Electronic Journal of Differential Equations, Vol. 2021 (2021), No. 49, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXACT BOUNDARY CONTROLLABILITY FOR THE WAVE EQUATION WITH MOVING BOUNDARY DOMAINS IN A STAR-SHAPED HOLE

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Abstract. We consider an exact boundary control problem for the wave equation in a moving bounded domain which has a star-shaped hole. The boundary domain is composed by two disjoint parts, one is the boundary of the hole, which is fixed, and the other one is the external boundary which is moving. The initial data has finite energy and the control obtained is square integrable and is obtained by means of the conormal derivative. We use the method of controllability presented by Russell in [20], and assume that the control acts only in the moving part of the boundary.

1. Introduction

There is a large number of works available for the exact boundary controllability problems of wave equations, because such problems have a great importance both from a practical and theoretical point of view. Since the 1970s, many works have appeared giving seminal contributions to the advancement of this branch of mathematics, among them we have [3, 12, 18] which are canonical references for the subject. Many works have appeared dealing with control problems for wave equations on Euclidean domains with diverse types of geometry, for example, domains with fixed and moving boundary and domains with perforated interior; see [2, 4, 6, 7, 11, 13, 17].

In this work we study an exact boundary control problem for the standard wave equation on a domain with moving boundary which has a single fixed hole. The boundary of such domains is composed by two disjoint parts: one it is the boundary on hole which is fixed, and the other one is the external boundary which is moving. We shall consider the control acting only on the moving boundary part. In practical situations, many processes involve domains with a geometry as described above. For example, a flexible body that is crossed by a cylindrical pillar and is fixed to it. Without any variation in the temperature of the environment the body has no dilation and thus its external boundary remains static. However, if there is a variation in the temperature, the body would have a dilation or a contraction, causing the mobility of the its external boundary. In this work when we deal with

2010 *Mathematics Subject Classification.* 35L05, 35L20, 35L53, 35B40, 93B05, 49J20.
Key words and phrases. Wave equation; energy decay; exact boundary controllability; non-cylindrical domains; holed domains.

Submitted August 16, 2020. Published June 1, 2021.

a domain with a hole, and we refer by external boundary as being the part of the boundary of the domain that does not coincide with that one of the hole.

To establish these concepts in more detail, we consider $B \subset \mathbb{R}^n$, $n \geq 2$, a convex compact set with the origin in its interior with smooth boundary Γ_0. We set $\Omega_\infty = \mathbb{R}^n - B$. Let $\Xi \subset \mathbb{R}^n$ be a simply connected bounded domain with piecewise smooth boundary Γ_1, with no cusps, such that $B \subset \Xi$. We assume that $\operatorname{dist}(\Gamma_0, \Gamma_1) \geq \epsilon > 0$ and set $\Omega = \Xi - B$. Hence, the boundary of Ω is $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Note that Ω is a holed domain whose hole has the shape of B.

We also consider the moving boundary domain $\Xi_t \subset \mathbb{R}^n$ where

$$\Xi_t = \{x \in \mathbb{R}^n : x = \alpha(t)y,\, y \in \Xi\}, \quad t \in [0, +\infty)$$

whose boundary is denoted by Γ_t and $\Xi_0 = \Xi$. Here $\alpha : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a piecewise bounded smooth function, where

$$\Xi_t \times \mathbb{R} \subset \cup_{\overline{x} \in \Xi} \{(x,t) \in \mathbb{R}^N \times \mathbb{R} : |x - \overline{x}|^2 \leq t^2\}, \tag{1.1}$$

with $B \subset \Xi_t$ and $\operatorname{dist}(\Gamma_0 \cap \Gamma_t) \geq \epsilon > 0$, for all $t > 0$. The boundedness of α implies in the existence of $r > 0$ such that $\Xi_t \subset B(0,r)$ for all $t \in [0, +\infty)$. Defining $\widetilde{\Omega} = B(0,r) - B$ and $\Omega_t = \Xi_t - B$ we can see that $\Omega_t \subset \widetilde{\Omega}$ for all $t \in [0, +\infty)$. The boundary of Ω_t is $\partial\Omega_t = \Gamma_0 \cup \Gamma_t$. Now, for $T > 0$, let us consider the non-cylindrical domain of $\mathbb{R}^{n+1}$,

$$Q_T = \cup_{0<t<T} \Omega_t \times \{t\}$$

whose the lateral boundary is $\Sigma_T \cup \Sigma_0$, where $\Sigma_T = \cup_{0<t<T} \Gamma_t \times \{t\}$ and $\Sigma_0 = \Gamma_0 \times [0,T]$.

We denote by (ν_x, ν_t) the outward unit normal vector defined almost all on $\Sigma_T \cup \Sigma_0$. Note that Q_T is a holed non-cylindrical domain in $\mathbb{R}^{n+1}$ whose the lateral boundary is composed by two disjoint parts Σ_T and Σ_0. Here, we requires that B be star-shaped with respect the origin, that is, $\{\nu_x \cdot x\} \leq 0$ for $x \in B$. The assumption (1.1) assures that the surface Σ_T is time-like. This is known to be sufficient to guarantee the well-posedness of the initial and boundary value problem studied here.

