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SOLVING SINGULAR EVOLUTION PROBLEMS IN SUB-RIEMANNIAN GROUPS VIA DETERMINISTIC GAMES

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Abstract. In this manuscript, we prove the existence of viscosity solutions to singular parabolic equations in Carnot groups. We develop the analysis by constructing appropriate deterministic games adapted to the algebraic and differential structures of Carnot groups. We point out that the proof of existence does not require a comparison principle and it is based on an Arzela-Ascoli-type theorem.

1. Introduction

In the previous decades, there has been a special interest in the study of partial differential equations in non-Euclidean frameworks. In this work, we study the existence of viscosity solutions for singular parabolic equations in Carnot groups, via two-person deterministic games.

To motivate the main results, we consider a family of surfaces $M_t \subset \mathbb{R}^N$, $t \geq 0$, given as the zero-level set of a function u:

$$M_t = \{p \in \mathbb{R}^N : u(t,p) = 0\}.$$

If we are interested in the movement of M_t by horizontal mean curvature, then the function u is a solution of the PDE

$$u_t(t,p) = \sum_{i,j=1}^{m_1} \Big(\delta_{ij} - \frac{X_i u X_j u}{\sum_{i=1}^{m_1}(X_i u)^2}\Big) X_i X_j u. \tag{1.1}$$

This model is known as the horizontal mean curvature flow equation (see [18, 13] for a derivation in Euclidean spaces, and [11, 12] for the corresponding discussion in Carnot groups). The main result of the paper is Theorem 2.1 below, where we establish the existence of viscosity solution to (1.1) by employing two-person deterministic games (see [15] for a stochastic approach to mean curvature flow in sub-Riemannian geometries). We point out that our results apply to a large class of singular equations including (1.1).

Existence and comparison results for Carnot groups are less usual than in the Euclidean framework. We highlight that even if one writes equations in Carnot groups in terms of the Euclidean gradient and the Hessian of the unknown and try to apply the Euclidean theory (for instance from [14]), this does not always work. Indeed, the above procedure may introduce degenerate points in the equations or

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may cause the loss of appropriate continuity. A classical example is the following: consider the sub-elliptic Laplacian operator in the Heisenberg group

$$\Delta_0 := X_1^2 + X_2^2$$

where X_1 and X_2 are given as in (3.3). In terms of Euclidean derivatives, the sub-elliptic Laplace equation may be written as

$$-\operatorname{tr}\left(\begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \\ 2y & -2x & 4(x^2+y^2) \end{pmatrix} \nabla^2 u\right) = 0.$$

Observe that the matrix

$$\begin{pmatrix} 1 & 0 & 2y \\ 0 & 1 & -2x \\ 2y & -2x & 4(x^2+y^2) \end{pmatrix}$$

is not uniformly elliptic for all (x, y, z). The interested reader may consult [27, Remark 1] where we exhibited an example for which the operator in terms of the Euclidean derivatives does not satisfies well-known assumptions on uniform continuity needed and frequently used in the theory of Euclidean viscosity solutions. However, the comparison derived intrinsically in the sub-Rimannian setting in [27] applied to the given example. We also refer the reader to Remark 2.3 in the present article.

In sub-Riemannian structures, the singularity of the equation may appear at points where part of the gradient (the horizontal gradient) vanishes. This is not the case of the Euclidean context, where the singularity comes from points where the full gradient vanishes. This facet of sub-Riemannian structures yields difficulties in the study of singular equations. In spite of these facts, there are some interested findings in Carnot groups. For elliptic and uniformly elliptic equations in the Heisenberg group $\mathbb{H}$, comparison results are given in [22], based on a sub-elliptic version of the Euclidean Crandall-Ishii Lemma (see [14] for details). At this point we quote the works [1] and [24] where the authors propose various form of partial nondegeneracy to weaken the uniform ellipticity assumption and apply their results to some sub-elliptic second order equations. Also, in [6], results related to infinite harmonic functions in the Heisenberg group were established. In the general case of Carnot groups, we find the work [5] for the p-Laplacian operator. In the setting of vector fields in $\mathbb{R}^N$ (with its standard group structure), we refer the reader to the paper [3]. Regarding parabolic equations in Carnot groups, we mention [4], where comparison results for *admissible* operators in the Heisenberg group where obtained. The word *admissible* refers to continuous and proper operators $\mathcal{F} = \mathcal{F}(t, p, u, \eta, \mathcal{X})$ which satisfy the following: for each $t \in [0, T]$, there is a modulus of continuity $\omega : [0, \infty] \to [0, \infty]$ so that

