi}(\Omega)$. *The solution* u_f *is the unique minimizer of the energy functional*

$$\hat{J}_f(u) = \int_\Omega (\Phi(|\nabla u|) - fu)\,\mathrm{d}x,$$

relative to $u \in W_0^{1,\Phi}(\Omega)$.

We define the functional $J_f : X \to \mathbb{R}$ by $J_f = -\hat{J}_f$ i.e.

$$J_f(u) = \int_\Omega (fu - \Phi(|\nabla u|))\,\mathrm{d}x.$$

We are interested in the minimization problem

$$\inf_{f\in A_\alpha} \gamma(f), \tag{1.9}$$

where $\gamma(f) = J_f(u_f)$. We note that for $f \in A_\alpha$, u_f is positive, see [8, Lemma 3.4], and that $u_f \in W^{2,\Phi}(\Omega)$, [3].

It is worth pointing out that when $p > 1$ and $\varphi(t) = |t|^{p-2}t$, (1.1) becomes the well-known Dirichlet p-Laplace boundary value problem

$$\begin{gathered} -\Delta_p u = f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.10}$$

In [14] the authors investigated the minimization problem (1.9) related to (1.10) for $p = 2$.

We close this section with some physical examples of function Φ.

(i) nonlinear elasticity: $\Phi(t) = (1 + t^2)^\delta - 1$, $\delta > \frac{1}{2}$;
(ii) plasticity: $\Phi(t) = t^\delta(\log(1 + t))^\epsilon$, $\delta \geq 1$, $\epsilon > 0$;
(iii) generalized Newtonian fluids: $\Phi(t) = \int_0^t s^{1-\delta}(\sinh^{-1} s)^\epsilon \,\mathrm{d}s$, $0 \leq \delta \leq 1$, $\epsilon > 0$.

For details, see [6, 7].

This article is organized as follows. In section 2, existence and uniqueness of optimal solutions to the minimization problem (1.9) are discussed. In section 3, we recite the definition of the tangent cones in $L^\infty(D)$, and use them to derive the optimality conditions satisfied by the optimal solutions of (1.9). Section 4 is devoted to further properties of the optimal solutions. In particular, we prove the optimal solutions increase as the parameter α increases. Also, when α is close to, say, β, the respective optimal solutions will be close to each other in the L^p-norm. The section is closed by showing that the optimal value grows linearly with respect to the parameter α.

2. Existence and uniqueness of optimal solutions

In this section we prove that the minimization problem (1.9) has a unique solution i.e. there is an $\hat{f} \in A_\alpha$ such that $\gamma(\hat{f}) = \inf_{f\in A_\alpha} \gamma(f)$. To this end, we first prove the following result.

Lemma 2.1. *The functional $\gamma : L^{\Phi^*}(\Omega) \to \mathbb{R}$ satisfies the following properties:*

(i) *γ is weakly sequentially continuous;*
(ii) *γ is strictly convex;*
(iii) *γ is Fréchet differentiable, and $\langle\gamma'(f), g\rangle = \int_\Omega g u_f \,\mathrm{d}x$, for all $g \in L^{\Phi^*}(\Omega)$.*

Proof. (i) Assume $f_n \rightharpoonup f$, in $L^{\Phi^*}(\Omega)$. We have

$$\begin{aligned} \gamma(f) + \int_\Omega (f_n - f)u_f \,\mathrm{d}x &= \int_\Omega f_n u_f \,\mathrm{d}x - \int_\Omega \Phi(|\nabla u_f|) \,\mathrm{d}x \\ &= J_{f_n}(u_f) \leq J_{f_n}(u_{f_n}) = \gamma(f_n) \\ &= J_f(u_{f_n}) + \int_\Omega (f_n - f)u_{f_n} \,\mathrm{d}x \\ &\leq J_f(u_f) + \int_\Omega (f_n - f)u_{f_n} \,\mathrm{d}x \\ &= \gamma(f) + \int_\Omega (f_n - f)u_{f_n} \,\mathrm{d}x. \end{aligned} \tag{2.1}$$

Since $f_n \rightharpoonup f$ in $L^{\Phi^*}(\Omega)$ we deduce that $\int_\Omega (f_n - f)u_f \,\mathrm{d}x \to 0$. Whence, to complete the proof of the assertion, it suffices to show

$$\int_\Omega (f_n - f)u_{f_n} \,\mathrm{d}x \to 0.$$

The sequence $\{f_n\}$ is bounded in $L^{\Phi^*}(\Omega)$. If $\|u_{f_n}\| > 1$ then by (1.6) we have

$$\|u_{f_n}\|^\lambda \le \int_\Omega \Phi(|\nabla u_{f_n}|)\,\mathrm{d}x \le \|u_{f_n}\|^\mu. \tag{2.2}$$

