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ENTIRE SOLUTIONS FOR THE HEAT EQUATION

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Abstract. We consider the solutions of the heat equation

$$\partial_t F = \partial_z^2 F$$

which are entire in z and t (caloric functions). We examine the relation of the z-order and z-type of an entire caloric function $F(t,z)$, viewed as function of z, to its t-order and t-type respectively, if it is viewed as function of t. Also, regarding the zeros $z_k(t)$ of an entire caloric function $F(t,z)$, viewed as function of z, we show that the points (t,z) at which

$$F(t,z) = \partial_z F(t,z) = 0$$

form a discrete set in $\mathbb{C}^2$ and, then, we derive the t-evolution equations of $z_k(t)$. These are differential equations that hold for all but countably many ts in $\mathbb{C}$.

1. Introduction

In this article we study certain properties of the solutions $F(t,z)$ of the standard heat equation $u_t = u_{zz}$ which are entire in z and t. We believe that such solutions deserve an independent study, since they exhibit some theoretically interesting phenomena, while, at the same time, they are computationally friendly.

Suppose that the function $F(t,z)$ is entire in z for every $t \in \mathbb{C}$ and entire in t for every $z \in \mathbb{C}$. Then, a consequence of the celebrated theorem of Hartogs (see [11]) is that, for any given pair (t_0, z_0) of complex numbers the function $F(t,z)$ equals to its Taylor expansion about (t_0, z_0), namely

$$F(t,z) = \sum_{j,k\geq 0} c_{jk}(t-t_0)^j (z-z_0)^k, \quad \text{where } c_{jk} = \frac{\partial_t^j \partial_z^k F(t_0,z_0)}{j!k!} \tag{1.1}$$

and the series converges absolutely for any $t, z \in \mathbb{C}$. A further consequence of the absolute convergence is that $F(t,z)$ can be expanded as

$$F(t,z) = \sum_{k\geq 0} a_k(t)(z-z_0)^k, \quad \text{as well as} \quad F(t,z) = \sum_{j\geq 0} b_j(z)(t-t_0)^j, \tag{1.2}$$

where $a_k(t)$, $k \geq 0$, and $b_j(z)$, $j \geq 0$, are entire functions.

Let us now assume that $F(t,z)$ is also *caloric*, namely it satisfies the heat equation

$$\partial_t F(t,z) = \partial_z^2 F(t,z) \tag{1.3}$$

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with "initial condition"

$$f(z) := F(0, z) \tag{1.4}$$

(notice that if (1.3) is satisfied in an open subset of $\mathbb{C}^2$, then, by analytic continuation of $\partial_t F(t, z)$ and $\partial_z^2 F(t, z)$ we have that (1.3) is automatically satisfied for every $(t, z) \in \mathbb{C}^2$). Since the operators ∂_t and ∂_z commute, by differentiating (1.3) with respect to t repeatedly we obtain

$$\partial_t^j F(t, z) = \partial_z^{2j} F(t, z), \quad j \geq 0. \tag{1.5}$$

Also, a rather trivial observation is that if $F(t, z)$ satisfies (1.5), so does

$$\tilde{F}(t, z) = F(t + t', z + z').$$

Using (1.5) in (1.1) yields

$$F(t_0 + t, z_0 + z) = \sum_{j,k \geq 0} \frac{\partial_z^{2j+k} F(t_0, z_0)}{j!k!} t^j z^k, \quad t, z \in \mathbb{C}. \tag{1.6}$$

It is convenient to write (1.6) as

$$F(t_0 + t, z_0 + z) = \sum_{m=0}^{\infty} \frac{\partial_z^m F(t_0, z_0)}{m!} P_m(t, z), \tag{1.7}$$

where

$$P_m(t, z) := \sum_{2j+k=m} \frac{m!}{j!k!} t^j z^k = \sum_{j=0}^{\lfloor m/2 \rfloor} \frac{m!}{j!(m-2j)!} t^j z^{m-2j}, \quad m \geq 0. \tag{1.8}$$

The quantity $P_m(t, z)$ is called the *m-th caloric polynomial* and it is clear from (1.8) that it is *parabolically m-homogeneous*, namely

$$P_m(\lambda^2 t, \lambda z) = \lambda^m P_m(t, z). \tag{1.9}$$

The first six caloric polynomials are

$$P_0(t, z) \equiv 1, \quad P_1(t, z) = z, \quad P_2(t, z) = z^2 + 2t, \quad P_3(t, z) = z(z^2 + 6t),$$
$$P_4(t, z) = z^4 + 12tz^2 + 12t^2, \quad P_5(t, z) = z(z^4 + 20tz^2 + 60t^2).$$

Let us review some important properties of the caloric polynomials. For each $m \geq 0$ the polynomial $P_m(t, z)$ satisfies the heat equation with initial condition

$$P_m(0, z) = z^m, \tag{1.10}$$

hence, the entire solution $F(t, z)$ of (1.3), with $f(z) = F(0, z) = 6 \sum_{m=0}^{M} a_m z^m$, is the polynomial $F(t, z) = \sum_{m=0}^{M} a_m P_m(t, z)$.

Also, from the standard integral formula, involving the heat kernel, which gives the solution of the heat equation in terms of the initial condition, we have

$$P_m(t, z) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-(z-\xi)^2/4t} \xi^m d\xi \quad \text{for } \Re(t) > 0. \tag{1.11}$$

Since

$$\frac{m!}{j!(m-2j)!} = \frac{m!}{(2j)!(m-2j)!}(j+1)(j+2)\cdots 2j = \binom{m}{2j}(j+1)(j+2)\cdots 2j$$

it follows that the coefficients of the caloric polynomials are positive integers.

It is easy to see from (1.8) that a crude bound of $P_m(t,z)$ is

$$|P_m(t,z)| \le \frac{m!(\lfloor m/2\rfloor+1)}{(\kappa_m)!(m-2\kappa_m)!} \max_{0\le j\le\lfloor m/2\rfloor} |t|^j|z|^{m-2j}, \tag{1.12}$$

where

$$\kappa_m = \Big\lfloor \frac{4m-1-\sqrt{8m+17}}{8} \Big\rfloor + 1. \tag{1.13}$$

If we differentiate (1.8) with respect to z we obtain

$$\begin{aligned} \partial_z P_m(t,z) &= \sum_{2j+k=m} \frac{m!}{j!(k-1)!} t^j z^{k-1} \\ &= m \sum_{2j+l=m-1} \frac{(m-1)!}{j!\,l!} t^j z^l = m P_{m-1}(t,z) \end{aligned} \tag{1.14}$$

for $m \ge 1$ (another way to see that $\partial_z P_m(t,z) = mP_{m-1}(t,z)$ is by observing that, since $P_m(t,z)$ is the solution of the heat equation (1.3) with $P_m(0,z) = z^m$, the derivative $\partial_z P_m(t,z)$ is the solution of (1.3) with initial condition mz^{m-1}).

The function

$$E_\lambda(t,z) := e^{\lambda^2 t+\lambda z} \tag{1.15}$$

is entire in (t,z) and satisfies the heat equation (1.3) for any value of the complex parameter λ (actually, even for the case of a square matrix λ with constant elements). Thus, we can apply (1.7) to $E_\lambda(t,z)$ (for $t_0 = z_0 = 0$) and obtain

$$E_\lambda(t,z) = e^{\lambda^2 t+\lambda z} = \sum_{m=0}^{\infty} \frac{\partial_z^m E_\lambda(0,0)}{m!} P_m(t,z) = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} P_m(t,z). \tag{1.16}$$

In other words, $e^{\lambda^2 t+\lambda z}$ is the generating function of the caloric polynomials.

Formula (1.8) also implies

$$P_m(-1,2z) = m! \sum_{j=0}^{\lfloor m/2\rfloor} \frac{(-1)^j}{j!(m-2j)!} (2z)^{m-2j} = H_m(z), \quad m \ge 0, \tag{1.17}$$

where $H_m(z)$ is the (physicists') Hermite polynomial of order m, i.e. of degree m (see, e.g., [19]). Thus, the parabolic homogeneity (1.9) yields

$$P_m(t,z) = \big(i\sqrt{t}\big)^m H_m\Big(\frac{z}{2i\sqrt{t}}\Big) \tag{1.18}$$

and equation (1.7) can be written as (for $t_0 = z_0 = 0$)

$$\begin{aligned} F(t,z) &= \sum_{m=0}^{\infty} \frac{\partial_z^m F(0,0)}{m!} \big(i\sqrt{t}\big)^m H_m\Big(\frac{z}{2i\sqrt{t}}\Big) \\ &= \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} \big(i\sqrt{t}\big)^m H_m\Big(\frac{z}{2i\sqrt{t}}\Big), \end{aligned} \tag{1.19}$$

where the second equality follows from (1.4). Thus, if $F(t,z)$ and $G(t,z)$ are two entire solutions of (1.3) with $F(0,z) \equiv G(0,z)$ (or $F(t_0,z) \equiv G(t_0,z)$ for some fixed $t_0 \in \mathbb{C}$), then (1.19) tells us that they have to be identical, namely $F(t,z) \equiv G(t,z)$. Equivalently, if for some fixed t_0 we have $F(t_0,z) \equiv 0$, then $F(t,z) \equiv 0$. On the other hand, the relation $F(t,z_0) \equiv 0$, for some z_0 does not imply that F is identically 0 (e.g., if $f(z) \not\equiv 0$ is odd, then $F(t,z)$ is a solution of (1.3) which is odd in z, and hence $F(t,0) \equiv 0$).

There are many known facts about the zeros of the Hermite polynomials. For instance, it is well known [19] that $H_m(z)$ is an even (odd) function if and only if m is even (odd). Furthermore, the zeros of $H_m(z)$ are real and simple. It, then, follows from (1.18) that [15] if $m = 2l$, the polynomial P_m is of the form

$$P_m(t,z) = (z^2 + \rho_{m,1} t) \cdots (z^2 + \rho_{m,l} t), \quad \text{with } 0 < \rho_{m,1} < \cdots < \rho_{m,l}, \tag{1.20}$$

while if $m = 2l + 1$, then P_m is of the form

$$P_m(t,z) = z(z^2 + \rho_{m,1} t) \cdots (z^2 + \rho_{m,l} t), \quad \text{with } \ 0 < \rho_{m,1} < \cdots < \rho_{m,l}. \tag{1.21}$$

From (1.20) and (1.21) we have that if $t \in \mathbb{C} \setminus \{0\}$, then the zeros of $P_m(t,z)$, viewed as a polynomial of z, are simple (the case $t = 0$ is exceptional since $P_m(0,z) = z^m$). Furthermore, by (1.14) and Rolle's Theorem we obtain the interlacing properties

$$0 < \rho_{m,1} < \rho_{m-1,1} < \rho_{m,2} < \cdots < \rho_{m-1,l-1} < \rho_{m,l}, \quad \text{if } m = 2l, \tag{1.22}$$

while

$$0 < \rho_{m-1,1} < \rho_{m,1} < \rho_{m-1,2} < \cdots < \rho_{m-1,l} < \rho_{m,l}, \quad \text{if } m = 2l + 1. \tag{1.23}$$

Let us also notice that (1.20) and (1.21) tell us that, if $m > 0$ is even, then the zeros of $P_m(t,z)$ (viewed as a function of z) are real if and only if $t \in \mathbb{R}^- := (-\infty, 0]$ and the same is true for the non-zero zeros of $P_m(t,z)$ in the case where m is odd ($z = 0$ is always a zero of $P_m(t,z)$, if m is odd).