$$\mathcal{F}(t, q, r, \tau\eta, \mathcal{Y}) - \mathcal{F}(t, p, r, \tau\eta, \mathcal{X}) \le \omega\big(d_C(p, q) + \tau\|\eta\|^2 + \|\mathcal{X} - \mathcal{Y}\|\big),$$

where $\tau > 0$. Hence, the results are not valid for singular equations. In [25], the author provides existence and uniqueness results for the Gauss curvature flow equation of graph in Carnot group in unbounded domains, generalizing the available results in the literature. For singular equations, in [8] it was covered the case of the parabolic p-Laplacian, but the structure of this equation in largely used in the derivation of the uniqueness principle. A remarkable progress was done in [16], where the authors proved existence and a comparison principle (needed for existence) for radially symmetric viscosity solutions for the horizontal mean curvature flow equation in the first-order Heisenberg group. The symmetry refers to solutions $u = u(t, p_1, p_2, p_3)$ for which it holds

$$u(t, p_1, p_2, p_3) = u(t, \tilde{p}_1, \tilde{p}_2, p_3) \quad \text{whenever } p_1^2 + p_2^2 = \tilde{p}_1^2 + \tilde{p}_2^2.$$

We also point out the reference [2] for uniqueness of viscosity solutions of mean curvature flow in two sub-Riemannian structures.

In [16], the existence of viscosity solutions is obtained via deterministic games. Here, we extend the findings of [16] to more general singular parabolic equations, by adapting the reasoning of [21] and [19] to the structure of Carnot groups. We point out that, unlike [16], our existence result does not require comparison of solutions and it is based on an Arzela-Ascoli-type theorem from [23]. As a final comment, we mention that the uniqueness of the solutions constructed here constitutes an open problem.

The organization of this article is as follows. In Section 2, we introduce the type of equations studied in the paper and the main result concerning existence of solutions. In the end of the section, we provide some applications of our results. In Section 3, we shall introduce the necessary background and notation on Carnot groups as well as the notion of viscosity solutions by means of parabolic jets. In the next Section 4, we provide the proof of the existence result. We end the paper with an Appendix where we prove a technical lemma needed in the analysis of the deterministic games.

2. Main result of the paper and assumptions

In this work, we study the existence of solutions for initial-value problems of the form

$$\begin{gathered} u_t + \mu u + \mathcal{F}(t, p, \nabla_{\mathcal{G},0} u, \nabla^{2,*}_{\mathcal{G},0} u) = 0, \quad \text{on } (0,T) \times \mathbb{G}, \\ u(0,p) = \psi(p), \quad \text{with } p \in \mathbb{G}. \end{gathered} \tag{2.1}$$

Here, $T > 0$ is fixed and $\mu \geq 0$ is a parameter. We refer the reader to Section 3 for notation and basic definitions involving Carnot groups. We use the following assumptions:

(A1) $\psi \in BC(\mathbb{G})$ and for each $\delta > 0$ there are smooth approximations $\psi_\delta^+, \psi_\delta^-$ with bounded right- and left-invariant horizontal derivatives of first and second order so that

$$\psi - \delta \leq \psi_\delta^- \leq \psi \leq \psi_\delta^+ \leq \psi + \delta.$$

(A2) $\mathcal{F} : [0,T] \times \mathbb{G} \times (\mathbb{R}^{m_1} \backslash \{0\}) \times \mathcal{S}^{m_1}(\mathbb{R}) \to \mathbb{R}$ is continuous.