From (1.3) and (2.2) we infer that

$$\begin{aligned}\int_\Omega f_n u_{f_n}\,\mathrm{d}x &= \int_\Omega a(|\nabla u_{f_n}|)|\nabla u_{f_n}|^2\,\mathrm{d}x \\ &= \int_\Omega \varphi(|\nabla u_{f_n}|)|\nabla u_{f_n}|\,\mathrm{d}x \\ &\ge \lambda \int_\Omega \Phi(|\nabla u_{f_n}|)\,\mathrm{d}x \\ &\ge \lambda \|u_{f_n}\|^\lambda.\end{aligned}$$

Now, by the Hölder inequality we have

$$\lambda\|u_{f_n}\|^\lambda \le \int_\Omega f_n u_{f_n}\,\mathrm{d}x \le C|f_n|_{\Phi^*}|u_{f_n}|_\Phi \le C\|u_{f_n}\|.$$

Thus, since $\lambda > 1$ we deduce that $\{u_{f_n}\}$ is bounded in X. Hence, up to a subsequence, there exists $w \in X$ such that $u_{f_n} \rightharpoonup w$ in X. By the Sobolev's embedding theorem, X is compactly embedded into $L^\Phi(\Omega)$, [1]. So, $u_{f_n} \to w$ in $L^\Phi(\Omega)$. So by the Hölder inequality we infer $\int_\Omega (f_n - f)u_{f_n}\,\mathrm{d}x \to 0$. Therefore, $\gamma(f_n) \to \gamma(f)$, as desired.

Remark 2.2. We point out that w is equal to u_f a.e. in Ω. Indeed, since the functional $u \mapsto \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x$ is weakly lower semi-continuous, see [15, Lemma 4.3], we have that

$$\begin{aligned}\gamma(f) &= J_f(u_f) \ge J_f(w) \\ &= \int_\Omega f w\,\mathrm{d}x - \int_\Omega \Phi(|\nabla w|)\,\mathrm{d}x \\ &\ge \limsup_{n\to\infty}\Big(\int_\Omega f_n u_{f_n}\,\mathrm{d}x - \int_\Omega \Phi(|\nabla u_{f_n}|)\,\mathrm{d}x\Big) \\ &= \limsup_{n\to\infty}\gamma(f_n) = \gamma(f).\end{aligned}$$

Hence $\gamma(f) = J_f(u_f) = J_f(w)$. Therefore the uniqueness of the maximizer yields $w = u_f$ a.e. in Ω.

(ii) The proof of this part is similar to that of [2, Lemma 3.2]. So we omit it.

(iii) Let $f, g \in L^{\Phi^*}(\Omega)$. For any $t \in (0,1)$ we set $h_t = f + tg$. By (2.1) we have

$$\gamma(f) + \int_\Omega (h_t - f)u_f\,\mathrm{d}x \le \gamma(h_t) \le \gamma(f) + \int_\Omega (h_t - f)u_{h_t}\,\mathrm{d}x.$$

By Remark 2.2, we infer that $u_{h_t} \to u_f$, as $t \to 0^+$, in $L^\Phi(\Omega)$. So

$$\langle \gamma'(f), g\rangle = \lim_{t\to 0^+}\frac{\gamma(h_t) - \gamma(f)}{t} = \int_\Omega g u_f\,\mathrm{d}x.$$

Therefore γ is Gâteaux differentiable; moreover, $\gamma'(f) = u_f$. Next, we show that γ' is continuous at $f \in L^{\Phi^*}(\Omega)$. Let $\{f_n\}$ be a sequence in $L^{\Phi^*}(\Omega)$ such that $f_n \to f$

in $L^{\Phi^*}(\Omega)$. By part *(i)* and Remark 2.2, we deduce that $u_{f_n} \to u_f$ in $L^{\Phi}(\Omega)$. Thus, for all $g \in L^{\Phi^*}(\Omega)$ we have

$$|\langle \gamma'(f_n) - \gamma'(f), g\rangle| = \Big| \int_\Omega g(u_{f_n} - u_f)\,\mathrm{d}x\Big| \to 0, \quad \text{as } n \to \infty.$$

Therefore γ is Fréchet differentiable in $L^{\Phi^*}(\Omega)$. □

The main result of this section reads as follows.