The rest of this article is organized as follows. In Section 2 we recall some general results regarding the order and the type of an entire function. These results are used in Section 3, where we study the relation of the orders ρ_z and ρ_t, as well as the types τ_z and τ_t, of an entire caloric function $F(t,z)$, viewed as function of z and t respectively. The main results of Section 3 are Theorems 3.4 and 3.5. Finally, in Section 4 we first show that the multiple zeros of $F(t,z)$ are isolated (Theorem 4.1). This enables us to get the dynamics of the zeros of $F(t,z)$, namely

$$z_k'(t) = 2 \sum_{j \neq k} \frac{1}{z_k(t) - z_j(t)} \tag{1.24}$$

for all but countably many $t \in \mathbb{C}$. The above equations were first derived by Csordas, Smith, and Varga [5] for the case where t is restricted in a real semiaxis (see [16, 18]), since one must use Theorem 4.1 of the present paper in order to analytically extend (1.24) to $t \in \mathbb{C}$. The equations (1.24) are a kind of "characteristics" for the heat equation and they also remind the equations which arise in the solution of the inverse spectral problem for the Hill operator (see [20]).

2. Order and type of an entire function

Let

$$g(z) = \sum_{n \geq 0} a_n z^n, \quad z \in \mathbb{C}, \tag{2.1}$$

be an entire function and

$$M(r) = M_g(r) := \sup_{|z| \leq r} |g(z)| = \max_{|z|=r} |g(z)|, \quad r > 0, \tag{2.2}$$

its maximum modulus.

We recall that the order of $g(z)$ is the quantity [7]

$$\rho = \rho(g) := \limsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln r}. \tag{2.3}$$

In other words, the order ρ of $g(z)$ is the smallest exponent $\rho' \geq 0$ such that for any given $\varepsilon > 0$ there is a $r_0 = r_0(\varepsilon) > 0$ for which

$$|g(z)| \leq \exp\left(|z|^{\rho'+\varepsilon}\right) \quad \text{whenever } r = |z| \geq r_0. \tag{2.4}$$

Clearly, $0 \leq \rho \leq \infty$.

Let us also recall [7] that if $0 < \rho < \infty$, the quantity

$$\tau = \tau(g) := \limsup_{r\to\infty} \frac{\ln M(r)}{r^\rho} \tag{2.5}$$

is the type of $g(z)$. In other words, τ is the smallest (extended) number $\tau' \geq 0$ such that for any given $\varepsilon > 0$ there is a $r_0 = r_0(\varepsilon) > 0$ for which

$$|g(z)| \leq \exp\left((\tau' + \varepsilon)|z|^\rho\right) \quad \text{whenever } r = |z| \geq r_0. \tag{2.6}$$

Clearly, $0 \leq \tau \leq \infty$. If $\tau = 0$, we say that $g(z)$ is of minimal type, whereas if $\tau = \infty$, we say that $g(z)$ is of maximal type. In the extreme cases where $\rho = 0$ or $\rho = \infty$ the type is not defined.

A well-known fact of complex analysis is [7] that the order ρ and the type τ of $g(z)$ are given by the formulas

$$\rho = \limsup_n \frac{n \ln n}{-\ln|a_n|} \tag{2.7}$$

and (in the case where $0 < \rho < \infty$)

$$\tau = \frac{1}{e\rho} \limsup_n n|a_n|^{\rho/n} \tag{2.8}$$

respectively, where a_n, $n = 0, 1, \dots$, are the coefficients of the power series of $g(z)$ as seen in (2.1).

Let us now set

$$a_n(z) := \frac{g^{(n)}(z)}{n!} \quad n = 0, 1, \dots \tag{2.9}$$

(so that $a_n(0) = a_n$). Then, in view of (2.3), formulas (2.7), (2.8), and (2.9) yield

$$\rho = \limsup_n \frac{n \ln n}{-\ln|a_n(z)|} \tag{2.10}$$

and (in the case where $0 < \rho < \infty$)

$$\tau = \frac{1}{e\rho} \limsup_n n|a_n(z)|^{\rho/n} = \frac{e^{\rho-1}}{\rho} \limsup_n n^{1-\rho} \left|g^{(n)}(z)\right|^{\rho/n}. \tag{2.11}$$

Sometimes, it is convenient to write (2.10) in the equivalent form (using that $\lim_n |a_n(z)| = 0$ and, hence, $-\ln|a_n(z)|$ is eventually positive)

$$e^{-1/\rho} = \limsup_n |a_n(z)|^{\frac{1}{n\ln n}} \tag{2.12}$$

or, in view of (2.9) and the fact that $\lim_n (n!)^{\frac{1}{n\ln n}} = e$, as

$$\theta := e^{1-(1/\rho)} = \limsup_n \left|g^{(n)}(z)\right|^{\frac{1}{n\ln n}}. \tag{2.13}$$

Notice that $\theta = \theta(\rho)$ is smooth and strictly increasing for $\rho \in [0, +\infty]$, with $\theta(0) := \theta(0^+) = 0$ and $\theta(+\infty) = e$.

2.1. **Additional properties of the order.** For the purposes of the present work we need to consider the cases where the lim sup appearing in (2.10) is taken over the subsequences $a_{2k}(z)$ and $a_{2k+1}(z)$, that is

$$\rho_0(z) := \limsup_k \frac{2k\ln(2k)}{-\ln|a_{2k}(z)|} = 2\limsup_k \frac{k\ln k}{-\ln|a_{2k}(z)|}, \quad z\in\mathbb{C}, \tag{2.14}$$

and

$$\rho_1(z) := \limsup_k \frac{(2k+1)\ln(2k+1)}{-\ln|a_{2k+1}(z)|} = 2\limsup_k \frac{k\ln k}{-\ln|a_{2k+1}(z)|}, \quad z\in\mathbb{C}, \tag{2.15}$$

respectively. Clearly, for every $z \in \mathbb{C}$ we have

$$\rho = \max\{\rho_0(z), \rho_1(z)\}. \tag{2.16}$$

It is convenient to introduce the quantities

$$\theta_0(z) := \exp\Big(1 - \frac{1}{\rho_0(z)}\Big) = \limsup_k \big|g^{(2k)}(z)\big|^{\frac{1}{2k\ln k}}, \quad z\in\mathbb{C}, \tag{2.17}$$

and

$$\theta_1(z) := \exp\Big(1 - \frac{1}{\rho_1(z)}\Big) = \limsup_k \big|g^{(2k+1)}(z)\big|^{\frac{1}{2k\ln k}}, \quad z\in\mathbb{C}. \tag{2.18}$$

The following theorem tells us that $\rho_0(z)$ and $\rho_1(z)$ are essentially the same as the order ρ of $g(z)$.

Theorem 2.1. *For an entire function $g(z)$ let θ, $\theta_0(z)$, and $\theta_1(z)$ be as in* (2.13), (2.17), *and* (2.18) *respectively. Then*

$$\theta_0(z) = \theta \quad \textit{and} \quad \theta_1(z) = \theta \quad \textit{for a.e. } z\in\mathbb{C}. \tag{2.19}$$

For a proof of the above theorem we refer to [11].

2.2. **Additional properties of the type.** We, now, turn our attention to the type of $g(z)$ for the case $0 < \rho < \infty$. In view of (2.11) we set (in the spirit of (2.14) and (2.15))

$$\begin{aligned}\tau_0(z) :=& \frac{1}{e\rho}\limsup_k 2k\,|a_{2k}(z)|^{\rho/2k}\\ =& \frac{2}{e\rho}\limsup_k k\,|a_{2k}(z)|^{\rho/2k}\\ =& \frac{(e/2)^{\rho-1}}{\rho}\limsup_k k^{1-\rho}\big|g^{(2k)}(z)\big|^{\rho/2k}, \quad z\in\mathbb{C}\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\tau_1(z) :=& \frac{1}{e\rho}\limsup_k (2k+1)\,|a_{2k+1}(z)|^{\frac{\rho}{2k+1}}\\ =& \frac{2}{e\rho}\limsup_k k\,|a_{2k+1}(z)|^{\frac{\rho}{2k}}\\ =& \frac{(e/2)^{\rho-1}}{\rho}\limsup_k k^{1-\rho}\big|g^{(2k+1)}(z)\big|^{\rho/2k}, \quad z\in\mathbb{C},\end{aligned} \tag{2.21}$$

where $a_n(z)$ is given by (2.9). From (2.11) it is obvious that

$$\tau = \max\{\tau_0(z), \tau_1(z)\}. \tag{2.22}$$

The following theorem gives a property of the type of $g(z)$ analogous of the property of the order of $g(z)$ given in Theorem 2.1.

Theorem 2.2. *Let τ be the type of the entire function $g(z)$, while $\tau_0(z)$ and $\tau_1(z)$ be as in* (2.20) *and* (2.21) *respectively.*

(i) *If $\tau < \infty$, then*
$$\tau_0(z) = \tau \quad \text{and} \quad \tau_1(z) = \tau \quad \text{for a.a. } z \in \mathbb{C}. \tag{2.23}$$
(ii) *If $\tau = \infty$, then there exists a dense G_δ subset $\mathcal{U}_\infty$ of $\mathbb{C}$ such that*
$$\tau_0(z) = \infty \quad \text{and} \quad \tau_1(z) = \infty \quad \text{for all } z \in \mathcal{U}_\infty. \tag{2.24}$$

For a proof of the above theorem we refer to [11]. In the case where $\tau = \infty$, the question whether $\tau_0(z) = \infty$ and $\tau_0(z) = \infty$ for a.a. $z \in \mathbb{C}$ remains open.

2.3. **Canonical products.** Here we give a brief review of some basic facts regarding canonical products. Suppose $z_1, z_2, \dots$ is a finite or infinite sequence of non-zero complex numbers such that
$$\sigma := \inf\Big\{s \geq 0 : \sum_{k\geq 1} \frac{1}{|z_k|^s} < \infty\Big\} < \infty. \tag{2.25}$$
Then, the *canonical product* (associated with $\{z_k\}_{k\geq 1}$) is the quantity
$$\Pi(z) := \prod_{k\geq 1} e_p\big(\frac{z}{z_k}\big), \tag{2.26}$$
where
$$e_0(z) := 1 - z, \quad e_p(z) := (1-z)\exp\Big(\frac{z}{1} + \frac{z^2}{2} + \dots + \frac{z^p}{p}\Big), \quad p > 0, \tag{2.27}$$
and p is related to σ as follows:

(i) If σ is not an integer, then $p = \lfloor \sigma \rfloor$.
(ii) If σ is an integer and
$$\sum_{k\geq 1} \frac{1}{|z_k|^\sigma} = \infty, \tag{2.28}$$
then $p = \sigma$.
(iii) If σ is an integer and
$$\sum_{k\geq 1} \frac{1}{|z_k|^\sigma} < \infty, \tag{2.29}$$
then $p = \max\{\sigma - 1, 0\}$.