(A3) $\lambda_0 := \sup_\eta |\mathcal{F}(t, p, \eta, \mathcal{O})| < \infty$.

(A4) There exists a positive constant λ_1 such that

$$\mathcal{F}(t, p, \eta, \mathcal{X}) - \mathcal{F}(t, p, \eta, \hat{\mathcal{X}}) \leq \frac{\lambda_1^2}{2} \mathcal{E}^+(\hat{\mathcal{X}} - \mathcal{X}),$$

where $\mathcal{E}^+(\mathcal{X}) := \max\{0, \text{ maximum eigenvalue of } \mathcal{X}\}$, and there exists a modulus of continuity $\omega = \omega(r) = O(r)$ as $r \to 0^+$ such that:

$$\mathcal{F}(t, p, \eta, \mathcal{X}) - \mathcal{F}(s, q, \eta, \mathcal{X}) \leq \omega\big(|s - t| + |p \cdot q^{-1}|_{\mathbb{G}}\big), \tag{2.2}$$

for all $(t,p), (s,q) \in [0,T] \times \mathbb{G}$, $\eta \in \mathbb{R}^{m_1} \setminus \{0\}$, and $\mathcal{X} \in S^{m_1}(\mathbb{R})$.

(A5) For any $r, R > 0$, there exists a modulus of continuity $\omega_{r,R}$ such that

$$\mathcal{F}(t, p, \hat{\eta}, \mathcal{X}) - \mathcal{F}(t, p, \eta, \mathcal{X}) \leq \omega_{r,R}(\|\hat{\eta} - \eta\|), \quad \text{if } \|\hat{\eta}\|, \ \|\eta\| \geq r, \ \|\mathcal{X}\| \leq R.$$

(A6) $\mathcal{F}_*(t,p,0,\mathcal{O}) = \mathcal{F}^*(t,p,0,\mathcal{O}) = 0$.

The main result of this article read as follows. of the paper is the following existence result.

Theorem 2.1. *Under assumptions* (A1)–(A6)*, equation* (2.1) *has a viscosity solution in* $BUC([0,T]\times\mathbb{G})$.

Remark 2.2. Assumption (A1) on the datum may be satisfied for $\psi \in BC(\mathbb{G})$ being constant outside a compact set. That is the setting of [16].

Remark 2.3. In this article, assumptions (A4) and (A5) will be usually applied to:

$$\eta = \nabla_{\mathcal{G},0}\phi(p) \quad \text{and} \quad \mathcal{X} = \nabla^{2,*}_{\mathcal{G},0}\phi(p),$$

for some smooth ϕ (we do not write the dependence on t). In terms of Euclidean derivatives, there are matrix fields A and M (see [8, Lemma 3.2]) so that

$$\nabla_{\mathcal{G},0}\phi(p) = A(p)\nabla\phi(p) \quad \text{and} \quad \nabla^{2,*}_{\mathcal{G},0}\phi(p) = A(p)\nabla^2\phi(p)A(p) + M(p),$$

where $\nabla\phi(p)$ and $\nabla^2\phi(p)$ denotes the Euclidean gradient and Hessian of ϕ at p. Hence in order to consider sub-elliptic equations in terms on Euclidean derivatives, it is natural to introduce the operator

$$\mathcal{G}(t,p,\eta_E,X_E) := \mathcal{F}(t,p,A(p)\eta_E, A(p)X_EA(p)+M(p)), \quad \eta_E\in\mathbb{R}^N, X_E\in S^N(\mathbb{R})$$

which necessarily depends on t and p (so the existence result from [19] is not applied). Moreover, hypothesis (A5) does not imply, in general, [19, (F4)], since the norm of $A(p)$ is not uniformly bounded in $\mathbb{G}$.