Theorem 2.3. *The minimization problem* (1.9) *has a unique solution.*

Proof. It's well known that A_α is w* closed in $L^\infty(\Omega)$ in addition to being convex; so it is weak* compact. Since the dual space of $L^\Phi(\Omega)$ is $L^{\Phi^*}(\Omega)$, by Lemma 2.1*(i)* and the inclusions $L^\Phi(\Omega) \subset L^1(\Omega)$ and $L^\infty(\Omega) \subset L^{\Phi^*}(\Omega)$ we infer that γ is weak* continuous in $L^\infty(\Omega)$. Therefore the minimization (1.9) has a solution. The uniqueness of the solution is a consequence of strict convexity of γ. □

3. Characterization of the optimal solution and its consequences

In this section we use tangent cones to derive the optimality condition satisfied by the optimal solutions, and obtain some qualitative results from this condition.

Definition 3.1. Let V be a normed linear space and K a nonempty subset of V. The inner (intermediate or derivable) tangent cone of K at $z \in K$, denoted by $T_K'(z)$, is defined as follows; $v \in T_K'(z)$ if and only if for each decreasing real numbers $t_n \downarrow 0$ there exists a sequence $\{v_n\}$ in V such that $\lim_{n\to\infty} v_n = v$ and $z + t_n v_n \in K$ for all $n \geq 1$.

The following two lemmas are useful for deriving the minimality conditions associated with problem (1.9). The proof of the following lemma is in [10, Theorem 4.14].

Lemma 3.2. *Let K be a nonempty subset of a real normed space V, and let F be a functional defined on an open superset of K. If z is a minimizer of F in K and if F is Fréchet differentiable at z, then*

$$\langle F'(z), v\rangle \geq 0, \quad \forall v \in T_K'(z), \tag{3.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between V and V', the dual of V. Here $F'(z)$ stands for the Gâteaux derivative of F at z. The condition (3.1) *is called the first order optimality condition.*

For the proof of the following Lemma see [14, Lemma 2.2].

Lemma 3.3. *Let V be a normed linear space, K a nonempty convex subset of V and $F : V \to \mathbb{R}$ a convex functional which is Gâteaux differentiable. If $\langle F'(z), v\rangle \geq 0$ for all $v \in T_K'(z)$, then z is a minimizer of F in K.*

For $f \in A_\alpha$ and $n \in \mathbb{N}$ we use the following notation:

- $\Omega_0 := \{x \in \Omega : f(x) = 0\}$,
- $\Omega^* := \{x \in \Omega : 0 < f(x) < 1\}$,
- $\Omega_1 := \{x \in \Omega : f(x) = 1\}$,
- $\Omega_n^0 = \{x \in \Omega : f(x) \leq 1/n\}$,
- $\Omega_n^1 = \{x \in \Omega : f(x) \geq 1 - 1/n\}$.

To determine the characteristics of tangent cones in A_α, we now state and prove some lemmas that are known but we have not been able to find their proofs.

Lemma 3.4. *Let $f \in A_\alpha$ and $h \in L^\infty(\Omega)$. If $h \in T'_{A_\alpha}(f)$ then*

(i) $\int_\Omega h \,\mathrm{d}x = 0$,
(ii) $\lim_{n\to\infty} \|\chi_{\Omega_n^0} h^-\|_\infty = 0$,
(iii) $\lim_{n\to\infty} \|\chi_{\Omega_n^1} h^+\|_\infty = 0$,

where h^+ (resp. h^-) is the positive (resp. negative) part of h.

Proof. (i) The proof of this part is simple.

(ii) Let $\varepsilon > 0$. We set $t_n = \frac{1}{\varepsilon}\|\chi_{\Omega_n^0} f\|_\infty$ for $n \in \mathbb{N}$. Thus there exists a sequence $\{h_n\}$ in $L^\infty(\Omega)$ such that $h_n \to h$ in $L^\infty(\Omega)$ and $f + t_n h_n \in A_\alpha$ for all $n \in \mathbb{N}$. So we have $h \geq (h - h_n) - \frac{f}{t_n}$ in Ω. Thus $h^- \leq \|h_n - h\|_\infty + \varepsilon$ a.e. in Ω_n^0 for all $n \in \mathbb{N}$. Hence $\limsup_{n\to\infty} \|\chi_{\Omega_n^0} h^-\|_\infty \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the result of this part is obtained.

(iii) The proof of this part is similar to (ii). $\square$

Lemma 3.5. *Let $f \in A_\alpha$ and $h \in L^\infty(\Omega)$ be such that*

(i) $\int_{\Omega_n^0} h^- \,\mathrm{d}x = \int_{\Omega_n^1} h^+ \,\mathrm{d}x$ *for all* $n \in \mathbb{N}$.
(ii) $\lim_{n\to\infty} \|\chi_{\Omega_n^0} h^-\|_\infty = 0$,
(iii) $\lim_{n\to\infty} \|\chi_{\Omega_n^1} h^+\|_\infty = 0$.

Then $h \in T'_{A_\alpha}(f)$.

