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ASYMPTOTIC BEHAVIOR OF LINEARIZED BOLTZMANN EQUATIONS FOR SOFT POTENTIALS WITH CUT-OFF

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Abstract. We consider the asymptotic behavior of the linearized Boltzmann equation for soft potentials with cut-off. By introducing a new decomposition of the linearized Boltzmann operator, we analyze the spectrum of the linearized Boltzmann operator and obtain the asymptotic behaviors of the linearized Boltzmann equation for $\gamma \in (-3, 0)$, extending the result in [12] for $\gamma \in (-1, 0)$.

1. Introduction

We consider the Boltzmann equation

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F = Q(F, F), \tag{1.1}$$

where $F = F(t, x, v)$ is the density distribution function of the particles with $(t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3$, $Q(F, G)$ is a bilinear collision operator given by

$$Q(F, G) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} q(|u - v|, \omega) \left(F(u')G(v') - F(u)G(v)\right) \, du \, d\omega$$

with $u' = u - [(u - v) \cdot \omega]\omega$, $v' = v + [(u - v) \cdot \omega]\omega$, $\omega \in \mathbb{S}^2$. For the case with inverse power interactions between particles in [4], the collision kernel $q(|u-v|, \omega)$ is taken as

$$q(|u - v|, \omega) = |u - v|^\gamma |\cos\theta|^{-\gamma'} q_0(\theta) \tag{1.2}$$

for $\gamma = 1 - \frac{4}{s}$, $\gamma' = 1 + \frac{2}{s}$, $s > 1$, where the function $q_0(\theta)$ is bounded, $q_0(\theta) \neq 0$ near $\theta = \pi/2$, and

$$\cos\theta = \frac{(u - v) \cdot \omega}{|u - v|}.$$

We study the Boltzmann equation (1.1) for soft potentials with cut-off. Namely, the collision kernel $q(|u - v|, \omega)$ is chosen as

$$q(|u - v|, \omega) = q(\theta)|u - v|^\gamma, \ \gamma \in (-3, 0), \tag{1.3}$$

where $q(\theta)$ satisfies $0 < q(\theta) \leq C|\cos\theta|$.

Considering the perturbation f of F around the global Maxwellian M as follows

$$F = M + M^{1/2} f,$$

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where

$$M = M(v) = \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}, \quad v \in \mathbb{R}^3, \tag{1.4}$$

then the Boltzmann equation (1.1) for F is reformulated in terms of f into

$$\frac{\partial f}{\partial t} = Bf + \Gamma(f, f),$$

where B the linearized Boltzmann operator

$$B = -v \cdot \nabla_x + L \tag{1.5}$$

with the linearized collision operator

$$Lf = M^{-1/2}\Big(Q(M, M^{1/2}f) + Q(M^{1/2}f, M)\Big), \tag{1.6}$$

and the nonlinear term $\Gamma(f, f)$ is

$$\Gamma(f, f) = M^{-1/2} Q(M^{1/2}f, M^{1/2}f).$$

There is a much important progress on the time decay estimates based on the spectral analysis for the linearized Boltzmann equation for hard potentials in [11, 13, 14, 15]. There have been a few researches on the time decay estimates with the help of the spectral analysis of the linearized Boltzmann equation for soft potentials with cut-off. The spectrum theory and time decay estimates for the linearized Boltzmann equation for $\gamma \in (-1, 0)$ with cut-off in spatially-periodic case were established in [1, 2]. The asymptotic behaviors of the semigroup based on the spectral analysis of the linearized Boltzmann equation for $\gamma \in (-1, 0)$ with cut-off in whole space were studied in [12].

In this article, we are concerned with the asymptotic behavior of the semigroup based on the spectral analysis of the linearized Boltzmann equation for $\gamma \in (-3, 0)$ with cut-off. The linearized Boltzmann collision operator L defined by (1.6) can be written as

$$L = -\nu + K,$$

where the operators ν and K with the kernel $k(u, v)$ are defined by (2.1) and (2.2) respectively. Ukai and Asano applied the upper bound of the kernel $k(u, v)$ for $\gamma \in (-1, 0)$ to obtain the following important estimate, cf. [12],

$$\int_{\mathbb{R}^3} |k(u, v)|^2 (1 + |u|)^{-\beta} du \le C_\beta (1 + |v|)^{-(\beta+1)}$$

for any $\beta \ge 0$, which implies that the integral operator K satisfies

$$K \in C(L^2_\theta(\mathbb{R}^3_v), L^2_\varsigma(\mathbb{R}^3_v)), \quad \text{if } \varsigma > \theta + \frac{2}{\gamma}. \tag{1.7}$$

The compactness of the integral operator K plays an important role in the spectral analysis of the linearized Boltzmann operator. Inspired by the work [5], we introduce a new decomposition of the linearized Boltzmann collision operator

$$L = \underbrace{-\nu + K_s}_{\text{as a whole}} + K_c, \tag{1.8}$$

where K_c is compact and the norm of K_s is small. In particular, it holds that

$$K_c \in C(L^2_{\theta_1}(\mathbb{R}^3_v), L^2_{\theta_2}(\mathbb{R}^3_v))$$

for any $\theta_1, \theta_2 \in \mathbb{R}$. Under the help of the decomposition, we can establish the time decay estimates of the semigroup e^{tB}.

We take the Fourier transform in (1.5) with respect to x, the linearized Boltzmann operator B is turned into

$$\widehat{B}(\xi) = -iv \cdot \xi + L. \tag{1.9}$$

By the Plancherel theorem, we only need to consider the time decay estimates of the semigroup $e^{t\widehat{B}(\xi)}$. To this end, we need to establish the estimates of the resolvent $(\lambda I - \widehat{B}(\xi))^{-1}$. We define the operator

$$\widehat{B}_0(\xi) = -iv \cdot \xi + L - P,$$

where the projection operator P is defined by (2.5). According to the result on the spectral analysis of $\widehat{B}_0(\xi)$ in Proposition 3.7 and the properties of the resolvent $(\lambda I - \widehat{B}_0(\xi))^{-1}$ in Lemma 3.10, we can study the spectrum of the operator $\widehat{B}(\xi)$ in $L^2_\theta(\mathbb{R}^3_v)$ for any $\theta \in \mathbb{R}$ and $\xi \neq 0$, and prove that (refer to Proposition 4.2)

$$\sigma(\widehat{B}(\xi)) \in \overline{\mathbb{C}}_-, \quad \sigma_p(\widehat{B}(\xi)) \in \mathbb{C}_-,$$

which is different from the spectrum of the linearized Boltzmann operator for hard potentials with cut-off in the case with $\theta = 0$ as [3, 14]. Combining the decomposition

$$(\lambda I - \widehat{B}(\xi))^{-1} = (I - (\lambda I - \widehat{B}_0(\xi))^{-1} P)^{-1} (\lambda I - \widehat{B}_0(\xi))^{-1},$$

and the properties of the resolvent $(\lambda I - \widehat{B}_0(\xi))^{-1}$ given by Lemma 3.10, we can obtain the properties of the resolvent $(\lambda I - \widehat{B}(\xi))^{-1}$ (refer to Lemma 4.7 for details). By the inverse Laplace transform and the properties of the resolvent $(\lambda I - \widehat{B}(\xi))^{-1}$, we can obtain the time decay estimates of the semigroup $e^{t\widehat{B}(\xi)}$ for any $|\xi| \geq r$ and $r > 0$ in a weighted velocity space, which is described by Theorem 4.8. Using resolvent identity, we have

$$\begin{aligned}
&(\lambda I - \widehat{B}(\xi))^{-1} \\
&= (\lambda I - \widehat{B}_0(\xi))^{-1} + (\lambda I - \widehat{B}_0(\xi))^{-1} P (I - P(\lambda I - \widehat{B}_0(\xi))^{-1} P)^{-1} P (\lambda I - \widehat{B}_0(\xi))^{-1}.
\end{aligned}$$

Then we analyze the singularities of $(\lambda I - \widehat{B}(\xi))^{-1}$ near $\xi = 0$, and point out that the singularities of $(\lambda I - \widehat{B}(\xi))^{-1}$ near $\xi = 0$ arise from $(I - P(\lambda I - \widehat{B}_0(\xi))^{-1} P)^{-1}$. We compute the eigenvalues of $P(\lambda I - \widehat{B}_0(\xi))^{-1} P$ near $\xi = 0$, and find that the singular points of

$$(I - P(\lambda I - \widehat{B}_0(\xi))^{-1} P)^{-1}$$

near $\xi = 0$ are $\mu_j(\kappa) = \sigma_j(\kappa) + i\tau_j(\kappa)$, $j = \pm 1, 0, 2, 3$, where $\sigma_j(\kappa), \tau_j(\kappa) \in C^\infty[-r_0, r_0]$ for some sufficiently small constant $r_0 > 0$ and $\kappa = |\xi|$, which satisfy the following asymptotic expansions for any $\kappa \in [-r_0, r_0]$,

$$\begin{gathered}
\sigma_j(\kappa) = \sigma_j^{(2)} \kappa^2 + O(\kappa^3), \quad j = \pm 1, 0, 2, 3, \\
\tau_j(\kappa) = \tau_j^{(1)} \kappa + O(\kappa^3), \quad j = \pm 1, 0, 2, 3,
\end{gathered}$$

where $\sigma_j^{(2)} < 0$ and $\tau_j^{(1)}$ are constants. For more details, we refer to Proposition 4.9. We obtain the time decay estimates of the semigroup $e^{t\widehat{B}(\xi)}$ near $\xi = 0$ under the help of the asymptotic analysis of $e^{t\widehat{B}(\xi)}$ near $\xi = 0$ given in Theorem 4.10.

For any $\xi \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$, $\widehat{B}(\xi)$ generates a semigroup $e^{t\widehat{B}(\xi)}$ on $L^2_\theta(\mathbb{R}^3_v)$ (refer to Lemma 4.4). Since the absence of the spectral gap for the linearized Boltzmann operator for soft potentials, we obtain the time decay estimates of the semigroup e^{tB} in a weighted Sobolev space. We state our main result below.

Theorem 1.1. *Let $\gamma \in (-3,0)$. For any $p \in [1,\frac{6}{5})$, $n \in [1,\frac{3}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2})$, $\theta \in \mathbb{R}$ and $l \in \mathbb{N}$, it holds that*

$$\begin{aligned} &\|e^{tB} f_0\|_{l,\theta,2} \\ &\le C\Big((1+t)^{-n}(\|f_0\|_{l,\theta-2n,2} + \|f_0\|_{L^{p,2}}) + (1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}\|Pf_0\|_{L^{p,2}}\Big) \end{aligned} \tag{1.10}$$

for any $t \ge 0$ and $f_0 \in H_{l,\theta-2n,2} \cap L^{p,2}$, where P is defined by (2.5).