2.1. **Some applications.**

Mean curvature flow equation. If $M \subset \mathbb{G}$ is a smooth hypersurface, we define $\Sigma(M)$ as the set of characteristic points of M, that is, the points $p\in M$ where the horizontal distribution at p is contained in the tangent space of M at p. The horizontal mean curvature flow is the flow $t\to M_t$ in which each point $p(t)\notin\Sigma(M_t)$ in the evolving surface moves along the horizontal normal with speed given by the horizontal mean curvature. The equation, outside the characteristic set, may be written as (1.1). For (1.1), the singular operator is given by

$$\mathcal{F}_{MCF}(\eta,\mathcal{X}) = -\operatorname{tr}\Big[\Big(I - \frac{\eta\otimes\eta}{\|\eta\|^2}\Big)\mathcal{X}\Big],$$

for $\eta\in\mathbb{R}^{m_1}\setminus\{0\}$ and $\mathcal{X}\in S^{m_1}(\mathbb{R})$. Hence, the existence of solutions follows from Theorem 2.1.

Parabolic infinite Laplacian. The infinite Laplacian is connected with the problem of finding minimal Lipschitz extensions, called absolute minimizers. In a Carnot group, we say that a Lipschitz function u in $\overline{\Omega}$ is an absolute minimizer if for every $V\subset\Omega$ and every Lipschitz function h in V such that $u=v$ on ∂V, it holds

$$\|\nabla_{\mathcal{G},0}u\|_{L^\infty(V)} \le \|\nabla_{\mathcal{G},0}v\|_{L^\infty(V)}.$$

It has been established independently in [2] and [28] that absolutely minimizers are viscosity solutions of the infinite Laplace equation. We consider the parabolic and normalized counterpart

$$\partial_t u(t,p) = \frac{1}{\sum_{i=1}^{m_1}(X_iu)^2}\sum_{i,j=1}^{m_1} X_iuX_jX_iX_ju. \tag{2.3}$$

Hence, the singular operator is

$$\mathcal{F}_{PIL}(\eta,\mathcal{X}) = -\frac{1}{\|\eta\|^2}\langle \mathcal{X}\eta,\eta\rangle,$$

for $\eta \in \mathbb{R}^{m_1} \setminus \langle 0\rangle$ and $\mathcal{X} \in S^{m_1}(\mathbb{R})$. Hence, the existence of solutions is derived from our analysis.

3. Preliminaries

3.1. **Carnot groups.** Let $\mathbb{G}$ be a connected and simply connected Lie group, whose Lie algebra $\mathcal{G}$ is real and N-dimensional. We say that $\mathbb{G}$ is a Carnot group of step $l \geq 1$ if $\mathcal{G}$ has a stratification, that is, there exist vector spaces $V_1,\dots,V_l$ such that

$$\mathcal{G} = V_1 \oplus \dots \oplus V_l, \quad [V_1,V_i] = V_{i+1}, \quad 1 \leq i \leq l-1, \quad [V_i,V_l] = 0, \quad i = 1,\dots,l.$$

Here, $[V_1,V_i]$ stands for the linear subspace generated by the vectors $[X,Y]$, where $X \in V_1$ and $Y \in V_i$. In particular, $\mathbb{G}$ is nilpotent. Choose a Riemannian metric with respect to which the V_i are mutually orthogonal. Let $m_i = \dim V_i$, for $i = 1,\dots,l$ and consider $h_r = m_1 + \dots + m_r$, $0 \leq r \leq l$, with $h_0 = 0$. Choose an orthonormal basis of V_i of left-invariant vector fields X_j, $j = h_{i-1}+1,\dots,h_i$. Thus, the dimension of $\mathbb{G}$ as a manifold is $N = h_l = m_1 + \dots m_l$. The exponential map $\exp : \mathcal{G} \to \mathbb{G}$ is a global diffeomorphism and may be used to define exponential coordinates φ in $\mathbb{G}$ as follows: any $p \in \mathbb{G}$ may be written uniquely as

$$p = \exp\big(p_1X_1 + \dots + p_NX_N\big),$$

and thus we may put $\varphi(p) = (p_1,\dots,p_N)$. In this way, we identify $\mathbb{G}$ with $(\mathbb{R}^N,\cdot)$, where the group law $\cdot$ is given by the Campbell-Hausdorff formula [9] as

