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PROPAGATING INTERFACE IN REACTION-DIFFUSION EQUATIONS WITH DISTRIBUTED DELAY

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Abstract. This article concerns the limiting behavior of the solution to a reaction-diffusion equation with distributed delay. We firstly consider the quasi-monotone situation and then investigate the non-monotone situation by constructing two auxiliary quasi-monotone equations. The limit behaviors of solutions of the equation can be obtained from the sandwich technique and the comparison principle of the Cauchy problem. It is proved that the propagation speed of the interface is equal to the minimum wave speed of the corresponding traveling waves. This makes possible to observe the minimum speed of traveling waves from a new perspective.

1. Introduction

We consider the limiting behavior (as $\varepsilon \to 0$) of the solution $u^\varepsilon(t,x) : [-\varepsilon\tau, \infty) \times \mathbb{R}^N \to \mathbb{R}$ for the reaction-diffusion equation with distributed delay:

$$\begin{gathered}\partial_t u^\varepsilon(t,x) = \varepsilon \Delta u^\varepsilon(t,x) + \frac{1}{\varepsilon}\Big[\int_0^\tau k(s) g(u^\varepsilon(t-\varepsilon s, x))ds - u^\varepsilon(t,x)\Big], \\ u^\varepsilon(s,x) = u_0^\varepsilon(\frac{s}{\varepsilon}, x),\end{gathered} \tag{1.1}$$

where $t > 0$, $x \in \mathbb{R}^N$, $s \in [-\varepsilon\tau, 0]$, and $\tau > 0$ is a given delay parameter, the function $k(\cdot)$ satisfies

$$k(\cdot) > 0, \quad \int_0^\tau k(s)ds = 1,$$

The kernel $k(\cdot)$ in model (1.1) has a biological explanation. Specifically, in population dynamics, it is sometimes assumed that all juveniles mature sexually at the exact same age τ. However, this approximate age τ is not always realistic. Because of individual differences and the influence of the external environment, the time required for an individual from birth to maturity is not a fixed constant. Therefore, many scholars [13, 14, 15, 16, 28] put forward the concept of distributed delay, and described such a dynamic process through the distribution of maturity time weighted by the probability density function. The birth function $g(\cdot) : [0, \infty) \to [0, \infty)$ is of the class C^2 and satisfies the following assumptions:

- (H1) $g(0) = g(1) - 1 = 0$, $g'(1) < 1 < g'(0)$, and $u < g(u) \le g'(0)u$ for any $u \in (0, 1)$.

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Clearly, assumption (H1) shows that (1.1) is a monostable system. There are typical examples of function $g(u)$ which satisfies the assumption above. One is $g(u) = \rho u e^{-au}$ with $\rho > 0$ and $a > 0$, and another is $g(u) = \frac{\rho u}{a + a u^{\varrho}}$ with $\rho > 0, \varrho > 1$ and $a > 0$.

As we know, when the diffusion coefficient is very small or the reaction term is very large, the solutions of some types of reaction diffusion equations usually generate internal transition layers (which is also called *interface*). This property is related to the traveling wave solutions of corresponding reaction diffusion equations. In particular, for the famous Fisher-KPP equation [11, 21]

$$u_t(t, x) = \Delta u(t, x) + u(t, x)(1 - u(t, x)), \quad \forall t \geq 0,\ x \in \mathbb{R}^N,$$

it admits the traveling wave solution connecting two equilibria 0 and 1 with the wave speed $c \geq c^* = 2$. Taking the scale

$$u^{\varepsilon}(t, x) \equiv u\Big(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\Big), \quad \varepsilon \in (0, 1),$$

we obtain

$$\begin{gathered} \partial_t u^{\varepsilon} = \varepsilon \Delta u^{\varepsilon} + \frac{1}{\varepsilon} u^{\varepsilon}(1 - u^{\varepsilon}), \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ u^{\varepsilon}(0, x) = u_0^{\varepsilon}(x), \quad \forall x \in \mathbb{R}^N. \end{gathered} \tag{1.2}$$

For the limiting behavior of u^{ε}, it seems reasonable to guess that the family $\{u^{\varepsilon}\}_{\varepsilon > 0}$ converges in some sense to 0 or 1 as $\varepsilon \to 0$. By a formal analysis, we can see that the diffusion term $\varepsilon \Delta u^{\varepsilon}$ can be negligible in the very early stage, since it is very small compared with the reaction term. As a result, (1.2) can be approximated by the ordinary differential equation

$$\frac{du^{\varepsilon}}{dt} = \varepsilon^{-1} u^{\varepsilon}(1 - u^{\varepsilon}).$$

Obviously, this ODE has the equilibria 0 and 1, the value of u^{ε} quickly becomes close to either 1 or 0 in most part of $\mathbb{R}^N$, which creates a steep transition layer. As soon as the interface develops, the diffusion term starts to increase gradually to balance the reaction term, then the interface ceases development and starts to propagate in a much slower time scale.

There are some studies about this phenomenon. For monostable case, Freidlin [12] investigated this asymptotic problem from the perspective of probability theory. Then Evans and Souganidis [10] studied it by a direct partial differential equation approach (i.e., geometric optics, Hamilton-Jacobi technics [5, 6]), similar methods were used in [9, 26, 27]. By the method of comparison principle, Alfaro and Ducrot [1, 2] proved the generation and motion of interface properties, and further estimated the thickness of the transition layers. Furthermore, Hilhorst et al. [18] applied the Hamilton Jacobi technics to consider the interface problem for the degenerate Fisher equation

$$\begin{gathered} u_t = \epsilon \Delta u^m + \frac{1}{\epsilon} u(1 - u), \quad \forall (t, x) \in [0, T] \times \Omega, \\ \frac{\partial (u^m)}{\partial \nu} = 0, \quad \forall (t, x) \in [0, T] \times \partial\Omega, \\ u(x, 0) = u_0(x), \quad \forall x \in \Omega, \end{gathered}$$

where $\Omega \in \mathbb{R}^N$ $(N \geq 1)$, $m \geq 2$. For the bistable case, Chen et al. [7] gave a rigorous analysis of both the generation and the motion of interface,

$$\begin{gathered}\partial_t u^\varepsilon = \Delta u^\varepsilon + \frac{1}{\varepsilon^2} g(u^\varepsilon), \quad \forall (t,x) \in (0,\infty) \times \mathbb{R}^N, \\ u^\varepsilon(0,x) = u_0(x), \quad \forall x \in \mathbb{R}^N.\end{gathered}$$

We refer the readers to [4, 8, 19, 23] for some other systems.

Note that, there is a key factor that can not be ignored in the mathematical model: time delay, which usually represents resource regeneration time, maturity cycle, breastfeeding time, feedback time in biological model and the latency in epidemic model. Consider this effect in the model, Alfaro and Ducrot [3] gave some results about the propagating interface of the monostable equation

$$\begin{gathered}\partial_t u^\varepsilon(t,x) = \varepsilon \Delta u^\varepsilon(t,x) + \frac{1}{\varepsilon}[g(u^\varepsilon(t-\varepsilon\tau, x)) - u^\varepsilon(t,x)], \quad \forall t > 0,\ x \in \mathbb{R}^N, \\ u^\varepsilon(s,x) = u_0^\varepsilon\big(\frac{s}{\varepsilon}, x\big), \quad \text{for } -\varepsilon\tau \leq s \leq 0,\ x \in \mathbb{R}^N,\end{gathered}$$

where $g(u)$ is an increasing function on the interval $(0,1)$. However, there seems no results for the distributed delay case. Therefore, this paper is devoted to investigating the propagating interface of (1.1) with distributed delay.

Before demonstrating the main theorem, we give some notation. For $c > 0$, denote $H_c := \mathrm{U}_{t\geq 0}(\{t\} \times H_{c,t})$ as the smooth solution of the free boundary problem

$$\begin{gathered}V = c \quad \text{on } H_{c,t}, \\ H_{c,t}\big|_{t=0} = H_0\end{gathered} \tag{1.3}$$

with V the normal velocity of $H_{c,t}$ in the exterior direction, the initial interface is defined as $H_0 = \partial H_0$. Assume the region enclosed by H_0, namely Ω_0, is convex, these solutions do exist for all $t \geq 0$. By a slight abuse of notation, we consider $H_{c,t}$ for all $t \geq -\varepsilon\tau$, with $\varepsilon > 0$ small enough. For each $t \geq -\varepsilon\tau$, we denote $\Omega_{c,t}$ as the region enclosed by the hypersurface $H_{c,t}$. In addition, assume that

(H2) $g(\cdot)$ is non-decreasing on $[0,1]$.

Next, we give the assumption on the initial condition.

Assumption 1.1 (Initial condition)**.** *Assume that $u_0(s,x) : [-\tau, 0] \times \mathbb{R}^N \to [0,1]$ is a uniformly continuous function satisfying the following conditions:*

(i) *there exists $w_0 \in BUC^2(\mathbb{R}^N, \mathbb{R})$ such that*
$$\Omega_0 := \{x \in \mathbb{R}^N : w_0(x) > 0\}$$
is a nonempty smooth bounded and convex domain, and
$$w_0(x) \leq u_0(s,x), \quad \forall (s,x) \in [-\tau, 0] \times \mathbb{R}^N; \tag{1.4}$$

(ii) *there exists $\delta_0 > 0$ such that*
$$|\nabla w_0(x) \nu_{\partial\Omega_0}(x)| \geq \delta_0, \quad \forall x \in H_0 := \partial\Omega_0, \tag{1.5}$$
where $\nu_{\partial\Omega_0}$ denotes the outward unit normal vector to Ω_0 at $x \in H_0$;

(iii) *there exists $v_0 \in BUC^2(\mathbb{R}^N, [0,1))$ such that*
$$\operatorname{supp} v_0 = \overline{\Omega}_0, \tag{1.6}$$
$$u_0(s,x) \leq v_0(x),\ \forall (s,x) \in [-\tau, 0] \times \mathbb{R}^N. \tag{1.7}$$

Theorem 1.2. *Suppose $g(u)$ satisfies* (H1) *and* (H2)*. Let the initial data $u_0(s,x)$ satisfy Assumption 1.1. For every $\varepsilon > 0$, let $u^\varepsilon(t,x) : [-\varepsilon\tau, \infty) \times \mathbb{R}^N \to \mathbb{R}$ be the solution of* (1.1)*. Then*

(i) *for each $c \in (0, c^*)$ and each $t_0 > 0$, we have*

$$\lim_{\varepsilon \to 0^+} \sup_{t \ge t_0} \sup_{x \in \overline{\Omega}_{c,t}} |u^\varepsilon(t,x;u_0) - 1| = 0;$$

(ii) *for each $c > c^*$ and each $t_0 > 0$, we have*

$$\lim_{\varepsilon \to 0^+} \sup_{t \ge t_0} \sup_{x \in \mathbb{R}^N \setminus \Omega_{c,t}} |u^\varepsilon(t,x;u_0)| = 0,$$

where c^* is the minimal wave speed of the corresponding traveling waves of (1.1)*.*

The proof of the above theorem is inspired by the method in [3]. Here we would like to emphasize that the sublinear condition $g(u) \le g'(0)u$ in assumption (H1) is not required explicitly in [3, Theorem 1.3], but it is an essential condition for the existence of monostable traveling wave solutions of (1.1).