Being $\mathcal{O} \subset \mathbb{R}^N$ an arbitrary domain, we denote by Sobolev spaces $L^2(\mathcal{O})$ and $H^1(\mathcal{O})$ the Lebesgue and Sobolev spaces, provided with theirs usual norms which will be denoted by $\|\cdot\|_{L^2(\mathcal{O})}$ and $\|\cdot\|_{H^1(\mathcal{O})}$ respectively (see [1]). Particularly, for $\mathcal{O} = \Omega$, we denote $\mathcal{H}^1(\Omega) = \{u(x) \in H^1(\Omega) : u(x) = 0 \text{ if } x \in \Gamma_0\}$. The topology of $\mathcal{H}^1(\Omega)$ is that one induced from $H^1(\Omega)$. Here, the space $H_0^1(\mathcal{O})$ is the closure C_0^∞ in $H^1(\mathcal{O})$ provided with the norm of $H^1(\mathcal{O})$. The purpose of this article is to study the exact boundary controllability problem

Theorem 1.1. *Let Ω be as defined above. Given $(f,g) \in \mathcal{H}^1(\Omega) \times L^2(\Omega)$, there exist $T > 0$ sufficiently large and a control function $h(\cdot,t) \in L^2(\Sigma_T)$ such that the solution $u \in H^1(Q_T)$ of the problem*

$$\begin{gathered} u_{tt} - \Delta u = 0 \quad \text{in } Q_T \\ u(\cdot,0) = f, \quad u_t(\cdot,0) = g, \quad \text{in } \Omega \\ u(\cdot,t) = 0, \quad \text{on } \Gamma_0 \times [0,T] \\ \nu_t u_t - \nabla u \cdot \nu_x = h(\cdot,t), \quad \text{on } \Sigma_T. \end{gathered} \tag{1.2}$$

satisfy the final condition

$$u(\cdot,T) = 0 = u_t(\cdot,T) \quad \text{in } \Omega_T. \tag{1.3}$$

In the literature some papers deal with exact boundary controllability problems for wave equation on domains with holes. However, as far as we know, there are no papers dealing with such problems in holed moving domains, as it is proposed in the present paper. In this paper [12] the author seeking to show the wide applicability of the HUM method, considers control problems on domains with different types of geometry. In particular, he has shown how the HUM can be applied to solve some exact boundary control problems on a fixed domain which has a hole. In [6] also using the HUM method the authors have considered exact boundary control problems for wave equation in domains which have one or a family of small holes.

In the literature there are also many works dealing with exact boundary control problems on domains that has a mobile boundary but with no hole; see [2, 4, 13] and citing references. At this point we highlight the contribution of the present paper by making available in the literature a work that deals with exact control problems on non-cylindrical holed domains.

Russell [20] developed a technique, based in [19], for studding an exact boundary control problem for wave equation with control acting only on a part of boundary of the domain. Here we shall utilize such technique to obtain the desirable exact boundary control problem proposed in Theorem 1.1.

The applicability of Russell's method requires some properties of the system to be considered, the principals are: linearity, time reversibility, local energy decay in a exterior domain (obtained via [10, 14]) and suitable traces theorems that are obtained in [21]. As seen above the Russell's method requires many properties of the system to be controlled but it has the advantage of requiring very little on the geometry of the domain. From this fact we can consider the limiting function α, defined above, to be only piecewise smooth.

Here, it is proposed that the boundary of the domain be comprise two disjoint parts: the internal part (the boundary of the hole) fixed, and the other part which is moving. An interesting point is to consider an exact boundary control problem for wave equation on a holed domain where both external and internal boundary moving. In the literature there are papers [8, 9] that obtain local energy decay estimates for the wave equation in the exterior of a domain with moving boundary. Another interesting point it is to study on exact boundary control problems, in holed domains, for systems of coupled waves equations as proposed in [5, 16]. We intend to return on this questions in posterior works.

The rest of this article is organized as follows. In Section 2 we presents a brief summary with respect to trace and extension properties. In Section 3 we obtain local energy decay estimates for the wave equation in exterior domain. In Section 4 we explore exact boundary controllability results in a holed domain with fixed boundary. In Section 5 we prove Theorem 1.1.

2. Extension and traces

In this section, we make a brief presentation about the trace and extension theorems, such properties are essential in the proof of Theorem 1.1. Remembering that we are considering Ω as a holed domain whose its boundary is performed by two disjoint parts which is denoted by Γ_0 and Γ_1 respectively, with $\operatorname{dist}(\Gamma_0, \Gamma_1) > 0$ and that Γ_0 is smooth surface and Γ_1 a surface smooth by parts with no cusps. In this work we shall use the following extension lemma.

Lemma 2.1. *Let $\Omega \subset \Omega_\infty$ as defined above and $V \subset \Omega_\infty$ an open set such that $\overline{\Omega} \subset V$. So, there exist a bounded linear operator $E : H^1(\Omega) \times L^2(\Omega) \to H^1(\Omega_\infty) \times L^2(\Omega_\infty)$ such that for $u \in H^1(\Omega) \times L^2(\Omega)$:*

Please define $A \Subset B$

1. *$Eu = u$ in Ω;*
2. *$\operatorname{supp} Eu \Subset V$;*
3. *$\|Eu\|_{H^1(\Omega_\infty)\times L^2(\Omega_\infty)} \leq C\|u\|_{H^1(\Omega)\times L^2(\Omega)}$, where C is a positive real constant independent on u.*

Proof. To sketch the proof let us firstly to define the projection operators

$$P_{i-1} : H^1(\Omega) \times L^2(\Omega) \to H^{i-1}(\Omega_\infty), \quad i = 1, 2.$$

As we are considering $\operatorname{dist}(\Gamma_0, \Gamma_1) > 0$ and Γ_1 is a piecewise smooth surface, we obtain a finite cover $(U_i)_{i=1}^k$ of Γ_1 where $U_i \subset \Omega_\infty$, $i = 1, \cdots, k$. So, we can adapt the proof of the [15, Theorems 3.9 and 3.10] to obtain the extension operators $E_1 : H^1(\Omega) \to H^1(\Omega_\infty)$, where $E_1 u = u$ in Ω with $\|E_1 u\|_{H^1(\Omega_\infty)} \leq C\|u\|_{H^1(\Omega)}$. On the other hand, using the classical extension by zero out Ω, we obtain the extension operator $E_0 : L^2(\Omega) \to L^2(\Omega_\infty)$ where $\|E_0 u\|_{L^2(\Omega_\infty)} \leq \|u\|_{L^2(\Omega)}$. Thus, we obtain the desirable extension operator by defines $E = (E_1 P_1, E_0 P_0)$. $\square$