It is a well-known fact in complex analysis [7] that the canonical product of (2.26) is entire in z of order σ and, furthermore, in the case $\sigma > 0$, this entire function is of minimal type if $\sum_{k\geq 1} |z_k|^{-\sigma} < \infty$.

Remark 2.3. In view of (2.27) formula (2.26) can be written as
$$\Pi(z) = \prod_{k\geq 1} \Big(1 - \frac{z}{z_k}\Big)\exp\Big(\frac{z}{z_k} + \frac{z^2}{2z_k^2} + \dots + \frac{z^p}{p z_k^p}\Big). \tag{2.30}$$
Obviously,
$$\Pi(0) = 1. \tag{2.31}$$

Also, in case where $p \geq 1$,

$$\frac{\Pi'(z)}{\Pi(z)} = \sum_{k\geq 1} \Big(\frac{1}{z - z_k} + \frac{1}{z_k} + \frac{z}{z_k^2} + \dots + \frac{z^{p-1}}{z_k^p} \Big). \tag{2.32}$$

Thus

$$\frac{\Pi'(0)}{\Pi(0)} = 0, \quad \text{which implies} \quad \Pi'(0) = 0 \tag{2.33}$$

and, more generally,

$$\frac{d^{r-1}}{dz^{r-1}}\Big[\frac{\Pi'(z)}{\Pi(z)}\Big]\Big|_{z=0} = 0 \quad \text{for } r = 1, 2, \dots, p,$$

which implies

$$\Pi^{(r)}(z) = 0 \quad \text{for } r = 1, 2, \dots, p. \tag{2.34}$$

Finally, let us notice that if

$$g(z) := e^{A_1 z + A_2 z^2 + \dots + A_m z^m} \Pi(z), \tag{2.35}$$

where $A_1, A_2, \dots, A_m$ are complex constants and $\Pi(z)$ is as in (2.30) with $p \geq 1$, then

$$g(0) = 1 \quad \text{and} \quad g'(0) = A_1 \tag{2.36}$$

(if $p \geq 2$, then $g''(0) = 2A_2 + A_1^2$).

2.4. **Some notation and terminology.** From now on we will use the following notation/terminology for typographical convenience.

Let $g(z)$ be an entire function of order $\rho \in (0, \infty)$ and type τ.

(i) If $\tau = 0$ we will say that the *exact order* of $g(z)$ is ρ^-.
(ii) If $0 < \tau < \infty$ we will say that the *exact order* of $g(z)$ is ρ.
(iii) If $\tau = \infty$ we will say that the *exact order* of $g(z)$ is ρ^+.

When $\rho = 0$ or $\rho = \infty$ we can consider the notion of exact order as equivalent to the order. For instance, the statement that the exact order of $g(z)$ is in $[0, 2^-]$ means that the order is between 0 and 2 (included), and in the case where it is equal to 2 the type of $g(z)$ is 0.

3. Order and type considerations for entire caloric functions

3.1. **t- and z-power series of a caloric function.** For $t_0 = 0$ the second equality in (1.2) can be written in the (Taylor) form

$$F(t, z) = \sum_{j\geq 0} \frac{\partial_t^j F(0, z)}{j!} \, t^j. \tag{3.1}$$

If $F(t, z)$ satisfies the heat equation (1.3), we can use (1.5) in (3.1) and get the following expansion in powers of t,

$$F(t, z) = \sum_{j\geq 0} \frac{\partial_z^{2j} F(0, z)}{j!} t^j = \sum_{j\geq 0} \frac{f^{(2j)}(z)}{j!} t^j, \tag{3.2}$$

where (recall (1.4)),

$$f(z) = F(0, z) = \sum_{k\geq 0} c_k z^k. \tag{3.3}$$

Let us also expand $F(t,z)$ in powers of z. For $z_0 = 0$ the first equality in (1.2) can be written in the (Taylor) form

$$F(t,z) = \sum_{k\geq 0} \frac{\partial_z^k F(t,0)}{k!} z^k = \sum_{k\geq 0} \frac{\partial_z^{2k} F(t,0)}{(2k)!} z^{2k} + \sum_{k\geq 0} \frac{\partial_z^{2k}[\partial_z F(t,0)]}{(2k+1)!} z^{2k+1}. \quad (3.4)$$

If $F(t,z)$ is caloric, so is $\partial_z F(t,z)$. Thus, we can use (1.5) in (3.4) and obtain the expansion

$$F(t,z) = \sum_{k\geq 0} \frac{\partial_t^k F(t,0)}{(2k)!} z^{2k} + \sum_{k\geq 0} \frac{\partial_t^k[\partial_z F(t,0)]}{(2k+1)!} z^{2k+1}, \quad (3.5)$$

or

$$F(t,z) = \sum_{k\geq 0} \frac{\phi^{(k)}(t)}{(2k)!} z^{2k} + \sum_{k\geq 0} \frac{\psi^{(k)}(t)}{(2k+1)!} z^{2k+1}, \quad (3.6)$$

where we have set

$$\phi(t) := F(t,0) \quad \text{and} \quad \psi(t) := \partial_z F(t,0). \quad (3.7)$$

We can say that formula (3.6) is the counterpart of (3.2).

Remark 3.1. If $F(t,z)$ satisfies the heat equation (1.3), so does $F(t,-z)$. Consequently, the z-even and z-odd parts of $F(t,z)$, namely

$$F_e(t,z) := \frac{F(t,z)+F(t,-z)}{2} \quad \text{and} \quad F_o(t,z) := \frac{F(t,z)-F(t,-z)}{2} \quad (3.8)$$

respectively, also satisfy (1.3), and, moreover, from (3.6) we obtain immediately that

$$F_e(t,z) = \sum_{k\geq 0} \frac{\phi^{(k)}(t)}{(2k)!} z^{2k} \quad \text{and} \quad F_o(t,z) = \sum_{k\geq 0} \frac{\psi^{(k)}(t)}{(2k+1)!} z^{2k+1}. \quad (3.9)$$

Finally, let us notice that if $\phi(t)$ and $\psi(t)$ are two arbitrary entire functions, then, for any fixed $t \in \mathbb{C}$ the radii of convergence of the two z-power series in (3.6) are infinite (and the same is true for the t-derivatives of those z-power series), hence both series are entire in (t,z) and satisfy the heat equation (1.3). Thus, every entire solution $F(t,z)$ of (1.3) is determined uniquely by a pair of (arbitrary) entire functions $\phi(t)$ and $\psi(t)$ via (3.6)-(3.7). As we will see in the next subsection, the situation is quite different if we want to determine an entire caloric solution $F(t,z)$ from the initial condition $f(z)$ via (3.2), since in this case the entire function $f(z)$ cannot be arbitrary.

3.2. **Order of the initial condition** $f(z)$**.** Recall that in view of (2.7) the order ρ of the entire function $f(z)$ can be expressed as

$$\rho = \limsup_k \frac{k \ln k}{-\ln |c_k|}, \quad (3.10)$$

where, in view of (3.3),

$$c_k = \frac{f^{(k)}(0)}{k!}, \quad k \geq 0. \quad (3.11)$$

The following theorem is not new (actually it probably goes back to Kovalevskaya, see [8]), but we include its proof here for the sake of completeness.

Theorem 3.2. *Suppose $f(z)$ is the initial condition of an entire solution $F(t,z)$ of the heat equation (1.3). Then the order ρ of $f(z)$ satisfies $0 \le \rho \le 2$. Furthermore, in the extreme case $\rho = 2$ the type τ of $f(z)$ must be minimal, i.e. zero. Thus, under the terminology introduced in Subsection 2.4, the exact order of $f(z)$ is in $[0, 2^-]$.*

Conversely, if $f(z)$ is an entire function whose exact order is in $[0, 2^-]$, then $F(t,z)$ given by (3.2) is entire in (t,z) and satisfies the heat equation (1.3) with $F(0,z) = f(z)$.

Proof. We need to consider the cases $k = 2j$ and $k = 2j+1$ separately. Observe that $\rho = \max\{\rho_0, \rho_1\}$ where

$$\rho_0 := \limsup_j \frac{2j\ln(2j)}{-\ln|c_{2j}|} = 2\limsup_j \frac{j\ln j}{-\ln|c_{2j}|}, \tag{3.12}$$

$$\rho_1 := \limsup_j \frac{(2j+1)\ln(2j+1)}{-\ln|c_{2j+1}|} = 2\limsup_j \frac{j\ln j}{-\ln|c_{2j+1}|}. \tag{3.13}$$

Let us first estimate ρ_0. For $z = 0$, in view of (3.11), formula (3.2) becomes

$$F(t,0) = \sum_{j\ge 0} \frac{t^j}{j!} f^{(2j)}(0) = \sum_{j\ge 0} \frac{(2j)!c_{2j}}{j!} t^j. \tag{3.14}$$

Since the second power series in (3.14) converges for all $t \in \mathbb{C}$, i.e. has an infinite radius of convergence, we must have

$$\limsup_j \big|\frac{(2j)!c_{2j}}{j!}\big|^{1/j} = 0 \quad \Big(\text{hence, } \lim_j \big|\frac{(2j)!c_{2j}}{j!}\big|^{1/j} = 0\Big) \tag{3.15}$$

equivalently, there is a sequence $\varepsilon_j \to 0$ such that

$$|c_{2j}|^{1/j} < \varepsilon_j \big[\frac{j!}{(2j)!}\big]^{1/j}. \tag{3.16}$$

By applying Stirling's asymptotic formula for $n!$ to (3.16) we obtain that there is a sequence $\varepsilon_j \to 0$ (not necessarily the same ε_j appearing in (3.16)) such that

$$|c_{2j}|^{1/j} < \frac{\varepsilon_j}{j}. \tag{3.17}$$

Now, (3.12) tells us that $2/\rho_0$ is the supremum of all exponents r such that

$$|c_{2j}|^{1/j} < \frac{1}{j^r} \quad \text{for every sufficiently large } j. \tag{3.18}$$

Therefore, by comparing (3.17) and (3.18) we obtain that

$$2/\rho_0 \ge 1, \quad \text{i.e.} \quad \rho_0 \le 2. \tag{3.19}$$

To determine ρ_1 we differentiate (3.2) with respect to z and arrive at

$$\partial_z F(t,z) = \sum_{j\ge 0} \frac{t^j}{j!} f^{(2j+1)}(z). \tag{3.20}$$

Then, in exactly the same way as in the case of ρ_0 we conclude that there is a sequence $\varepsilon_j \to 0$ such that

$$|c_{2j+1}|^{1/j} < \frac{\varepsilon_j}{j}. \tag{3.21}$$

Hence

$$\rho_1 \le 2 \tag{3.22}$$

and, therefore, the order of $f(z)$ satisfies $\rho \le 2$.