Remark 1.2. If $Pf_0 = 0$, then the time decay rate in (1.10) could reach $(1+t)^{-n}$, which is faster than $(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})}$ for any $n \in [1,\frac{3}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2})$ and $p \in [1,\frac{6}{5})$.

Notation. We will use C as a general positive constant. Denote $\langle\cdot,\cdot\rangle$ as the inner product on $L^2(\mathbb{R}^3_v)$. We write T^* for the adjoint operator of the operator T. $B(X,Y)$ stands for the class of linear bounded operators defined on the space X with the range in Y, the norm of $T \in B(X,Y)$ is expressed as $\|T\|_{B(X,Y)}$, we will use $B(X)$ for $B(X,X)$. $C(X,Y)$ represents the class of compact operators defined on the space X with the range in Y, we will write $C(X)$ for $C(X,X)$. Let Σ be a metric space and $\mathscr{L}$ be a normed space, we define $L^\infty(\Sigma,\mathscr{L})$ and $C^0(\Sigma,\mathscr{L})$ as follows

$$L^\infty(\Sigma,\mathscr{L}) = \{f : \Sigma \to \mathscr{L} : \sup_{x\in\Sigma}\|f\|_{\mathscr{L}} < \infty\},$$
$$C^0(\Sigma,\mathscr{L}) = \{f : \Sigma \to \mathscr{L} : f \text{ is continuous from } \Sigma \text{ to } \mathscr{L}\}.$$

We denote by $\sigma(T)$, $\sigma_p(T)$ and $\sigma_e(T)$ the spectrum, point spectrum and essential spectrum for the operator T. We denote by $\varrho(T)$ the resolvent set, and by $(\lambda I - T)^{-1}$ the resolvent with $\lambda \in \varrho(T)$. We define $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ and $\mathbb{C}_- = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$. We define the Fourier transform $\hat{f}(\xi)$ of $f(x)$ as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx.$$

For $\theta \in \mathbb{R}$, we define a weighted L^2-Lebesgue space $L^2_\theta(\mathbb{R}^3_v) = \{f(v) : \nu^{\theta/2}(v)f(v) \in L^2(\mathbb{R}^3_v)\}$ with the norm

$$\|f\|_{L^2_\theta(\mathbb{R}^3_v)} = \Big(\int_{\mathbb{R}^3} \nu(v)^\theta |f(v)|^2 dv\Big)^{1/2},$$

where $\nu(v)$ is given by (2.1). For $\theta \in \mathbb{R}$, we introduce the weighted Sobolev space of the function $f(x,v)$ by $H_{l,\theta,2} = L^2_\theta(\mathbb{R}^3_v; H^l(\mathbb{R}^3_x))$ with the norm

$$\|f\|_{l,\theta,2} = \Big(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \nu(v)^\theta (1+|\xi|)^{2l} |\hat{f}(\xi,v)|^2 d\xi dv\Big)^{1/2}.$$

For $p \ge 1$, we also need the space $L^{p,2} = L^2(\mathbb{R}^3_v; L^p(\mathbb{R}^3_x))$ with the norm

$$\|f\|_{L^{p,2}} = \Big(\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3} |f(x,v)|^p dx\Big)^{2/p} dv\Big)^{1/2}.$$

The rest of the paper is organized as follows. In Section 2, we introduce some properties of the linear collision operator. In Section 3, we present the results on spectral analysis of the operator $\widehat{B}_0(\xi)$ for any $\xi \in \mathbb{R}^3$ and some properties of the resolvent $(\lambda - \widehat{B}_0(\xi))^{-1}$. In Section 4, we give the spectral analysis of the operator $\widehat{B}(\xi)$ for any $\xi \in \mathbb{R}^3$ and the time decay estimates of the semigroup e^{tB}.

2. Preliminaries

In this section, we introduce a new decomposition of the linearized Boltzmann collision operator, then give some properties of the collision operator and some lemmas, which will be used later.

The linearized collision operator L defined by (1.6) satisfies

$$(Lf)(v) = -\nu(v)f(v) + (Kf)(v),$$

where

$$\nu(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(|u-v|,\omega)M(u)\,du\,d\omega, \tag{2.1}$$

$$\begin{aligned}(Kf)(v) &= \int_{\mathbb{R}^3} k(u,v)f(u)du\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(|u-v|,\omega)M^{1/2}(u)\\ &\quad\times\Big(M^{1/2}(u')f(v') + M^{1/2}(v')f(u') - M^{1/2}(v)f(u)\Big)\,du\,d\omega.\end{aligned} \tag{2.2}$$

We will describe some properties of the operator L. For more details, we refer to [5]. The null space N_0 of the operator L is a subspace spanned by the orthonormal basis $\{M_j,\ j=0,1,2,3,4\}$ with

$$M_0 = M^{1/2}, \quad M_j = v_jM^{1/2}\ (j=1,2,3), \quad M_4 = \frac{(|v|^2-3)}{\sqrt{6}}M^{1/2}, \tag{2.3}$$

where M is defined by (1.4). The operator $-L$ is nonnegative and self-adjoint on $L^2(\mathbb{R}^3_v)$, and satisfies

$$\langle -Lf,f\rangle \geq \delta\|(I-P)f\|^2_{L^2_1} \tag{2.4}$$

for some constant $\delta>0$, where the projection operator P is defined in $L^2(\mathbb{R}^3_v)$ as

$$Pf = \sum_{j=0}^{4}\langle f,M_j\rangle M_j. \tag{2.5}$$

$\nu(v)$ is called the collision frequency, and satisfies

$$C_1(1+|v|)^\gamma \leq \nu(v) \leq C_2(1+|v|)^\gamma \tag{2.6}$$

for $\gamma\in(-3,0)$ and some constants $C_1,\ C_2>0$.

We use a crucial decomposition of the operator K introduced by [5]. For convenience to the readers, we write it here. For any $\epsilon>0$, define a smooth cut-off function $\chi_\epsilon(r)$ satisfying

$$\chi_\epsilon(r) = 1 \text{ for } r\geq 2\epsilon, \quad \chi_\epsilon(r)=0 \text{ for } r\leq\epsilon. \tag{2.7}$$

The operator K is decomposed as follows

$$\begin{gathered}K = K_c + K_s, \quad K_c = K_{2c} - K_{1c},\\ K_s = K_{2s} - K_{1s} = (K_{2s}^{1-\chi} + K_{2s}^{\chi}) - (K_{1s}^{1-\chi} + K_{1s}^{\chi}),\end{gathered} \tag{2.8}$$

where

$$K_{1c}f = \int_{|u|+|v|\leq m}\int_{\mathbb{S}^2} |u-v|^\gamma\chi_\epsilon(|u-v|)q(\theta)M^{1/2}(u)M^{1/2}(v)f(u)\,du\,d\omega,$$

$$K_{1s}^{1-\chi}f = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |u-v|^\gamma\{1-\chi_\epsilon(|u-v|)\}q(\theta)M^{1/2}(u)M^{1/2}(v)f(u)\,du\,d\omega,$$

$$
\begin{aligned}
K_{1s}^{\chi} f &= \int_{|u|+|v|\geq m}\int_{\mathbb{S}^2} |u-v|^{\gamma}\chi_\epsilon(|u-v|)q(\theta)M^{1/2}(u)M^{1/2}(v)f(u)\,du\,d\omega,\\
K_{2s}^{1-\chi} f &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |u-v|^{\gamma}\{1-\chi_\epsilon(|u-v|)\}q(\theta)M^{1/2}(u)\\
&\quad\times \Big(M^{1/2}(u')f(v')+M^{1/2}(v')f(u')\Big)\,du\,d\omega,\\
K_{2c} f &= 4\int_{|v|+|v+u_\parallel|\leq m}\frac{1}{|u_\parallel|}e^{-\frac14|u_\parallel|^2-|\zeta_\parallel|^2}f(v+u_\parallel)k(u_\parallel,\zeta_\perp)du_\parallel,\\
K_{2s}^{\chi} f &= 4\int_{|v|+|v+u_\parallel|\geq m}\frac{1}{|u_\parallel|}e^{-\frac14|u_\parallel|^2-|\zeta_\parallel|^2}f(v+u_\parallel)k(u_\parallel,\zeta_\perp)du_\parallel
\end{aligned}
$$

with

$$
k(u_\parallel,\zeta_\perp)=\int_{\mathbb{R}^2} e^{-|u_\perp+\zeta_\perp|^2}[|u_\parallel|^2+|u_\perp|^2]^{\frac{\gamma-1}{2}}\chi\Big(\sqrt{|u_\parallel|^2+|u_\perp|^2}\Big)\frac{q(\theta)}{|\cos\theta|}du_\perp,
$$

and the integration variables

$$
u_\parallel=(u\cdot\omega)\omega,\quad u_\perp=u-(u\cdot\omega)\omega, \tag{2.9}
$$

$$
\zeta_\parallel+\zeta_\perp=\frac12(2v+u_\parallel),\quad \zeta_\parallel\|u_\parallel,\quad \zeta_\perp\|u_\perp. \tag{2.10}
$$

We list some properties of the operators K and P, which will be used later. For the simplicity of expression, for any $\theta\in\mathbb{R}$, we write L^2_θ for $L^2_\theta(\mathbb{R}^3_v)$.

Lemma 2.1 ([5]). *For $\theta\in\mathbb{R}$, it holds that*

$$
|\langle \nu^\theta K f, g\rangle|\leq C\|\nu^{\theta/2}f\|_{L^2_1}\|\nu^{\theta/2}g\|_{L^2_1}, \tag{2.11}
$$

where $\nu(v)$ is given by (2.1).

Lemma 2.2 ([5]). *It holds that*

$$
|\langle \nu^\theta K_s f, g\rangle|\leq \eta\|\nu^{\theta/2}f\|_{L^2_1}\|\nu^{\theta/2}g\|_{L^2_1} \tag{2.12}
$$

for any $\theta\in\mathbb{R}$ and $\eta>0$.

Lemma 2.3. *For P defined by* (2.5)*, we have*

$$
P\in C(L^2_{\theta_1},L^2_{\theta_2}) \tag{2.13}
$$

for any $\theta_1,\theta_2\in\mathbb{R}$.

A proof of the above lemma can be found in [12, Lemma 4.3], we omit it here.