Next we mention a result on the regularity of the traces of the solution of the wave equation which it is essential in the proof of Theorem 1.1. Let us begin with some notation and definitions. Let $P(\xi, D)$ be a linear second order hyperbolic partial differential equation with C^∞ coefficients depending on ξ in some open bounded domain $\Xi \subset \mathbb{R}^N$. Being $\Sigma \subset \Xi$ an oriented smooth hypersurface which is time-like and non-characteristic with respect to $P(\xi, D)$. Let $\eta = (\eta_1, \cdots, \eta_N)$ be a unit normal to Σ. If $\sum a^{ij} \frac{\partial^2}{\partial \xi_i \partial \xi_j}$ is the principal part of $P(\xi, D)$, then the expression $\frac{\partial u}{\partial \eta} = \sum a^{ij} \frac{\partial u}{\partial \xi_i} \eta_j$ defines the conormal derivative of u relative to the $P(\xi, D)$ along Σ. An important fact it is to know what the regularity of the traces of the conormal derivative on surfaces, for this purpose we turn to [21]. Considering $\Xi \subset \mathbb{R}^N$, with $N \geq 2$, [21, Theorem 2] proves that if $u \in H^1_{\rm loc}(\Xi)$ is such that $P(\xi, D)u \in L^2_{\rm loc}(\Xi)$ then $\frac{\partial u}{\partial \eta} \in L^2_{\rm loc}(\Sigma)$.

Particularly, if we consider $P(\xi, D)$ as being the standard wave operator, its principal part will be $\frac{\partial^2}{\partial t^2} - \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$. Now, if γ is a smooth hypersurface in $\mathbb{R}^N$ we consider the surface $\gamma \times \mathbb{R}$ whose unit normal vector is $\nu = (\nu_x, \nu_t)$, where $\nu_x = (\nu_1, \cdots, \nu_N)$. In this case the conormal derivative of u along $\gamma \times \mathbb{R}$ is $\frac{\partial u}{\partial \nu} = \nu_t u_t - \nabla u \cdot \nu_x$. Particularly, if we apply the trace result mentioned in the previous paragraph for the wave operator we obtain the following result.

Lemma 2.2. *Let u be the solution of the initial-boundary value problem*

$$\begin{gathered} u_{tt} - \Delta u = 0 \quad \text{in } \Omega_\infty \times \mathbb{R} \\ u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1, \quad \text{in } \Omega_\infty \\ u(\cdot, t) = 0, \quad \text{on } \Gamma_0 \times \mathbb{R} \end{gathered} \tag{2.1}$$

with initial data $(u_0, u_1) \in H^1(\Omega_\infty) \times L^2(\Omega_\infty)$, where $\operatorname{supp}(u_0), \operatorname{supp}(u_1) \Subset \Omega_\infty$. Let γ be a smooth hypersurface in Ω_∞, with no self intersection and considers the surface $\gamma \times \mathbb{R}$ which the unit normal vector is $\nu = (\nu_x, \nu_t)$. Then the conormal derivative of u along $\gamma \times \mathbb{R}$ has trace $\nu_t u_t - \nabla u \cdot \nu_x \in L^2_{\rm loc}(\gamma \times \mathbb{R})$.

3. Local energy decay

There are many publications dedicated to obtain local energy decay estimates for hyperbolic equations in exterior domains; see [10, 14, 22] and references there in. In this paper such estimates play a fundamental role in the proof of Theorem 1.1. So, we dedicate this section to such subject.

Let us consider the initial initial-boundary value problem

$$\begin{gathered} u_{tt} - \Delta u = 0 \quad \text{in } \Omega_\infty \times (0,+\infty) \\ u(\cdot,0) = f, \quad u_t(\cdot,0) = g, \quad \text{in } \Omega_\infty \\ u(\cdot,t) = 0, \quad \text{in } \Gamma_0 \times (0,+\infty). \end{gathered} \tag{3.1}$$

Let $\mathcal{O} \subset \Omega_\infty$ a bounded domain, the energy of the solution u of the (3.1) confined in $\mathcal{O}$ is defined by

$$E(t,\mathcal{O},u) = \frac{1}{2}\int_{\mathcal{O}} [|\nabla u|^2 + u_t^2](x,t)dx. \tag{3.2}$$

If there exist a positive constant C and a function $p(t)$ such that

$$E(t,\mathcal{O},u) \le Cp(t)E(0,\mathcal{O},u), \tag{3.3}$$

with $p(t) \to 0$, as $t \to +\infty$, we say the energy of (3.1) decays locally. Adapting process in [14, 10] we show the validity of the following local energy decay estimate.

Lemma 3.1. *Let $(f,g) \in H^1(\Omega_\infty) \times L^2(\Omega_\infty)$ with $\operatorname{supp}(f), \operatorname{supp}(g) \subset \mathcal{O} \subset \Omega_\infty$, then there exist a positive real constant K, independent of f and g, such that the solution u of* (3.1) *satisfies*

$$\|u(\cdot,t)\|^2_{H^1(\mathcal{O})} + \|u_t(\cdot,t)\|^2_{L^2(\mathcal{O})} \le \frac{K}{t-R}\big\{\|u(\cdot,0)\|^2_{H^1(\mathcal{O})} + \|u_t(\cdot,0)\|^2_{L^2(\mathcal{O})}\big\}, \tag{3.4}$$

for $t > R$ sufficiently large, where R is such that $\mathcal{O} \subset \Omega_R$ being $B(0,R)$ the ball of center 0 and radius R and $\Omega_R = B(0,R) \cap \Omega_\infty$.