Next, let us estimate the type τ of $f(z)$ in the case where $\rho = 2$. Recall that if $\rho = 2$, then

$$\tau = \frac{1}{2e} \limsup_k k|c_k|^{2/k}. \tag{3.23}$$

As in the case of ρ we, again, need to consider the cases $k = 2j$ and $k = 2j+1$ odd separately. We, thus, observe that $\tau = \max\{\tau_0, \tau_1\}$ where

$$\tau_0 := \frac{1}{e} \limsup_j j|c_{2j}|^{1/j} \quad \text{and} \quad \tau_1 := \frac{1}{e} \limsup_j j|c_{2j+1}|^{1/j}. \tag{3.24}$$

Then, by (3.17) and (3.21) we obtain immediately that $\tau_0 = \tau_1 = 0$ and, consequently $\tau = 0$. In other words, if $f(z)$ is of order 2, then it is of minimal type.

The second part of the theorem is easy (see also Remark 3.7 below). $\square$

Using the Hadamard Factorization Theorem [7] we obtain immediately the following corollary of Theorem 3.2.

Corollary 3.3. *Suppose $f(z) \not\equiv 0$ is the initial condition of an entire solution $F(t,z)$ of the heat equation* (1.3)*. Then $f(z)$ has the form*

$$f(z) = e^{\lambda z + \beta} z^d \, \Pi(z), \tag{3.25}$$

where λ and β are complex constants, d is a nonnegative integer and $\Pi(z)$ is a canonical product whose exact order is in $[0, 2^-]$.

Finally note that from Subsection 2.1 it follows that the canonical product $\Pi(z)$ of (3.25) must be of the form

$$\Pi(z) = \prod_{k\ge 1} e_p\Big(\frac{z}{a_k}\Big), \quad \text{where } p = 0 \text{ or } 1 \text{ or } 2. \tag{3.26}$$

The case $p = 2$ can happen only if the exact order of $\Pi(z)$ is 2^- and $\sum_{k\ge1} |z_k|^{-2} = \infty$.

3.3. **Order and type of a caloric function $F(t,z)$ when t or z is fixed.** Suppose we freeze $t \in \mathbb{C}$ and we consider

$$\rho_z := \operatorname{ord}_z F(t,z) \quad \text{and} \quad \tau_z := \operatorname{type}_z F(t,z), \tag{3.27}$$

namely the order and the type of $F(t,\cdot)$, which for convenience we call the z-order and z-type of $F(t,z)$. One might think that these quantities depend on t. However, by applying [4, Theorem 1.1] for the choice $f(D) = e^{tD^2}$ (in the notation of [4, Theorem 1.1]) and noticing that $F(t,z) = e^{tD^2} F(0,z)$ with $D = \partial_z$, we obtain that ρ_z and τ_z are, actually, independent of t (actually, this also follows from (3.6)). We can, therefore, choose $t = 0$ and conclude that

$$\rho_z = \rho \quad \text{and} \quad \tau_z = \tau, \tag{3.28}$$

where ρ and τ are the order and type of $f(z) = F(0,z)$ respectively. And then, Theorem 3.2 implies immediately that the exact z-order of $F(t,z)$ is in $[0, 2^-]$.

The following open question envisages a refinement of formula (3.28).

Open Question. Let $z_1(t), z_2(t), \dots$ be the zeros of $F(t,z)$ (viewed as an entire function of z). Is it true that for any $t_1, t_2 \in \mathbb{C}$ and any $\alpha > 0$ we have that

$$\sum_{k\ge1} \frac{1}{|z_k(t_1)|^\alpha} < \infty \quad \text{if and only if} \quad \sum_{k\ge1} \frac{1}{|z_k(t_2)|^\alpha} < \infty?$$

More specifically, if the zeros are infinitely many and arranged so that $|z_k(t)| \le |z_{k+1}(t)$ for all $k \ge 1$, how close are the asymptotics of the sequences $\{z_k(t_1)\}_{k\ge 1}$ and $\{z_k(t_2)\}_{k\ge 1}$ as $k \to \infty$?

Regarding the order and the type of $F(t,z)$ viewed as a function of t things are more complicated, since these quantities may depend on z. For instance it is clear from (3.6)-(3.7) that there are nontrivial caloric functions $F(t,z)$ such that, for a given z_0 the quantity $F(t,z_0)$ can vanish for all $t \in \mathbb{C}$ (e.g., just think of $F(t,z) = e^{-\lambda^2 t}\sin(\lambda z)$, for $z_0 = k\pi/\lambda$, $k \in \mathbb{Z}$).

The goal of this subsection is to clarify the notions of t-order and t-type of an entire caloric function $F(t,z)$ and to relate them to its z-order and its z-type respectively.

Theorem 3.4. *Let $F(t,z)$ be an entire solution of the heat equation* (1.3)*. For a fixed $z \in \mathbb{C}$ we consider the orders of $F(t,z)$ and $\partial_z F(t,z)$ viewed as functions of t, namely*
$$\rho_{t,0}(z) := \operatorname{ord}_t F(t,z), \quad \rho_{t,1}(z) := \operatorname{ord}_t \partial_z F(t,z), \tag{3.29}$$
and we set
$$\rho_t := \max\{\rho_{t,0}(z),\ \rho_{t,1}(z)\}. \tag{3.30}$$
Then:

(i) *The quantity ρ_t is independent of z and is related to the z-order ρ_z $(=\rho)$ of $F(t,z)$ via the formula*
$$\rho_t = \frac{\rho}{2-\rho} \tag{3.31}$$
(in particular, if $\rho = 2$, then $\rho_t = \infty$, while ρ_t is finite in the case where $\rho \in [0,2)$, and $\rho_t = 0$ if and only if $\rho = 0$).

(ii) *We have*
$$\rho_{t,0}(z) = \rho_t \quad \text{and} \quad \rho_{t,1}(z) = \rho_t \quad \text{for a.a. } z \in \mathbb{C}. \tag{3.32}$$

Proof. Let us set
$$c_k(z) := \frac{f^{(k)}(z)}{k!}, \quad k \ge 0, \quad z \in \mathbb{C} \tag{3.33}$$
(so that, in view of (3.11), $c_k(0) = c_k$). Then, by substituting (3.33) in (3.2) we obtain
$$F(t,z) = \sum_{j\ge 0} \frac{(2j)!}{j!} c_{2j}(z) t^j \tag{3.34}$$
$$\partial_z F(t,z) = \sum_{j\ge 0} \frac{(2j)!}{j!} c_{2j}'(z) t^j = \sum_{j\ge 0} \frac{(2j+1)!}{j!} c_{2j+1}(z) t^j. \tag{3.35}$$
Then, by (3.34) and (2.10) we obtain
$$\rho_{t,0}(z) = \operatorname{ord}_t F(t,z) = \limsup_j \frac{j\ln j}{-\ln|c_{2j}(z)(2j)!/j!|}\,, \tag{3.36}$$
which implies
$$\rho_{t,0}(z) = \limsup_j \frac{1}{\frac{-\ln|c_{2j}(z)|}{j\ln j} - \frac{\ln((2j)!)}{j\ln j} + \frac{\ln(j!)}{j\ln j}}\,. \tag{3.37}$$

Now, by Stirling's formula we have that $\ln((2j)!)/(j\ln j) \to 2$ and $\ln(j!)/(j\ln j) \to 1$ as $j \to \infty$. Thus (3.37) becomes

$$\rho_{t,0}(z) = \limsup_j \frac{1}{\frac{-\ln|c_{2j}(z)|}{j\ln j} - 1} = \limsup_j \frac{\frac{j\ln j}{-\ln|c_{2j}(z)|}}{1 - \frac{j\ln j}{-\ln|c_{2j}(z)|}} = \frac{\rho_0(z)}{2-\rho_0(z)}\,, \tag{3.38}$$

where

$$\rho_0(z) := 2\limsup_j \frac{j\ln j}{-\ln|c_{2j}(z)|}. \tag{3.39}$$

Likewise, from (3.35) we have

$$\rho_{t,1}(z) = \operatorname{ord}_t \partial_z F(t,z) = \limsup_j \frac{j\ln j}{-\ln|c_{2j+1}(z)(2j+1)!/j!|} \tag{3.40}$$

and in the same way as above we obtain

$$\rho_{t,1}(z) = \frac{\rho_1(z)}{2-\rho_1(z)}, \tag{3.41}$$

where

$$\rho_1(z) := 2\limsup_j \frac{j\ln j}{-\ln|c_{2j+1}(z)|}. \tag{3.42}$$

But in view of (3.33), the quantities $\rho_0(z)$ and $\rho_1(z)$ of (3.39) and (3.42) respectively, associated with $f(z)$, are the analogs of the quantities introduced in (2.14) and (2.15), associated to $g(z)$. Hence, by Theorem 2.1 we obtain that $\rho_0(z) = \rho_1(z) = \rho$ for a.a. $z \in \mathbb{C}$ and, therefore, formulas (3.38) and (3.41), together with (3.30), imply (3.31) and (3.32). $\square$

We will refer to ρ_t of (3.30) as the *caloric t-order of* $F(t,z)$. Also, let us set

$$\mathcal{E}_0 := \{z \in \mathbb{C} : \rho_{t,0}(z) = \rho_t\} \quad \text{and} \quad \mathcal{E}_1 := \{z \in \mathbb{C} : \rho_{t,1}(z) = \rho_t\}, \tag{3.43}$$

so that, in view of (3.30), $\mathcal{E}_0 \cup \mathcal{E}_1 = \mathbb{C}$. Theorem 3.4 tells us that both $\mathcal{E}_0$ and $\mathcal{E}_1$ have full measure. Consequently,

$$\mathcal{E} := \mathcal{E}_0 \cap \mathcal{E}_1 \quad \text{is a subset of } \mathbb{C} \text{ of full measure.} \tag{3.44}$$

In general it might happen that $\rho_{t,0}(z_0) < \rho_t$ for some z_0 and $\rho_{t,1}(z_1) < \rho_t$ for some $z_1 \neq z_0$. For instance, if $F(t,z) = e^{-\lambda^2 t}\sin(\lambda z)$, then $\rho_t = 1$. However, $F(t,z_0) \equiv 0$ for $z_0 = k\pi/\lambda$, $k \in \mathbb{Z}$, while $\partial_z F(t,z_1) \equiv 0$ for $z_1 = [k+(1/2)]\pi/\lambda$, $k \in \mathbb{Z}$, and, consequently, $\rho_{t,0}(z_0) = \rho_{t,1}(z_1) = 0$ for those values of z_0 and z_1.

Next, we present the analog of Theorem 3.4 regarding the t-type of $F(t,z)$.