Lemma 2.4. *Let $\gamma\in(-3,0)$. We have*

(i) *For any $\theta\in\mathbb{R}$, it holds that*

$$
K\in B(L^2_\theta,L^2_{\theta-2}). \tag{2.14}
$$

(ii) *For any $\theta\in\mathbb{R}$ and $\eta>0$, it holds that*

$$
\|K_s\|_{B(L^2_\theta,L^2_{\theta-2})}\leq\eta. \tag{2.15}
$$

(iii) *For any $\theta_1,\theta_2\in\mathbb{R}$, it holds that*

$$
K_c\in C(L^2_{\theta_1},L^2_{\theta_2}). \tag{2.16}
$$

Proof. (i) From (2.11), for any $\theta \in \mathbb{R}$, we have

$$|\langle \nu^{-1/2}\nu^{\theta/2}(v)Kf, \nu^{1/2}\nu^{\theta/2}(v)g\rangle| \le C\|\nu^{1/2}\nu^{\theta/2}(v)f\|_{L^2}\|\nu^{1/2}\nu^{\theta/2}(v)g\|_{L^2},$$

which implies that

$$\|Kf\|_{L^2_{\theta-1}} \le C\|f\|_{L^2_{\theta+1}}.$$

Thus, we can get (2.14).

(ii) By (2.12), for any $\theta \in \mathbb{R}$ and $\eta > 0$, we have

$$|\langle \nu^{-1/2}\nu^{\theta/2}(v)K_s f, \nu^{1/2}\nu^{\theta/2}(v)g\rangle| \le \eta\|\nu^{1/2}\nu^{\theta/2}(v)f\|_{L^2}\|\nu^{1/2}\nu^{\theta/2}(v)g\|_{L^2},$$

which implies that

$$\|K_s f\|_{L^2_{\theta-1}} \le \eta\|f\|_{L^2_{\theta+1}}.$$

Thus, we can obtain (2.15).

(iii) Since $\frac{1}{|u_\parallel|} \in L^2_{loc}(\mathbb{R}^3)$, the kernel $k(u_\parallel, \zeta_\perp)$ is bounded for the chosen $\epsilon > 0$ and any given $m > 0$. The Hilbert-Schmidt theorem clearly shows that K_c is a compact operator from $L^2_{\theta_1}$ to $L^2_{\theta_2}$ for any θ_1, $\theta_2 \in \mathbb{R}$. Thus, we have proved (2.16). The proof of Lemma 2.4 is complete. □

3. Spectrum and resolvent

To analyze the spectrum of the operator $\widehat{B}(\xi)$ on L^2_θ for any $\xi \ne 0$ and $\theta \in \mathbb{R}$, we introduce some auxiliary operators as follows

$$\widehat{A}_0(\xi) = -iv\cdot\xi, \tag{3.1}$$

$$\widehat{A}(\xi) = -iv\cdot\xi - \nu(v), \tag{3.2}$$

$$\widehat{A}_s(\xi) = -iv\cdot\xi - \nu(v) + K_s, \tag{3.3}$$

$$\widehat{B}_0(\xi) = \widehat{B}(\xi) - P = \widehat{A}_s(\xi) + K_0 \tag{3.4}$$

with

$$K_0 = K_c - P, \tag{3.5}$$

where P, K_s, and K_c are defined by (2.5) and (2.8). Let

$$D(T(\xi)) = \{f \in L^2_\theta : v\cdot\xi f(v) \in L^2_\theta\}, \tag{3.6}$$

where $T(\xi) = \widehat{A}_0(\xi)$, $\widehat{A}(\xi)$, $\widehat{A}_s(\xi)$, $\widehat{B}_0(\xi)$ or $\widehat{B}(\xi)$ for any $\theta \in \mathbb{R}$ and $\xi \in \mathbb{R}^3$. It is obvious that

$$D(\widehat{B}(\xi)) = D(\widehat{B}_0(\xi)) = D(\widehat{A}_s(\xi)) = D(\widehat{A}(\xi)) = D(\widehat{A}_0(\xi)).$$

Lemma 3.1. *The operator $\widehat{A}(\xi)$ generates a strongly continuous contraction semigroup on L^2_θ for any $\theta \in \mathbb{R}$ and $\xi \in \mathbb{R}^3$.*

Proof. It holds for $f \in D(\widehat{A}(\xi))$ that

$$\mathrm{Re}\langle \nu^\theta \widehat{A}(\xi)f, f\rangle = \mathrm{Re}\langle \nu^\theta(-iv\cdot\xi - \nu)f, f\rangle = \langle \nu^\theta(-\nu)f, f\rangle \le 0,$$
$$\mathrm{Re}\langle \nu^\theta \widehat{A}^*(\xi)f, f\rangle = \mathrm{Re}\langle \nu^\theta(iv\cdot\xi - \nu)f, f\rangle = \langle \nu^\theta(-\nu)f, f\rangle \le 0,$$

which implies that the operators $\widehat{A}(\xi)$ and $\widehat{A}^*(\xi)$ are dissipative on L^2_θ. Since $D(\widehat{A}^*(\xi))$ and $D(\widehat{A}(\xi))$ are dense in L^2_θ, then $\widehat{A}(\xi)$ is a densely defined closed operator on L^2_θ by [9, Theorem VIII.1]. Thus, with the help of Corollary 4.4 on p.15 of [8], we obtain that the operator $\widehat{A}(\xi)$ generates a strongly continuous contraction semigroup on L^2_θ. The proof is complete. □

Let

$$\Sigma = \Sigma_{(\lambda,\xi)} = \mathbb{C}_+ \times \mathbb{R}^3. \tag{3.7}$$

Based on Lemma 3.1, we can obtain the following properties for the resolvent $(\lambda I - \widehat{A}(\xi))^{-1}$.

Lemma 3.2. *Let $\gamma \in (-3, 0)$. For any $\theta \in \mathbb{R}$, the following statements hold.*

(i) $(\lambda I - \widehat{A}(\xi))^{-1} \in L^\infty(\overline{\Sigma}, B(L^2_{\theta-2}, L^2_\theta))$.

(ii) $(\lambda I - \widehat{A}(\xi))^{-1} \in C^0(\overline{\Sigma}, B(L^2_{\theta-2-\zeta}, L^2_\theta))$ *for any* $\zeta > 0$.

(iii) *For any fixed $r > 0$ and $f \in L^2_{\theta-2}$, it holds that*

$$\sup_{\lambda \in \overline{\mathbb{C}_+}, |\lambda| \geq a, |\xi| \leq r} \|(\lambda I - \widehat{A}(\xi))^{-1} f\|_{L^2_\theta} \to 0, \quad \text{as } a \to \infty.$$

(iv) *Write $\lambda = \sigma + i\tau$ with $\sigma, \tau \in \mathbb{R}$ and let $f \in L^2_{\theta-1}$. Then*

$$\sup_{\sigma \geq 0, \xi \in \mathbb{R}^3} \int_{-\infty}^{+\infty} \|((\sigma + i\tau)I - \widehat{A}(\xi))^{-1} f\|^2_{L^2_\theta} d\tau \leq C \|f\|^2_{L^2_{\theta-1}}.$$

Here Σ is defined by (3.7).

A proof of the above lemma can be found in [12, Lemma 5.1], we omit it here.

Remark 3.3. Since $L^2_{\theta_1}$ is dense in $L^2_{\theta_2}$ for $\theta_1, \theta_2 \in \mathbb{R}$ and $\theta_1 < \theta_2$, by the aid of (i) and (ii) in Lemma 3.2, it holds that $(\lambda I - \widehat{A}(\xi))^{-1} f \in C^0(\overline{\Sigma}, L^2_\theta)$ for any $f \in L^2_{\theta-2}$.

Thanks to (2.15), we can analyze the resolvent set of the operator $\widehat{A}_s(\xi)$ on L^2_θ for any $\xi \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$.

Lemma 3.4. *Let $\gamma \in (-3, 0)$. We have*

$$\varrho(\widehat{A}_s(\xi)) \supset \mathbb{C}_+, \quad \sigma(\widehat{A}_s(\xi)) \subset \overline{\mathbb{C}}_-. \tag{3.8}$$

Proof. For $\lambda \in \varrho(\widehat{A}(\xi))$, we decompose $\lambda I - \widehat{A}_s(\xi)$ as follows

$$(\lambda I - \widehat{A}_s(\xi)) = (\lambda I - \widehat{A}(\xi))(I - (\lambda I - \widehat{A}(\xi))^{-1} K_s). \tag{3.9}$$

Combining (i) in Lemma 3.2 and (2.15), and choosing sufficiently small η, we are able to show that

$$\begin{aligned} &\|(\lambda I - \widehat{A}(\xi))^{-1} K_s\|_{B(L^2_\theta)} \\ &\leq \sup_{(\lambda,\xi) \in \overline{\mathbb{C}_+} \times \mathbb{R}^3} \|(\lambda I - \widehat{A}(\xi))^{-1}\|_{B(L^2_{\theta-2}, L^2_\theta)} \cdot \|K_s\|_{B(L^2_\theta, L^2_{\theta-2})} \leq \frac{1}{2}, \end{aligned}$$

which implies that

$$\|(I - (\lambda I - \widehat{A}(\xi))^{-1} K_s)^{-1}\|_{B(L^2_\theta)} \leq 2. \tag{3.10}$$

According to Lemma 3.1, (3.9) and the Hille-Yosida theorem, we have

$$\mathbb{C}_+ \subset \varrho(\widehat{A}(\xi)) \subset \varrho(\widehat{A}_s(\xi)).$$

The proof is complete. $\square$

For any $\lambda \in \varrho(\widehat{A}_s(\xi)) \cap \varrho(\widehat{A}(\xi))$, we have

$$(\lambda I - \widehat{A}_s(\xi))^{-1} = (I - (\lambda I - \widehat{A}(\xi))^{-1} K_s)^{-1} (\lambda I - \widehat{A}(\xi))^{-1}. \tag{3.11}$$

From Lemma 3.4, we can obtain the following properties for the resolvent of $(\lambda I - \widehat{A}_s(\xi))^{-1}$.

Lemma 3.5. *Let $\gamma \in (-3,0)$. For any $\theta \in \mathbb{R}$, the following statements hold.*

(i) $(\lambda I - \widehat{A}_s(\xi))^{-1} \in L^\infty(\overline{\Sigma}, B(L^2_{\theta-2}, L^2_\theta))$.

(ii) $(\lambda I - \widehat{A}_s(\xi))^{-1} \in C^0(\overline{\Sigma}, B(L^2_{\theta-2-\zeta}, L^2_\theta))$ *for any* $\zeta > 0$.