Proof. From (i) for $n = 1$ we infer $\int_\Omega h \,\mathrm{d}x = 0$. Assume $\|h\|_\infty \neq 0$. Let $t_n \in (0, \frac{1}{n\|h\|_\infty})$ for $n \geq 1$. For each n we define $h_n := h + \chi_{\Omega_n^0} h^- - \chi_{\Omega_n^1} h^+$ in Ω. From (ii) and (iii) we deduce $h_n \to h$ in $L^\infty(\Omega)$. Also, for any n, $\int_\Omega h_n \,\mathrm{d}x = 0$ by (i). Thus $\int_\Omega (f + t_n h_n)\,\mathrm{d}x = \alpha$ for all $n \geq 1$. Since $\Omega_n^0 \cap \Omega_n^1 = \emptyset$ for $n \geq 3$, it is easy to check that $0 \leq f + t_n h_n \leq 1$ in Ω for all $n \geq 3$. Therefore $h \in T'_{A_\alpha}(f)$. $\square$

Lemma 3.6. *Let $f \in A_\alpha$. If $h \in T'_{A_\alpha}(f)$, then*

$$h(x) \geq 0 \text{ a.e. in } \Omega_0, \quad h(x) \leq 0 texta.e. in \Omega_1.$$

Proof. Since $\Omega_0 \subset \Omega_n^0$ and $\Omega_1 \subset \Omega_n^1$ for all $n \in \mathbb{N}$, the assertion readily follows from Lemma 3.4. $\square$

The following Theorems are the main results of this section.

Theorem 3.7. *$\hat{f}$ minimizes $\gamma(f)$ relative to A_α if and only if*

(i) $|\Omega^*| = 0$,
(ii) $u_{\hat{f}}(x_0) \geq u_{\hat{f}}(x_1)$ *for all* $(x_0, x_1) \in \Omega_0 \times \Omega_1$.

Proof. Let $\hat{f} \in A_\alpha$ be the solution of (1.9). We have $\Omega^* = \cup_{n=1}^\infty \Omega_n^*$, where

$$\Omega_n^* := \big\{x \in \Omega : \frac{1}{n} \leq \hat{f}(x) \leq 1 - \frac{1}{n}\big\}.$$

Note that $\Omega_n^* \subset \Omega_{n+1}^*$. We show that $u_{\hat{f}}$ is constant on Ω^*. To derive a contradiction, assume not. Hence, $u_{\hat{f}}$ is not constant on Ω_n^* for some $n \in \mathbb{N}$. Thus, there exist two measurable sets ω_1 and ω_2 in Ω_n^* such that

$$|\omega_1| = |\omega_2| \quad \text{and} \quad \int_{\omega_1} u_{\hat{f}} \,\mathrm{d}x < \int_{\omega_2} u_{\hat{f}} \,\mathrm{d}x. \tag{3.2}$$

Let

$$h(x) := \begin{cases} 1 & x \in \omega_1, \\ -1 & x \in \omega_2, \\ 0 & x \in (\omega_1 \cup \omega_2)^c. \end{cases}$$

So $h \in T'_{A_\alpha}(\hat{f})$ by Lemma 3.5. From Lemma 2.1 (iii) and (3.2) we deduce

$$\langle \gamma'(\hat{f}), h \rangle = \int_\Omega h u_{\hat{f}} \,\mathrm{d}x = \int_{\omega_1} u_{\hat{f}} \,\mathrm{d}x - \int_{\omega_2} u_{\hat{f}} \, dx < 0,$$

which is a contradiction by Lemma 3.2. Thus, $u_{\hat{f}}$ is constant on Ω^*. To show that the measure of Ω^* is zero we proceed as follows. Using the regularity of $u_{\hat{f}}$, the differential equation in (1.1) holds almost everywhere. So restricting that equation to the set $\hat{\Omega}^*$ will give a contradiction unless the measure of Ω^* is zero.

(ii) To derive a contradiction, suppose there exist two measurable sets $\omega_0 \subset \Omega_0$ and $\omega_1 \subset \Omega_1$ such that

$$|\omega_0| = |\omega_1| \quad \text{and} \quad \int_{\omega_0} u_{\hat{f}} \,\mathrm{d}x < \int_{\omega_1} u_{\hat{f}} \,\mathrm{d}x. \tag{3.3}$$

Let

$$h(x) := \begin{cases} 1 & x \in \omega_0, \\ -1 & x \in \omega_1, \\ 0 & x \in (\omega_0 \cup \omega_1)^c \end{cases}$$

which belongs to $T'_{A_\alpha}(\hat{f})$. By Lemma 2.1 (iii) and (3.3) we have

$$\langle \gamma'(\hat{f}), h \rangle = \int_\Omega h u_{\hat{f}} \,\mathrm{d}x = \int_{\omega_0} u_{\hat{f}} \,\mathrm{d}x - \int_{\omega_1} u_{\hat{f}} \, dx < 0,$$

which is a contradiction by Lemma 3.2. Therefore $u_{\hat{f}}(x_0) \geq u_{\hat{f}}(x_1)$ for all $(x_0, x_1) \in \Omega_0 \times \Omega_1$.