$$\varphi_i(p\cdot q) = \varphi_i(p) + \varphi_i(q) + R_i(p,q), \quad i = 1,\dots,N, \tag{3.1}$$

where R_i depends only on φ_k for $k < i$. In what follows, we just write p_i for $\varphi_i(p)$. We sometimes use the notation

$$p = (p_{1,1},\dots,p_{l,m_l}).$$

The first layer V_1 spanned by the vector fields $X_1,\dots,X_{m_1}$ plays an important role in the theory and it is called the horizontal distribution. Thus, for every p in $\mathbb{G}$:

$$V_{1,p} = \operatorname{span}\big\{X_{1,p},\dots,X_{m_1,p}\big\}.$$

Metric structure on $\mathbb{G}$. If $\gamma : I = [0,1] \to \mathbb{G}$ is an absolutely continuous curve in $\mathbb{G}$ that satisfies

$$\gamma'(t) \in V_{1,\gamma(t)}, \quad \text{for a. e. } t \in I,$$

we call γ a horizontal path. We define the Carnot-Carathéodory distance on $\mathbb{G}$ by

$$d_C(p,q) = \inf\big\{\|\gamma'(t)\| : \gamma'(t) \in V_{1,\gamma(t)},\ \forall t \in I,\ \gamma(0) = p,\ \gamma(1) = q\big\},$$

where $\|\cdot\|$ is the norm induced by the Riemann structure on $\mathcal{G}$. Since Carnot groups satisfy the Hörmander's condition, we get by Chow's Theorem, that d is well-defined. It is well-known that the topology induced by d_C is equivalent to the Euclidean topology. However, d_C is not bi-Lipschitz equivalent to the Euclidean distance (see [26]).

Calculus on Carnot groups. Consider the Carnot group $\mathbb{R} \times \mathbb{G}$ where we add $\partial/\partial t$ to the horizontal frame as X_0. For any $1 \leq k \leq l$, we say that $u : \mathbb{R} \times \mathbb{G} \to \mathbb{R}$ belongs to $\mathcal{C}^k_{\rm sub}$ if it is continuous and $X^I u$ is continuous for $I = (i_0, i_1, \dots, i_N)$ so that

$$d(I) = i_0 d_0 + i_1 d_1 + \dots + i_N d_N \leq k,$$

where $d_0 = 1$ and $d_m = j$ if the corresponding vector field belongs to V_j, $j = 1, \dots, l$.

The full (spacial) gradient with respect to the Carnot frame $\{X_1, \dots, X_N\}$ will be given by

$$\nabla_{\mathcal{G}} u(p) = \sum_{i=1}^{N} (X_{i,p} u) X_{i,p}, \quad p \in \mathbb{G}.$$

We shall also consider the horizontal and second order horizontal gradients of u,

$$\nabla_{\mathcal{G},0} u(p) = \sum_{i=1}^{h_1} (X_{i,p} u) X_{i,p}, \quad \text{and} \quad \nabla_{\mathcal{G},1} u(p) = \sum_{i=h_1+1}^{h_2} (X_{i,p} u) X_{i,p}.$$

The symmetrized horizontal Hessian matrix, denoted by $\nabla^{2,*}_{\mathcal{G},0} u$, has entries

$$\nabla^{2,*}_{\mathcal{G},0} u_{ij} = \frac{1}{2}\big(X_i X_j u + X_j X_i u\big), \quad i, j = 1, \dots, m_1,$$

The stratified Taylor expansion of a $\mathcal{C}^2_{\rm sub}$-function u at $(t,p) \in \mathbb{R} \times \mathbb{G}$ reads as (see [17, Theorem 1.42] or[9, Exercise 6, Chapter 20]):

$$\begin{aligned} &u(s,q) \\ &= u(t,p) + u_t(t,p)(s-t) + \big\langle \nabla_{\mathcal{G},0} u(t,p), (p^{-1}\cdot q)_1 \big\rangle + \big\langle \nabla_{\mathcal{G},1} u(t,p), (p^{-1}\cdot q)_2 \big\rangle \\ &\quad + \frac{1}{2}\big\langle \nabla^{2,*}_{\mathcal{G},0} u(t,p)(p^{-1}\cdot q)_1, (p^{-1}\cdot q)_1 \big\rangle + o(d_C(p,q)^2 + |s-t|). \end{aligned} \tag{3.2}$$