The proof of Lemma 3.1 follows the ideas presented by Morawetz [14], so it is essential to prove some preliminaries results. The compactness of the initial data f and g implies finite propagation speed property for solution of the system (3.1), that is the solution u of (3.1), in the instant t, has support contained in some bounded region of the space Ω_∞. So if we take $R > 0$ and the ball $B(0,R)$ of center 0 and radius R, such that $\operatorname{supp}(f) \cup \operatorname{supp}(g) \subset \Omega_R = B(0,R) \cap \Omega_\infty$, then from some $t > R$ the function $u(x,t)$ as well as its derivatives are null for $x \in \Omega_\infty$ with $|x| \ge t$.

Other important and classical property of the solution u of the system (3.1) is that his total energy

$$E(t,\Omega_\infty,u) = \frac{1}{2}\int_{\Omega_\infty} [|\nabla u|^2 + u_t^2](x,t)dx, \tag{3.5}$$

is conserved with respect to t, that is,

$$E(t,\Omega_\infty,u) = E(0,\Omega_\infty,u), \quad \text{for all } t > 0. \tag{3.6}$$

In the proof of estimate (3.4) we will used the following result.

Lemma 3.2. *Let u be the solution of the boundary-value problem* (3.1)*, if w is the solution of the problem*

$$\begin{gathered} w_{tt}-\Delta w=0 \quad \text{in } \Omega_\infty\times(0,+\infty)\\ w(\cdot,0)=h,\quad w_t(\cdot,0)=f,\quad \text{in } \Omega_\infty\\ w(\cdot,t)=0,\quad \text{in } \Gamma_0\times(0,+\infty) \end{gathered} \tag{3.7}$$

with $\Delta h=g$ and $h=0$ in B. Then $w_t=u$.

Proof. Note that $v=w_t$ satisfies

$$\begin{gathered} v_{tt}-\Delta v=0 \quad \text{in } \Omega_\infty\times(0,+\infty)\\ v(\cdot,0)=f,\quad v_t(\cdot,0)=\Delta h,\quad \text{in } \Omega_\infty\\ v(\cdot,t)=0,\quad \text{in } \Gamma_0\times(0,+\infty) \end{gathered} \tag{3.8}$$

with $\Delta h=g$ and $h=0$ on B. Then $v=w_t$ satisfies (3.1) and by the uniqueness of solution we conclude that $v=w_t=u$ $\square$

From Lemma 3.2 we obtain the estimate

$$\begin{aligned} \int_{\Omega_\infty}|u(x,t)|^2dx&=\int_{\Omega_\infty}|w_t(x,t)|^2dx\\ &\le E(t,\Omega_\infty,w_t)=E(0,\Omega_\infty,w_t)\\ &\le CE(0,\Omega_\infty,u). \end{aligned} \tag{3.9}$$

Now, for $T>0$, let $D=\Omega_\infty\times[0,T]$ be an exterior region which boundary is $\partial D=\partial D_1\cup\partial D_2\cup\partial D_3$, where

$$\partial D_1=\Omega_\infty\times\{0\},\quad \partial D_2=\Omega_\infty\times\{T\},\quad \partial D_3=\Gamma_0\times[0,T].$$

Multiplying the equality $u_{tt}-\Delta u=0$ by $(x\cdot\nabla u)+tu_t+\frac{N-1}{2}u$ and integrating it on D, from the Gauss' divergence formula, as in [10], we obtain

$$\begin{aligned} tE(t,\Omega_\infty,u)&=\int_{\Omega_R}(x\cdot\nabla u(\cdot,0))\,u_t(\cdot,0)dx+\frac{(N-1)}{2}\int_{\Omega_R}u(\cdot,0)u_t(\cdot,0)dx\\ &\quad-\int_{\Omega_\infty}(x\cdot\nabla u(\cdot,t))\,u_t(\cdot,t)dx-\frac{(N-1)}{2}\int_{\Omega_\infty}u(\cdot,t)u_t(\cdot,t)dx\\ &\quad+\frac12\int_{D_3}\{x\cdot\nu(\cdot)\}\big|\frac{\partial u}{\partial\nu}(.,s)\big|^2\,dx\,ds. \end{aligned} \tag{3.10}$$

Since $B=\Omega_\infty^c$ is star-shaped with respect to origin, that is $\{x\cdot\nu(\cdot)\}\le 0$, it follow that

$$\frac12\int_{D_3}\{x\cdot\nu(\cdot)\}\big|\frac{\partial u}{\partial\nu}(.,s)\big|^2\,dx\,ds\le 0.$$

So, from (3.10) we obtain the estimate

$$\begin{aligned} &tE(t,\Omega_\infty,u)\\ &\le\int_{\Omega_R}(x\cdot\nabla u(\cdot,0))\,u_t(\cdot,0)dx+\frac{(N-1)}{2}\int_{B(0,R)}u(\cdot,0)u_t(\cdot,0)dx\\ &\quad-\int_{\Omega_\infty}(x\cdot\nabla u(\cdot,t))\,u_t(\cdot,t)dx-\frac{(N-1)}{2}\int_{\Omega_\infty}u(\cdot,t)u_t(\cdot,t)dx. \end{aligned} \tag{3.11}$$