Conversely, assume (i) and (ii) hold. Thus

$$c = \sup_{x \in \Omega_1} u_{\hat{f}}(x) = \inf_{x \in \Omega_0} u_{\hat{f}}(x) > 0.$$

Fix $h \in T'_{A_\alpha}(\hat{f})$, and apply Lemmas 2.1 (iii), 3.4 and 3.6 to obtain

$$\langle \gamma'(\hat{f}), h \rangle = \int_{\Omega_0} h u_{\hat{f}} \,\mathrm{d}x + \int_{\Omega_1} h u_{\hat{f}} \, dx \geq \int_{\Omega_0} hc \,\mathrm{d}x + \int_{\Omega_1} hc \, dx = c \int_\Omega h \,\mathrm{d}x = 0.$$

Therefore, we deduce from Lemma 3.3 that $\hat{f}$ is a minimizer. □

Henceforth, we suppose that Ω is simply connected. Also, we will make the following assumptions on the functions $a(t)$ and $\varphi(t)$:

(A1) $a \in C^1(0, +\infty)$ and there exist positive constant λ_1 and μ_1 such that

$$0 < \lambda_1 < \frac{t\varphi'(t)}{\varphi(t)} \leq \mu_1, \quad \forall t > 0.$$

Theorem 3.8. *Let $\hat{f}$ be the minimizer of $\gamma(f)$ relative to A_α. Then $\hat{f}$ is a characteristic function which is equal to $\chi_{\{u_{\hat{f}} < \hat{c}\}}$ where $\hat{c} = \max_{x \in \bar{\Omega}} u_{\hat{f}}(x)$. Moreover, the set $\{u_{\hat{f}} < \hat{c}\}$ is connected and contains a layer around $\partial\Omega$. Also, the boundary of it has measure zero.*

Proof. From assumption (A1) we deduce that $u_{\hat{f}} \in C^{1,\delta}(\bar{\Omega})$ for some $\delta > 0$, see [12] and [13, Theorem 1.7]. From $|\Omega^*| = 0$ we infer that there exists $\hat{\Omega} \subset \Omega_1$ such that $|\hat{\Omega}| = \alpha$ and $\hat{f} = \chi_{\hat{\Omega}}$. Note that Ω_1 contains a neighborhood of $\partial\Omega$. We set

$$\hat{c} = \sup_{x\in\Omega_1} u_{\hat{f}}(x) = \inf_{x\in\Omega_0} u_{\hat{f}}(x) > 0.$$

From the continuity of $u_{\hat{f}}$ we deduce that $u_{\hat{f}} = \hat{c}$ on $\partial\Omega_0$. Restricting the differential equation in (1.1) to the set Ω_0 we get

$$\nabla \cdot (a(|\nabla u_{\hat{f}}|)\nabla u_{\hat{f}}) = 0, \quad \text{in } \Omega_0, \quad \text{and} \quad u_{\hat{f}} = \hat{c}, \quad \text{on } \partial\Omega_0.$$

Let $w = u_{\hat{f}} - \hat{c}$; so we have

$$\nabla \cdot (a(|\nabla w|)\nabla w) = 0, \quad \text{in } \Omega_0, \quad \text{and} \quad w = 0, \quad \text{on } \partial\Omega_0.$$

Thus, $\int_{\Omega_0} a(|\nabla w|)|\nabla w|^2 \,\mathrm{d}x = 0$. Since a is a positive function we infer that $\nabla w = 0$ a.e. in Ω_0. Therefore, $w = 0$ on $\partial\Omega_0$ implies $u_{\hat{f}} = \hat{c}$ in Ω_0. Whence, $\hat{\Omega} = \{x \in \Omega : u_{\hat{f}}(x) < \hat{c}\}$, where $\hat{c} = \max_{\bar{\Omega}} u_{\hat{f}}$. We know that $\partial\hat{\Omega} \subset \{x \in \Omega : u_{\hat{f}} = \hat{c}\} \cap \Omega_1$. If

$$|\{x \in \Omega : u_{\hat{f}} = \hat{c}\} \cap \Omega_1| > 0,$$

then $\hat{f} = 0$ in this set, which leads to a contradiction. Thus $|\partial\hat{\Omega}| = 0$.