Theorem 3.5. *Let $F(t,z)$ be an entire solution of the heat equation* (1.3) *whose z-order ρ_z $(=\rho)$ satisfies $0 < \rho_z < 2$ (or, equivalently, by* (3.31) *the caloric t-order of $F(t,z)$ satisfies $0 < \rho_t < \infty$). Thinking of z as a parameter, we consider the types of $F(t,z)$ and $\partial_z F(t,z)$ viewed as functions of t, namely*

$$\tau_{t,0}(z) := \operatorname{type}_t F(t,z), \quad z \in \mathcal{E}_0, \tag{3.45}$$

$$\tau_{t,1}(z) := \operatorname{type}_t \partial_z F(t,z), \quad z \in \mathcal{E}_1, \tag{3.46}$$

where the sets $\mathcal{E}_0$ and $\mathcal{E}_1$ are the full measure sets defined by (3.43)-(3.29)-(3.30). *We also set*

$$\tau_t := \max\{\tau_{t,0}(z),\, \tau_{t,1}(z)\}, \quad z \in \mathcal{E} = \mathcal{E}_0 \cap \mathcal{E}_1. \tag{3.47}$$

Then

(i) *The quantity τ_t is independent of $z \in \mathcal{E}$ and is related to the z-order ρ_z $(= \rho)$ and the z-type τ_z $(= \tau)$ of $F(t,z)$ via the formula*

$$\tau_t = \Big(1 - \frac{\rho}{2}\Big)(2\rho)^{\frac{\rho}{2-\rho}} \tau^{\frac{2}{2-\rho}}. \tag{3.48}$$

In particular, $\tau_t = 0$ if and only if $\tau = 0$ and $\tau_t = \infty$ if and only if $\tau = \infty$.

(ii) *If $\tau_t < \infty$, then*

$$\tau_{t,0}(z) = \tau_t \quad \text{and} \quad \tau_{t,1}(z) = \tau_t \quad \text{for a.a. } z \in \mathbb{C}. \tag{3.49}$$

Proof. (i) In view of (3.33) and (2.11) the type τ of $f(z) = F(0,z)$ is given by

$$\tau = \frac{1}{e\rho} \limsup_k k\,|c_k(z)|^{\rho_z/k} \quad \text{for any } \; z \in \mathbb{C}. \tag{3.50}$$

If we set

$$\tau_0(z) := \frac{2}{e\rho} \limsup_j j|c_{2j}(z)|^{\rho_z/2j}, \quad \tau_1(z) := \frac{2}{e\rho} \limsup_j j|c_{2j+1}(z)|^{\rho_z/2j}, \tag{3.51}$$

then, obviously (3.50) implies

$$\tau = \max\{\tau_0(z), \tau_1(z)\} \quad \text{for any } z \in \mathbb{C}, \tag{3.52}$$

while by Theorem 2.2 , if $\tau_t < \infty$, we have

$$\tau_0(z) = \tau \quad \text{and} \quad \tau_1(z) = \tau \quad \text{for a.a. } z \in \mathbb{C}. \tag{3.53}$$

Next, let us set

$$\hat{\tau}_{t,0}(z) := \frac{1}{e\rho_t} \limsup_j j\Big|\frac{(2j)!\,c_{2j}(z)}{j!}\Big|^{\rho_t/j}, \quad z \in \mathbb{C}, \tag{3.54}$$

$$\hat{\tau}_{t,1}(z) := \frac{1}{e\rho_t} \limsup_j j\Big|\frac{(2j+1)!c_{2j+1}(z)}{j!}\Big|^{\rho_t/j}, \quad z \in \mathbb{C}. \tag{3.55}$$

Then, formulas (3.34), (3.35), (3.43), (3.45), and (3.46) imply

$$\hat{\tau}_{t,0}(z) = \tau_{t,0}(z) \quad \text{for all } z \in \mathcal{E}_0, \tag{3.56}$$

$$\hat{\tau}_{t,1}(z) = \tau_{t,1}(z) \quad \text{for all } z \in \mathcal{E}_1. \tag{3.57}$$

Application of Stirling's asymptotic formula for the factorial to (3.54) gives

$$\begin{aligned} \hat{\tau}_{t,0}(z) &= \frac{1}{e\rho_t} \limsup_j j\big(\frac{4j}{e}\big)^{\rho_t} \big(|c_{2j}(z)|^{1/j}\big)^{\rho_t} \\ &= \frac{4^{\rho_t}}{\rho_t\, e^{\rho_t+1}} \big[\limsup_j j\big(|c_{2j}(z)|^{1/j}\big)^{\frac{\rho_t}{\rho_t+1}}\big]^{\rho_t+1}, \quad z \in \mathbb{C}. \end{aligned} \tag{3.58}$$

Since (3.31) can be written as

$$\frac{\rho_t}{\rho_t+1} = \frac{\rho}{2}, \tag{3.59}$$

in view of (3.31), (3.59), and (3.51), formula (3.58) yields

$$\begin{aligned} \hat{\tau}_{t,0}(z) &= \frac{4^{\rho_t}}{\rho_t e^{\rho_t+1}} \big[\limsup_j j|c_{2j}(z)|^{\rho/2j}\big]^{\rho_t+1} \\ &= \frac{4^{\rho_t}}{\rho_t\, e^{\rho_t+1}} \big[\frac{e\rho\tau_0(z)}{2}\big]^{\rho_t+1} \\ &= \big(1 - \frac{\rho}{2}\big)(2\rho)^{\frac{\rho}{2-\rho}} \tau_0(z)^{\frac{2}{2-\rho}}, \quad z \in \mathbb{C}. \end{aligned} \tag{3.60}$$

In the same way, starting from (3.55) we obtain

$$\hat{\tau}_{t,1}(z) = \big(1 - \frac{\rho}{2}\big)(2\rho)^{\frac{\rho}{2-\rho}}\tau_1(z)^{\frac{2}{2-\rho}}, \quad z \in \mathbb{C}. \tag{3.61}$$

Therefore (3.48) follows by using (3.56) and (3.57) in (3.47), and then invoking (3.60), (3.61) and (3.52). As for (3.49), it follows from (3.53), (3.56), (3.57), (3.60), (3.61), and (3.47). □

We will refer to τ_t of (3.47) as the *caloric t-type of* $F(t,z)$. The question whether (3.49) remains valid in the case where $\tau_t = \infty$ remains open.

As an example let us observe that in the special case $\rho_z = \rho = 1$ formula (3.31) implies that $\rho_t = 1$ and then (3.48) yields $\tau_t = \tau_z^2 = \tau^2$ (e.g., this is the case of the special solution $E_\lambda(t,z) = e^{\lambda^2 t + \lambda z}$, where, clearly, $\rho_z = \rho_t = 1$, $\tau_z = |\lambda|$, and $\tau_t = |\lambda|^2$).

Remark 3.6. From the definition (3.30) of the caloric t-order ρ_t we know that the t-orders $\rho_{t,0}(z) = \operatorname{ord}_t F(t,z)$ and $\rho_{t,1}(z) = \operatorname{ord}_t \partial_z F(t,z)$ are $\le \rho_t$ for every $z \in \mathbb{C}$. However, the t-type $\operatorname{type}_t F(t,z)$ of $F(t,z)$ or the t-type $\operatorname{type}_t \partial_z F(t,z)$ of $\partial_z F(t,z)$ can become bigger than the caloric t-type τ_t of $F(t,z)$ for some exceptional values of z. For instance, suppose $f(z) = f_1(z) + f_2(z)$, where the order of $f_1(z)$ is smaller than the order of $f_2(z)$, while the type of $f_1(z)$ is bigger than the type of $f_2(z)$. Furthermore, let us assume that $f_2(z)$ is an odd function. Then, we can easily construct examples where $\operatorname{type}_t F(t,0) > \tau_t$.

Remark 3.7. Let $f(z)$ be an entire function whose exact order is in $[0, 2^-]$. Then

$$F(t,z) := \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-(z-\xi)^2/4t} f(\xi)\, d\xi \quad \text{for } \Re(t) > 0 \tag{3.62}$$

satisfies the heat equation (1.3) for $\Re(t) > 0$. Actually,

$$F(t - t_0, z) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-t_0)}} e^{-(z-\xi)^2/4(t-t_0)} F(t_0, \xi)\, d\xi \tag{3.63}$$

for $\Re(t - t_0) > 0$. Also $F(t,z)$ satisfies (1.3).

If t and z are real, with $t > 0$, then we can use the substitution $\eta = (z-\xi)/\sqrt{t}$ or $\eta = (\xi - z)/\sqrt{t}$ in the integral of (3.62) and obtain

$$\begin{aligned} F(t,z) &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2/4} f\big(z + \eta\sqrt{t}\big)\, d\eta \\ &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2/4} f\big(z - \eta\sqrt{t}\big)\, d\eta\,. \end{aligned} \tag{3.64}$$

Then, from our assumption for the order and type of $f(z)$, the integral in the right-hand side of (3.64) is entire in (t,z), satisfies the heat equation for every $t, z \in \mathbb{C}$ (e.g., by analytic continuation) and it is clear from (3.64) that $F(0,z) = f(z)$.

Example 3.8. (i) Suppose

$$f(z) = e^{az^2}, \quad \text{where } a \in \mathbb{C} \setminus \{0\}, \tag{3.65}$$

so that $\rho = 2$ and $\tau = |a| > 0$ (thus the exact order of $f(z)$ is 2). Here, if $F(t,z)$ is as in (3.62), and hence it satisfies the heat equation with initial condition $F(0,z) = f(z)$, then

$$F(t,z) = \frac{1}{\sqrt{1 - 4at}} \exp\Big(\frac{az^2}{1 - 4at}\Big). \tag{3.66}$$

Obviously, this $F(t,z)$ is not entire since it has a strong singularity at $t = 1/4a$ (independent of z), a combination of an essential singularity and a square-root branch point (this singularity "disappears" as $a \to 0$). If t starts at 0, makes a loop around $1/4a$ and comes back to 0, then we obtain a different value of $F(0,z)$. Thus, if initially $F(0,z) = e^{az^2}$, then, after one loop we obtain the other branch of $F(0,z)$, namely $F(0,z) = -e^{az^2}$. This counterexample is in agreement with the previous discussion. Another observation is that for $t \neq 1/4a$ the function $F(t,z)$ is entire in z and its z-order is $\rho_z = \rho = 2$, independent of t. However, its z-type is $\tau_z = |a||1-4at|^{-1}$ (thus depends on t). Finally, notice that $F(t,z)$ is never 0.

(ii) As a variant of the above case, we consider

$$f(z) = \cos(az^2) = \frac{e^{iaz^2} + e^{-iaz^2}}{2}, \quad \text{where } a \in \mathbb{C} \setminus \{0\}, \tag{3.67}$$

so that, again, $\rho = 2$ and $\tau = |a| > 0$. Then $F(t,z)$ of (3.62) becomes

$$F(t,z) = \frac{1}{2\sqrt{1-4iat}} \exp\Big(\frac{iaz^2}{1-4iat}\Big) + \frac{1}{2\sqrt{1+4iat}} \exp\Big(\frac{-iaz^2}{1+4iat}\Big). \tag{3.68}$$

Here $F(t,z)$ has a strong singularities at $t = \pm 1/4ia$; furthermore, it has infinitely many zeros. Actually, z is a zero of $F(t,z)$ if and only if

$$z^2 = (1+16a^2t^2)\Big[\frac{1}{4ia} \ln\Big(\frac{1-4iat}{1+4iat}\Big) + \frac{\pi}{2a}\Big],$$

where $\ln(\cdot)$ stands for the multivalued logarithmic function.