Now observe that

$$
\begin{aligned}
&\Big|\int_{\Omega_\infty} (x\cdot\nabla u(\cdot,t))\, u_t(\cdot,t)dx\Big| \\
&\le R\int_{\Omega_R} |\nabla u(\cdot,t)|\, |u_t(\cdot,t)|dx + \int_{|x|\ge R} |x||\nabla u(\cdot,t)||u_t(\cdot,t)|dx.
\end{aligned}
$$

Because of the finite propagation speed property, we can take $t > R$ such that $u(x,t)$ and its derivatives are null, so

$$
\begin{aligned}
\int_{|x|\ge R} |x||\nabla u(\cdot,t)||u_t(\cdot,t)|dx &= \int_{t\ge|x|\ge R} |x||\nabla u(\cdot,t)||u_t(\cdot,t)|dx \\
&\le \frac{t}{2}\int_{|x|\ge R} \{|\nabla u(\cdot,t)|^2 + |u_t(\cdot,t)|^2\}dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Big|\int_{\Omega_\infty} (x\cdot\nabla u(\cdot,t))\, u_t(\cdot,t)dx\Big| \\
&\le \frac{R}{2}\int_{\Omega_R} \{|\nabla u(\cdot,t)|^2 + |u_t(\cdot,t)|^2\}dx + \frac{t}{2}\int_{|x|\ge R} \{|\nabla u(\cdot,t)|^2 + |u_t(\cdot,t)|^2\}dx.
\end{aligned}
\tag{3.12}
$$

On the other hand,

$$
\Big|\int_{\Omega_\infty} u(\cdot,t)u_t(\cdot,t)dx\Big| \le \frac{1}{2}\int_{\Omega_\infty} \left[|u(\cdot,t)|^2 + |u_t(\cdot,t)|^2\right] dx. \tag{3.13}
$$

Joining (3.9) and (3.13) we obtain

$$
\Big|\int_{\Omega_\infty} u(\cdot,t)u_t(\cdot,t)dx\Big| \le CE(0,\Omega_\infty,u), \tag{3.14}
$$

where C is a positive constant independent of the initial data f and g.

The following inequalities are also valid

$$
\Big|\int_{\Omega_R} (x\cdot\nabla u(\cdot,0))\, u_t(\cdot,0)dx\Big| \le C(R)E(0,\Omega_R,u), \tag{3.15}
$$

$$
\Big|\int_{\Omega_R} u(\cdot,0)u_t(\cdot,0)dx\Big| \le C(R)E(0,\Omega_R,u), \tag{3.16}
$$

where $C(R)$ is a positive constant which vary from line to line and depend on R but not on of the initial data f and g.

Now, joining and manipulating the the inequalities (3.11)-(3.16) we obtain

$$
\begin{aligned}
&tE(t,\Omega_R,u) + tE(t,|x|\ge R,u) \\
&\qquad \le CE(0,\Omega_R,u) + RE(t,\Omega_R,u) \qquad + tE(t,|x|\ge R,u),
\end{aligned}
$$

which implies

$$
E(t,\Omega_R,u) \le \frac{K}{t-R}E(0,\Omega_R,u), \tag{3.17}
$$

where K is a positive constant independent of the initial data.

Applying (3.17) to the solution w of (3.7), and by Lemma 3.2 we obtain

$$\begin{aligned}\int_{\Omega_R} |u|^2 dx &= \int_{B(0,R)} |w_t|^2 dx \\ &\le E(t,\Omega_R,w_t) \le E(t,\Omega_R,u) \\ &\le \frac{K}{t-R} E(0,\Omega_R,u).\end{aligned} \tag{3.18}$$

Joining (3.17) and (3.18) and considering $\mathcal{O} \subset \Omega_R$, we obtain (3.4), completes the proof. Lemma 3.1.

4. Control in a holed domain with fixed boundary

In this section we prove an exact boundary control problem for the wave equation on a holed domain with fixed boundary. This section plays a important rule in the proof Theorem 1.1. Let Ω be as define in the first section and $T > 0$. The boundary of Ω is $\partial\Omega = \Gamma_0 \cup \Gamma_1$.

Lemma 4.1. *Given $(v_0, v_1) \in \mathcal{H}^1(\Omega) \times L^2(\Omega)$ and $T > 0$, there exists a control function $h(\cdot,t) \in L^2(\Gamma_1 \times [0,T])$ such that the solution $v \in H^1(\Omega \times [0,T])$ of the problem*

$$\begin{gathered} v_{tt} - \Delta v = 0 \quad \text{in } \Omega \times [0,T] \\ v(\cdot,T) = v_0, \quad v_t(\cdot,T) = v_1, \quad \text{in } \Omega \\ v(\cdot,t) = 0, \quad \text{on } \Gamma_0 \times [0,T] \\ \nu_t v_t - \nabla v \cdot \nu_x = h(\cdot,t), \quad \text{on } \Gamma_1 \times [0,T]. \end{gathered} \tag{4.1}$$

satisfies the condition

$$v(\cdot,0) = 0 = v_t(\cdot,0) \quad \text{in } \Omega. \tag{4.2}$$

Proof. Let δ be a positive number, and $\Omega_\delta = \{y \in \Omega_\infty : \exists x \in \Omega; |x-y| < \delta\}$ be an open neighborhood of Ω. Given an arbitrary $(w_0, w_1) \in \mathcal{H}^1(\Omega) \times L^2(\Omega)$, according Lemma 2.1 we can extend (w_0, w_1), for all Ω_∞. Let $(\widetilde{w}_0, \widetilde{w}_1)$ be the extension of (w_0, w_1), that is, $(\widetilde{w}_0, \widetilde{w}_1) = E(w_0, w_1)$. Let w the solution of the backward initial boundary value problem

$$\begin{gathered} w_{tt} - \Delta w = 0 \quad \text{in } \Omega_\infty \times (0,+\infty) \\ w(\cdot,T) = \widetilde{w}_0, \quad w_t(\cdot,T) = \widetilde{w}_1, \quad \text{in } \Omega_\infty \\ w(\cdot,t) = 0, \quad \text{in } \Gamma_0 \times (0,+\infty). \end{gathered} \tag{4.3}$$