We now prove that $\hat{\Omega}$ is connected. Suppose not, and consider E an open component of $\hat{\Omega}$ whose boundary does not intersect the boundary of Ω. Since $u_{\hat{f}} = \hat{c}$ on ∂E and

$$-\nabla \cdot (a(|\nabla u_{\hat{f}}|)\nabla u_{\hat{f}}) = 1 \quad \text{in } E,$$

we obtain

$$\begin{aligned}\int_E (u_{\hat{f}} - \hat{c}) \,\mathrm{d}x &= \int_E a(|\nabla u_{\hat{f}}|)\nabla u_{\hat{f}} \cdot \nabla(u_{\hat{f}} - \hat{c}) \,\mathrm{d}x \\ &= \int_E a(|\nabla u_{\hat{f}}|)|\nabla u_{\hat{f}}|^2 \,\mathrm{d}x \geq 0.\end{aligned}$$

This is a contradiction, because $u_{\hat{f}} < \hat{c}$ in E. Therefore, $\hat{\Omega}$ is connected. □

4. Monotonicity, stability and regularity

Let $\alpha, \beta \in (0, |\Omega|)$. Let $\hat{f}_\alpha \in A_\alpha$ and $\hat{f}_\beta \in A_\beta$ be the solutions of

$$\inf_{f\in A_\alpha} \gamma(f) \quad \text{and} \quad \inf_{f\in A_\beta} \gamma(f),$$

respectively. By Theorem 2.3, we know that $\hat{f}_\alpha = \chi_{\hat{\Omega}_\alpha}$ and $\hat{f}_\beta = \chi_{\hat{\Omega}_\beta}$. Moreover, we have

$$\hat{\Omega}_\alpha = \{x \in \Omega : u_\alpha(x) < c_\alpha\} \quad \text{and} \quad \hat{\Omega}_\beta = \{x \in \Omega : u_\beta(x) < c_\beta\}, \tag{4.1}$$

where $c_\alpha = \max_{\bar{\Omega}} u_\alpha$ and $c_\beta = \max_{\bar{\Omega}} u_\beta$. Recall that

$$\begin{gathered}-\nabla \cdot (a(|\nabla u_\alpha|)\nabla u_\alpha) = \chi_{\hat{\Omega}_\alpha} \quad \text{in } \Omega, \\ u_\alpha = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{4.2}$$

and

$$\begin{gathered}-\nabla \cdot (a(|\nabla u_\beta|)\nabla u_\beta) = \chi_{\hat{\Omega}_\beta} \quad \text{in } \Omega, \\ u_\beta = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{4.3}$$

We now state the monotonicity results. The proof of the following lemma is similar to [14, Theorems 4.1 and 4.2], so we omit it.

Lemma 4.1. *If $0 < \beta < \alpha < |\Omega|$, then $\hat{\Omega}_\beta \subset \hat{\Omega}_\alpha$, and $c_\beta < c_\alpha$.*

Now we state a stability result. Let $0 < \alpha_n < |\Omega|$, $n \in \mathbb{N}$, and $\chi_{\hat{\Omega}_{\alpha_n}}$ denote the unique solution of the minimization problem

$$\inf_{f \in A_{\alpha_n}} \gamma(f).$$

Lemma 4.2. *Let $\chi_{\hat{\Omega}_\alpha}$ denotes the minimizer of problem (1.9), satisfying $|\hat{\Omega}_\alpha| = \alpha$. If $\alpha_n \to \alpha$, then $\chi_{\hat{\Omega}_{\alpha_n}} \to \chi_{\hat{\Omega}\alpha}$ in $L^p(\Omega)$ for any $p \geq 1$. Moreover, $|\hat{\Omega}_{\alpha_n} \triangle \hat{\Omega}_\alpha| \to 0$. Here Δ denotes the symmetric difference of sets.*

The proof of the above lemma is similar to that of [14, Theorem 5.1]. Next we prove the continuity of the mapping $\alpha \to c_\alpha$, compared with [9, Theorem 2.4].

Lemma 4.3. *For $\alpha \in (0, |\Omega|)$, the map $\alpha \mapsto c_\alpha$ is continuous.*

Proof. Let $\alpha \in (0, |\Omega|)$. We only prove continuity from the left at α. The right continuity is proved similarly. To this end, consider $\{\alpha_n\}$, a sequence in $(0, |\Omega|)$ such that $\alpha_n \uparrow \alpha$. By Lemma 4.2 we infer $\chi_{\hat{\Omega}_{\alpha_n}} \to \chi_{\hat{\Omega}_\alpha}$ in $L^{\lambda'}(\Omega)$, hence $\chi_{\hat{\Omega}_{\alpha_n}} \to \chi_{\hat{\Omega}_\alpha}$ in $L^{\Phi^*}(\Omega)$ by (1.7). From Lemma 2.1 (i), Remark 2.2, we deduce $u_{\alpha_n} \to u_\alpha$ in $L^\Phi(\Omega)$, so by (1.7), $u_{\alpha_n} \to u_\alpha$ in $L^\lambda(\Omega)$. Assume c_{α_n} does not convergent to c_α. In that case, there exists a constant $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there is $m_n > n$ such that $c_\alpha - c_{\alpha_{m_n}} > \varepsilon$. From Lemma 4.1 we have $\hat{\Omega}_{\alpha_{m_n}} \subset \hat{\Omega}_\alpha$, so $u_\alpha = c_\alpha$ and $u_{\alpha_{m_n}} = c_{\alpha_{m_n}}$ in $\Omega \setminus \hat{\Omega}_\alpha$. Hence for all $n \geq 1$ we deduce