Example 3.9. Here we examine the case where the initial condition is

$$f(z) = z^\alpha, \quad \alpha \in \mathbb{C}. \tag{3.69}$$

For this $f(z)$ the solution of the heat equation (1.3) cannot be entire in (t,z), unless, of course, $\alpha = m$, a nonnegative integer. We consider the function

$$F(t,z) := F(t,z;\alpha) := \frac{i^\alpha}{\Gamma(-\alpha)} t^{\alpha/2} \int_0^\infty \frac{e^{-\xi^2 + it^{-1/2}z\xi}}{\xi^{\alpha+1}}\, d\xi, \quad \Re(\alpha) < 0, \tag{3.70}$$

where $\Gamma(\cdot)$ is the gamma function. Notice that $F(t,z;\alpha)$ is entire in z for any complex $t \neq 0$ and analytic in $t \neq 0$ for any complex z. The singularity at $t = 0$ it is a combination of a branch point and an essential singularity. Also, it is not hard to check that $F(t,z)$ satisfies the heat equation for every $z \in \mathbb{C}$, $t \in \mathbb{C} \setminus \{0\}$. Furthermore, if $t \to 0$ in a way so that $\Im(t^{-1/2}z) \geq 0$, then $F(t,z;\alpha)$ approaches (some branch of) z^α. However, if $t \to 0$ in an arbitrary way, then $\lim_{t\to 0} F(t,z;\alpha)$ may not be equal to z^α. This is the sense in which the initial condition $F(0,z;\alpha) = z^\alpha$ is satisfied.

The integral representation of $F(t,z;\alpha)$ given in (3.70) makes sense only for $\Re(\alpha) < 0$. We can give other (contour) integral representations of $F(t,z;\alpha)$ over contours in the complex plane avoiding the positive semiaxis, which are valid for any $\alpha \in \mathbb{C}$. But, instead of doing that, we continue the analysis of $F(t,z;\alpha)$ as follows.

We observe that $F(t,z) = F(t,z;\alpha)$ in (3.70) can be written as

$$F(t,z) = \frac{i^\alpha t^{\alpha/2}}{\Gamma(-\alpha)} h\Big(\frac{it^{-1/2}z}{2}\Big), \quad \text{where } h(x) := h(x;\alpha) := \int_0^\infty \frac{e^{-\xi^2+2x\xi}}{\xi^{\alpha+1}}\, d\xi, \tag{3.71}$$

where the function $h(x)$ of (3.71) satisfies the *Hermite equation*, namely

$$u''(x) - 2xu'(x) = -2\alpha u(x), \tag{3.72}$$

with

$$h(0) = \frac{1}{2}\Gamma\big(-\frac{\alpha}{2}\big) \quad \text{and} \quad h'(0) = \Gamma\big(\frac{1-\alpha}{2}\big). \tag{3.73}$$

Clearly, every solution of (3.72) is entire in x. Actually, the general solution of (3.72) can be expressed as

$$u(x) = \sum_{n=0}^{\infty} a_n x^k, \tag{3.74}$$

where the coefficients a_k satisfy the recursion

$$\frac{a_{n+2}}{a_n} = \frac{2(n-\alpha)}{(n+1)(n+2)}. \tag{3.75}$$

We can single out two linearly independent solutions of (3.72) in a convenient way. By taking $a_0 = 1$ and $a_1 = 0$ we obtain the solution $u_e(x) = u_e(x;\alpha)$ which is even in x and satisfies $u_e(0) = 1$, while by taking $a_0 = 0$ and $a_1 = 1$ we obtain the solution $u_o(x) = u_o(x;\alpha)$ which is odd in x and satisfies $u_o'(0) = 1$. Thus,

$$u_e(x) = 1 + \sum_{k=1}^{\infty}(-1)^k \frac{2^k\alpha(\alpha-2)\cdots\big(\alpha-2(k-1)\big)}{(2k)!}x^{2k} \tag{3.76}$$

and

$$u_o(x) = x + \sum_{k=1}^{\infty}(-1)^k \frac{2^k(\alpha-1)(\alpha-3)\cdots(\alpha-2k+1)}{(2k+1)!}x^{2k+1}. \tag{3.77}$$

Evidently (because of the analytic dependence on the parameter α), $u_e(x;\alpha)$ and $u_o(x;\alpha)$ are also entire in α. Furthermore, in view of (3.73),

$$h(x) = h(x;\alpha) = \frac{1}{2}\Gamma\big(-\frac{\alpha}{2}\big)u_e(x;\alpha) + \Gamma\big(\frac{1-\alpha}{2}\big)u_o(x;\alpha). \tag{3.78}$$

An additional consequence of formula (3.78) is that, by analytic continuation in α, we obtain the meromorphic extension of $h(x;\alpha)$, which we also denote by $h(x;\alpha)$, for all complex α.

Using (3.78) in (3.71) yields

$$F(t,z;\alpha) = \frac{i^\alpha t^{\alpha/2}}{\Gamma(-\alpha)}\Big[\frac{1}{2}\Gamma\big(-\frac{\alpha}{2}\big)u_e\big(\frac{it^{-1/2}z}{2};\alpha\big) + \Gamma\big(\frac{1-\alpha}{2}\big)u_o\big(\frac{it^{-1/2}z}{2};\alpha\big)\Big]. \tag{3.79}$$

With the help of the well-known Legendre's duplication formula for the Gamma function, namely the formula

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\big(z+\frac{1}{2}\big),$$

equation (3.79) simplifies as

$$\begin{aligned} &F(t,z;\alpha) \\ &= \sqrt{\pi}(2i)^\alpha t^{\alpha/2}\Big[\frac{1}{\Gamma(\frac{1-\alpha}{2})}u_e\Big(\frac{it^{-1/2}z}{2};\alpha\Big) + \frac{2}{\Gamma(-\alpha/2)}u_o\Big(\frac{it^{-1/2}z}{2};\alpha\Big)\Big]. \end{aligned} \tag{3.80}$$

It follows that for $t \neq 0$ the function $F(t,z;\alpha)$ is entire in z and α, while for any z and α it is analytic in t, except for $t = 0$, where it may have a strong singularity (branch point combined with an essential singularity). Thus, if t makes a loop around 0 in the complex t-plane, then we may arrive at a different value of $F(t, z; \alpha)$. Notice, however, that if $m \geq 0$ is an integer, $F(t, z; m)$ becomes the m-th caloric polynomial, i.e.

$$F(t, z; m) = P_m(t, z). \tag{3.81}$$

Generically, the z-order of $F(t, z; \alpha)$ is 2, while its z-type depends on t.

The case $\alpha = -1$ is of particular interest. The solutions $u_e(x; \alpha)$ and $u_o(x; \alpha)$ become respectively

$$u_e(x; -1) = e^{x^2} \quad \text{and} \quad u_o(x; -1) = e^{x^2} \int_0^x e^{-\xi^2} d\xi \tag{3.82}$$

for every $x \in \mathbb{C}$. Then, with the help of (3.82) and (3.80) we can construct the solution of the heat equation

$$\begin{aligned} F(t, z) &= \frac{i}{\sqrt{t}} e^{-z^2/4t} \int_{-\infty}^{-iz/2\sqrt{t}} e^{-\zeta^2} d\zeta \\ &= \frac{i}{\sqrt{t}} e^{-z^2/4t} \Big(\frac{\sqrt{\pi}}{2} + \int_0^{-iz/2\sqrt{t}} e^{-\zeta^2} d\zeta \Big), \end{aligned} \tag{3.83}$$

where the first contour integral is taken over a contour which approaches the negative real axis at $-\infty$. Formula (3.83) implies that

$$F(t, z) \to \frac{1}{z} \quad \text{as } t \to 0 \text{ in certain directions.} \tag{3.84}$$

From $F(t, z)$ of (3.83) we can also obtain the solution of the heat equation with "initial condition" $f(z) = \ln z$ as $\int_1^z F(t, \zeta) d\zeta$ (the quotations here remind us that the initial condition is satisfied in a certain sense). Furthermore, the solutions of the heat equation with "initial conditions" $f(z) = z^{-m}$, $m \in \mathbb{N}$, can be also obtained from $F(t, z)$ by differentiating it $m - 1$ times with respect to z (or $(m-1)/2$ times with respect to t, in m is odd).

4. Zeros of entire caloric functions

We start with a result stating that the multiple zeros of a (nontrivial) entire caloric function $F(t, z)$, viewed as a function of z, cannot accumulate in $\mathbb{C}^2$.

Theorem 4.1. *Suppose $F(t, z) \not\equiv 0$ is entire in (t, z) and satisfies the heat equation (1.3). If*

$$F(t^*, z^*) = \partial_z F(t^*, z^*) = 0 \quad \textit{for some } (t^*, z^*) \in \mathbb{C}^2, \tag{4.1}$$

then there is a ($\mathbb{C}^2$-open) neighborhood U of (t^*, z^*) such that

$$|F(t, z)| + |\partial_z F(t, z)| > 0 \quad \textit{for every } (t, z) \in U \setminus \{(t^*, z^*)\}. \tag{4.2}$$

Proof. Without loss of generality and for typographical convenience we take $t^* = z^* = 0$. If the statement of the theorem is false, then there exists a sequence of points $(t_n, z_n) \neq (0, 0)$, $n = 1, 2, \dots$, such that $(t_n, z_n) \to (0, 0)$ and

$$F(t_n, z_n) = \partial_z F(t_n, z_n) = 0 \quad \text{for every } n \geq 1. \tag{4.3}$$

Let μ be the smallest value of m such that $\partial_z^m F(0, 0) \neq 0$ (since $F(t, z) \not\equiv 0$, formula (1.7) guarantees that μ exists; of course, due to the assumption (4.1) we have $\mu \geq 2$). Then, by taking $t_0 = z_0 = 0$ in (1.7) we obtain

$$F(t,z) = \frac{\partial_z^\mu F(0,0)}{\mu!} P_\mu(t,z) + \sum_{m=\mu+1}^{\infty} \frac{\partial_z^m F(0,0)}{m!} P_m(t,z), \tag{4.4}$$

which also implies

$$\partial_z F(t,z) = \frac{\partial_z^\mu F(0,0)}{\mu!} \partial_z P_\mu(t,z) + \sum_{m=\mu+1}^{\infty} \frac{\partial_z^m F(0,0)}{m!} \partial_z P_m(t,z). \tag{4.5}$$

In view of (1.8), formulas (4.4) and (4.5) imply that, given an open ball $B \subset \mathbb{C}^2$ centered at $(0,0)$ there is an $C > 0$ (i.e. depending only on B) such that for every $(t,z) \in B$ we have

$$\big|F(t,z) - \frac{\partial_z^\mu F(0,0)}{\mu!} P_\mu(t,z)\big| \leq C \max\big\{|z|^{\mu+1}, |z|^{\mu-1}|t|, \dots, |t|^{\lfloor(\mu+2)/2\rfloor}\big\} \tag{4.6}$$

and

$$\big|\partial_z F(t,z) - \frac{\partial_z^\mu F(0,0)}{\mu!} \partial_z P_\mu(t,z)\big| \leq C \max\big\{|z|^{\mu}, |z|^{\mu-2}|t|, \dots, |t|^{\lfloor(\mu+1)/2\rfloor}\big\}. \tag{4.7}$$