Now, for $T > 0$ we define the bounded linear operator

$$\overline{S}_T : \mathcal{H}^1_0(\Omega_\delta) \times L^2(\Omega_\delta) \to \mathcal{H}^1(\Omega_\infty) \times L^2(\Omega_\infty)$$

such that $\overline{S}_T(w(\cdot,T), w_t(\cdot,T)) = (w(\cdot,0), w_t(\cdot,0))$, where w is the solution of (4.3). From the decay estimate (3.4), with $\mathcal{O} = \Omega_\delta$, applied to w we obtain the estimate

$$\|(w(\cdot,0), w_t(\cdot,0))\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)} \le \frac{K}{T-R} \|(\widetilde{w}_0, \widetilde{w}_1)\|^2_{\mathcal{H}^1(\Omega_\infty)\times L^2(\Omega_\infty)}, \tag{4.4}$$

for $T > R$ sufficiently large and K is a constant independent on data $(\widetilde{w}_0, \widetilde{w}_1)$.

Now we consider the cut off function $\phi \in C_0^\infty(\Omega_\infty)$ such that $\phi \equiv 1$ in $\Omega_{\delta/2}$, and $\phi \equiv 0$ out side of Ω_δ. Then we solve the forward initial boundary value problem

$$\begin{gathered} z_{tt} - \Delta z = 0 \quad \text{in } \Omega_\infty \times (0,+\infty) \\ z(\cdot,0) = \phi w(\cdot,0), \quad z_t(\cdot,0) = \phi w_t(\cdot,0), \quad \text{in } \Omega_\infty \\ z(\cdot,t) = 0, \quad \text{in } \Gamma_0 \times (0,+\infty). \end{gathered} \tag{4.5}$$

We define the linear operator $S_T : \mathcal{H}_0^1(\Omega_\delta) \times L^2(\Omega_\delta) \to \mathcal{H}^1(\Omega_\infty) \times L^2(\Omega_\infty)$ by $S_T(z(\cdot,0), z_t(\cdot,0)) = (z(\cdot,T), z_t(\cdot,T))$. Applying again the decay estimate (3.4), with $\mathcal{O} = \Omega_\delta$, for function z, we obtain

$$\|(z(\cdot,T), z_t(\cdot,T))\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)} \le \frac{K}{T-R}\|(z(\cdot,0), z_t(\cdot,0)))\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)}\,, \tag{4.6}$$

for $T > R$ sufficiently large and K is a constant independent on data (z_0, z_1).

We define $\widetilde{v}(\cdot,t) = w(\cdot,t) - z(\cdot,t)$ and see that $\widetilde{v}$ satisfies

$$\begin{gathered} \widetilde{v}_{tt} - \Delta\widetilde{v} = 0 \quad \text{in } \Omega_\infty \times (0,+\infty) \\ \widetilde{v}(\cdot,T) = w(\cdot,T) - z(\cdot,T), \quad \widetilde{v}_t(\cdot,T) = w_t(\cdot,T) - z_t(\cdot,T) \quad \text{in } \Omega_\infty \\ \widetilde{v}(\cdot,t) = 0, \quad \text{in } \Gamma_0 \times (0,+\infty) \end{gathered} \tag{4.7}$$

and

$$\begin{gathered} \widetilde{v}(\cdot,0) = w(\cdot,0) - \phi w(\cdot,0) = 0 \quad \text{in } \Omega, \\ \widetilde{v}_t(\cdot,0) = w_t(\cdot,0) - \phi w_t(\cdot,0) = 0 \quad \text{in } \Omega, \end{gathered}$$

since $\phi = 1$ in Ω.

Note that the function $\widetilde{v}$ solves the homogeneous wave equation and has the desirable final state $(\widetilde{v}(\cdot,0), \widetilde{v}_t(\cdot,0)) = (0,0)$ in Ω. Now an important step it is to know if we may obtain $T > 0$ such that $(\widetilde{v}(\cdot,T), \widetilde{v}_t(\cdot,T))$ extend the initial data (v_0, v_1). That is, we wish establish solution for the equations

$$w(\cdot,T) - z(\cdot,T) = v_0, \quad w_t(\cdot,T) - z_t(\cdot,T) = v_1 \quad \text{in } \Omega.$$

The last equations can be rewriting as

$$E(w_0, w_1) - (z(\cdot,T), z_t(\cdot,T)) = (v_0, v_1) \quad \text{in } \Omega. \tag{4.8}$$

We want to solve (4.8) for unknown $(w_0, w_1) \in \mathcal{H}^1(\Omega) \times L^2(\Omega)$. For this purpose we rewrite equation (4.8) in terms of the operators S_T and $\overline{S}_T$. Note that

$$\begin{aligned} (z(\cdot,T), z_t(\cdot,T)) &= S_T(\phi w(\cdot,0), \phi w_t(\cdot,0)) \\ &= S_T M_\phi(w(\cdot,0), w_t(\cdot,0)) \\ &= S_T M_\phi \overline{S}_T(w(\cdot,T), w_t(\cdot,T)) \\ &= [S_T M_\phi \overline{S}_T E](w_0, w_1), \end{aligned}$$

where M_ϕ is the operator multiplication by ϕ. Thus, (4.8) becomes

$$(w_0, w_1) - \mathcal{R} S_T M_\phi \overline{S}_T E(w_0, w_1) = (v_0, v_1) \quad \text{in } \Omega, \tag{4.9}$$

where $\mathcal{R}$ denotes the restriction to Ω. Denoting $\mathcal{R} S_T M_\phi \overline{S}_T E$ by K_T, equation (4.9) can be rewritten as

$$(I - K_T)(w_0, w_1) = (v_0, v_1) \quad \text{in } \Omega, \tag{4.10}$$

where I is the identity operator in $\mathcal{H}^1(\Omega) \times L^2(\Omega)$.