$$\int_\Omega |u_\alpha - u_{\alpha_{m_n}}|^\lambda \, dx \geq \int_{\Omega \setminus \hat{\Omega}_\alpha} |c_\alpha - c_{\alpha_{m_n}}|^\lambda \, dx > \varepsilon^\lambda (|\Omega| - \alpha).$$

This is a contradiction, because $u_{\alpha_n} \to u_\alpha$ in $L^\lambda(\Omega)$. This completes the proof. $\square$

We prove now our first results related to the functional ℓ.

Theorem 4.4. *For $\alpha \in (0, |\Omega|)$, let $\ell(\alpha) = \inf_{f \in A_\alpha} \gamma(f)$. The mapping $\alpha \mapsto \ell(\alpha)$ is Lipschitz continuous, strictly convex and differentiable, with derivative c_α.*

Proof. Since $\ell(\alpha) = \gamma(\chi_{\hat{\Omega}_\alpha})$, we have

$$\begin{aligned} \ell(\alpha) &= \int_\Omega (\chi_{\hat{\Omega}_\alpha} u_\alpha - \Phi(|\nabla u_\alpha|)) \, dx \\ &= \min_{|D|=\alpha} \max_{v \in X} \int_\Omega (\chi_D v - \Phi(|\nabla v|)) \, dx \\ &\geq \min_{|D|=\alpha} \int_\Omega (\chi_D v_0 - \Phi(|\nabla v_0|)) \, dx, \end{aligned} \tag{4.4}$$

for any positive function $v_0 \in X$. By the Bathtub Lemma, see [11], we have

$$\min_{|D|=\alpha} \int_\Omega \chi_D v_0 \, dx = \int_\Omega \chi_{\tilde{D}} v_0 \, dx,$$

where $\tilde{D}$ is such that $|\tilde{D}| = \alpha$ and

$$\{x \in \Omega : v_0(x) < t\} \subset \tilde{D} \subset \{x \in \Omega : v_0(x) \leq t\},$$

for a suitable $t > 0$. Thus, from (4.4) we deduce

$$\ell(\alpha) \geq \int_\Omega (\chi_{\tilde{D}} v_0 - \Phi(|\nabla v_0|)) \, dx. \tag{4.5}$$

Let $0<\beta<\alpha<|\Omega|$. We know

$$\ell(\beta)=\int_\Omega (\chi_{\hat{\Omega}_\beta}u_\beta-\Phi(|\nabla u_\beta|))\,\mathrm{d}x. \tag{4.6}$$

From (4.5) with $v_0=u_\beta$ we infer

$$\ell(\alpha)\geq\int_\Omega (\chi_{\tilde{D}_\alpha}u_\beta-\Phi(|\nabla u_\beta|))\,\mathrm{d}x, \tag{4.7}$$

where

$$\{x\in\Omega: u_\beta(x)<c_\beta\}\subset\tilde{D}_\alpha\subset\{x\in\Omega: u_\beta(x)\leq c_\beta\},\quad |\tilde{D}_\alpha|=\alpha.$$

Since $\hat{\Omega}_\beta\subset\tilde{D}_\alpha$ we have $|\tilde{D}_\alpha\setminus\hat{\Omega}_\beta|=\alpha-\beta$. Now, since $u_\beta=c_\beta$ outside $\hat{\Omega}_\beta$, from (4.6) and (4.7) we deduce

$$\ell(\alpha)-\ell(\beta)\geq\int_{\tilde{D}_\alpha\setminus\hat{\Omega}_\beta}u_\beta\,\mathrm{d}x=(\alpha-\beta)c_\beta. \tag{4.8}$$

By a similar argument we can derive

$$\ell(\alpha)-\ell(\beta)\leq(\alpha-\beta)c_\alpha. \tag{4.9}$$

Thus, from (4.8) and (4.9) we obtain

$$c_\beta\leq\frac{\ell(\alpha)-\ell(\beta)}{\alpha-\beta}\leq c_\alpha. \tag{4.10}$$