Now, given B there is an n_0 such that $(t_n, z_n) \in B$ for all $n \geq n_0$. Hence, by using (4.3) in (4.6) and (4.7) we obtain

$$|P_\mu(t_n, z_n)| \leq C' \max\big\{|z_n|^{\mu+1}, |z_n|^{\mu-1}|t_n|, \dots, |t_n|^{\lfloor(\mu+2)/2\rfloor}\big\}, \tag{4.8}$$

$$|\partial_z P_\mu(t_n, z_n)| \leq C' \max\big\{|z_n|^{\mu}, |z_n|^{\mu-2}|t_n|, \dots, |t_n|^{\lfloor(\mu+1)/2\rfloor}\big\} \tag{4.9}$$

for every $n \geq n_0$, where for typographical convenience we have set

$$C' := \frac{\mu! C}{|\partial_z^\mu F(0,0)|}. \tag{4.10}$$

Let us consider the case where $\mu = 2l$. Then, substituting (1.20) in (4.8) yields

$$\begin{aligned} &|z_n^2 + \rho_{\mu,1} t_n| \cdots |z_n^2 + \rho_{\mu,l} t_n| \\ &\leq C' \max\big\{|z_n|^{2l+1}, |z_n|^{2l-1}|t_n|, \dots, |z_n||t_n|^l, |t_n|^{l+1}\big\} \end{aligned} \tag{4.11}$$

for every $n \geq n_0$, while, in view of (1.14), substituting (1.21) in (4.9) yields

$$\begin{aligned} &|z_n||z_n^2 + \rho_{\mu-1,1} t_n| \cdots |z_n^2 + \rho_{\mu-1,l-1} t_n| \\ &\leq \frac{C'}{\mu} \max\big\{|z_n|^{2l}, |z_n|^{2l-2}|t_n|, \dots, |z_n|^2|t_n|^{l-1}, |t_n|^l\big\} \end{aligned} \tag{4.12}$$

for every $n \geq n_0$.

If $z_n = 0$ (hence $t_n \neq 0$), then (4.11) becomes $|\rho_{\mu,1} \cdots \rho_{\mu,l}\, t_n^l| \leq C'|t_n|^{l+1}$, which, in view of (1.22) and the assumption $t_n \to 0$, cannot be satisfied for any sufficiently large n. Thus, without loss of generality we can assume $z_n \neq 0$.

We set

$$\lambda_n := \frac{t_n}{z_n^2}. \tag{4.13}$$

Then, (4.11) and (4.12) become respectively

$$|1 + \rho_{\mu,1}\lambda_n| \cdots |1 + \rho_{\mu,l}\lambda_n| \leq C'|z_n| \max\big\{1, |\lambda_n|, \dots, |\lambda_n|^l, |\lambda_n|^{l+1}|z_n|\big\} \tag{4.14}$$

and

$$|1 + \rho_{\mu-1,1}\lambda_n| \cdots |1 + \rho_{\mu-1,l-1}\lambda_n| \leq \frac{C'}{\mu}|z_n| \max\big\{1, |\lambda_n|, \dots, |\lambda_n|^l\big\} \tag{4.15}$$

for sufficiently large n.

If $|\lambda_n|$ becomes arbitrarily large, then, in view of (1.22), formula (4.14) should imply that there is a constant $C'' > 0$ such that

$$|\lambda_n|^l \leq C''|z_n| \max\left\{|\lambda_n|^l, |\lambda_n|^{l+1}|z_n|\right\} = C'' \max\left\{|\lambda_n|^l|z_n|, |\lambda_n|^l|t_n|\right\}, \tag{4.16}$$

i.e. $1 \leq C'' \max\{|z_n|, |t_n|\}$, which is, obviously, impossible since $z_n, t_n \to 0$. Therefore, the sequence λ_n must be bounded and, hence, by (4.14) and (4.15) there must exist a constant $M > 0$ such that

$$|1 + \rho_{\mu,1}\lambda_n| \cdots |1 + \rho_{\mu,l}\lambda_n| \leq M|z_n|. \tag{4.17}$$

$$|1 + \rho_{\mu-1,1}\lambda_n| \cdots |1 + \rho_{\mu-1,l-1}\lambda_n| \leq M|z_n| \tag{4.18}$$

for every sufficiently large n. Let λ_{n_k} be a convergent subsequence of the sequence λ_n, with $\lim \lambda_{n_k} = \lambda \in \mathbb{C}$. However, if take limits in (4.17) and (4.18) as $n_k \to \infty$ we obtain

$$|1 + \rho_{\mu,1}\lambda| \cdots |1 + \rho_{\mu,l}\lambda| = 0, \tag{4.19}$$

$$|1 + \rho_{\mu-1,1}\lambda| \cdots |1 + \rho_{\mu-1,l-1}\lambda| = 0 \tag{4.20}$$

which contradict (1.22).

The remaining case is $\mu = 2l + 1$. Here, by substituting (1.21) in (4.8) we obtain

$$|z_n||z_n^2 + \rho_{\mu,1}\, t_n| \cdots |z_n^2 + \rho_{\mu,l}\, t_n| \leq C' \max\left\{|z_n|^{2l+2}, |z_n|^{2l}|t_n|, \dots, |t_n|^{l+1}\right\} \tag{4.21}$$

for every $n \geq n_0$, while, in view of (1.14), substituting (1.20) in (4.9) yields

$$\begin{aligned} &|z_n^2 + \rho_{\mu-1,1}t_n| \cdots |z_n^2 + \rho_{\mu-1,l}t_n| \\ &\leq \frac{C'}{\mu} \max\left\{|z_n|^{2l+1}, |z_n|^{2l-1}|t_n|, \dots, |z_n||t_n|^l, |t_n|^{l+1}\right\} \end{aligned} \tag{4.22}$$

for every $n \geq n_0$.

By proceeding in the same manner as in the case $\mu = 2l$, we again arrive at a contradiction. Therefore, our assumption of the existence of the sequence (t_n, z_n) is false. □

An immediate consequence of Theorem 4.1 is that, if $F(t,z) \not\equiv 0$ is entire in (t,z) and satisfies the heat equation, then the set

$$\mathcal{M}_F := \{(t,z) \in \mathbb{C}^2 \,:\, F(t,z) = \partial_z F(t,z) = 0\} \tag{4.23}$$

is discrete in $\mathbb{C}^2$. For example, if F is a caloric polynomial, then by (1.20) and (1.21) we have

$$\mathcal{M}_{P_m} = \{(0,0)\}, \quad m \geq 2, \quad \text{while} \quad \mathcal{M}_{P_0} = \mathcal{M}_{P_1} = \emptyset. \tag{4.24}$$

One peculiar consequence of Theorem 4.1 (together with the fact [11] that zeros of entire functions of two or more complex variables are never isolated) is that if an entire function $A(t,z)$ can be written as

$$A(t,z) = A_1(t,z)^2 A_2(t,z), \tag{4.25}$$

where $A_1(t,z)$ and $A_2(t,z)$ are entire and $A_1(t_0,z_0) = 0$ for some point $(t_0,z_0) \in \mathbb{C}^2$, then $A(t,z)$ cannot satisfy the heat equation (1.3).

Finally, let us notice that the analog to Theorem 4.1 in the case where $F(t,z)$ is viewed as a function of t does not hold. For example if we consider the caloric function $F(t,z) = e^{-\lambda^2 t}\sin(\lambda z)$, then $F(t,0) \equiv 0$, hence $\partial_t^j F(t,0) \equiv 0$ for every $j \in \mathbb{N}$.

4.1. **Dynamics of the zeros.** The following lemma is well known (see [5, Lemma 2.3]). Since its proof is very short, we include it here for the sake of completeness.

Lemma 4.2. *Let $g(z)$ be analytic in a domain D of $\mathbb{C}$. If $z_0 \in D$ is such that $g(z_0) \neq 0$ and we set $G(z) := (z - z_0)g(z)$, then*
$$\frac{G''(z_0)}{G'(z_0)} = 2\,\frac{g'(z_0)}{g(z_0)}.$$

Proof. For $z \in D$ We have
$$\frac{G''(z)}{G'(z)} = \frac{(z - z_0)g''(z) + 2g'(z)}{(z - z_0)g'(z) + g(z)}$$
and the statement follows by setting $z = z_0$. □

Corollary 4.3. *Suppose $G(z)$ is an entire function with zeros $z_0, z_1, z_2, \dots,$ where z_0 is a simple zero of $G(z)$. Furthermore, let us also assume (essentially without loss of generality) that $G(0) \neq 0$.*

(i) If $\Sigma_{k\geq 0}|z_k|^{-1} < \infty$ and
$$G(z) = Ce^{Az}\prod_{k\geq 0}\Big(1 - \frac{z}{z_k}\Big), \tag{4.26}$$
where A and $C \neq 0$ are complex constants, then
$$\frac{G''(z_0)}{G'(z_0)} = 2A + 2\sum_{k\geq 1}\frac{1}{z_0 - z_k}. \tag{4.27}$$

(ii) If $\Sigma_{k\geq 0}|z_k|^{-2} < \infty$ and
$$G(z) = Ce^{Az}\prod_{k\geq 0}\Big(1 - \frac{z}{z_k}\Big)e^{z/z_k}, \tag{4.28}$$
where, again, A and $C \neq 0$ are complex constants, then
$$\frac{G''(z_0)}{G'(z_0)} = 2A + 2\sum_{k\geq 1}\Big(\frac{1}{z_0 - z_k} + \frac{1}{z_k}\Big). \tag{4.29}$$

Proof. Set
$$g(z) := \frac{G(z)}{z - z_0} \tag{4.30}$$
(thus $G(0) \neq 0$ implies $g(0) \neq 0$). Then, by Lemma 4.2 we have
$$\frac{G''(z_0)}{G'(z_0)} = 2\,\frac{g'(z_0)}{g(z_0)}. \tag{4.31}$$
Using (4.26) in (4.30) yields
$$g(z) = ce^{Az}\prod_{k\geq 1}\Big(1 - \frac{z}{z_k}\Big), \tag{4.32}$$
where $c \neq 0$. Thus, the Mittag-Leffler expansion of $g'(z)/g(z)$ is [7]
$$\frac{g'(z)}{g(z)} = A + \sum_{k\geq 1}\frac{1}{z - z_k}.$$
Therefore, formula (4.27) follows by setting $z = z_0$ in the above formula and substituting in (4.31).

The proof of formula (4.29) is very similar. □

Notice that formula (4.29) differs from (4.27) only if $\Sigma_{k\geq 0}|z_k|^{-1} = \infty$. Let us now consider the set

$$\Gamma := \{F(t,z) = 0\} := \big\{(t,z) \in \mathbb{C}^2 : F(t,z) = 0\big\}\,. \tag{4.33}$$

If Γ is empty, that is if $F(t,z)$ is never 0, then by the Hadamard Factorization Theorem and by Theorems 3.2 and 3.4 of the previous section it follows that (recall (1.15))

$$F(t,z) = cE_\lambda(t,z) = ce^{\lambda^2 t + \lambda z}, \tag{4.34}$$

for some constants $c, \lambda \in \mathbb{C}$, with $c \neq 0$.