Now, for solving equation (4.9) it is sufficient to show that K_T is a contraction in $\mathcal{H}^1(\Omega)\times L^2(\Omega)$. It is in this point where the energy decay takes place, by considering inequalities (4.4) and (4.6) note that

$$\begin{aligned}
\|K_T(w_0,w_1)\|_{\mathcal{H}^1(\Omega)\times L^2(\Omega)} &= \|(z(\cdot,T),z_t(\cdot,T))\|_{\mathcal{H}^1(\Omega)\times L^2(\Omega)}\\
&\le \|(z(\cdot,T),z_t(\cdot,T))\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)}\\
&\le \frac{K}{T-R}\|(z(\cdot,0),z_t(\cdot,0))\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)}\\
&= \frac{K}{T-R}\|(\phi w(\cdot,0),\phi w_t(\cdot,0))\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)}\\
&\le \frac{\widetilde{K}}{T-R}\|(w(\cdot,0),w_t(\cdot,0))\|^2_{\mathcal{H}^1(\Omega_\delta)\times L^2(\Omega_\delta)}\\
&\le \frac{K\widetilde{K}}{(T-R)^2}\|E(w_0,w_1)\|^2_{\mathcal{H}^1(\Omega)\times L^2(\Omega)}\\
&\le \frac{K\widetilde{K}}{(T-R)^2}\|(w_0,w_1)\|^2_{\mathcal{H}^1(\Omega)\times L^2(\Omega)}.
\end{aligned}$$

From the inequalities above we obtain

$$\|K_T(w_0,w_1)\|_{\mathcal{H}^1(\Omega)\times L^2(\Omega)} \le \frac{\sqrt{K\widetilde{K}}}{T-R}\|(w_0,w_1)\|_{\mathcal{H}^1(\Omega)\times L^2(\Omega)}, \tag{4.11}$$

for $T>R$ sufficiently large, where $\widetilde{K}$ is a positive constant depending only K and Ω. So, we choose a $T>R$ such that $\frac{\sqrt{K\widetilde{K}}}{T-R}\le c<1$ and for such T, K_T is a contraction. Thus, we take the solution (w_0,w_1) for (4.10) and take it to the begin of the proof in order to obtain w, z and $\widetilde{v}=w-z$, where $\widetilde{v}$ solves (4.7) and has the desirable final condition $(\widetilde{v}(\cdot,0),\widetilde{v}(\cdot,0))=(0,0)$. To complete the proof we define $v=\widetilde{v}_{|\Omega}$ and apply Lemma 2.2 from the previous section to read off the trace of $\nu_t u_t-\nabla u\cdot\nu_x$ as a function of $\in L^2(\Gamma_1\times[0,T])$ completing the proof. □

5. Proof of Theorem 1.1

Let Ω, Ω_∞ and $\widetilde{\Omega}$ as defined in the initial section. See that the $\widetilde{\Omega}$ is a holed domain with fixed boundary $\partial\widetilde{\Omega}=\widetilde{\Gamma}\cup\Gamma_0$, where Γ_0 is the boundary of the hole and $\widetilde{\Gamma}$ is the external boundary. Let $(\widetilde{f},\widetilde{g})\in\mathcal{H}^1(\Omega_\infty)\times L^2(\Omega_\infty)$ such that $\widetilde{f}$ and $\widetilde{g}$ are extensions of f and g respectively, with $\operatorname{supp}(\widetilde{f})\subset B(0,r)\cap\Omega_\infty$ and $\operatorname{supp}(\widetilde{g})\subset B(0,r)\cap\Omega_\infty$. Here the number r is that one defined in the Section 1. Let $\widetilde{u}$ be the solution of the initial-boundary value problem

$$\begin{gathered}
\widetilde{u}_{tt}-\Delta\widetilde{u}=0 \quad \text{in } \Omega_\infty\times(0,+\infty)\\
\widetilde{u}(\cdot,0)=\widetilde{f},\quad \widetilde{u}_t(\cdot,0)=\widetilde{g},\quad \text{in } \Omega_\infty\\
\widetilde{u}(\cdot,t)=0,\quad \text{in } \Gamma_0\times(0,+\infty).
\end{gathered}\tag{5.1}$$

Now, for a $T>r$, we take the state $(\widetilde{u}(\cdot,T),\widetilde{u}_t(\cdot,T))\in\mathcal{H}^1(\widetilde{\Omega})\times L^2(\widetilde{\Omega})$ and according to Lemma 4.1, changing Ω by $\widetilde{\Omega}$, we solve the exact boundary control problem

$$
\begin{gathered}
v_{tt} - \Delta v = 0 \quad \text{in } \widetilde{\Omega} \times [0,T] \\
v(\cdot,T) = \widetilde{u}(\cdot,T), \quad v_t(\cdot,T) = \widetilde{u}_t(\cdot,T), \quad \text{in } \widetilde{\Omega} \\
v(\cdot,t) = 0, \quad \text{on } \Gamma_0 \times [0,T] \\
\nu_t v_t - \nabla v \cdot \nu_x = h(\cdot,t), \quad \text{on } \widetilde{\Gamma} \times [0,T],
\end{gathered} \tag{5.2}
$$

which satisfies, at the instant $t = 0$, the condition

$$
v(\cdot,0) = 0 = v_t(\cdot,0) \quad \text{in } \widetilde{\Omega}. \tag{5.3}
$$