Furthermore, we extend the above result to the non-monotone case. Since the equation is lack of the monotonicity, the comparison is invalid, the method in [3] is not applicable. To overcome this issue, we first construct two auxiliary quasi-monotone equations. The propagating interface of the equation is estimated by using the sandwich technique and the comparison theorems of the Cauchy problem.

We modify assumption (H2) into the following one:

- (H2') There exists $\alpha \in (0,1)$ such that $g(\cdot)$ is increasing on $[0,\alpha]$, non-increasing and positive on $[\alpha, +\infty)$. Furthermore, $g(u)$ satisfies
  - (a) $g(u) \le g'(0)u - ku^2$ on $[0,\alpha]$, where $k > 0$ is a fixed constant;
  - (b) there exists a positive constant δ^* such that $g(u) \ge g'(0)u - \rho u^2$ on $[0, \delta^*]$, where $\rho \in (0, \frac{g'(0)}{2\delta^*}]$ is a fixed constant.

Then we construct a function

$$g^+(u) = g'(0)\frac{Mu}{M+u}, \quad \forall u \in [0, +\infty),$$

where the constant $M > 0$ will be given later. By calculating, for any $u \in [0,\alpha]$, it holds

$$\begin{aligned} g^+(u) - g(u) \ge & g'(0)\frac{Mu}{M+u} - g'(0)u + ku^2 \\ &= -\frac{g'(0)u^2}{M+u} + ku^2 \\ &= u^2\Big(k - \frac{g'(0)}{M+u}\Big) \ge 0, \end{aligned}$$

if and only if $M > 0$ is chosen sufficiently large. This implies $g^+(\alpha) \ge g(\alpha) := \max_{u \ge 0} g(u)$. Since $g^+(u)$ is increasing on $[0, +\infty)$, we immediately obtain that $g^+(u) \ge g^+(\alpha) \ge g(u)$ on $[\alpha, +\infty)$. Obviously, $g^+(u) = u$ has a unique positive root $u_+^* = M(g'(0) - 1)$.

Furthermore, we construct another function

$$g^-(u) = g'(0)\frac{Nu}{N+u}, \quad \forall u \in [0, +\infty),$$

where the constant $N > 0$ will be given later. For each $0 \le u \le \delta^*$, it holds

$$\begin{aligned} g(u) - g^-(u) &\ge g'(0)u - \rho u^2 - g'(0)\frac{Nu}{N+u} \\ &= \frac{g'(0)u^2}{N+u} - \rho u^2 \\ &= u^2\Big(\frac{g'(0)}{N+u} - \rho\Big) \\ &\ge u^2\Big(\frac{g'(0)}{N+u} - \frac{g'(0)}{2\delta^*}\Big) \\ &= g'(0)u^2\frac{2\delta^* - N - u}{2\delta^*(N+u)} \ge 0, \end{aligned}$$

if and only if $N > 0$ is chosen sufficiently small. Since $g^-(u)$ is increasing in $u \in [0, +\infty)$, we can further take $N > 0$ small enough such that

$$g^-(u_+^*) = g'(0)\frac{Nu_+^*}{N+u_+^*} \le g'(0)N < g(u), \quad \forall u \in [\delta^*, u_+^*].$$

Thus, we have that $g^-(u) \le g(u) \le g^+(u)$ for all $u \in [0, u_+^*]$. Clearly, $g^-(u) = u$ has a unique positive root u_-^* satisfying $u_-^* < 1 < u_+^*$. Besides, it follows from the definition of $g^\pm(u)$ that $g(u)$ and $g^\pm(u)$ have the same linearization at zero.

From the definition of $g^\pm(u)$, we can obtain the following two auxiliary systems:

$$\begin{gathered} \partial_t u^\varepsilon(t,x) = \varepsilon\Delta u^\varepsilon(t,x) + \frac{1}{\varepsilon}\Big[\int_0^\tau k(s)g^+(u^\varepsilon(t-\varepsilon s,x))ds - u^\varepsilon(t,x)\Big], \\ t > 0,\ x \in \mathbb{R}^N, \\ u^\varepsilon(s,x) = \bar{u}_0\big(\frac{s}{\varepsilon}, x\big), \quad s \in [-\varepsilon\tau, 0],\ x \in \mathbb{R}^N \end{gathered} \tag{1.8}$$

and

$$\begin{gathered} \partial_t u^\varepsilon(t,x) = \varepsilon\Delta u^\varepsilon(t,x) + \frac{1}{\varepsilon}\Big[\int_0^\tau k(s)g^-(u^\varepsilon(t-\varepsilon s,x))ds - u^\varepsilon(t,x)\Big], \\ t > 0,\ x \in \mathbb{R}^N, \\ u^\varepsilon(s,x) = \underline{u}_0\big(\frac{s}{\varepsilon}, x\big), \quad s \in [-\varepsilon\tau, 0],\ x \in \mathbb{R}^N. \end{gathered} \tag{1.9}$$

Theorem 1.3. *Suppose $g(u)$ satisfies* (H1) *and* (H2'). *Let the initial data $u_0(s,x)$ satisfy Assumption 1.1. For every $\varepsilon > 0$, let $u^\varepsilon(t,x) : [-\varepsilon\tau, \infty) \times \mathbb{R}^N \to \mathbb{R}$ be the solution of* (1.1). *Then*

(i) *for each $c \in (0, c^*)$ and each $t_0 > 0$, we have*

$$u_-^* \le \lim_{\varepsilon\to 0^+} \inf_{t\ge t_0} \inf_{x\in\overline{\Omega}_{c,t}} u^\varepsilon(t,x;u_0) \le \lim_{\varepsilon\to 0^+} \sup_{t\ge t_0} \sup_{x\in\overline{\Omega}_{c,t}} u^\varepsilon(t,x;u_0) \le u_+^*;$$

(ii) *for each $c > c^*$ and each $t_0 > 0$, we have*

$$\lim_{\varepsilon\to 0^+} \sup_{t\ge t_0} \sup_{x\in\mathbb{R}^N\setminus\Omega_{c,t}} |u^\varepsilon(t,x;u_0)| = 0,$$

where c^* is the minimal wave speed of the corresponding traveling waves of (1.1).

Note that, the well-posedness of equations (1.1), (1.8) and (1.9) can be proved by a method similar to that of [33, Theorem 2.3], which mainly use the theory of abstract functional differential equations [25, 35], we omit the proof. This paper is organized as follows: In Section 2, we prove the generation and the motion of interface of system (1.1) when the birth function $g(\cdot)$ is non-decreasing. In Section 3, on the basis of the above result, when $g(\cdot)$ is non-monotone, the propagating interface in (1.1) is estimated.

Next we give some notation. Let $X = BUC(\mathbb{R}^N, \mathbb{R})$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}^N$ to $\mathbb{R}$ with the supremum norm $\|\cdot\|_X$. Let $X^+ := \{v \in X : \varphi(x) \geq 0, x \in \mathbb{R}^N\}$. Then X is a Banach lattice under the partial ordering induced by X^+. Let $\mathcal{C} = C([-\tau, 0], X)$ be the Banach space of continuous functions from $[-\tau, 0]$ into X with the supremum norm and let $\mathcal{C}^+ = \{\varphi \in \mathcal{C} : \varphi(s) \in X^+, s \in [-\tau, 0]\}$. Then $\mathcal{C}^+$ is a positive cone of $\mathcal{C}$. Let $\mathcal{C}_0 = C([-\tau, 0], \mathbb{R})$. Usually, we identify an element $\varphi \in \mathcal{C}$ as a function from $[-\tau, 0] \times \mathbb{R}^N$ into $\mathbb{R}$ defined by $\varphi(s, x) = \varphi(s)(x)$. We define

$$[a, b]_{\mathcal{C}} := \{\varphi \in \mathcal{C} : a \leq \varphi(s, x) \leq b, \ \forall (s, x) \in [-\tau, 0] \times \mathbb{R}^N\}.$$

In addition, $[a, b]_{\mathcal{C}_0} := \mathcal{C}_0 \cap [a, b]_{\mathcal{C}}$. For any continuous function $w : [-\tau, T) \to X, T > 0$, we define $w_t \in \mathcal{C}, t \in [0, T)$, by $w_t(s) = w(t + s), \ s \in [-\tau, 0]$. Then $t \mapsto w_t$ is a continuous function from $[0, T)$ to $\mathcal{C}$.