Therefore, ℓ is Lipschitz continuous and from Lemma 4.3 we deduce that ℓ is differentiable and $\ell'(\alpha)=c_\alpha$. Since the mapping $\alpha\mapsto c_\alpha$ is strictly increasing, Lemma 4.1 implies that ℓ is strictly convex. □

Let $u_1\in W_0^{1,\Phi}(\Omega)$ be the solution of (1.1) for $f=1$. Let

$$\gamma_1:=\frac{1}{|\Omega|}\gamma(\chi_\Omega)=\frac{1}{|\Omega|}\int_\Omega(u_1-\Phi(|\nabla u_1|))\,\mathrm{d}x.$$

Our final result is an upper bound for $\ell(\alpha)/\alpha$.

Theorem 4.5. *For each $\alpha\in(0,|\Omega|)$ we have $\ell(\alpha)\leq\gamma_1\alpha$.*

Proof. Let $K:=\{f\in L^\infty(\Omega): 0\leq f\leq 1\}$. Define the linear functional $\Lambda: L^\infty(\Omega)\to\mathbb{R}$ by $\Lambda(f):=\int_\Omega f\,\mathrm{d}x$. Thus A_α and $K\cap\Lambda^{-1}(\{\alpha\})$ are identical. Let $f_\alpha\in A_\alpha$ be the solution of $\ell(\alpha)=\min_{f\in A_\alpha}\gamma(f)$. Hence $\gamma(f)-\ell(\Lambda(f))\geq 0$ for all $f\in K$ and $\gamma(f_\alpha)-\ell(\Lambda(f_\alpha))=0$. Thus by enforcing a standard minimality condition we obtain

$$0\in\partial(\gamma-\ell(\Lambda))(f_\alpha)+N_K(f_\alpha), \tag{4.11}$$

where $N_K(f_\alpha)$ denotes the normal cone to K at f_α. By (4.11), for $g\in N_K(f_\alpha)$ we have

$$\gamma'(f_\alpha)(f_\alpha-f)-\ell'(\alpha)(\alpha-\Lambda(f))=\langle g,f-f_\alpha\rangle\leq 0, \tag{4.12}$$

for all $f\in K$. Hence, we obtain

$$\gamma'(f_\alpha)(f_\alpha)-\int_\Omega\Phi(|\nabla u_{f_\alpha}|)\,\mathrm{d}x-\gamma'(f_\alpha)(f)-\alpha\ell'(\alpha)+\Lambda(f)\ell'(\alpha)\leq 0, \tag{4.13}$$

for all $f\in K$. Since $\ell(\alpha)=\gamma(f_\alpha)$, by Lemma 2.1 (iii) and (4.13) we infer that

$$\ell(\alpha)-\gamma'(f_\alpha)(f)-\alpha\ell'(\alpha)+\Lambda(f)\ell'(\alpha)\leq 0,\quad\forall f\in K. \tag{4.14}$$

In particular, setting $f = 0$ in (4.14) yields $\alpha\ell'(\alpha) - \ell(\alpha) \geq 0$. Thus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\Big(\frac{\ell(\alpha)}{\alpha}\Big) \geq 0 \quad \text{in } (0, |\Omega|).$$

Integrating both sides of the last inequality above, on the interval $(\alpha, |\Omega|)$, we obtain

$$\frac{\ell(\alpha)}{\alpha} \leq \frac{\ell(|\Omega|)}{|\Omega|} \quad \text{in } (0, |\Omega|).$$

Therefore, we obtain the desired conclusion. □

5. Conclusions

In this work, an elliptic partial differential equation with zero Dirichlet boundary condition is considered. The differential operator is of elliptic type, and the external force only depends on the space variables. The structure of the equation organically suggests that a suitable function space to find solutions would be the Orlicz-Sobolev space. Next, an energy functional is introduced which depends on the force function that itself belongs to an α-admissible set of measurable functions taking values between 0 and 1 while its integral is equal to a prescribed value. The energy functional is minimized over the admissible set, and existence of optimal solutions are verified. Moreover, by proving strict convexity of the functional, uniqueness of optimal solutions are guaranteed. The remaining of the paper focusses on derivation of qualitative properties of the optimal solution. To this end, we have used the tangent cones in order to derive the optimality condition which, in turn, is utilized to show that the optimal solution is indeed classical i.e. it is $\{0, 1\}$-valued. We have shown that the optimal solution grows when the parameter α increases. Furthermore, a stability result has been shown in the sense that if α is close to β, then their corresponding optimal solutions are close in the L^p-norm. Our final result concerns the optimal value $\ell(\alpha)$. More precisely, we have shown that $\ell(\alpha)/\alpha$ is bounded from above, so the growth of the optimal value is linear with respect to α.

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