(iii) *For any fixed $r > 0$ and $f \in L^2_{\theta-2}$, it holds that*

$$\sup_{\lambda \in \overline{\mathbb{C}_+}, |\lambda| \geq a, |\xi| \leq r} \|(\lambda I - \widehat{A}_s(\xi))^{-1} f\|_{L^2_\theta} \to 0, \text{ as } a \to \infty.$$

(iv) *Write $\lambda = \sigma + i\tau$ with $\sigma, \tau \in \mathbb{R}$ and let $f \in L^2_{\theta-1}$. Then*

$$\sup_{\sigma \geq 0, \xi \in \mathbb{R}^3} \int_{-\infty}^{+\infty} \|((\sigma + i\tau)I - \widehat{A}_s(\xi))^{-1} f\|^2_{L^2_\theta} d\tau \leq C\|f\|^2_{L^2_{\theta-1}}.$$

Here Σ is defined by (3.7).

Proof. By (3.10), it holds that

$$(I - (\lambda I - \widehat{A}(\xi))^{-1} K_s)^{-1} \in L^\infty(\overline{\Sigma}, B(L^2_\theta)). \tag{3.12}$$

Combining (3.11) and (3.12), we can respectively obtain (i), (iii) and (iv) from (i), (iii) and (iv) in Lemma 3.2. We next prove (ii).

Let $S_1(\lambda,\xi) = (\lambda I - \widehat{A}_s(\xi))^{-1}$. For any $(\lambda_0,\xi_0), (\lambda_1,\xi_1) \in \overline{\Sigma}$, we have

$$\begin{aligned} &\|S_1(\lambda_1,\xi_1)f - S_1(\lambda_0,\xi_0)f\|_{L^2_\theta} \\ &\leq \|S_1(\lambda_1,\xi_1)\chi_m(|v|)f - S_1(\lambda_0,\xi_0)\chi_m(|v|)f\|_{L^2_\theta} \\ &\quad + \|S_1(\lambda_1,\xi_1)\{1-\chi_m(|v|)\}f - S_1(\lambda_0,\xi_0)\{1-\chi_m(|v|)\}f\|_{L^2_\theta} \\ &=: I_1 + I_2, \end{aligned} \tag{3.13}$$

where $\chi_m(|v|)$ is defined by (2.7). For any $\epsilon > 0$ and $\zeta > 0$, it holds that

$$\begin{aligned} I_1 &\leq 2 \sup_{(\lambda,\xi) \in \overline{\mathbb{C}_+} \times \mathbb{R}^3} \|S_1(\lambda,\xi)\chi_m(|v|)f\|_{L^2_\theta} \\ &\leq C(1+m)^{\frac{\zeta\gamma}{2}} \|f\|_{L^2_{\theta-2-\zeta}} < \epsilon, \end{aligned} \tag{3.14}$$

where $m > 0$ is chosen large enough. For any $\epsilon > 0$, assuming $|\lambda_1 - \lambda_0| < \epsilon$ and $|\xi_1 - \xi_0| < \epsilon$, we have

$$\begin{aligned} I_2 &= \|(\lambda_1 I + iv\cdot\xi_1 + \nu(v) - K_s)^{-1}(\lambda_0 - \lambda_1 + iv\cdot\xi_0 - iv\cdot\xi_1) \\ &\quad \times (\lambda_0 I + iv\cdot\xi_0 + \nu(v) - K_s)^{-1}\{1-\chi_m(|v|)\}f\|_{L^2_\theta} \\ &\leq C\|(\lambda_1 I + iv\cdot\xi_1 + \nu(v) - K_s)^{-1}\{1-\chi_m(|v|)\}\|_{B(L^2_\theta)}(|\lambda_0 - \lambda_1| \\ &\quad + m|\xi_0 - \xi_1|)\|(\lambda_0 I + iv\cdot\xi_0 + \nu(v) - K_s)^{-1}\{1-\chi_m(|v|)\}f\|_{L^2_\theta} \\ &\leq C\epsilon\|f\|_{L^2_\theta}, \end{aligned} \tag{3.15}$$

which together with (3.13) and (3.14) yields (ii). The proof is complete. □

For $\widehat{B}_0(\xi)$, we have the following similar result to [12, Lemma 5.2].

Lemma 3.6. *Let $\gamma \in (-3,0)$. For any $\xi \in \mathbb{R}^3$, $\widehat{B}_0(\xi)$ generates a strongly continuous contraction semigroup on L^2.*

Proof. Since $D(\widehat{B}_0^*(\xi)) = D(\widehat{B}_0(\xi))$ is dense in L^2, it holds that $\widehat{B}_0(\xi)$ is a densely defined closed operator on L^2 by [9, Theorem VIII.1]. Thanks to (2.4), for any $f \in D(\widehat{B}_0(\xi))$, it holds that

$$\begin{aligned}\mathrm{Re}\langle \widehat{B}_0(\xi)f, f\rangle &= \mathrm{Re}\langle \widehat{B}_0^*(\xi)f, f\rangle = \langle (L-P)f, f\rangle \\ &= -(\langle -Lf, f\rangle + \langle Pf, f\rangle) \\ &\le -(\delta\|(I-P)f\|_{L^2_1}^2 + \|Pf\|_{L^2}^2) < 0,\end{aligned} \tag{3.16}$$

which implies that $\widehat{B}_0(\xi)$ and $\widehat{B}_0^*(\xi)$ are dissipative operators on L^2. Thus, with the help of [8, Corollary 4.4 on p.15], the operator $\widehat{B}_0(\xi)$ generates a strongly continuous contraction semigroup on L^2. The proof is complete. □

Based on Lemma 3.6, we can analyze the spectrum of the operator $\widehat{B}_0(\xi)$ in L^2_θ for any $\xi \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$.

Proposition 3.7. *Let $\gamma \in (-3, 0)$. We have the following results.*

- (i) $\sigma(\widehat{B}_0(\xi)) \subset \overline{\mathbb{C}}_-$, $\varrho(\widehat{B}_0(\xi)) \supset \mathbb{C}_+$.
- (ii) $\sigma_e(\widehat{B}_0(\xi)) = \sigma_e(\widehat{A}_s(\xi))$.
- (iii) $\sigma_p(\widehat{B}_0(\xi)) \subset \mathbb{C}_-$.

Proof. According to Lemma 2.3 and (2.16), we know that the operator $K_0 : L^2_\theta \to L^2_\theta$ is compact. By [6, Theorem 5.35 on p.244], we have $\sigma_e(\widehat{B}_0(\xi)) = \sigma_e(\widehat{A}_s(\xi))$. Thus, we have proved (ii). By Lemma 3.4, we have $\sigma_e(\widehat{A}_s(\xi)) \subset \sigma(\widehat{A}_s(\xi)) \subset \overline{\mathbb{C}}_-$. Combining this and (ii), (iii), we can gain (i). We next prove (iii). Let $\lambda \in \sigma_p(\widehat{B}_0(\xi))$, there exists $f \in D(\widehat{B}_0(\xi))$ and $f \neq 0$, we have

$$\lambda f = \widehat{B}_0(\xi) f. \tag{3.17}$$

For $\theta \le 0$, then $f \in L^2$, we can apply (3.16) to derive $\mathrm{Re}\,\lambda < 0$. For $\theta > 0$, assume $\mathrm{Re}\,\lambda \ge 0$. According to Lemma 2.3, (2.16) and (3.5), K_0 is bounded from L^2_θ to L^2_{-2}, which together with Lemma 3.5 leads to

$$\lambda f = \widehat{B}_0(\xi) f \Rightarrow f = (\lambda I - \widehat{A}_s(\xi))^{-1} K_0 f \in L^2.$$

Then, by (3.16), we have $\lambda \in \mathbb{C}_-$, which is a contraction to the assumption. Thus, we have proved (iii). The proof is complete. □

For any $\lambda \in \varrho(\widehat{B}_0(\xi)) \cap \varrho(\widehat{A}_s(\xi))$, we have

$$(\lambda I - \widehat{B}_0(\xi))^{-1} = (I - (\lambda I - \widehat{A}_s(\xi))^{-1} K_0)^{-1} (\lambda I - \widehat{A}_s(\xi))^{-1}. \tag{3.18}$$

Let

$$M(\lambda, \xi) = (\lambda I - \widehat{A}_s(\xi))^{-1} K_0, \tag{3.19}$$

where $\widehat{A}_s(\xi)$ and K_0 is defined by (3.3) and (3.5) respectively. We state some properties of $M(\lambda, \xi)$ below.

Lemma 3.8. *Let $\gamma \in (-3, 0)$. For any $\theta \in \mathbb{R}$, the following statements hold.*

- (i) $M(\lambda, \xi) \in L^\infty(\overline{\Sigma}, C(L^2_\theta))$.
- (ii) $M(\lambda, \xi) \in C^0(\overline{\Sigma}, C(L^2_\theta))$.
- (iii) *For any $r > 0$, it holds that*

$$\sup_{\lambda \in \overline{\mathbb{C}_+}, |\lambda| \ge a, |\xi| \le r} \|M(\lambda, \xi)\|_{B(L^2_\theta)} \to 0, \quad \text{as } a \to \infty.$$

(iv) *It holds that*
$$\sup_{\lambda\in\overline{\mathbb{C}_+},|\xi|\geq r}\|M(\lambda,\xi)\|_{B(L^2_\theta)}\to 0,\quad \text{as } r\to\infty.$$
Here Σ *is defined by* (3.7).

Proof. (i) According to Lemma 2.3, (2.16) and (3.5), we have
$$K_0\in C(L^2_\theta,L^2_{\theta-2}). \tag{3.20}$$
Combining (i) in Lemma 3.5 and (3.20), we can obtain (i).

(ii) By Lemma 2.3, (2.16) and (3.5), for any $\zeta>0$, it holds that
$$K_0\in C(L^2_\theta,L^2_{\theta-2-\zeta}), \tag{3.21}$$
which together with (ii) in Lemma 3.5 and (3.21) leads to (ii).

(iii) For any $f\in L^2_\theta$, it holds that
$$\|M(\lambda,\xi)\|_{B(L^2_\theta)}=\sup_{\|f\|_{L^2_\theta}=1}\|M(\lambda,\xi)f\|_{L^2_\theta}. \tag{3.22}$$
Combining (iii) in Lemma 3.5, (3.20) and (3.22), we can get (iii).