Here, $(p^{-1}\cdot q)_1$ and $(p^{-1}\cdot q)_2$ denote the projection of $p^{-1}\cdot q$ onto V_1 and V_2, respectively. We remark that if instead of choosing a left-invariant frame, we consider a right-invariant basis of the Lie algebra, then we may also define right horizontal derivatives of first and second order (see [17]).

Example 3.1. The simplest example of a Carnot group is the Euclidean space with the usual norm $(\mathbb{R}^N, |\cdot|)$. This is a Carnot group of step 1.

Example 3.2. One of the most familiar Carnot groups is the Heisenberg group $\mathbb{H}$, whose background manifold is $\mathbb{R}^{2n+1}$. Given two points $p = (p_1, \dots, p_{2n+1})$ and $q = (q_1, \dots, q_{2n+1})$ in $\mathbb{H}$, we define a group operation by

$$p \cdot q = \Big(p_1 + q_1, \dots, p_{2n} + q_{2n}, p_{2n+1} + q_{2n+1} + \frac{1}{2}\sum_{i=1}^{2n}(p_i q_{i+n} - p_{i+n} q_i)\Big)$$

The standard basis of left invariant vector fields of the Heisenberg Lie algebra, denoted by $\mathcal{H}$, is given by

$$\begin{aligned} X_j &= \partial_j - \frac{p_{n+j}}{2}\partial_{2n+1}, \quad j = 1, \dots, n; \\ Y_j &= \partial_{j+n} + \frac{p_j}{2}\partial_{2n+1}, \quad j = 1, \dots, n; \\ T &= \partial_{2n+1}. \end{aligned} \tag{3.3}$$

Note that $[X_j, Y_j] = T$ for $j = 1, \dots, n$. Then the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$ satisfy the Hörmander's condition. In this case, the stratification of the Lie algebra is given by

$$\mathcal{H} = V_1 \oplus V_2,$$

where $V_1 = \operatorname{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ and $V_2 = \operatorname{span}\{T\}$. Hence, $\mathbb{H}$ is a step-2 group. For applications of the Heisenberg group to quantum mechanics, the interested reader may consult the monograph [10].

Example 3.3. The Engel group $\mathbb{E}^4$ is the Carnot group whose Lie algebra is

$$\mathcal{G} = V_1 \oplus V_2 \oplus V_3,$$

where $V_1 = \operatorname{span}\{X_1, X_2\}$, $V_2 = \operatorname{span}\{X_3\}$ and $V_3 = \operatorname{span}\{X_4\}$, with the Carnot frame

$$\begin{aligned}
X_1 &= \partial_1 - \frac{p_2}{2}\partial_3 - \Big(\frac{p_3}{2} + \frac{p_2}{12}(p_1 + p_2)\Big)\partial_4 \\
X_2 &= \partial_2 + \frac{p_1}{2}\partial_3 - \Big(\frac{p_3}{2} - \frac{p_1}{12}(p_1 + p_2)\Big)\partial_4 \\
X_3 &= \partial_3 + \frac{1}{2}(p_1 + p_2)\partial_4 \\
X_4 &= \partial_4,
\end{aligned} \tag{3.4}$$

$p = (p_1, \dots, p_4)$. The Engel group is a Carnot group of step 3. For more details on the Engel group see [20].

3.2. **Jets and viscosity solutions in Carnot groups.** We shall recall the definition of parabolic jets in Carnot groups and the notion of viscosity solutions.