Considering Ω_t as defined in the Section 1, note that $\Omega_t \subset \widetilde{\Omega}$ for all $t > 0$, so follows that for each $T > 0$ we have $Q_T = \cup_{0<t<T} \Omega_t \times \{t\} \subset \widetilde{\Omega} \times [0,T]$. Defining $u = \widetilde{u} - v$ we can see that the restriction of u to Q_T satisfies

$$
\begin{gathered}
u_{tt} - \Delta u = 0 \quad \text{in } Q_T \\
u(\cdot,0) = f, \quad u_t(\cdot,0) = g, \quad \text{in } \Omega \\
u(\cdot,t) = 0, \quad \text{on } \Gamma_0 \times [0,T]
\end{gathered} \tag{5.4}
$$

and the condition

$$
u(\cdot,T) = 0 = u_t(\cdot,T) \quad \text{in } \Omega_T. \tag{5.5}
$$

Now, to conclude the proof, we read the trace of the conormal derivative of u on the surface Σ_T. Note that the components $\widetilde{u}$ and v of u are under the conditions for applying Lemma 2.2. So, we read the trace of the conormal derivative of $\widetilde{u}$ and v on surface Σ_T obtaining a L^2 function. So, the desirable control function is obtained taking $\nu_t u_t - \nabla u \cdot \nu_x = h(\cdot,t)$ on Σ_T. This completes the proof.

Remark 5.1. Note that Theorem 1.1 establishes only the existence of the control time T. It does not provide a lower bound from which the control time can be taken. On the other hand, using the HUM method, the authors in [6] showed the existence of a T_0 from which the system is controllable. A way for we obtain lower estimates for the control time, using the Russell's controllability method, is to follow the ideas of analytic extension given by Lagnese [11]. But here we have a difficulty applying it because we do not have the explicit formulas for the solution to the initial-boundary value problem for the wave equation in a exterior domain.

References

[1] R. Adams, J. Fournier; *obolev spaces,* second ed., Elsevier Science, Oxisford UK, 2003.

[2] C. Bardos, G. Chen; *Control and stabilization for wave equation, part III: domain with moving boundary,* SIAM J. Control Optim., **19** (1981), 123–138.

[3] C. Bardos, G. Lebeau, J. Rauch; *harp sufficient conditions for the observation, control, and stabilization of waves from the boundary,* SIAM J. Control and Optimization, **30** (5)(1992), 1024–1065.

[4] W. D. Bastos, J. Ferreira; *Exact boundary control for the wave equation in a polyhedral time-dependent domain,* Appl Math Lett., **12** (1999), 1–5.

[5] W. D. Bastos, R. S. O. Nunes, J. M. Pitot; *Exact boundary controllability for a series of membranes elastically connected,* Electron. J. Differential Equations, **2017** No. 6 (2017), 1–16.

[6] D. Cioranescu, P. Donato, E. Zuazua; *Exact boundary controlability for the wave equation in domains with small holes,* J. Math. Pures et Appl., **71** (1992), 343–357.

[7] D. Cioranescu, P. Donato, E. Zuazua; *Aproximate boundary controllability for the wave equation in perforated domains,* SIAM J. Control Optm., **32** (1994), 35-50.

[8] J. Cooper; *Local Decay of Solutions of the Wave Equation in the Exterior of a Moving Body,* J. Math Anal Appl., **49** (1975), 130–153.

[9] J. Cooper, W. A. Strauss; *Energy Boundedness and Decay of Waves Reflecting off a Moving Obstacle,* Indiana University Mathematics Journal, **25** (7) (1976), 671–690.
[10] R. Ikehata; *Local energy decay for linear wave equations with non-compactly supported initial data,* Math. Meth. Appl. Sci., **27** (2004), 1881–1892.
[11] J. Lagnese; *Boundary value control of a class of hyperbolic equations in a general region,* SIAM J Control Optim., **15**(6) (1977), 973–983.
[12] J-L. Lions; *Exact controllability, stabilization and perturbations for distributed system,* SIAM Rev., **30**(1) (1988), 1–68.
[13] M. M. Miranda; *Exact controllability for the wave equation in domains with variable boundary,* Rev. Mat. Univ. Complut. Madr., **9** (1996) 435–457.
[14] C. S. Morawetz; *The decay of solutions of the exterior initial-boundary value problem for the wave equation,* Comm. Pure Appl. Math., **14** (1961), 561–568.
[15] J. Necas; *Direct Methods in the Theory of Elliptic Equations,* Springer-Verlag, Berlin, 2012.
[16] R. S. O. Nunes; *Exact Boundary Controllability and Energy Decay for a System of Wave Equations Linearly Coupled,* Mediterr. J. Math., **18** (30) (2021).
[17] R. S. O. Nunes, W. D. Bastos; *Energy decay for the linear Klein-Gordon equation and boundary control,* J. Math. Anal. Appl., **414** (2014), 934–944.
[18] D. L. Russell; *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations,* Stud. Appl. Math., **52** (1973), 189–211.
[19] D. L. Russell; *Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions,* SIAM Review, **20** no. 4, (1978), 639–739.
[20] D. L. Russell; *Exact boundary value controllability theorems of wave and heat processes in star-complemented regions,* Differential Games and Control Theory, Roxin, Liu and Sternberg, ed., Marcel Dekker, New York, 1974.
[21] D. Tataru; *On regularity of the boundary traces for the wave.* Ann. Scuola Norm. Pisa, C. L. Sci. (4) **26** (1) (1998), 185–206.
[22] E. C. Zachmanoglou; *The decay of the initial-boundary value problem for hyperbolic equations,* J. Math. Anal. Appl., **13** (1966), 504-515.

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