(iv) For any $f\in L^2_\theta$, we have
$$\begin{aligned}\|M(\lambda,\xi)f\|_{L^2_\theta}&\leq\|(\lambda I-\widehat{A}_s(\xi))^{-1}\chi_m(|v|)K_0f\|_{L^2_\theta}\\&\quad+\|(\lambda I-\widehat{A}_s(\xi))^{-1}\{1-\chi_m(|v|)\}K_0f\|_{L^2_\theta}\\&=:J_1+J_2,\end{aligned}$$
where $\chi_m(|v|)$ is defined by (2.7). By (3.21), it holds for any $\epsilon>0$ and $\zeta>0$ that
$$J_1\leq C(1+m)^{\frac{\zeta\delta}{2}}\|(\lambda I-\widehat{A}_s)^{-1}\|_{B(L^2_{\theta-2},L^2_\theta)}\|K_0\|_{B(L^2_\theta,L^2_{\theta-2-\zeta})}\|f\|_{L^2_\theta}\leq\epsilon, \tag{3.23}$$
where $m>0$ is chosen large enough. We next estimate J_2. Write $S_1=\{v\in\mathbb{R}^3: |v|\leq m,\ |\operatorname{Im}\lambda+v\cdot\xi|\leq\frac{|\xi|}{\sqrt{r}}\}$, $S_2=\{v\in\mathbb{R}^3:|v|\leq m\}\backslash S_1$. We use $\frac{\xi}{|\xi|}$, ξ_1, ξ_2 as an orthonormal basis, then
$$v=\langle v,\frac{\xi}{|\xi|}\rangle\frac{\xi}{|\xi|}+\Big(v-\langle v,\frac{\xi}{|\xi|}\rangle\frac{\xi}{|\xi|}\Big)=L\frac{\xi}{|\xi|}+L_1\xi_1+L_2\xi_2.$$
It holds that
$$\operatorname{meas}S_1=\int_{S_1}1dv\leq\int_{-m}^m dL_1\int_{-m}^m dL_2\int_{-\frac{1}{\sqrt{r}}-\frac{\operatorname{Im}\lambda}{|\xi|}}^{\frac{1}{\sqrt{r}}-\frac{\operatorname{Im}\lambda}{|\xi|}}dL\leq\frac{8m^2}{\sqrt{r}}.$$
For any $\lambda\in\overline{\mathbb{C}_+}$ and $|\xi|\geq r$, we have
$$\begin{aligned}&\|(\lambda I-\widehat{A}(\xi))^{-1}\{1-\chi_m(|v|)\}K_0f\|^2_{L^2_\theta}\\&=\int_{S_1}\nu^\theta(v)\frac{1}{|\operatorname{Re}\lambda+\nu(v)|^2+|\operatorname{Im}\lambda+v\cdot\xi|^2}\{1-\chi_m(|v|)\}|K_0f|^2dv\\&\quad+\int_{S_2}\nu^\theta(v)\frac{1}{|\operatorname{Re}\lambda+\nu(v)|^2+|\operatorname{Im}\lambda+v\cdot\xi|^2}\{1-\chi_m(|v|)\}|K_0f|^2dv\\&\leq C\|f\|^2_{L^2_\theta(S_1)}+\frac{1}{r}\|f\|^2_{L^2_\theta}\\&\to 0,\quad \text{as } r\to\infty.\end{aligned} \tag{3.24}$$

Combining (3.24) and (3.10) yields that

$$\begin{aligned} J_2 &= \|(I-(\lambda I-\widehat{A}(\xi))^{-1}K_s)^{-1}(\lambda I-\widehat{A}(\xi))^{-1}\{1-\chi_m(|v|)\}K_0 f\|_{L^2_\theta} \\ &\le \|(I-(\lambda I-\widehat{A}(\xi))^{-1}K_s)^{-1}\|_{B(L^2_\theta)}\cdot\|(\lambda I-\widehat{A}(\xi))^{-1}\{1-\chi_m(|v|)\}K_0 f\|_{L^2_\theta} \\ &\to 0, \quad \text{as } r\to\infty, \end{aligned}$$

which together with (3.23) verifies (iv). The proof is complete. □

Lemma 3.9. *Let $\gamma\in(-3,0)$. For any $\theta\in\mathbb{R}$, we have the following results.*

(i) $1\in\varrho((\lambda I-\widehat{A}_s(\xi))^{-1}K_0)$ *for any* $(\lambda,\xi)\in\overline{\Sigma}$.
(ii) $(I-(\lambda I-\widehat{A}_s(\xi))^{-1}K_0)^{-1}\in C^0(\overline{\Sigma},B(L^2_\theta))$.
(iii) $(I-(\lambda I-\widehat{A}_s(\xi))^{-1}K_0)^{-1}\in L^\infty(\overline{\Sigma},B(L^2_\theta))$.

Here Σ is defined by (3.7).

Proof. From (i) in Lemma 3.8, $(\lambda I-\widehat{A}_s(\xi))^{-1}K_0: L^2_\theta\to L^2_\theta$ is compact. By the aid of the spectral theory of the compact operator, if $1\in\sigma((\lambda I-\widehat{A}_s(\xi))^{-1}K_0)$, then $1\in\sigma_p((\lambda I-\widehat{A}_s(\xi))^{-1}K_0)$. There exists $f\in L^2_\theta$ and $f\ne 0$, it holds that

$$(\lambda I-\widehat{A}_s(\xi))^{-1}K_0 f = f \Rightarrow \widehat{B}_0(\xi)f=\lambda f,$$

which implies that $\lambda\in\sigma_p(\widehat{B}_0(\xi))$. It is a contradiction to (iii) in Proposition 3.7. Thus, we have proved (i). Combining (i), (ii) in Lemma 3.8 and (i) in Lemma 3.9, it holds that $(I-(\lambda I-\widehat{A}_s(\xi))^{-1}K_0)^{-1}\in C^0(\overline{\Sigma},B(L^2_\theta))$. Thus, we obtain (ii). Making use of (iii), (iv) in Lemma 3.8, there exists a constant r_0 which is large enough, it holds for $(\lambda,\xi)\in\overline{\Sigma}$ and $|\lambda|+|\xi|\ge r_0$ that

$$\|(\lambda I-\widehat{A}_s(\xi))^{-1}K_0\|_{B(L^2_\theta)}\le\frac{1}{2}.$$

Then

$$\|(I-(\lambda I-\widehat{A}_s(\xi))^{-1}K_0)^{-1}\|_{B(L^2_\theta)}\le 2. \tag{3.25}$$

In view of (ii) in Lemma 3.9, we know that $(I-(\lambda I-\widehat{A}_s(\xi))^{-1}K_0)^{-1}$ is uniformly bounded for $(\lambda,\xi)\in\overline{\Sigma}$ and $|\lambda|+|\xi|\le r_0$. Combining this and (3.25), we have proved (iii). The proof is complete. □

With the help of (3.18), Lemma 3.5, Lemma 3.8, and Lemma 3.9, we can obtain the following properties of the resolvent $(\lambda I-\widehat{B}_0(\xi))^{-1}$. The proof is omitted here.

Lemma 3.10. *Let $\gamma\in(-3,0)$. For any $\theta\in\mathbb{R}$, the following statements hold.*

(i) $(\lambda I-\widehat{B}_0(\xi))^{-1}\in L^\infty(\overline{\Sigma},B(L^2_{\theta-2},L^2_\theta))$.
(ii) $(\lambda I-\widehat{B}_0(\xi))^{-1}\in C^0(\overline{\Sigma},B(L^2_{\theta-2-\zeta},L^2_\theta))$ *for any* $\zeta>0$.
(iii) *For any fixed $r>0$ and $f\in L^2_{\theta-2}$, it holds that*

$$\sup_{\lambda\in\overline{\mathbb{C}_+},|\lambda|\ge a,|\xi|\le r}\|(\lambda I-\widehat{B}_0(\xi))^{-1}\|_{L^2_\theta}\to 0, \quad \text{as } a\to\infty.$$

(iv) *Write $\lambda=\sigma+i\tau$ with $\sigma,\tau\in\mathbb{R}$ and let $f\in L^2_{\theta-1}$. Then*

$$\sup_{\sigma\ge 0,\xi\in\mathbb{R}^3}\int_{-\infty}^{+\infty}\|((\sigma+i\tau)I-\widehat{B}_0(\xi))^{-1}f\|^2_{L^2_\theta}d\tau\le C\|f\|^2_{L^2_{\theta-1}}.$$

Here Σ is defined by (3.7).

4. Decay estimates of semigroup

In this section, we give the spectrum structure of the operator $\widehat{B}(\xi)$ on L^2_θ for any $\xi \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$, and some properties of the resolvent $(\lambda I - \widehat{B}(\xi))^{-1}$. Finally, we obtain the time decay estimates of the semigroup e^{tB} on the space $H_{l,\theta,2}$.

4.1. **Estimates at high frequency.**

We recall the definition of $\widehat{B}(\xi)$ given by (1.9) and (3.4)

$$\widehat{B}(\xi) = -iv \cdot \xi + L = \widehat{B}_0(\xi) + P. \tag{4.1}$$

We first state the result on the spectrum of the operator $\widehat{B}(\xi)$ on L^2 for any $\xi \in \mathbb{R}^3$.

Lemma 4.1. *Let $\gamma \in (-3, 0)$. For any $\xi \in \mathbb{R}^3$, the following statements hold.*

(i) *$\widehat{B}(\xi)$ generates a strongly continuous contraction semigroup on L^2. Consequently,*

$$\varrho(\widehat{B}(\xi)) \supset \mathbb{C}_+, \quad \sigma(\widehat{B}(\xi)) \subset \overline{\mathbb{C}}_-. \tag{4.2}$$

(ii)

$$\sigma_p(\widehat{B}(\xi)) \cap \{\mathrm{Re}\,\lambda = 0\} = \begin{cases} \emptyset, & \text{if } \xi \neq 0, \\ \{0\}, & \text{if } \xi = 0. \end{cases} \tag{4.3}$$

Proof. Since $D(\widehat{B}^*(\xi)) = D(\widehat{B}(\xi))$ is dense in L^2, it holds that $\widehat{B}(\xi)$ is a densely defined closed operator on L^2 by [9, Theorem VIII.1]. Thanks to (2.4), for any $f \in D(\widehat{B}(\xi))$, we have

$$\mathrm{Re}\langle \widehat{B}(\xi)f, f\rangle = \mathrm{Re}\langle \widehat{B}^*(\xi)f, f\rangle = \langle Lf, f\rangle \leq -(\delta\|(I-P)f\|^2_{L^2_1}) \leq 0, \tag{4.4}$$

which implies that $\widehat{B}(\xi)$ and $\widehat{B}^*(\xi)$ are dissipative operators on L^2. Thus, with the help of Corollary 4.4 on p.15 of [8], the operator $\widehat{B}(\xi)$ generates a strongly continuous contraction semigroup on L^2. Thus, we have proved (i). Let $\lambda \in \sigma_p(\widehat{B}(\xi))$, there exists $f \in L^2$ and $f \neq 0$, it holds that

$$\widehat{B}(\xi)f = \lambda f. \tag{4.5}$$

By (4.4), we have

$$\mathrm{Re}\,\lambda\langle f, f\rangle = \mathrm{Re}\langle \widehat{B}(\xi)f, f\rangle = \langle Lf, f\rangle \leq 0, \tag{4.6}$$

which implies $\mathrm{Re}\,\lambda \leq 0$. If $\mathrm{Re}\,\lambda = 0$, it holds from (4.6) that $\langle Lf, f\rangle = 0$, which implies that $f \in kerL$. Then (4.5) is turned into

$$(\mathrm{Im}\,\lambda + v \cdot \xi)Pf = 0,$$

which is impossible for $f \neq 0$ unless $\mathrm{Im}\,\lambda = 0$ and $\xi = 0$. Thus, we have proved (ii). The proof is complete. □

We have the following results about the spectrum of the operator $\widehat{B}(\xi)$ on L^2_θ for any $\xi \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$.