Parabolic jets. Let u be an upper-semicontinuous function in $[0, T] \times \mathbb{G}$. We define the parabolic superjet of u at the point (t, p) as

$$\begin{aligned}
\mathcal{P}^{2,+}u(t,p) = \Big\{&(a, \eta, \xi, \mathcal{X}) \in \mathbb{R} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times S^{m_1}(\mathbb{R}) \quad \text{such that} \\
&u(s,q) \le u(t,p) + a(s-t) + \langle \eta, (p^{-1} \cdot q)_1 \rangle + \langle \xi, (p^{-1} \cdot q)_2 \rangle \\
&+ \frac{1}{2}\langle \mathcal{X}(p^{-1} \cdot q)_1, (p^{-1} \cdot q)_1 \rangle + o\big(d_c(p,q)^2 + |s-t|\big) \text{ as } (s,q) \to (t,p) \Big\}.
\end{aligned}$$

Similarly, if v is lower semicontinuous in $[0, T] \times \mathbb{G}$, we define the parabolic subjet $\mathcal{P}^{2,-}v(t,p)$. It is known that parabolic jets may be seen as appropriate derivatives of test functions touching the given function by above or below. More precisely, if u is upper semicontinuous, we consider

$$\begin{aligned}
\mathcal{K}^{+,2}u(t,p) = \Big\{&\big(\phi_t(t,p), \nabla_{\mathcal{G},0}\phi(t,p), \nabla_{\mathcal{G},1}\phi(t,p), \nabla^{2,*}_{\mathcal{G},0}\phi(t,p)\big) \text{ so that } \phi \text{ is } \mathcal{C}^2_{\rm sub} \\
&\text{and } (u-\phi)(s,q) \le (u-\phi)(t,p) \text{ for all } (s,q) \text{ close to } (t,p)\Big\}
\end{aligned}$$

and similarly define $\mathcal{K}^{-,2}v(t,p)$ for test function touching the lower semicontinuous function v from below. Hence it follows that

$$\begin{aligned}
\mathcal{P}^{2,+}u(t,p) &= \mathcal{K}^{2,+}u(t,p), \\
\mathcal{P}^{2,-}v(t,p) &= \mathcal{K}^{2,-}v(t,p).
\end{aligned} \tag{3.5}$$

Finally, we shall also consider the theoretic closure of the sets defined above. We define $\overline{\mathcal{P}}^{2,+}u(t,p)$ as the set of $(a, \eta, \xi, \mathcal{X})$ in $\mathbb{R} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times S^{m_1}(\mathbb{R})$ so that there exists a sequence $(t_n, p_n, a_n, \eta_n, \xi_n, \mathcal{X}_n)$ converging to $(t, p, a, \eta, \mathcal{X})$ satisfying $(a_n, \eta_n, \mathcal{X}_n) \in \mathcal{P}^{2,+}u(t_n, p_n)$ for all n. In a similar way, we define $\overline{\mathcal{P}}^{2,-}v(t,p)$.

Viscosity solutions. Let $\Omega \subset \mathbb{G}$ be a domain. Observe that in the following definition, the operator $\mathcal{F}$ does not depend on $\xi \in \mathbb{R}_{m_2}$.

Definition 3.4. An upper semicontinuous function $u : [0,T] \times \overline{\Omega} \to \mathbb{R}$ is a subsolution to the equation

$$u_t + \mathcal{F}(t, p, u, \nabla_{\mathcal{G},0} u, \nabla^{2,*}_{\mathcal{G},0} u) = 0 \tag{3.6}$$

in $(0,T) \times \Omega$ if for every $(t,p) \in (0,T) \times \Omega$ and every subjet $(a, \eta, \xi, \mathcal{X}) \in \overline{\mathcal{P}}^{2,+}u(t,p)$ it holds

$$a + \mathcal{F}_*(t, p, u(t,p), \eta, \mathcal{X}) \le 0. \tag{3.7}$$

Here, the subscript $*$ stands for the lower semicontinuous envelope of $\mathcal{F}$. Similarly, we say that a lower semicontinuous function $v : [0,T] \times \overline{\Omega} \to \mathbb{R}$ is a supersolution to (3.6) if for every $(t,p) \in (0,T) \times \Omega$ and every superjet $(a, \eta, \xi, \mathcal{Y}) \in \overline{\mathcal{P}}^{2,-}v(t,p)$ it holds

$$a + \mathcal{F}^*(t, p, u(t,p), \eta, \mathcal{Y}) \ge 0.$$

Analogously, the superscript $*$ stands for the upper semicontinuous envelope of $\mathcal{F}$. Finally, we say that a continuous function is a viscosity solution to (3.6) if it is a viscosity sub and supersolution.