2. Monotone case

In this section we investigate the generation and propagation of interface in equation (1.1), when $g(\cdot)$ is non-decreasing, namely, (H1) and (H2) hold. Some important notation is given at first. We defined the function

$$\tilde{\mathfrak{D}}(t, x) := \begin{cases} -\operatorname{dist}(x, H_{c,t}) & \text{for } x \in \Omega_{c,t}, \\ \operatorname{dist}(x, H_{c,t}) & \text{for } x \in \mathbb{R}^N \backslash \Omega_{c,t}, \end{cases} \tag{2.1}$$

where $\operatorname{dist}(x, H_{c,t})$ is the distance from x to the hypersurface $H_{c,t}$. We remark that $\tilde{\mathfrak{D}} = 0$ on H_c, $|\frac{\partial \tilde{\mathfrak{D}}(t,x)}{\partial x}| = 1$ in a neighborhood of H_c. Let $T > 0$ be given. Choose $\mathfrak{D}_0 > 0$ small enough so that $\tilde{\mathfrak{D}}$ is smooth in the tubular neighborhood of H_c

$$\{(t, x) \in [-\varepsilon\tau, T] \times \mathbb{R}^N : |\tilde{\mathfrak{D}}(t, x)| < 3\mathfrak{D}_0\}.$$

Then, for any $s \in \mathbb{R}$, define a smooth increasing function $\Upsilon(s)$ as

$$\Upsilon(s) := \begin{cases} s & \text{if } |s| \leq \mathfrak{D}_0, \\ -2\mathfrak{D}_0 & \text{if } s \leq -2\mathfrak{D}_0, \\ 2\mathfrak{D}_0 & \text{if } s > 2\mathfrak{D}_0. \end{cases}$$

On the basis of the above definitions, the cut-off signed distance function $\mathfrak{D}(t, x)$ is represented as follows:

$$\mathfrak{D}(t, x) := \Upsilon(\tilde{\mathfrak{D}}(t, x)). \tag{2.2}$$

From the definition of $\mathfrak{D}(t, x)$, we know

$$|\mathfrak{D}(t, x)| < \mathfrak{D}_0 \Rightarrow |\nabla \mathfrak{D}(t, x)| = 1, \tag{2.3}$$

furthermore, the free boundary problem (1.3) implies

$$|\mathfrak{D}(t, x)| < \mathfrak{D}_0 \Rightarrow \partial_t \mathfrak{D}(t, x) + c = 0. \tag{2.4}$$

By the mean value theorem, there exists a constant $\overline{N} > 0$ such that

$$|\partial_t \mathfrak{D}(t, x) + c| \leq \overline{N} |\mathfrak{D}(t, x)|, \quad \forall (t, x) \in [-\varepsilon\tau, T] \times \mathbb{R}^N. \tag{2.5}$$

In addition, there exists a constant $C > 0$ such that

$$|\nabla\mathfrak{D}(t,x)| + |\Delta\mathfrak{D}(t,x)| \le C, \quad \forall (t,x) \in [-\varepsilon\tau, T] \times \mathbb{R}^N. \tag{2.6}$$

2.1. Preliminaries.

Proposition 2.1 (Comparison principle [33, Theorem 2.3.]). *Let $\tau > 0$, $T > 0$ and $g : \mathbb{R} \to \mathbb{R}$ a non-decreasing and continuous function be given. Let $u, v \in C([-\tau, T] \times \mathbb{R}^N)$ be two bounded functions. Assume*

$$\begin{aligned} &(\partial_t - \Delta + 1)u(t,x) - \int_0^\tau k(s)g(u(t-s,x))ds \le 0, \\ &(\partial_t - \Delta + 1)v(t,x) - \int_0^\tau k(s)g(v(t-s,x))ds \ge 0, \end{aligned} \tag{2.7}$$

for almost every $(t,x) \in (0,T) \times \mathbb{R}^N$, and

$$u(s,x) \le v(s,x), \quad \forall (s,x) \in [-\tau, 0] \times \mathbb{R}^N. \tag{2.8}$$

Then $u(t,x) \le v(t,x)$ for all $(t,x) \in [-\tau, T] \times \mathbb{R}^N$.

Bistable approximations of g. For $\eta \in (0,1]$, we introduce a non-decreasing and bounded map $g_\eta : \mathbb{R} \to \mathbb{R}$ of the class C^2 such that

$$\begin{gathered} g_\eta(u) = g(u), \quad \forall u \in [0,1], \\ g_\eta(-\eta) = -\eta, \quad g_\eta'(-\eta) < 1, \\ g_\eta(u) < u, \quad \forall u \in (-\eta, 0) \cup (1, \infty), \\ g_\eta(u) > u, \quad \forall u \in (-\infty, -\eta) \cup (0,1). \end{gathered} \tag{2.9}$$

In addition, we assume that g_η satisfies

$$\forall (\eta, \eta') \in (0,1]^2, \quad \eta < \eta' \ \Rightarrow\ g_{\eta'}(u) \le g_\eta(u), \quad \forall u \in \mathbb{R}. \tag{2.10}$$

Traveling waves. We consider the one dimensional bistable system

$$\begin{gathered} (\partial_t - \Delta + 1)u(t,x) = \int_0^\tau k(s)g_\eta(u(t-s,x))ds, \quad t > 0,\ x \in \mathbb{R}, \\ u(s,x) = \varphi_0 \in [-\eta, 1]_{\mathcal{C}}, \quad \forall s \in [-\tau, 0], \end{gathered} \tag{2.11}$$

which admits the solution $u_\eta \equiv u_\eta(t,x;\varphi_0) : [-\tau, \infty) \times \mathbb{R} \to [-\eta, 1]$. System (2.11) generates a strongly continuous and non-decreasing semiflow $\{Q_\eta(t)\}_{t\ge 0}$ as

$$[Q_\eta(t)\varphi](s,x) = (u_\eta)_t(s,x;\varphi_0),\ \forall (s,x) \in [-\tau, 0] \times \mathbb{R}.$$

From the definition of g_η, for $t \ge 0$, there is $Q_\eta(t)[0,1]_{\mathcal{C}} \subset [0,1]_{\mathcal{C}}$, where $Q(t) := Q_\eta(t)|_{[0,1]_{\mathcal{C}}}$ does not depend upon η, Q_η presents a semiflow generated by a bistable dynamic and Q is a semiflow generated by the corresponding monostable type.

Lemma 2.2 (Bistable traveling waves). *For each $\eta \in (0,1]$, the following results hold:*

- (i) *there exists a unique speed c_η such that (2.11) has a traveling wave solution $(U_\eta, c_\eta) \in C^2(\mathbb{R}) \times \mathbb{R}$ whose profile U_η is non-increasing and satisfies*

$$\begin{gathered} U_\eta''(z) + c_\eta U_\eta'(z) + \int_0^\tau k(s)g_\eta(U_\eta(z + c_\eta s))ds - U_\eta(z) = 0, \quad \forall z \in \mathbb{R}, \\ U_\eta(-\infty) = 1, \quad U_\eta(+\infty) = -\eta; \end{gathered} \tag{2.12}$$

(ii) *there exist two positive constants μ and M such that*

$$\begin{gathered} |1-U_\eta(z)|+|-\eta-U_\eta(-z)|\le Me^{\mu z},\ \forall z\le 0,\\ |U_\eta'(z)|+|U_\eta''(z)|\le Me^{-\mu|z|},\ \forall z\in\mathbb{R}; \end{gathered} \tag{2.13}$$

(iii) *there exists some constant $\gamma>0$ such that, for any $\varphi_0\in[-\eta,1]_{\mathcal{C}}$ with*

$$\liminf_{x\to-\infty}\min_{s\in[-\tau,0]}\varphi_0(s,x)>0,\quad \limsup_{x\to+\infty}\max_{s\in[-\tau,0]}\varphi_0(s,x)<0, \tag{2.14}$$

one can find $C=C(\varphi_0)>0$ and $\xi=\xi(\varphi_0)\in\mathbb{R}$ such that

$$|u_\eta(t,x;\varphi_0)-U_\eta(x-c_\eta t+\xi)|\le Ce^{-\gamma t},\quad \forall(t,x)\in[0,+\infty)\times\mathbb{R}.$$

Proof. Item (i) can be found in [33, Theorem 5.5] (also in [22, Theorem 5.1(iii)]). The proof of (ii), can be found in [34, Theorem 3.5]. The proof of (iii) can be found in [33, Theorem 4.5]. □

We refer to [24] for the existence of the monotone traveling waves of the one dimensional monostable system

$$(\partial_t-\Delta+1)u(t,x)=\int_0^\tau k(s)g(u(t-s,x))ds,\quad t>0,\ x\in\mathbb{R}. \tag{2.15}$$

Lemma 2.3 (Monostable traveling waves [24]). *There exists $c^*>0$ such that equation (2.15) has a traveling wave solution $(U_c,c)\in C^2(\mathbb{R})\times(0,\infty)$ with $0<U_c<1$, if and only if $c\ge c^*$. In addition, the waves are non-increasing for $c\ge c^*$.*

Let (U^*,c^*) be the traveling wave of (2.15) with minimal wave speed, i.e.

$$\begin{gathered} (U^*)''(z)+c^*(U^*)'(z)+\int_0^\tau k(s)g(U^*(z+c^*s))ds-U^*(z)=0,\quad \forall z\in\mathbb{R},\\ U^*(-\infty)=1,\quad U^*(+\infty)=0. \end{gathered} \tag{2.16}$$

Lemma 2.4 (Convergence of speeds). *Let $\{g_\eta^+\}_{\eta\in(0,1]}$ satisfy (2.9) and (2.10). Then the family $\{c_\eta\}_{\eta\in(0,1]}$ is decreasing and*

$$c_\eta\nearrow c^*\quad \text{as } \eta\searrow 0.$$

The proof of the above lemma is similar to that of [3, Lemma 2.4.], we omit it.

2.2. **Generation of interface.** We considering the two differential equations with delay

$$\begin{gathered} \frac{d}{dt}v(t)=\int_0^\tau k(s)g(v_t(-s))ds-v(t),\quad t>0,\\ v_0(\cdot)=\phi(\cdot)\in[0,1]_{\mathcal{C}_0}, \end{gathered} \tag{2.17}$$

and

$$\begin{gathered} \frac{d}{dt}v(t)=\int_0^\tau k(s)g_\eta(v_t(-s))ds-v(t),\quad t>0,\\ v_0(\cdot)=\phi(\cdot)\in[-\eta,1]_{\mathcal{C}_0}. \end{gathered} \tag{2.18}$$

Lemma 2.5. *For each $\phi\in\mathcal{C}_0$, (2.18) has a unique global (mild) solution $v_\eta=v_\eta(\cdot;\phi):[-\tau,\infty)\to\mathbb{R}$ and the semiflow $V_\eta(t)\phi=V_\eta(t;\phi):=(v_\eta)_t(\cdot;\phi)$ is strongly continuous and monotone increasing on $\mathcal{C}_0$. It further satisfies the following properties:*

(i) *for each $t\ge0$, $V_\eta(t)[-\eta,1]_{\mathcal{C}_0}\subset[-\eta,1]_{\mathcal{C}_0}$;*

(ii) *for each* $t \geq 0$, $V_\eta(t)[0,1]_{\mathcal{C}_0} \subset [0,1]_{\mathcal{C}_0}$. *The restriction* $V(t) = V_\eta(t) \mid_{[0,1]_{\mathcal{C}_0}}$ *does not depend upon* η *and, for* $\phi \in [0,1]_{\mathcal{C}_0}$, *the map* $t \mapsto V(t)\phi = V(t;\phi)$ *is the mild solution* $v_t(\cdot;\phi)$ *of* (2.17).