Proposition 4.2. *Let $\gamma \in (-3, 0)$. The following statements hold.*

(i) *$\sigma(\widehat{B}(\xi)) \subset \overline{\mathbb{C}}_-$, $\varrho(\widehat{B}(\xi)) \supset \mathbb{C}_+$.*
(ii) *$\sigma_e(\widehat{B}(\xi)) = \sigma_e(\widehat{B}_0(\xi))$.*
(iii) *$\sigma_p(\widehat{B}(\xi)) \subset \mathbb{C}_-$ for $\xi \neq 0$.*

Proof. We only give the proof of (iii). The proof of (i) and (ii) can be given using arguments similar to those in (i) and (ii) of Proposition 3.7. For any $\xi \neq 0$, let $\lambda \in \sigma_p(\widehat{B}(\xi))$, there exists $f \in D(\widehat{B}(\xi))$ and $f \neq 0$. It holds that

$$\lambda f = \widehat{B}(\xi) f.$$

For $\theta \leq 0$, then $f \in L^2$, by (ii) in Lemma 4.1, we have $\operatorname{Re} \lambda < 0$ for $\xi \neq 0$. For $\theta > 0$, assume $\operatorname{Re} \lambda \geq 0$. By Lemma 2.3 and applying the boundness of the operator P from L^2_θ to L^2_{-2} and (i) in Lemma 3.10, we have

$$\lambda f = \widehat{B}(\xi) f \Rightarrow f = (\lambda I - \widehat{B}_0(\xi))^{-1} P f \in L^2.$$

By (4.3), $\lambda \in \mathbb{C}_-$ for $\xi \neq 0$, which is a contradiction to the assumption. Thus, we have proved (iii). The proof is complete. □

Remark 4.3. By Lemma 4.1 and applying similar arguments to those in the proof of Proposition 4.2, we have the following result about the spectrum of the operator $\widehat{B}(\xi)$ on L^2_θ for any $\theta \in \mathbb{R}$ and $\xi \in \mathbb{R}^3$,

$$\sigma_p(\widehat{B}(\xi)) \subset \mathbb{C}_- \cup \{0\}.$$

Lemma 4.4. *$\widehat{B}(\xi)$ generates a strongly continuous semigroup on L^2_θ for any $\theta \in \mathbb{R}$ with*

$$\|e^{t\widehat{B}(\xi)}\|_{B(L^2_\theta)} \leq e^{t\|K\|_{B(L^2_\theta)}}. \tag{4.7}$$

Proof. Based on Lemma 3.1, $\widehat{A}(\xi)$ generates a strongly continuous contraction semigroup on L^2_θ for any $\theta \in \mathbb{R}$ and $\xi \in \mathbb{R}^3$, which implies that $\|e^{t\widehat{A}(\xi)}\|_{B(L^2_\theta)} \leq 1$. By (2.14), we have $K \in B(L^2_\theta)$. By the theory of the bounded perturbation of semigroup in [8], we obtain that $\widehat{B}(\xi) = \widehat{A}(\xi) + K$ generates a strongly continuous semigroup on L^2_θ and $e^{t\widehat{B}(\xi)}$ satisfies (4.7). The proof is complete. □

For any $\lambda \in \varrho(\widehat{B}_0(\xi)) \cap \varrho(\widehat{B}(\xi))$, we have

$$(\lambda I - \widehat{B}(\xi))^{-1} = (I - (\lambda I - \widehat{B}_0(\xi))^{-1} P)^{-1} (\lambda I - \widehat{B}_0(\xi))^{-1}. \tag{4.8}$$

We define the set

$$\Sigma_r = \{(\lambda, \xi) \in \mathbb{C}_+ \times \mathbb{R}^3 : |\lambda| + |\xi| \geq r\} \tag{4.9}$$

for any $r > 0$. Let

$$M_1(\lambda, \xi) = (\lambda I - \widehat{B}_0(\xi))^{-1} P. \tag{4.10}$$

Then we obtain a similar results as in Lemma 3.8.

Lemma 4.5. *Let $\gamma \in (-3, 0)$. For any $\theta \in \mathbb{R}$, we have the following results.*

(i) $M_1(\lambda, \xi) \in L^\infty(\overline{\Sigma}, C(L^2_\theta))$.
(ii) $M_1(\lambda, \xi) \in C^0(\overline{\Sigma}, C(L^2_\theta))$.
(iii) *For any $r > 0$, it holds that*

$$\sup_{\lambda \in \overline{\mathbb{C}_+}, |\lambda| \geq a, |\xi| \leq r} \|M_1(\lambda, \xi)\|_{B(L^2_\theta)} \to 0, \quad \text{as } a \to \infty.$$

(iv) *It holds that*

$$\sup_{\lambda \in \overline{\mathbb{C}_+}, |\xi| \geq r} \|M_1(\lambda, \xi)\|_{B(L^2_\theta)} \to 0, \quad \text{as } r \to \infty.$$

Here Σ is defined by (3.7).

Proof. It holds that

$$M_1(\lambda,\xi) = (I - (\lambda I - \widehat{A}_s(\xi))^{-1}K_0)^{-1}(\lambda I - \widehat{A}_s(\xi))^{-1}P$$

for any $\lambda \in \varrho(\widehat{A}_s(\xi))$. Thus, under the help of Lemma 2.3, Lemma 3.8 and Lemma 3.9, we can prove Lemma 4.5. We omit the details here. □

Similar to Lemma 3.9, we have the following results.

Lemma 4.6. *Let $\gamma \in (-3,0)$. For any $\theta \in \mathbb{R}$, the following statements hold.*

(i) $1 \in \varrho((\lambda I - \widehat{B}_0(\xi))^{-1}P)$ *for any* $(\lambda,\xi) \in \overline{\Sigma_r}$.
(ii) $(I - (\lambda I - \widehat{B}_0(\xi))^{-1}P)^{-1} \in C^0(\overline{\Sigma_r}, B(L^2_\theta))$.
(iii) $(I - (\lambda I - \widehat{B}_0(\xi))^{-1}P)^{-1} \in L^\infty(\overline{\Sigma_r}, B(L^2_\theta))$.

Here Σ_r is defined by (4.9).

Proof. The proof is similar to the one of Lemma 3.9. We just sketch it. In terms of the compactness of the operator $(\lambda I - \widehat{B}_0(\xi))^{-1}P$ and the spectrum of the operator $\widehat{B}(\xi)$ stated in Proposition 4.2 and Remark refrem41, we can prove (i). By the aid of (i), (ii) in Lemma 4.5 and (i) in Lemma 4.6, we can obtain the continuity of $(I - (\lambda I - \widehat{B}_0(\xi))^{-1}P)^{-1}$ on $\overline{\Sigma_r}$. Finally, combining this and (iii), (vi) in Lemma 4.5, we obtain the uniformly boundness of $(I - (\lambda I - \widehat{B}_0(\xi))^{-1}P)^{-1}$ on $\overline{\Sigma_r}$. □

According to Lemma 3.10, Lemma 4.5, Lemma 4.6, Proposition 4.2, and (4.8), we can obtain the following properties of the resolvent of $(\lambda I - \widehat{B}(\xi))^{-1}$. The details are omitted here.

Lemma 4.7. *For any $\gamma \in (-3,0)$ and $\theta \in \mathbb{R}$, the following statements hold.*

(i) $(\lambda I - \widehat{B}(\xi))^{-1} \in L^\infty(\overline{\Sigma}_r, B(L^2_{\theta-2}, L^2_\theta))$.
(ii) $(\lambda I - \widehat{B}(\xi))^{-1} \in C^0(\overline{\Sigma}_r, B(L^2_{\theta-2-\zeta}, L^2_\theta))$ *for any* $\zeta > 0$.
(iii) *For any fixed $r > 0$ and $f \in L^2_{\theta-2}$, it holds that*

$$\sup_{\lambda \in \overline{\mathbb{C}_+}, |\lambda| \geq a, |\xi| \leq r} \|(\lambda I - \widehat{B}(\xi))^{-1}f\|_{L^2_\theta} \to 0, \quad \text{as } a \to \infty.$$

(iv) *Write $\lambda = \sigma + i\tau$ with $\sigma,\tau \in \mathbb{R}$ and let $f \in L^2_{\theta-1}$. It holds for any $r > 0$ that*

$$\sup_{\sigma \geq 0, |\xi| \geq r} \int_{-\infty}^{+\infty} \|((\sigma + i\tau)I - \widehat{B}(\xi))^{-1}f\|^2_{L^2_\theta} d\tau \leq C\|f\|^2_{L^2_{\theta-1}}$$

with a constant $C > 0$ depending on r.

Here Σ_r is defined by (4.9).

With the help of Lemma 4.7, we can evaluate the time decay estimates of the semigroup $e^{t\widehat{B}(\xi)}$ for any $|\xi| \geq r$ and $r > 0$.

Theorem 4.8. *Let $\gamma \in (-3,0)$. For any $|\xi| \geq r$, $r > 0$, $\theta \in \mathbb{R}$ and $n \geq 1$, it holds that*

$$\|e^{t\widehat{B}(\xi)}\|_{B(L^2_{\theta-2n}, L^2_\theta)} \leq C(1+t)^{-n} \tag{4.11}$$

for any $t \geq 0$.

Proof. Denote the semigroup $e^{t\widehat{B}(\xi)}$ by the inverse Laplace transform of the resolvent $(\lambda I - \widehat{B}(\xi))^{-1}$ as follows

$$e^{t\widehat{B}(\xi)}f_0 = \lim_{a\to\infty}\frac{1}{2\pi i}\int_{\sigma-ia}^{\sigma+ia} e^{\lambda t}(\lambda I - \widehat{B}(\xi))^{-1}f_0 d\lambda \tag{4.12}$$

for any $f_0 \in D(\widehat{B}(\xi))$, where $\sigma > 0$ can be chosen arbitrarily.