Suppose now that $F(t,z)$ is a nontrivial entire solution of the heat equation, which is not of the form (4.34) and, hence, it vanishes for some values of t and z. Then Γ is a nonempty set in $\mathbb{C}^2$ (and, of course, $\Gamma \neq \mathbb{C}^2$). If $F(t,z)$ is *irreducible*, namely it cannot be written as

$$F(t,z) = A_1(t,z)A_2(t,z), \tag{4.35}$$

where both $A_1(t,z)$ and $A_2(t,z)$ are entire and assume the value 0, then we can say that Γ is a "curve" in $\mathbb{C}^2$. Otherwise Γ is a union of such component-curves (as we have seen, a consequence of Theorem 4.1 is that multiple components do not exist). For example, if F is a caloric polynomial, then, in view of (1.20) and (1.21) we have

$$\Gamma = \cup_{j=1}^{l}\{z^2 + \rho_{m,j}t = 0\}, \quad \text{if } m = 2l, \tag{4.36}$$

$$\Gamma = \{z = 0\} \cup \cup_{j=1}^{l}\{z^2 + \rho_{m,j}t = 0\}, \quad \text{if } m = 2l+1. \tag{4.37}$$

Let $z_1(t), z_2(t), \dots$ be the zeros of $F(t,z)$. These zeros can be seen as branches of a global analytic function, say $Z(T)$ defined on a Riemann surface which can be identified with Γ. In general, this will be an infinitely sheeted surface. The ramification points of Γ are the points (t^*, z^*) satisfying (4.1) and Theorem 4.1 assures us that they form a discrete set in $\mathbb{C}^2$. Thus, for every zero $z_k(t)$ we have that $\partial_z F(t, z_k(t)) \neq 0$ for a.a. $t \in \mathbb{C}$, where here "a.a." means "almost all," namely all except for a discrete subset of $\mathbb{C}$. Then, by differentiating (implicitly) $F(t, z_k(t)) = 0$ with respect to t we obtain

$$\partial_t F(t, z_k(t)) + \partial_z F(t, z_k(t)) z_k'(t) = 0 \tag{4.38}$$

or, in view of (1.3),

$$z_k'(t) = -\frac{\partial_t F(t, z_k(t))}{\partial_z F(t, z_k(t))} = -\frac{\partial_z^2 F(t, z_k(t))}{\partial_z F(t, z_k(t))} \quad \text{for a.a. } t \in \mathbb{C}. \tag{4.39}$$

Thus, if for $F(t,z)$ we have that $\rho_z < 1$, then we can apply Corollary 4.3 to (4.39) and obtain

$$z_k'(t) = -2\sum_{j\neq k} \frac{1}{z_k(t) - z_j(t)} \quad \text{for a.a. } t \in \mathbb{C}. \tag{4.40}$$

The above derivation of (4.40) is an imitation of the derivation of (1.24) as presented in [5].

As an application, let us consider the system of ordinary differential equations

$$z_k'(t) = -2\sum_{j\neq k} \frac{1}{z_k(t) - z_j(t)}, \quad 1 \leq k \leq N, \tag{4.41}$$

with initial condition

$$z_k(0) = a_k \quad 1 \leq k \leq N, \tag{4.42}$$

where $a_1, \dots, a_N$ are distinct non-zero complex numbers. To solve this system, we form the polynomial

$$f(z) := \Big(1 - \frac{z}{a_1}\Big) \cdots \Big(1 - \frac{z}{a_N}\Big) = 1 + A_1 z + \cdots + A_N z^N. \tag{4.43}$$

Then, the solution $z_1(t), \dots, z_N(t)$ of the system is the set of zeros of the polynomial in (t, z) given by

$$F(t, z) := 1 + \sum_{k=1}^{N} A_k P_k(t, z), \tag{4.44}$$

where $P_k(t, z)$ is the k-th caloric polynomial.

We expect that this application extends to the infinite case (i.e. $N = \infty$) under the restriction that $\sum_k |a_k|^{-1} < \infty$.

4.1.1. *Even caloric functions.* Suppose that the initial condition $f(z)$ is even, i.e. it satisfies $f(-z) = f(z)$, Then, by formula (3.2) we have that the solution $F(t, z)$ of the heat equation also satisfies $F(t, -z) = F(t, z)$. Hence $F(t, z) = \Phi(t, z^2)$, where $\Phi(t, \mu)$ is entire in (t, μ) and if ρ_z is the z-order of $F(t, z)$, then the order of $\Phi(t, \mu)$ with respect to μ (the μ-order) is $\rho_z/2$. Furthermore, $\Phi(t, \mu)$ satisfies the heat-type equation

$$\partial_t \Phi(t, \mu) = 4\mu \partial_\mu^2 \Phi(t, \mu) + 2\partial_\mu \Phi(t, \mu). \tag{4.45}$$

Let $\pm z_1(t), \pm z_2(t), \dots$ be the zeros of $F(t, z)$. Then the zeros of $\Phi(t, \mu)$ are $\mu_1(t) = z_1(t)^2, \mu_2(t) = z_2(t)^2, \dots$, and by imitating the derivation of (4.39) we now have, in view of (4.45)

$$\mu_k'(t) = -\frac{\partial_t \Phi(t, \mu_k(t))}{\partial_\mu \Phi(t, \mu_k(t))} = -4\mu_k(t) \frac{\partial_\mu^2 \Phi(t, \mu_k(t))}{\partial_\mu \Phi(t, \mu_k(t))} - 2 \quad \text{for a.a. } t \in \mathbb{C}. \tag{4.46}$$

If $\rho_z < 2$, then the μ-order of $\Phi(t, \mu)$ is less than 1. We can, therefore apply Corollary 4.3 to (4.46) and obtain

$$\mu_k'(t) = -2 - 8\mu_k(t) \sum_{j \neq k} \frac{1}{\mu_k(t) - \mu_j(t)} \quad \text{for a.a. } t \in \mathbb{C}. \tag{4.47}$$

4.1.2. *Odd caloric functions.* Now, suppose that the initial condition $f(z)$ satisfies $f(-z) = -f(z)$, i.e. is odd. Then, by formula (3.2) we have that the solution $F(t, z)$ of the heat equation also satisfies $F(t, -z) = -F(t, z)$. Hence $F(t, z) = z\Psi(t, z^2)$, where $\Psi(t, \mu)$ is entire in (t, μ) and if ρ_z is the z-order of $F(t, z)$, then the μ-order of $\Psi(t, \mu)$ is $\rho_z/2$. Furthermore, $\Psi(t, \mu)$ satisfies the heat-type equation

$$\partial_t \Psi(t, \mu) = 4\mu \partial_\mu^2 \Psi(t, \mu) + 6\partial_\mu \Psi(t, \mu). \tag{4.48}$$

Let $z_0(t) \equiv 0, \pm z_1(t), \pm z_2(t), \dots$ be the zeros of $F(t, z)$. Then the zeros of $\Psi(t, \mu)$ are $\mu_1(t) = z_1(t)^2, \mu_2(t) = z_2(t)^2, \dots$, and by imitating the derivation of (4.39) we now have, in view of (4.48)

$$\mu_k'(t) = -\frac{\partial_t \Psi(t, \mu_k(t))}{\partial_\mu \Psi(t, \mu_k(t))} = -4\mu_k(t) \frac{\partial_\mu^2 \Psi(t, \mu_k(t))}{\partial_\mu \Psi(t, \mu_k(t))} - 6 \quad \text{for a.a. } t \in \mathbb{C}. \tag{4.49}$$

If $\rho_z < 2$, then the μ-order of $\Psi(t, \mu)$ is less than 1. We can, therefore apply Corollary 3.3 to (4.49) and obtain

$$\mu_k'(t) = -6 - 8\mu_k(t) \sum_{j \neq k} \frac{1}{\mu_k(t) - \mu_j(t)}. \tag{4.50}$$

4.1.3. *General examples.* Let us start with an observation whose validity is easily checked.

Observation. Suppose $F(t,z)$ satisfies the heat equation (1.3), with $F(0,z) = f(z)$. Then

$$G(t,z) := F(t, z+2\lambda t)\, E_\lambda(t,z) = F(t, z+2\lambda t)\, e^{\lambda^2 t + \lambda z}, \tag{4.51}$$

where λ is a complex constant, also satisfies (1.3), with $G(0,z) = e^{\lambda z} f(z)$ (actually, we can even take λ to be a square matrix with constant elements).

As an example, let us take $f(z) = z^m$, where m is a positive integer. Then $F(t,z) = P_m(t,z)$, the m-th caloric polynomial, and

$$G(t,z) = e^{\lambda^2 t + \lambda z} P_m(t, z+2\lambda t), \tag{4.52}$$

is the solution of the heat equation with initial condition $G(0,z) = z^m e^{\lambda z}$.

(i) Let $F(t,z)$ and $G(t,z)$ be entire solutions of the heat equation (1.3) with initial conditions

$$f(z) = \prod_k \Big(1 - \frac{z}{a_k}\Big) \quad \text{and} \quad g(z) = e^{\lambda z} \prod_k \Big(1 - \frac{z}{a_k}\Big) \tag{4.53}$$

respectively, where the order of the product $\Pi_k[1-(z/a_k)]$ is $\sigma < 1$ (in other words, there is an $\alpha < 1$ such that $\Sigma_k |a_k|^{-\alpha} < \infty$). Then, as we have seen the z-order of $F(t,z)$ is σ. It follows that if $z_1(t), z_2(t), \dots$ are the zeros of $F(t,z)$, then $\Sigma_k' |z_k(t)|^{-\alpha} < \infty$ for some $\alpha < 1$ (the prime on the sum indicates that we omit the $z_k(t)$'s which are equal to 0).

The relation of $G(t,z)$ and $F(t,z)$ is given by (4.51). Thus, if $w_1(t), w_2(t), \dots$ are the zeros of $G(t,z)$ (viewed as a function of z), then (4.51) implies that

$$w_k(t) = z_k(t) - 2\lambda t, \quad k \ge 1, \tag{4.54}$$

and it follows that $\Sigma_k' |w_k(t)|^{-\alpha} < \infty$.

Finally, since (4.52) implies $w_k'(t) = z_k'(t) - 2\lambda$, while $z_k(t)$, $k \ge 1$, satisfies (4.40), we have

$$w_k'(t) = -2\lambda - 2 \sum_{j \ne k} \frac{1}{w_k(t) - w_j(t)} \quad \text{for a.a. } t \in \mathbb{C}. \tag{4.55}$$

(ii) Let $F(t,z)$ be an entire solution of the heat equation (1.3) with initial condition

$$f(z) = e^{\lambda z} z^d \prod_k \Big(1 - \frac{z}{a_k}\Big) e^{z/a_k}, \tag{4.56}$$

where, $d \ge 0$ is an integer and the order ρ of $f(z)$ is < 2. Then, by applying Corollary 4.3(ii) we can get that the zeros $z_1(t), z_2(t), \dots$ of $F(t,z)$ satisfy

$$z_k'(t) = -2\lambda - 2 \sum_{j \ne k} \Big[\frac{1}{z_k(t) - z_j(t)} + \frac{1}{z_j(t)}\Big] \quad \text{for a.a. } t \in \mathbb{C}.$$

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