The above lemma follows straightforwardly from [29], we omit its proof.

Lemma 2.6. *The following holds:*

(i) *for* $\phi \in [0,1]_{\mathcal{C}_0} \backslash \{0\}$, *we have* $\lim_{t\to\infty} V(t)\phi = 1$ *in* $\mathcal{C}_0$;

(ii) *there exists* $\delta_1 > 0$, $M > 0$ *and* $\lambda > 0$ *such that, for all* $\phi \in \mathcal{C}_0$,

$$\|1-\phi\|_{L^\infty(-\tau,0)} \leq \delta_1 \Rightarrow \|1 - V(t)\phi\|_{L^\infty(-\tau,0)} \leq Me^{-\lambda t}, \ \forall t \geq 0.$$

Proof. Firstly, we give the proof of case (i). Consider a special situation, if there exists $\zeta \in (0,1)$ such that $\phi(s) \geq \zeta$, for all $s \in [-\tau, 0]$. Since the semiflow corresponding to (2.17) is monotonically increasing and satisfies $V(t)[0,1]_{\mathcal{C}_0} \subset [0,1]_{\mathcal{C}_0}$, we can find a solution with the initial data ζ, i.e., $V(t;\zeta) = v_t(\cdot;\zeta)$. Since $g(\zeta) > \zeta$ and the mapping $t \mapsto v(t;\zeta)$ is non-decreasing, then $\lim_{t\to+\infty} v(t,x) = 1$, which also indicates that $\|V(t)\zeta - 1\|_\infty = \sup_{-\tau\leq s\leq 0} |v(t+s,\zeta)-1| \to 0$ as $t \to +\infty$. Next, we consider the general situation. Since $\phi \in [0,1]_{\mathcal{C}_0}\backslash\{0\}$, there exists $-\tau \leq A < B \leq 0$ and $\beta > 0$ such that

$$\phi(s) \geq \beta \mathbf{1}_{[A,B]}(s), \quad \forall s \in [-\tau, 0].$$

It follows from (2.17) that

$$\frac{d}{dt}(e^t v(t;\phi)) = e^t \int_0^\tau k(s) g(v_t(-s)) ds \geq 0,$$

which implies that $e^t v(t;\phi)$ is non-decreasing with respect to $t > 0$. In addition, for $t \in \big[0, \frac{B-A}{2}\big]$, we have

$$\begin{aligned}\frac{d}{dt}(e^t v(t;\phi)) &= e^t \int_0^\tau k(s) g(v_t(-s)) ds \\ &\geq e^t \int_{-B}^{-\frac{A+B}{2}} k(s) g(\beta) ds \\ &> g(\beta) \int_{-B}^{-\frac{A+B}{2}} k(s) ds.\end{aligned}$$

Integrating the above inequality on $[0, \frac{B-A}{2}]$, we have

$$v\Big(\frac{B-A}{2}, \phi\Big) > \frac{B-A}{2} g(\beta) e^{\frac{A-B}{2}} \int_{-B}^{-\frac{A+B}{2}} k(s) ds > 0.$$

Consequently,

$$e^t v(t;\phi) \geq e^{\frac{B-A}{2}} v\Big(\frac{B-A}{2}, \phi\Big) > 0, \quad \forall t > \frac{B-A}{2} > 0,$$

that is,

$$v(t;\phi) \geq e^{-(t-\frac{B-A}{2})} v\Big(\frac{B-A}{2}, \phi\Big) > 0, \quad \forall t > \frac{B-A}{2} > 0.$$

Through the special case discussed above, we can verify that (i) is holds.

Now we prove (ii). The characteristic equation corresponding to (2.17) around 1 is

$$\Delta(\lambda) := \lambda + 1 - g'(1) \int_0^\tau k(s) e^{-\lambda s} ds = 0.$$

Since $g'(1) < 1$, all roots of $\Delta(\lambda) = 0$ have strictly negative real parts, then we obtain the conclusion (see [17, 31]). This completes the proof. $\square$

Proposition 2.7. *Let $\phi \geq 0$ in $\mathcal{C}_0 \backslash \{0\}$ be given. There exists $\lambda > 0$ such that, for all $\alpha > 0$, there exists $\varepsilon_0 = \varepsilon_0(\alpha) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,*

$$1 - \varepsilon^{\alpha\lambda/2} \leq V(\alpha \ln|\varepsilon| + t; \varepsilon \ln|\varepsilon|\phi)(s) \leq 1, \quad \forall (s,t) \in [-\tau, 0] \times [0, \infty).$$

Proof. Let $\phi \geq 0$ belong to $\mathcal{C}_0 \backslash \{0\}$. Since $g'(0) > 1$, let $\delta \in (0,1)$ and $\rho > 1$ satisfy

$$g(u) \geq \rho u, \ \forall u \in [0, \delta]. \tag{2.19}$$

Choose δ as an initial data, it follows from Lemma 2.4 that, there exist $M > 0$ and $\lambda > 0$ such that

$$0 \leq 1 - V(t; \delta)(s) \leq M e^{-\lambda t}, \quad \forall (s,t) \in [-\tau, 0] \times [0, +\infty). \tag{2.20}$$

Let $\alpha > 0$, choose a sufficiently small $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, there is $\varepsilon|\ln\varepsilon|\phi \in [0, \delta]_{\mathcal{C}_0}$ and $\varepsilon^{\frac{\lambda\alpha}{2}} M < 1$. Note that $\phi \geq 0$ holds in $\mathcal{C}_0 \backslash \{0\}$, then there exists $-\tau < A < B < 0$ and $\beta > 0$ such that

$$\varepsilon|\ln\varepsilon|\phi \geq \varepsilon|\ln\varepsilon|\beta \mathbf{1}_{[A,B]}(s), \quad \forall s \in [-\tau, 0].$$

Applying the argument in Lemma 2.4 and (2.19), there exists $\zeta > 0$ such that, for sufficiently small $\varepsilon > 0$, it holds

$$v_\varepsilon(t) := v(t; \varepsilon|\ln\varepsilon|\phi) \geq \zeta\varepsilon|\ln\varepsilon|, \quad \forall t \in [\tau, 2\tau]. \tag{2.21}$$

Next, for all $t \in (0, 2\tau]$, there is

$$\frac{d}{dt}(e^t v_\varepsilon(t)) = e^t \int_0^\tau k(s) g(\varepsilon|\ln\varepsilon|\phi(t-s)) ds \leq e^{2\tau}\varepsilon|\ln\varepsilon|\|\phi\|_\infty \|g'\|_\infty := C\varepsilon|\ln\varepsilon|.$$

Integrating the above inequality from 0 to $t \in (0, 2\tau]$, for sufficiently small $\varepsilon > 0$, it yields

$$v_\varepsilon(t) \leq 2\tau(\phi(0) + C)\varepsilon|\ln\varepsilon| < \delta.$$

Thus, we define

$$t^\varepsilon := \sup\{t > 2\tau : v_\varepsilon(\tilde{t}) \leq \delta, \quad \forall \tilde{t} \in [2\tau, t]\}.$$

It follows from (2.17) and (2.19) that

$$v'_\varepsilon(t) \geq \rho \int_0^\tau k(s) v_\varepsilon(t-s) ds - v_\varepsilon(t), \quad \forall t \in [2\tau, t^\varepsilon]. \tag{2.22}$$

Since $\rho > 1$, there exists $a > 0$ such that $a + 1 = \rho \int_0^\tau k(s) e^{-as} ds$. Then the mapping $h : \ t \mapsto \mathcal{A}\varepsilon|\ln\varepsilon|e^{at}$, $\mathcal{A} := \zeta/e^{2a\tau}$ satisfies

$$h'(t) = \rho \int_0^\tau k(s) h(t-s) ds - h(t), \quad \forall t \in [2\tau, t^\varepsilon], \tag{2.23}$$

where $h(t) \leq \zeta\varepsilon|\ln\varepsilon|$, $t \in [\tau, 2\tau]$. From (2.21)–(2.23), we obtain

$$v_\varepsilon(t) \geq \mathcal{A}\varepsilon|\ln\varepsilon|e^{at}, \quad \forall t \in [2\tau, t^\varepsilon].$$

It follows from $v_\varepsilon(t^\varepsilon) = \delta$ that

$$t^\varepsilon \leq \frac{1}{a} \ln \frac{\delta}{\mathcal{A}\varepsilon|\ln\varepsilon|}. \tag{2.24}$$

Since the mapping $t \mapsto v_\varepsilon(t)$ is increasing, then, $v_\varepsilon(t^\varepsilon) = \delta$ yields

$$v_\varepsilon(t^\varepsilon + t) \geq \delta, \quad \forall t \in [0, +\infty).$$

From (2.24), for sufficiently small $\varepsilon > 0$, there holds $t^\varepsilon \le \alpha|\ln\varepsilon|$, thus,

$$v_\varepsilon(\alpha|\ln\varepsilon| + t + s) \ge \delta, \quad \forall (s,t) \in [-\tau, 0] \times [0, +\infty).$$

Note that the semiflow corresponding to (2.17) is increasing in $\mathcal{C}_0$, then one has

$$0 \le 1 - v_\varepsilon(\alpha|\ln\varepsilon| + t + s) \le 1 - V_\varepsilon(\alpha|\ln\varepsilon| + t; \delta)(s).$$

Combining this with (2.20), we obtain

$$0 \le 1 - v_\varepsilon(\alpha|\ln\varepsilon| + t + s) \le M e^{-\lambda(\alpha|\ln\varepsilon|+t)} \le M\varepsilon^{\lambda\alpha} \le \varepsilon^{\frac{\lambda\alpha}{2}}.$$

This completes the proof. $\square$

Lemma 2.8. *For each $t > 0$, the map $\phi \in \mathcal{C}_0 \mapsto V_\eta(t;\phi) \in \mathcal{C}_0$ provided in Lemma 2.4 is of class C^2. For each $\phi_0 \in \mathcal{C}_0$ and each $\phi \in \mathcal{C}_0$, the map $t \in [0,\infty) \mapsto \partial_\phi V_\eta(t;\phi_0)\cdot\phi \in \mathcal{C}_0$ is the mild solution of the non-autonomous equation*

$$\begin{gathered} \frac{dv}{dt}(t) = L(t,\phi_0)v_t, \quad t > 0, \\ v(s) = \phi(s), \quad s \in [-\tau, 0], \end{gathered} \tag{2.25}$$

wherein, for each $t > 0$, $L(t,\phi_0) : \mathcal{C}_0 \to \mathbb{R}$ is defined by

$$L(t,\phi_0)\phi := \int_0^\tau k(s) g_\eta'\left(V_\eta(t;\phi_0)(-s)\right)\phi(-s)ds - \phi(0). \tag{2.26}$$

Moreover, for each $\phi_0 \in \mathcal{C}_0$ and each $\phi \in \mathcal{C}_0$, the map $t \in [0,\infty) \mapsto \partial^2_{\phi,\phi} V_\eta(t;\phi_0)\cdot(\phi,\phi)$ is the solution of

$$\begin{gathered} \frac{dv}{dt}(t) = L\left(t,\phi_0\right)v_t + \mathcal{G}\left(t;\phi_0;\phi\right), \quad t > 0, \\ v(s) = 0, \quad s \in [-\tau, 0], \end{gathered}$$

where the map $t \to G(t;\phi_0;\phi)$ is defined by

$$\mathcal{G}(t;\phi_0;\phi) := \int_0^\tau k(s) g_\eta''\left(V_\eta(t;\phi_0)(-s)\right)\left[\partial_\phi V_\eta(t;\phi_0)(-s)\cdot\phi(-s)\right]^2. \tag{2.27}$$

The proof of the above lemma is similar to that of [17, 31], we omit it.