Let $S_2(\lambda,\xi) = (\lambda I - \widehat{B}(\xi))^{-1}$ and $\lambda = s + i\tau$. According to Proposition 4.2 and Lemma 4.7, we can use the Cauchy's theorem in (4.12) to shift the path of the integration from $s = \sigma$ to $s = 0$ and obtain

$$\begin{aligned} &e^{t\widehat{B}(\xi)}f_0 \\ &= \lim_{a\to\infty}\frac{1}{2\pi i}\int_{-ia}^{+ia} e^{\lambda t}(\lambda I - \widehat{B}(\xi))^{-1}f_0 d\lambda \\ &\quad + \lim_{a\to\infty}\frac{1}{2\pi i}\Big(\int_{\sigma}^{0} e^{(s-ia)t}S_2(s-ia,\xi)f_0 ds + \int_0^{\sigma} e^{(s+ia)t}S_2(s+ia,\xi)f_0 ds\Big). \end{aligned} \tag{4.13}$$

From (iii) in Lemma 4.7, for any $f_0 \in L^2_{\theta-2}$, we have

$$\|S_2(s \mp ia,\xi)f_0\|_{L^2_\theta} \to 0, \quad \text{as } a \to \infty. \tag{4.14}$$

Thus, the last two terms on the right-hand side of (4.13) vanish, and (4.13) is reduced to

$$e^{t\widehat{B}(\xi)}f_0 = \lim_{a\to\infty}\frac{1}{2\pi}\int_{-a}^{a} e^{i\tau t}(i\tau I - \widehat{B}(\xi))^{-1}f_0 d\tau, \tag{4.15}$$

where we make the variable substitution $\lambda = i\tau$. Applying the integration by parts on the right-hand side of (4.15) yields

$$\begin{aligned} e^{t\widehat{B}(\xi)}f_0 &= \lim_{a\to\infty}\frac{1}{2\pi}\int_{-a}^{a} e^{i\tau t}(i\tau I - \widehat{B}(\xi))^{-1}f_0 d\tau \\ &= \lim_{a\to\infty}\Big(\frac{1}{2\pi}\sum_{k=1}^{n} e^{i\tau t}\frac{(k-1)!}{it^k}(i\tau I - \widehat{B}(\xi))^{-k}f_0\Big)\Big|_{\tau=-a}^{\tau=a} \\ &\quad + \lim_{a\to\infty}\frac{1}{2\pi}\frac{n!}{t^n}\int_{-a}^{a} e^{i\tau t}(i\tau I - \widehat{B}(\xi))^{-(n+1)}f_0 d\tau \end{aligned} \tag{4.16}$$

for any $f_0 \in L^2_{\theta-2(n+1)}$, where we have used

$$\frac{d^l}{ds^l}S_2(s+i\tau,\xi)f_0 = \frac{1}{i^l}\frac{d^l}{d\tau^l}S_2(s+i\tau,\xi)f_0 = (-1)^l l! S_2(\lambda,\xi)^{l+1}f_0,$$

which is valid at $s = 0$ for any $f_0 \in L^2_{\theta-2(l+1)}$ from (i) in Lemma 4.7. Owing to (iii) in Lemma 4.7, the first term on the right-hand side of (4.16) tends to 0 as $a \to \infty$, (4.16) is reduced to

$$e^{t\widehat{B}(\xi)}f_0 = \lim_{a\to\infty}\frac{1}{2\pi}\frac{n!}{t^n}\int_{-a}^{a} e^{i\tau t}(i\tau I - \widehat{B}(\xi))^{-(n+1)}f_0 d\tau. \tag{4.17}$$

For any $f_0 \in D(\widehat{B}(\xi)) \cap L^2_{\theta-2(n+1)}$ and $g \in L^2_\theta$, by (4.17) and (iv) in Lemma 4.7, it holds that

$$|\langle \nu^\theta e^{t\widehat{B}(\xi)}f_0, g\rangle| = \Big|\lim_{a\to\infty}\int_{\mathbb{R}^3}\nu^\theta\frac{n!}{2\pi t^n}g\int_{-a}^{a} e^{i\tau t}(i\tau I - \widehat{B}(\xi))^{-(n+1)}f_0 d\tau dv\Big|$$

$$
\begin{aligned}
&\le \frac{C}{t^n}\int_{-\infty}^{\infty} |\langle \nu^\theta (i\tau I-\widehat{B}(\xi))^{-(n+1)}f_0, g\rangle| d\tau\\
&\le \frac{C}{t^n}\int_{-\infty}^{\infty} |\langle \nu^\theta (i\tau I-\widehat{B}(\xi))^{-n}f_0, (-i\tau I-\widehat{B}(-\xi))^{-1}g\rangle| d\tau\\
&\le \frac{C}{t^n}\int_{-\infty}^{\infty} \|(i\tau I-\widehat{B}(\xi))^{-n}f_0\|_{L^2_{\theta-1}} \|(-i\tau I-\widehat{B}(-\xi))^{-1}g)\|_{L^2_{\theta+1}} d\tau\\
&\le \frac{C}{t^n}\Big(\int_{-\infty}^{\infty} \|(i\tau I-\widehat{B}(\xi))^{-1}f_0\|^2_{L^2_{\theta-2n+1}} d\tau\Big)^{1/2}\\
&\quad \times \Big(\int_{-\infty}^{\infty} \|(-i\tau I-\widehat{B}(-\xi))^{-1}g)\|^2_{L^2_{\theta+1}} d\tau\Big)^{1/2}\\
&\le \frac{C}{t^n}\|f_0\|_{L^2_{\theta-2n}}\|g\|_{L^2_\theta},
\end{aligned}
$$

which implies that

$$
\|e^{t\widehat{B}(\xi)}\|_{B(L^2_{\theta-2n},L^2_\theta)} \le Ct^{-n}. \tag{4.18}
$$

By Lemma 4.4 and (4.18), we can obtain for any $n\in\mathbb{N}^*$ and $t\ge 0$ that

$$
\|e^{t\widehat{B}(\xi)}\|_{B(L^2_{\theta-2n},L^2_\theta)} \le C(1+t)^{-n}. \tag{4.19}
$$

By applying the interpolation theorem, we can obtain (4.19) for any $n\ge 1$. The proof is complete. □

4.2. **Estimates at low frequency.** In this subsection, we analyze the singularities of $(\lambda I-\widehat{B}(\xi))^{-1}$ near $\xi=0$. We decompose $(\lambda I-\widehat{B}(\xi))^{-1}$ as follows

$$
\begin{aligned}
(\lambda I-\widehat{B}(\xi))^{-1} &= (\lambda I-\widehat{B}_0(\xi))^{-1}\\
&\quad + (\lambda I-\widehat{B}_0(\xi))^{-1}(I-P(\lambda I-\widehat{B}_0(\xi))^{-1})^{-1}P(\lambda I-\widehat{B}_0(\xi))^{-1}.
\end{aligned} \tag{4.20}
$$

We will check that

$$
(I-P(\lambda I-\widehat{B}_0(\xi))^{-1})^{-1}Pf = P(I-P(\lambda I-\widehat{B}_0(\xi))^{-1}P)^{-1}Pf. \tag{4.21}
$$

Write

$$
g = (I-P(\lambda I-\widehat{B}_0(\xi))^{-1})^{-1}Pf. \tag{4.22}
$$

By (4.22), it holds that

$$
g = P(\lambda I-\widehat{B}_0(\xi))^{-1}g + Pf \in kerL,
$$

which, from (4.22), implies

$$
Pg = P(\lambda I-\widehat{B}_0(\xi))^{-1}Pg + Pf.
$$

Thus, we obtain

$$
g = Pg = P(I-P(\lambda I-\widehat{B}_0(\xi))^{-1}P)^{-1}Pf. \tag{4.23}
$$

Substituting (4.21) into (4.20), we have

$$
\begin{aligned}
(\lambda I-\widehat{B}(\xi))^{-1} &= (\lambda I-\widehat{B}_0(\xi))^{-1}\\
&\quad + (\lambda I-\widehat{B}_0(\xi))^{-1}P(I-P(\lambda I-\widehat{B}_0(\xi))^{-1}P)^{-1}P(\lambda I-\widehat{B}_0(\xi))^{-1}.
\end{aligned} \tag{4.24}
$$

Combining (i) in Lemma 3.10 and (i) in Lemma 4.5, we obtain that the singularities of the resolvent $(\lambda I-\widehat{B}(\xi))^{-1}$ near $\xi=0$ arise from $(I-P(\lambda I-\widehat{B}_0(\xi))^{-1}P)^{-1}$.

We next analyze the singularities of $(I - P(\lambda I - \widehat{B}_0(\xi))^{-1}P)^{-1}$ near $\xi = 0$. Write $\lambda = \sigma + i\tau$, $|\xi| = \kappa$, let

$$W(\sigma, \tau, \xi) = P(\lambda I - \widehat{B}_0(\xi))^{-1}P. \tag{4.25}$$

By using the C^∞ extension theorem in [10], we can make C^∞ extension of $W(\sigma, \tau, \xi)$ for $\sigma \le 0$, which is still written as $W(\sigma, \tau, \xi)$ for the simplicity. Denote respectively the eigenvalues and the corresponding eigenfunctions of the operator $W(\sigma, \tau, \xi)$ by $\mu_j(\sigma, \tau, |\xi|)$ and $\phi_j(\sigma, \tau, |\xi|)$, and the point spectrum of the operator $W(\sigma, \tau, \xi)$ by $\sigma_p(W(\sigma, \tau, \xi))$. We have the following results on the spectral analysis for the operator $W(\sigma, \tau, \xi)$ near $\xi = 0$.