Proposition 2.9. *Assume $g(\cdot)$ satisfies* (H1) *and* (H2).

(i) *There exist $\hat{C}_1 > 0$ and $\gamma_1 > 0$ such that, for all $\phi_0 \in \mathcal{C}_0$,*

$$e^{-\tau-(t+s)} \le \partial_\phi V_\eta(t;\phi_0)1(s) \le \hat{C}_1 e^{\gamma_1(t+s)},$$

for all $(s,t) \in [-\tau,0] \times [0,\infty)$.

(ii) *There exist $\hat{C}_2 > 0$ and $\gamma_2 > 0$ such that, for all $\phi_0 \in \mathcal{C}_0$,*

$$|\partial_{\phi\phi} V_\eta(t;\phi_0)\cdot(1,1)(s)| \le \hat{C}_2 e^{\gamma_2}(t+s),$$

for all $(s,t) \in [-\tau,0] \times [0,\infty)$.

(iii) *There exist $\hat{C}_3 > 0$ and $\gamma_3 > 0$ such that, for all $\phi_0 \in \mathcal{C}_0$,*

$$|\partial_{\phi\phi} V_\eta(t;\phi_0)\cdot(1,1)(s)| \le \hat{C}_3 e^{\gamma_3 t}\partial_\phi V_\eta(t;\phi_0)\cdot 1(s),$$

for all $(s,t) \in [-\tau,0] \times [0,\infty)$.

Proof. (i) Let $\phi_0 \in \mathcal{C}_0$. Firstly, $V_\eta(t)$ is monotonically increasing and satisfies

$$\partial_\phi V_\eta(t;\phi_0)\cdot 1(s) \geq 0, \quad \forall (s,t)\in[-\tau,0]\times[0,+\infty). \tag{2.28}$$

From (2.25) and (2.26), there exists $w(t) := \partial_\phi V_\eta(t;\phi_0)\cdot 1(0)$ satisfying

$$w'(t) \geq -w(t), \quad \forall t \geq 0,$$

i.e., $w(t) \geq e^{-t}$ for all $t \geq 0$. Then, for any $(s,t)\in[-\tau,0]\times[0,+\infty)$ which satisfies $t+s\geq 0$, we have

$$\partial_\phi V_\eta(t;\phi_0)1(s) \geq e^{-(t+s)}.$$

For any $(s,t)\in[-\tau,0]\times[0,+\infty)$ satisfying $t+s<0$, we have

$$\partial_\phi V_\eta(t;\phi_0)1(s) \geq e^{-(\tau+t+s)}.$$

Therefore, we prove that the inequality on the left is valid.

Next, we choose a constant $\ell \gg 1$ such that

$$0 \leq g_\eta'(u) \leq \ell, \quad \forall u \in \mathbb{R}. \tag{2.29}$$

It follows from (2.25) and (2.26) that

$$w'(t) \leq \ell\int_0^\tau k(s)w(t-s)ds - w(t), \quad t>0,\ w(s)=1,\ s\in[-\tau,0]. \tag{2.30}$$

Let $h(t) = e^{(\ell-1)t}$ for all $t>0$. Obviously, $h(t)$ is an increasing function which satisfies $0<h(t-s)\leq h(t)$ for all $s>0$. Note that $\int_0^\tau k(s)\left[h(t-s)-h(t)\right]ds \leq 0$, then we obtain

$$\begin{aligned} h'(t) - \ell\int_0^\tau k(s)h(t-s)ds + h(t) &\geq h'(t) - \ell\int_0^\tau k(s)h(t)ds + h(t) \\ &\geq h'(t) - \ell h(t) + h(t) = 0, \end{aligned} \tag{2.31}$$

where $t>0$, $h(s)\geq 1$, $s\in[-\tau,0]$. Thus, (2.30) and (2.31) indicates $w(t)\leq e^{(\ell-1)t}$ for all $t\geq 0$. As discussed above, the inequality on the right holds.

(ii) It follows from (2.27) and (i) that, there exists a constant $A>0$ such that for any $\phi_0\in\mathcal{C}_0$,

$$|\mathcal{G}(t;\phi;1)| \leq Ae^{2\gamma_1(t-\tau)}, \quad \forall t\geq 0.$$

Thus, function $w(t) := \partial_{\phi\phi}V_\eta(t;\phi_0)\cdot(1,1)(0)$ satisfies

$$w'(t) \leq \ell\int_0^\tau k(s)w(t-s)ds - w(t) + Ae^{2\gamma_1(t-\tau)}, \tag{2.32}$$

where, $t>0$, $w(s)=1$, $s\in[-\tau,0]$. If $\tilde{K}e^{\tilde{\mu}t}$ is a supper solution of (2.32), where the constant $\tilde{K}>0$, $\tilde{\mu}>0$ will be given later, then we have

$$\tilde{\mu} \geq \ell\int_0^\tau k(s)e^{-\tilde{\mu}s}ds - 1 + \frac{A}{\tilde{K}}e^{(2\gamma_1-\tilde{\mu})t-2\gamma_1\tau}, \quad \forall t>0, \tag{2.33}$$

if and only if $\tilde{\mu}>2\gamma_1$ and $\tilde{K}>0$ is chosen sufficiently large. By an argument as in (i), there exist constants $K>0$ and $\mu>0$ such that, for any $\phi_0\in\mathcal{C}_0$, $s\in[-\tau,0]$ and $t\geq 0$, it holds

$$\partial_{\phi\phi}V_\eta(t;\phi_0)\cdot(1,1)(s) \leq Ke^{\mu(t+s)}.$$

Furthermore, since $g_\eta'(u)\geq 0$ for all $u\in\mathbb{R}$, We can get

$$\begin{aligned} w'(t) &\geq \int_0^\tau k(s)g_\eta'\left(V_\eta(t;\phi_0)(-s)\right)\phi(-s)ds - w(t) - Ae^{2\gamma_1(t-\tau)} \\ &\geq -w(t) - Ae^{2\gamma_1(t-\tau)}. \end{aligned}$$

Constructing a sub-solution $-\tilde{K}e^{\tilde{\mu}t}$, it is a lower bound of $w(t)$. In conclusion, we obtain the boundedness of the second derivative.

(iii) This result can be obtained directly from (i) and (ii). $\square$

Proposition 2.10 (Sub-solution)**.** *Assume g satisfies* (H1) *and* (H2). *Let $u_0(s,x)$ and $w_0(x)$ satisfy Assumption 1.1. Then there exist $K>0$, $\alpha>0$ and $\varepsilon_0>0$ such that, for all $\varepsilon\in(0,\varepsilon_0)$,*

$$\max\Big\{0; v_\eta\Big(\frac{t}{\varepsilon}; w_0(x)-\varepsilon K\tau - Kt\Big)\Big\} \le u^\varepsilon(t,x)$$

for all $(t,x)\in[-\varepsilon\tau, \alpha\varepsilon\ln|\varepsilon|]\times\mathbb{R}^N$. Here, $u^\varepsilon(t,x)$ denotes the solution of (1.1), *$v_\eta = v_\eta(\cdot;\phi):[-\tau,\infty)\to\mathbb{R}$ denotes the solution of* (2.18).

Proof. Define

$$\mathcal{L}^\varepsilon_\eta[u](t,x) := \partial_t u(t,x) - \varepsilon\Delta u(t,x) - \frac{1}{\varepsilon}\Big[\int_0^\tau k(s)g_\eta(u(t-\varepsilon s,x))ds - u(t,x)\Big].$$

Obviously, $\mathcal{L}^\varepsilon_\eta[u^\varepsilon](t,x)\equiv 0$. Let

$$\underline{u}(t,x) := v_\eta\Big(\frac{t}{\varepsilon}; w_0(x)-\varepsilon K\tau - Kt\Big),$$

then, for any $t>0$, $x\in\mathbb{R}^N$, we have

$$\begin{aligned}
\mathcal{L}^\varepsilon_\eta[\underline{u}](t,x) &= -V^\varepsilon(t,x)\Big[K+\varepsilon\Delta w_0(x) + \varepsilon\frac{W^\varepsilon(t,x)}{V^\varepsilon(t,x)}|\nabla w_0(x)|^2\Big]\\
&\quad + \frac{1}{\varepsilon}\Big[\Big(\frac{dv_\eta}{dt}+v_\eta\Big)\Big(\frac{t}{\varepsilon}; w_0(x)-\varepsilon K\tau - Kt\Big)\\
&\quad - \int_0^\tau k(s)g_\eta\Big(v_\eta\Big(\frac{t}{\varepsilon}-s; w_0(x)-\varepsilon K\tau - K(t-\varepsilon s)\Big)\Big)ds\Big],
\end{aligned}$$