Proposition 4.9. *There exists a constant $r_0 > 0$ and functions $\mu_j(\sigma, \tau, |\xi|)$, $j = \pm 1, 0, 2, 3$ defined on $\Sigma^0 = \{(\sigma, \tau, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 : |\sigma| + |\tau| \le r_0,\ |\xi| \le r_0\}$, and functions $\sigma_j(\kappa)$, $\tau_j(\kappa)$, $j = \pm 1, 0, 2, 3$ defined on $I_0 = [-r_0, r_0]$, such that*

- (i) (a) $\sigma_p(W(\sigma, \tau, \xi)) = \{\mu_j(\sigma, \tau, |\xi|),\ j = \pm 1, 0, 2, 3\}$, $(\sigma, \tau, \xi) \in \Sigma^0$.
  - (b) $\mu_j \in C^\infty(\Sigma^0)$, $-1 \le j \le 3$.
  - (c) $\mu_j(0, 0, 0) = 1$, $-1 \le j \le 3$.
  - (d) $\frac{\partial \mu_j}{\partial \sigma}(0, 0, 0) = \frac{1}{i}\frac{\partial \mu_j}{\partial \tau}(0, 0, 0) = -1$, $-1 \le j \le 3$.
- (ii) (a) $\sigma_j(\kappa),\ \tau_j(\kappa) \in C^\infty(I_0)$, $-1 \le j \le 3$.
  - (b) $\mu_j(\sigma_j(\kappa), \tau_j(\kappa), \kappa) \equiv 1$, $\kappa \in I_0$ *and* $-1 \le j \le 3$.
  - (c) $\sigma_j(\kappa)$, $\tau_j(\kappa)$ *satisfy the following asymptotic expansions for* $\kappa \in I_0$ *and* $-1 \le j \le 3$,

$$\sigma_j(\kappa) = \sigma_j^{(2)}\kappa^2 + O(\kappa^3), \tag{4.26}$$

$$\tau_j(\kappa) = \tau_j^{(1)}\kappa + O(\kappa^3), \tag{4.27}$$

where the constants $\sigma_j^{(2)} < 0$, and $\tau_j^{(1)}$ with explicit expression as

$$\sigma_j^{(2)} = \begin{cases} \frac{3}{5}\langle L^{-1}P^\perp(v_1\mu_4), P^\perp(v_1\mu_4)\rangle, & \text{if } j = 0, \\ \frac{1}{2}\langle L^{-1}P^\perp(v_1\mu_1), P^\perp(v_1\mu_1)\rangle & \\ +\frac{1}{5}\langle L^{-1}P^\perp(v_1\mu_4), P^\perp(v_1\mu_4)\rangle, & \text{if } j = \pm 1, \\ \langle L^{-1}v_1\mu_j, v_1\mu_j\rangle, & \text{if } j = 2, 3 \end{cases}$$

where $P^\perp = I - P$, P is defined by (2.5), *and*

$$\tau_j^{(1)} = \begin{cases} 0, & \text{if } j = 0, 2, 3, \\ \mp\sqrt{\frac{5}{3}}, & \text{if } j = \pm 1. \end{cases}$$

Moreover, the eigen-projections $P_j(\sigma, \tau, \xi)$, $-1 \le j \le 3$ defined by

$$P_j(\sigma, \tau, \xi)f = \langle f, \phi_j(\sigma, \tau, \kappa)\rangle \phi_j(\sigma, \tau, \kappa) \quad \text{for any } f \in L^2_\theta \tag{4.28}$$

satisfy

- (iii) (a) $P_j \in C^\infty(\Sigma^0, B(L^2_\theta))$, $-1 \le j \le 3$.
  - (b) $\sum_{j=-1}^{3} P_j(0, 0, 0) = P$.

We omit the proof of the above proposition. We mention that the method of the asymptotic analysis for σ_j and $\tau_j, -1 \le j \le 3$ is different from that in [7]. For more details, please refer to [12].

Thanks to Proposition 4.9, for any $(\sigma,\tau,\xi)\in\Sigma^0$ it holds that

$$[I-W(\sigma,\tau,\xi)]^{-1}P=\sum_{j=-1}^{3}\frac{1}{1-\mu_j(\sigma,\tau,\kappa)}P_j(\sigma,\tau,\xi). \tag{4.29}$$

Substituting (4.29) into (4.24), it holds for any $(\sigma,\tau,\xi)\in\Sigma^0$ that

$$(\lambda I-\widehat{B}(\xi))^{-1}=(\lambda I-\widehat{B}_0(\xi))^{-1}+\sum_{j=-1}^{3}(1-\mu_j(\sigma,\tau,\kappa))^{-1}U_j(\sigma,\tau,\xi), \tag{4.30}$$

where $U_j(\sigma,\tau,\xi)=(\lambda I-\widehat{B}_0(\xi))^{-1}P_j(\sigma,\tau,\xi)(\lambda I-\widehat{B}_0(\xi))^{-1}$. Taking the derivation with respect to τ on (4.30), we obtain

$$\begin{aligned}&(\lambda I-\widehat{B}(\xi))^{-(n+1)}\\&=(\lambda I-\widehat{B}_0(\xi))^{-(n+1)}+\sum_{j=-1}^{3}\sum_{m=0}^{n}(1-\mu_j(\sigma,\tau,\kappa))^{-(m+1)}U_{j,m}^{(n)}(\sigma,\tau,\xi)\end{aligned} \tag{4.31}$$

for any $n\in\mathbb{N}^*$, where we have used

$$\begin{aligned}\frac{d^m}{d\sigma^m}((\sigma+i\tau)I-\widehat{B}(\xi))^{-1}&=\frac{d^m}{i^m d\tau^m}((\sigma+i\tau)I-\widehat{B}(\xi))^{-1}\\&=(-1)^m m!((\sigma+i\tau)I-\widehat{B}(\xi))^{-(m+1)}\end{aligned}$$

for any $0\le m\le n$, and $U_{j,m}^{(n)}(\sigma,\tau,\xi)$ are given as the linear combinations of products of μ_j, U_j and their derivatives and satisfy

$$U_{j,m}^{(n)}(\sigma,\tau,\xi)\in C^\infty(\Sigma^0,C(L^2_\theta)) \tag{4.32}$$

for any $\theta\in\mathbb{R}$. In particular, $U_{j,n}^{(n)}=i^{-n}(\frac{\partial\mu_j}{\partial\tau})^nU_j$. By using (i)(d) in Proposition 4.9, it holds that

$$U_{j,n}^{(n)}(0,0,0)=P_j(0,0,0). \tag{4.33}$$

With the help of Proposition 4.9, (4.31), (4.32), (4.33) and by repeating the similar arguments as proving Theorem 7.1 in [12], we have the following asymptotic behavior of $e^{t\widehat{B}(\xi)}$ near $\xi=0$.

Theorem 4.10. *Let $\gamma\in(-3,0)$. Then there exist two constants $r_1>0$ and $\eta_0>0$, such that for any $|\xi|\le r_1$, $\theta\in\mathbb{R}$, $n\ge1$ and $t\ge0$ it holds*

$$\begin{aligned}&\|e^{t\widehat{B}(\xi)}f_0\|_{L^2_\theta}\\&\le C\Big((1+t)^{-n}(\|f_0\|_{L^2_{\theta-2n}}+\rho_{n-\frac12}(\kappa)\|f_0\|_{L^2})+e^{-\eta_0\kappa^2t}\|Pf_0\|_{L^2}\Big),\end{aligned} \tag{4.34}$$

where $\rho_{n-\frac12}(\kappa)=|\kappa|^{-2(n-\frac12)}$ and P is defined by (2.5).

By Theorem 4.8 and Theorem 4.10, we can obtain the time decay estimates on e^{tB} on the space $H_{l,\theta,2}$.

Proof of Theorem 1.1. It holds that

$$
\begin{aligned}
&\|e^{tB}f_0\|_{l,\theta,2}^2 \\
&= \int_{\mathbb{R}^3} (1+|\xi|)^{2l} \|e^{t\widehat{B}(\xi)}\hat{f}_0\|_{L^2_\theta}^2 d\xi \\
&= \int_{|\xi|\geq r_1} (1+|\xi|)^{2l} \|e^{t\widehat{B}(\xi)}\hat{f}_0\|_{L^2_\theta}^2 d\xi + \int_{|\xi|\leq r_1} (1+|\xi|)^{2l} \|e^{t\widehat{B}(\xi)}\hat{f}_0\|_{L^2_\theta}^2 d\xi \\
&=: I_1 + I_2,
\end{aligned} \tag{4.35}
$$

where r_1 is given by Theorem 4.10. Applying Theorem 4.8, we have

$$
I_1 \leq C(1+t)^{-2n} \|f_0\|_{l,\theta-2n,2}^2. \tag{4.36}
$$

Substituting (4.34) in Theorem 4.10 into I_2, we obtain

$$
\begin{aligned}
I_2 &\leq C((1+t)^{-2n}(\|f_0\|_{l,\theta-2n,2}^2 + \int_{|\xi|\leq r_1} \rho_{n-\frac{1}{2}}(|\xi|)^2 \|\hat{f}_0(\xi)\|_{L^2}^2 d\xi) \\
&\quad + \int_{|\xi|\leq r_1} e^{-2\eta_0|\xi|^2 t} \|P\hat{f}_0(\xi)\|_{L^2}^2 d\xi) \\
&\leq C((1+t)^{-2n}(\|f_0\|_{l,\theta-2n,2}^2 + (\int_{|\xi|\leq r_1} \rho_{n-\frac{1}{2}}(|\xi|)^{2q'} d\xi)^{\frac{1}{q'}} \|\hat{f}_0(\xi)\|_{L^{2q,2}}^2) \\
&\quad + (\int_{|\xi|\leq r_1} e^{-2q'\eta_0|\xi|^2 t} d\xi)^{\frac{1}{q'}} \|P\hat{f}_0(\xi)\|_{L^{2q,2}}^2) \\
&\leq C((1+t)^{-2n}(\|f_0\|_{l,\theta-2n,2}^2 + \|f_0\|_{L^{p,2}}^2) + (1+t)^{-3(\frac{1}{p}-\frac{1}{2})} \|Pf_0\|_{L^{p,2}}^2),
\end{aligned} \tag{4.37}
$$

where we have used the Hölder inequality and the Hausdorff-Young inequality with $\frac{1}{q}+\frac{1}{q'}=1$, $\frac{1}{p}+\frac{1}{2q}=1$, $q\geq 1$, $p\in[1,6/5)$, and

$$
\int_{|\xi|\leq r_1} \rho_{n-\frac{1}{2}}(|\xi|)^{2q'} d\xi < \infty,
$$

if $n < \frac{3}{4q'}+\frac{1}{2} = \frac{3}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}$, and

$$
\Big(\int_{|\xi|\leq r_1} e^{-2q'\eta_0|\xi|^2 t} d\xi\Big)^{1/q'} \leq C(1+t)^{-3(\frac{1}{p}-\frac{1}{2})},
$$

and

$$
\|\hat{f}_0\|_{L^{2q,2}}^2 \leq C\|f_0\|_{L^{p,2}}^2, \quad \|P\hat{f}_0\|_{L^{2q,2}}^2 \leq C\|Pf_0\|_{L^{p,2}}^2.
$$

Then combining (4.35), (4.36) and (4.37), we can obtain (1.10). The proof is complete. □

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