where

$$\begin{aligned}
V^\varepsilon(t,x) &= \partial_\phi V_\eta\Big(\frac{t}{\varepsilon}; w_0(x)-\varepsilon K\tau - Kt\Big)\cdot 1(0),\\
W^\varepsilon(t,x) &= \partial_{\phi\phi} V_\eta\Big(\frac{t}{\varepsilon}; w_0(x)-\varepsilon K\tau - Kt\Big)\cdot (1,1)(0).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\Big(\frac{dv_\eta}{dt}+v_\eta\Big)\Big(\frac{t}{\varepsilon}; w_0(x)-\varepsilon K\tau - Kt\Big)\\
&\quad - \int_0^\tau k(s)g_\eta\Big(v_\eta\Big(\frac{t}{\varepsilon}-s; w_0(x)-\varepsilon K\tau - K(t-\varepsilon s)\Big)\Big)ds\\
&\le \Big(\frac{dv_\eta}{dt}+v_\eta\Big)\Big(\frac{t}{\varepsilon}; w_0(x)-\varepsilon K\tau - Kt\Big)\\
&\quad - \int_0^\tau k(s)g_\eta\Big(v_\eta\Big(\frac{t}{\varepsilon}-s; w_0(x)-\varepsilon K\tau - Kt\Big)\Big)ds\\
&= \int_0^\tau k(s)g_\eta\Big(v_\eta\Big(\frac{t}{\varepsilon}-s; w_0(x)-\varepsilon K\tau - Kt\Big)\Big)ds\\
&\quad - \int_0^\tau k(s)g_\eta\Big(v_\eta\Big(\frac{t}{\varepsilon}-s; w_0(x)-\varepsilon K\tau - Kt\Big)\Big)ds = 0,
\end{aligned}$$

where we used the monotonicity of v_η and $g_\eta(\cdot)$. It follows from Proposition 2.9 that, for all $\varepsilon \in (0,1)$, $t > 0$, $x \in \mathbb{R}^{\mathbb{N}}$,

$$\begin{aligned}\mathcal{L}^\varepsilon_\eta[\underline{u}](t,x) &\leq -V^\varepsilon(t,x)\Big[K + \varepsilon\Delta w_0(x) + \varepsilon\frac{W^\varepsilon(t,x)}{V^\varepsilon(t,x)}|\nabla w_0(x)|^2\Big] \\ &\leq -V^\varepsilon(t,x)\Big[K - \varepsilon\|\Delta w_0\|_{L^\infty} - \varepsilon\|\nabla w_0\|^2_{L^\infty}\hat{K}e^{\gamma\frac{t}{\varepsilon}}\Big].\end{aligned}$$

For any $\varepsilon \in (0,1)$, $t \in (0, \gamma^{-1}\varepsilon\|\ln\varepsilon\|)$ and $x \in \mathbb{R}^N$, we have

$$\mathcal{L}^\varepsilon_\eta[\underline{u}](t,x) \leq -V^\varepsilon(t,x)\Big[K - \varepsilon\|\Delta w_0\|_{L^\infty} - \varepsilon\|\nabla w_0\|^2_{L^\infty}\hat{C}\Big] \leq 0,$$

if $K > 0$ is sufficiently large.

Next, for all $s \in [-\varepsilon\tau, 0]$, from (1.4), it holds that

$$\underline{u}(s,x) = w_0(x) - \varepsilon K\tau - Ks \leq w_0(x) \leq u_0\big(\frac{s}{\varepsilon}, x\big) = u^\varepsilon(s,x).$$

Finally, the comparison principle in Proposition 2.1 indicates that

$$\underline{u}(t,x) \leq u^\varepsilon(t,x),\ \forall (t,x) \in [-\varepsilon\tau, \gamma^{-1}\varepsilon|\ln\varepsilon| \times \mathbb{R}^N].$$

Using that $u^\varepsilon(t,x) \geq 0$, we then complete the proof. □

Proposition 2.11 (Generation of interface from below)**.** *Let the initial data $u_0(x)$ satisfy Assumption 1.1. Denote by $\mathfrak{D}(0,x)$ the smooth cut-off signed distance function to H_0 (where $d(0,x) < 0$ if and only if $x \in \Omega_0$). Then there exists $\delta_0 > 0$, $\alpha_0 > 0$, $\rho_0 > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and all $(s,x) \in [-\tau, 0] \times \mathbb{R}^N$, we have*

$$1 - \varepsilon^{\rho_0} \leq u^\varepsilon(\alpha_0\varepsilon|\ln\varepsilon| + \varepsilon\tau + \varepsilon s, x) \leq 1,$$

provided that $\mathfrak{D}(0,x) \leq -\delta_0\varepsilon|\ln\varepsilon|$.

Proof. Choose $K > 0$ and $\alpha > 0$ as in Proposition 2.10. Let $\alpha_0 = \alpha/2$, $\rho_0 = \alpha_0\lambda$. For $\phi = \alpha_0 \in \mathcal{C}_0\backslash\{0\}$, choose λ as in Proposition 2.7. From Assumption (ii), the mean value theorem provides the existence of a constant $\delta_0 > 0$, such that for sufficiently small $\varepsilon > 0$, there is

$$\mathfrak{D}(0,x) \leq -\delta_0\varepsilon|\ln\varepsilon| \Rightarrow w_0(x) \geq 4\alpha_0\varepsilon|\ln\varepsilon|$$

For any $-\tau <\leq s \leq 0$, define $\hat{T} = \alpha_0\varepsilon|\ln\varepsilon| + \varepsilon\tau + \varepsilon s \in [\alpha_0\varepsilon|\ln\varepsilon|, \alpha_0\varepsilon|\ln\varepsilon| + \varepsilon\tau]$. Choose x satisfying $\mathfrak{D}(0,x) \leq -\delta_0\varepsilon|\ln\varepsilon|$. When ε is sufficiently small, it holds that $0 \leq \hat{T} \leq 2\alpha_0\varepsilon|\ln\varepsilon| = \alpha\varepsilon|\ln\varepsilon|$ and $w_0(x) - \varepsilon K\tau - K\hat{T} \geq \alpha_0\varepsilon|\ln\varepsilon|$. Then

$$v_\eta(\hat{T}/\varepsilon, w_0(x) - \varepsilon K\tau - K\hat{T}) \geq v_\eta(\hat{T}/\varepsilon, \alpha_0\varepsilon|\ln\varepsilon|) \geq 1 - \varepsilon^{\rho_0},$$

since $v_\eta(t;\phi)$ is an increasing semiflow. By Proposition 2.10, we obtain that

$$u^\varepsilon(\hat{T}, x) \geq v_\eta(\hat{T}/\varepsilon, \alpha_0\varepsilon|\ln\varepsilon|) \geq 1 - \varepsilon^{\rho_0}.$$

This completes the proof. □

2.3. **Propagation of interface. Lower barriers via bistable approximation:** For $\eta \in (0,1]$, we denote (U_η, c_η) as the traveling wave solution of system (2.11), which satisfy

$$\begin{gathered} U_\eta''(z) + c_\eta U_\eta'(z) + \int_0^\tau k(s) g_\eta(U_\eta(z + c_\eta s))ds - U_\eta(z) = 0, \quad \forall z \in \mathbb{R}, \\ U_\eta(-\infty) = 1, \quad U_\eta(0) = 0, \quad U_\eta(+\infty) = -\eta. \end{gathered} \tag{2.34}$$

Define the sub-solution of (2.11) as

$$u_\eta^-(t,x) := U_\eta\Big(\frac{\mathfrak{D}_\eta(t,x) + \varepsilon|\ln\varepsilon| p(t)}{\varepsilon}\Big) - q(t), \tag{2.35}$$

where

$$p(t) = -e^{-\beta t/\varepsilon} + e^{Qt} + P, \tag{2.36}$$

$$q(t) = \sigma\left(\beta e^{-\beta t/\varepsilon} + \varepsilon Q e^{Qt}\right), \tag{2.37}$$

the positive constants β, σ, P, Q are determined in the proof.

Proposition 2.12 (Sub-solution)**.** *There exist positive constants β, σ, Q, for all $P > 1$ and sufficiently small $\varepsilon > 0$, it holds*

$$\varepsilon\mathcal{L}_\eta^\varepsilon[u_\eta^-](t,x) = \varepsilon\partial_t u_\eta^-(t,x) - \varepsilon^2\Delta u_\eta^-(t,x) - \int_0^\tau k(s) g_\eta(u_\eta^-(t-\varepsilon s, x))ds + u_\eta^-(t,x) \le 0,$$

for all $t > 0$, $x \in \mathbb{R}^N$.

Proof. For simplicity we denote

$$z := \frac{\mathfrak{D}_\eta(t,x) + \varepsilon|\ln\varepsilon| p(t)}{\varepsilon}.$$

From the definition of $\mathfrak{D}(t,x)$ in (2.4), we have

$$\mathfrak{D}(t - \varepsilon s, x) = \mathfrak{D}(t,x) + c\varepsilon s + \varepsilon\Theta_\varepsilon(t,x),$$

where $\Theta_\varepsilon(t,x)$ vanishes close to the interface and is $\mathcal{O}(1)$:

$$|\mathfrak{D}(t,x)| \le \mathfrak{D}_0, \|\Theta_\varepsilon\|_{L^\infty} \le B \Rightarrow \Theta_\varepsilon(t,x) = 0 \tag{2.38}$$

for some constant $B > 0$. Next, since $p(t)$ is increasing and $U_\eta(z)$ is decreasing, it holds

$$\begin{aligned} u^-(t - \varepsilon s, x) &= U_\eta\Big(\frac{\mathfrak{D}_\eta(t - \varepsilon s, x) + \varepsilon|\ln\varepsilon| p(t - \varepsilon s)}{\varepsilon}\Big) - q(t - \varepsilon s) \\ &\ge U_\eta\Big(\frac{\mathfrak{D}_\eta(t,x) + \varepsilon|\ln\varepsilon| p(t)}{\varepsilon} + cs + \Theta_\varepsilon(t,x)\Big) - q(t - \varepsilon s). \end{aligned}$$

Since $g_\eta(u)$ is non-decreasing, we have

$$\begin{aligned} &g_\eta(u^-(t - \varepsilon s, x)) \\ &\ge g_\eta\Big(U_\eta\Big(\frac{\mathfrak{D}_\eta(t,x) + \varepsilon|\ln\varepsilon| p(t)}{\varepsilon} + cs + \Theta_\varepsilon(t,x)\Big) - q(t - \varepsilon s)\Big) \\ &= g_\eta\Big(U_\eta\Big(\frac{\mathfrak{D}_\eta(t,x) + \varepsilon|\ln\varepsilon| p(t)}{\varepsilon} + cs + \Theta_\varepsilon(t,x)\Big)\Big) - q(t - \varepsilon s)(g_\eta)'(\theta), \end{aligned}$$

for some constant θ satisfying

$$U_\eta\left(z' + cs + \Theta_\varepsilon\right) - q(t - \varepsilon s) \le \theta \le U_\eta\left(z' + cs\right),$$

where $z' := \frac{\mathfrak{D}_\eta(t,x)+\varepsilon|\ln\varepsilon|p(t)}{\varepsilon}$. Hence, it yields

$$\begin{aligned} g_\eta(u^-(t-\varepsilon s,x)) &\geq g_\eta\left(U_\eta\left(z'+cs+\Theta_\varepsilon\right)\right)-q(t-\varepsilon s)(g_\eta)'(\theta)\\ &\geq g_\eta(U(z'+cs))-q(t-\varepsilon s)(g_\eta)'(\theta)\\ &\quad+\Theta_\varepsilon(t,x)(g_\eta\circ U)'(z'+cs+\omega\Theta_\varepsilon(t,x)) \end{aligned}$$

for some $\omega\in[0,1]$. By calculations, we have

$$\begin{aligned} \varepsilon\mathcal{L}^\varepsilon_\eta[u^-_\eta](t,x) &= (\varepsilon|\ln\varepsilon|p'(t)+\partial_t\mathfrak{D})\cdot U'-\varepsilon q'(t)-U''\cdot(\nabla\mathfrak{D})^2-\varepsilon U'\cdot\Delta\mathfrak{D}\\ &\quad-\Big(\int_0^\tau k(s)g_\eta(u^-_\eta(t-\varepsilon s,x))ds-u^-_\eta(t,x)\Big)\\ &\leq(\varepsilon|\ln\varepsilon|p(t)+\partial_t\mathfrak{D})\cdot U'-\varepsilon q'(t)-U''\cdot(\nabla\mathfrak{D})^2-\varepsilon U'\cdot\Delta\mathfrak{D}\\ &\quad-\int_0^\tau k(s)\Big[g_\eta(U(z'+cs))-q(t-\varepsilon s)(g_\eta)'(\theta)\\ &\quad+\Theta_\varepsilon(t,x)(g_\eta\circ U)'(z'+cs+\omega\Theta_\varepsilon(t,x))\Big]ds+U_\eta(z)-q(t)\\ &=E_1+E_2+E_3, \end{aligned}$$

where

$$\begin{aligned} E_1&=\varepsilon|\ln\varepsilon|p'(t)\cdot U'(z)+\int_0^\tau k(s)q(t-\varepsilon s)g'_\eta(\theta)ds-q(t)-\varepsilon q'(t),\\ E_2&=(\partial_t\mathfrak{D}(t,x)+c-\varepsilon\Delta\mathfrak{D}(t,x))U'(z)+(1-|\nabla\mathfrak{D}(t,x)|^2)U''(z),\\ E_3&=-\int_0^\tau k(s)\Theta_\varepsilon(t,x)(g_\eta\circ U)'(z'+cs+\omega\Theta_\varepsilon(t,x))ds. \end{aligned}$$

Using (2.36) and (2.37), we have

$$\begin{aligned} E_1&=\beta e^{-\frac{\beta}{\varepsilon}t}\Big[|\ln\varepsilon|U'+\sigma\Big(\int_0^\tau k(s)g'_\eta(\theta)\varepsilon e^{\beta s}ds-1+\beta\Big)\Big]\\ &\quad+\varepsilon Qe^{Qt}\Big[|\ln\varepsilon|U'+\sigma\Big(\int_0^\tau k(s)g'_\eta(\theta)\varepsilon e^{-Qs}ds-1-\varepsilon Q\Big)\Big]\\ &=\beta e^{-\beta t/\varepsilon}\mathfrak{e}_1+\varepsilon Qe^{Qt}\mathfrak{e}_2. \end{aligned}$$

From the definition of g_η, we have $g'_\eta(-\eta)<1$ and $g'_\eta(1)<1$. Consequently, we can fix small $\varrho>0$ and $\beta>0$ such that

$$\int_0^\tau k(s)g'_\eta(u)\varepsilon e^{\beta s}ds-1+\beta<0,\quad\forall u\in[-\eta-\varrho,-\eta+\varrho]\cup[1-\varrho,1+\varrho].$$

On the one hand, since $U(-\infty)=1$, $U(+\infty)=-\eta$ and $U_\eta(z'+cs+\Theta_\varepsilon)-q(t-\varepsilon s)\leq\theta\leq U_\eta(z'+cs)$, there exists a sufficient large z_* such that $\theta\in[-\eta-\varrho,-\eta+\varrho]\cup[1-\varrho,1+\varrho]$ once $|z|\geq z_*$ (In order to control $-q(t-\varepsilon\tau)$, we choose a sufficiently small σ). Since $U'(z)\leq 0$, we obtain $\mathfrak{e}_1\leq-\sigma\beta$ in the region $\{|z|>z_*\}$. On the other hand, in the region $\{|z|\leq z_*\}$, we have $U'(z)\leq-\varsigma$ for some $\varsigma>0$, then $\mathfrak{e}_1\leq-\varsigma|\ln\varepsilon|+C$. As a result, it yields $\mathfrak{e}_1\leq-\sigma\beta$. We could get $\mathfrak{e}_2\leq-\sigma\beta$ by a similar argument. Therefore, it holds

$$E_1\leq\Big(\beta e^{-\frac{\beta}{\varepsilon}t}+\varepsilon Qe^{Qt}\Big)(-\sigma\beta)\leq-\sigma\beta\varepsilon Q.$$

To show $\varepsilon\mathcal{L}^\varepsilon_\eta[u^-_\eta](t,x)\leq 0$, we divide the discussion into two situations. On the one hand, when $|\mathfrak{D}(t,x)|<\mathfrak{D}_0$, it follows from (2.3) and (2.4) that $E_2=$

$-\varepsilon\Delta\mathfrak{D}(t,x)U'(z)$. In addition, (2.38) yields $E_3 = 0$. Hence,

$$\varepsilon\mathcal{L}^{\varepsilon}_{\eta}[u^-_\eta](t,x) \le -\sigma\beta\varepsilon Q + \varepsilon\|U'\|_{L^\infty(\mathbb{R})}\cdot\|\Delta\mathfrak{D}(t,x)\|_{L^\infty(\mathbb{R})} \le 0,$$

provided that $Q > 0$ is large enough. On the other hand, when $|\mathfrak{D}(t,x)| \ge \mathfrak{D}_0$, we can use the exponential decay of the derivatives of U to control E_2 and E_3. Indeed in this region, $|z| \ge \mathfrak{D}_0/(2\varepsilon)$. Hence, combining the exponential decay of U' and U'', (2.5) and (2.6), we have a bound

$$|E_2| \le C_2 e^{-C_2\mathfrak{D}_0/(2\varepsilon)} \quad \text{for some } C_2 > 0.$$

Also (2.38) indicates that

$$|z' + cs + \omega\Theta_\varepsilon(t,s)| \ge \mathfrak{D}_0/(2\varepsilon) - c\tau - \omega B \ge \frac{\mathfrak{D}_0}{4\varepsilon},$$

which yields $|E_3| \le C_3 e^{-C_3\frac{\mathfrak{D}_0}{4\varepsilon}}$ for some $C_3 > 0$. Hence,

$$\varepsilon\mathcal{L}^{\varepsilon}_{\eta}[u^-_\eta](t,x) \le -\sigma\beta\varepsilon Q + Ce^{-C\frac{\mathfrak{D}_0}{4\varepsilon}} \le 0,$$

if $0 < \varepsilon \ll 1$. This completes the proof. □

Lemma 2.13. *There exists $P > 1$ such that, for sufficiently small $\varepsilon > 0$, it holds*

$$u^-_\eta(t,x) \le u^\varepsilon(t + \alpha_0\varepsilon|\ln\varepsilon| + \varepsilon\tau, x), \quad \forall -\varepsilon\tau \le t \le 0,\ x \in \mathbb{R}^N,$$

where $\alpha_0\varepsilon|\ln\varepsilon|$ denotes the "generation of interface from below time" appearing in Proposition 2.11.

Proof. We consider two cases. On the one hand, if $\mathfrak{D}(t,x) \ge -\varepsilon|\ln\varepsilon|p(t)$, from the definition of U_η, we have $u^-_\eta(t,x) \le 0$. On the other hand, for any $(t,x) \in [-\varepsilon\tau, 0]\times\mathbb{R}^N$, if $\mathfrak{D}(t,x) < -\varepsilon|\ln\varepsilon|p(t)$, it follows from Proposition 2.11 (Generation of interface) that

$$\mathfrak{D}(0,x) \le -\delta_0\varepsilon|\ln\varepsilon| \Rightarrow 1 - \varepsilon^{\rho_0} \le u^\varepsilon(\alpha_0\varepsilon|\ln\varepsilon| + \varepsilon\tau + \varepsilon t, x) \le 1, \quad t \in [-\varepsilon\tau, 0]. \tag{2.39}$$

Then it holds

$$\begin{aligned}
\mathfrak{D}(0,x) &= \mathfrak{D}(t,x) + \mathcal{O}(t)\\
&\le -\varepsilon|\ln\varepsilon|p(t) + C\varepsilon\tau\\
&\le -\varepsilon(-e^{\beta\tau} + e^{-A\tau} + P) + C\varepsilon\tau\\
&\le -\delta_0\varepsilon|\ln\varepsilon|,
\end{aligned}$$

where $\varepsilon > 0$ is sufficiently small and P is sufficiently large. From (2.39), we just need to prove that $u^-_\eta(t,x) \le 1 - \varepsilon^{\rho_0}$. From the definition of $q(t)$, we obtain that

$$u^-_\eta(t,x) \le 1 - \varepsilon^{\rho_0}.$$

This completes the proof. □

Proof of Theorem 1.2(i). From Proposition 2.12 and Lemma 2.13, by the comparison principle, we obtain

$$u^-_\eta(t - \alpha_0\varepsilon|\ln\varepsilon| - \varepsilon\tau, x) \le u^\varepsilon(t,x), \quad \forall t \ge \alpha_0\varepsilon|\ln\varepsilon| + \varepsilon\tau,\ x \in \mathbb{R}^N. \tag{2.40}$$

Note that $u^-_\eta(t,x)$ is defined in (2.35) and $U_\eta(-\infty) = 1$, then the conclusion in Theorem 1.2(i) can be immediately obtained by Lemma 2.6 and (2.40). □

Upper barriers: Let (U^*, c^*) denote the traveling wave of (2.15) with minimal wave speed, which is given in Lemma 2.3. It satisfies

$$\begin{gathered}(U^*)''(z) + c^*(U^*)'(z) + \int_0^\tau k(s)g(U^*(z+c^*s))ds - U^*(z) = 0, \quad \forall z \in \mathbb{R},\\ (U^*)'(z) \le 0, \quad \forall z \in \mathbb{R},\\ U^*(-\infty) = 1, \quad U^*(+\infty) = 0.\end{gathered} \tag{2.41}$$

Next, we study the upper estimate on $u^\varepsilon(t,x)$ of system (1.1).

Proposition 2.14 (Super-solution)**.** *There exists $\kappa \in \mathbb{R}$ such that, for all $\varepsilon > 0$ small enough,*

$$u^\varepsilon(t,x) \le U^*\Big(\frac{\mathfrak{D}(0,x) - c^*t}{\varepsilon} - \kappa\Big), \quad \forall (t,x) \in [-\varepsilon\tau, \infty) \times \mathbb{R}^N.$$

Proof. From Assumption 1.1(iii), we know that $\|v_0\|_\infty < 1$, so there exists $\kappa \in \mathbb{R}$ such that $\|v_0\|_\infty \le U^*(c^*\tau - \kappa)$. Without loss of generality, here we choose $\kappa = 0$, then

$$\|v_0\|_\infty \le U^*(c^*\tau). \tag{2.42}$$

Let $x_0 \in \partial\Omega_0 = H_0$ and n_0 be the outward unit normal vector to H_0 at x_0, then define

$$u^+(t,x) := U^*\Big(\frac{(x-x_0)\cdot n_0 - c^*t}{\varepsilon}\Big)$$

and $z = \frac{(x-x_0)\cdot n_0 - c^*t}{\varepsilon}$. By calculating, it yields

$$\begin{aligned} &\varepsilon\mathcal{L}_\eta^\varepsilon[u^+](t,x)\\ &= \partial_t u^+(t,x) - \varepsilon\Delta u^+(t,x) - \frac{1}{\varepsilon}\left(\int_0^\tau k(s)g(u^+(t-\varepsilon s, y))ds - u^+(t,x)\right)\\ &= -\frac{c^*}{\varepsilon}(U^*)'(z) - \frac{1}{\varepsilon}(U^*)''(z) - \frac{1}{\varepsilon}\Big(\int_0^\tau k(s)g(U^*(z+c^*s))ds - U^*(z)\Big) = 0, \end{aligned}$$

where $(t,x) \in (0,+\infty) \times \mathbb{R}^N$.

Next, we prove that

$$u^\varepsilon(s,x) = u_0\big(\frac{s}{\varepsilon}, x\big) \le U^*\Big(\frac{(x-x_0)\cdot n_0 - c^*s}{\varepsilon}\Big) = u(s,x),$$

for all $(s,x) \in [-\varepsilon\tau, 0] \times \mathbb{R}^N$. It follows from Assumption 1.1(iii) that

$$\bar{u}_0\big(\frac{s}{\varepsilon}, x\big) \le v_0(x).$$

With the decrease of U^*, we have

$$U^*\Big(\frac{(x-x_0)\cdot n_0}{\varepsilon} + c^*\tau\Big) \le U^*\Big(\frac{(x-x_0)\cdot n_0 - c^*s}{\varepsilon}\Big).$$

Thus, we need only to prove that

$$v_0(x) \le U^*\Big(\frac{(x-x_0)\cdot n_0}{\varepsilon} + c^*\tau\Big), \quad \forall x \in \mathbb{R}^N. \tag{2.43}$$

When $(x-x_0)\cdot n_0 \le 0$, (2.42) implies

$$\|v_0\|_\infty \le U^*(c^*\tau) \le U^*\Big(\frac{(x-x_0)\cdot n_0}{\varepsilon} + c^*\tau\Big).$$

When $(x-x_0)\cdot n_0 > 0$, unequality (1.6) and the convexity of Ω_0 imply that $v_0(x) = 0$, thus (2.43) is obvious.

Finally, from the comparison principle, we obtain that

$$u^\varepsilon(t,x) \le U^*\Big(\frac{(x-x_0)\cdot n_0 - c^* t}{\varepsilon}\Big), \quad \forall (t,x) \in [-\varepsilon\tau, \infty] \times \mathbb{R}^N$$

for every $x_0 \in \partial\Omega_0$. This completes the proof. □

Proof of Theorem 1.2(ii). We obtain the conclusion from Proposition 2.14. □

3. Non-monotone case

Since the auxiliary systems (1.8) and (1.9) are monotonically increasing, the conclusion in Section 2 is applicable. Hence, we can get the following lemmas.

Lemma 3.1 (Upper system)**.** *Suppose $g(u)$ satisfies* (H1) *and* (H2')*, and the initial data $\bar u_0(s,x) : [-\tau, 0] \times \mathbb{R}^N \to [0, u_+^*]$ is continuous and satisfies Assumption 1.1. For each $\varepsilon > 0$, let $u_+^\varepsilon(t,x) : [-\varepsilon\tau, \infty) \times \mathbb{R}^N \to \mathbb{R}$ be the solution of* (1.8)*. Then the following convergence results hold:*

(i) *for each $c \in (0, c^*)$ and each $t_0 > 0$, we have*

$$\lim_{\varepsilon\to 0^+} \sup_{t\ge t_0} \sup_{x\in\overline{\Omega}_{c,t}} |u_+^* - u_+^\varepsilon(t,x;\bar u_0)| = 0, \quad c \in (0,c^*),$$

(ii) *for each $c > c^*$ and each $t_0 > 0$, we have*

$$\lim_{\varepsilon\to 0^+} \sup_{t\ge t_0} \sup_{x\in\mathbb{R}^N\setminus\Omega_{c,t}} |u_+^\varepsilon(t,x;\bar u_0)| = 0, \quad c > c^*,$$

where c^* is the minimal wave speed of the corresponding traveling waves of (1.8).

Lemma 3.2 (Lower system)**.** *Suppose g satisfies* (H1) *and* (H2)$'$*, and the initial data $\underline{u}_0(s,x) : [-\tau, 0] \times \mathbb{R}^N \to [0, u_-^*]$ is continuous and satisfies Assumption 1.1. For each $\varepsilon > 0$, let $u_-^\varepsilon(t,x) : [-\varepsilon\tau, \infty] \times \mathbb{R}^N \to \mathbb{R}$ be the solution of* (1.9)*. Then the following convergence results hold:*

(i) *for each $c \in (0, c^*)$ and each $t_0 > 0$, we have*

$$\lim_{\varepsilon\to 0^+} \sup_{t\ge t_0} \sup_{x\in\overline{\Omega}_{c,t}} |u_-^* - u_-^\varepsilon(t,x;\underline{u}_0)| = 0, \quad c \in (0,c^*),$$

(ii) *for each $c > c^*$ and each $t_0 > 0$, we have*

$$\lim_{\varepsilon\to 0^+} \sup_{t\ge t_0} \sup_{x\in\mathbb{R}^N\setminus\Omega_{c,t}} |u_-^\varepsilon(t,x;\underline{u}_0)| = 0, \quad c > c^*,$$

where c^* is the minimal wave speed of the corresponding traveling waves of (1.9).

Next, we give a comparison principle whose proof can be found in [25, 35, 32].

Lemma 3.3. *Suppose g satisfies* (H1) *and* (H2')*, and for any $u_0 \in \mathcal{C}_{[0,u_+^*]}$,* (1.1)*,* (1.8) *and* (1.9) *have unique solutions $u^\varepsilon(t,x;u_0)$, $u_+^\varepsilon(t,x;u_0)$ and $u_-^\varepsilon(t,x;u_0)$ with $u^\varepsilon, u_+^\varepsilon, u_-^\varepsilon \in C([-\varepsilon\tau,\infty] \times \mathbb{R}^N)$, respectively. In addition, for any $u_0, \underline{u}_0, \bar u_0 \in \mathcal{C}_{[0,u_+^*]}$, if $\underline{u}_0 \le u_0 \le \bar u_0$, then $0 \le u_-^\varepsilon(t,x;\underline{u}_0) \le u^\varepsilon(t,x;u_0) \le u_+^\varepsilon(t,x;\bar u_0)$ for all $t \ge 0$, $x \in \mathbb{R}^N$.*

Proof of Theorem 1.3. Let $\underline{u}_0(s,x) = \min\{u_0(s,x), u_-^\varepsilon\}$ for all $(s,x) \in [-\tau,0]\times\mathbb{R}^N$. Then by Lemma 3.3, we have

$$u_-^\varepsilon(t,x;\underline{u}_0) \le u_-^\varepsilon(t,x;u_0) \le u^\varepsilon(t,x;u_0) \le u_+^\varepsilon(t,x;u_0)$$

for any $(t,x) \in [-\varepsilon\tau,\infty)\times\mathbb{R}$. Now we consider two cases to complete the proof.

Case (i): For each $c \in (0,c^*)$ and each $t_0 > 0$, it follows from Lemma 3.1(i) and Lemma 3.2(i) that

$$u_-^* \le \lim_{\varepsilon\to 0^+} \inf_{t\ge t_0} \inf_{x\in\overline{\Omega}_{c,t}} u_-^\varepsilon(t,x;u_0),$$

$$\lim_{\varepsilon\to 0^+} \sup_{t\ge t_0} \sup_{x\in\overline{\Omega}_{c,t}} u_+^\varepsilon(t,x;u_0) \le u_+^*,$$

which further implies that

$$u_-^* \le \lim_{\varepsilon\to 0^+} \inf_{t\ge t_0} \inf_{x\in\overline{\Omega}_{c,t}} u_-^\varepsilon(t,x;u_0) \le \lim_{\varepsilon\to 0^+} \sup_{t\ge t_0} \sup_{x\in\overline{\Omega}_{c,t}} u_+^\varepsilon(t,x;u_0) \le u_+^*.$$

Case (ii): For each $c > c^*$ and each $t_0 > 0$, it follows from Lemma 3.1(ii) that

$$\lim_{\varepsilon\to 0^+} \sup_{t\ge t_0} \sup_{x\in\mathbb{R}^N\setminus\Omega_{c,t}} |u_+^\varepsilon(t,x;u_0)| = 0.$$

Then it is clear that

$$\lim_{\varepsilon\to 0^+} \sup_{t\ge t_0} \sup_{x\in\mathbb{R}^N\setminus\Omega_{c,t}} |u^\varepsilon(t,x;u_0)| = 0.$$

This completes the proof. □

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