4. Deterministic games: proof of Theorem 2.1

In this section we employ two-person deterministic games to prove the existence of solutions to

$$u_t + \mu u + \mathcal{F}(t, p, \nabla_{\mathcal{G},0} u, \nabla^{2,*}_{\mathcal{G},0} u) = 0, \quad \text{on } (0,T) \times \mathbb{G},$$
$$u(0,p) = \psi(p), \quad \text{with } p \in \mathbb{G}.$$

under the assumptions listed in Theorem 2.1.

Remark 4.1. From (A3) and (A4), we conclude that $\mathcal{F}$ has at most linear growth (and at least linear decay). In fact, there exists $C = C(\lambda_0, \lambda_1)$ such that

$$|\mathcal{F}(t, p, \eta, \mathcal{X})| \le C(1 + \|\mathcal{X}\|),$$

for $(t, p, \eta, \mathcal{X}) \in [0,T] \times \mathbb{G} \times (\mathbb{R}^{m_1} \backslash \{0\}) \times \mathcal{S}^{m_1}(\mathbb{R})$. In addition, from (A4), $\mathcal{F}$ is (degenerate) elliptic, since $\mathcal{F}(\cdot, \hat{\mathcal{X}}) \le \mathcal{F}(\cdot, \mathcal{X})$ for $\hat{\mathcal{X}} \ge \mathcal{X}$.

We first describe the setting of the game. There are two players, Player I and Player II. Let $T > 0$ be the maturity of the game. For each $\varepsilon \in (0,1)$ let m be the number of steps,

$$m := \Big[\frac{T}{\varepsilon^2}\Big],$$

where $[\cdot]$ is the integer part function. The players' choices are the following:

- the initial position is $p_0 = p$ at $t_0 = 0$;
- Player I chooses a pair $(\eta_0, \mathcal{X}_0) \in (\mathbb{R}^{m_1} \backslash \{0\}) \times \mathcal{S}^{m_1}(\mathbb{R})$, with

$$\|\eta_0\| \le \varepsilon^{-1/4}, \quad \|\mathcal{X}_0\| \le \varepsilon^{-1/2};$$

- for these choices of Player I, Player II chooses a horizontal direction:
$$q_0 = (\nu_0, 0) \in \mathbb{G}, \quad \text{with } \nu_0 \in \mathbb{R}^{m_1} \text{ and } |q_0|_{\mathbb{G}} \le \varepsilon^{-1/4},$$
where the Carnot gauge is defined as:
$$|p|_{\mathbb{G}} := \sum_{j=1}^{l} \sum_{i=1}^{m_j} |p_{i,j}|^{1/j} \approx d_C(p,0). \tag{4.1}$$
- Player I moves from p_0 to $p_1 := p_0 \cdot \delta_\varepsilon(q_0)$, where the dilatation δ_ε is given by $[\delta_\varepsilon(p)]_{i,j} = \varepsilon^i p_{i,j}$;
- the above steps are repeated m times;
- at the maturity time T, Player I is at the final position p_m and pays to Player II the amount:
$$\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^m \psi(p_m) + \sum_{i=0}^{m-1} \Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{i+1} R^\varepsilon(T - i\varepsilon^2, p_i, q_i, \eta_i, \mathcal{X}_i), \tag{4.2}$$
where R^ϵ is the running cost defined in (4.5), and p_i, q_i, η_i and $\mathcal{X}_i$ are the choices of the players at the i-th step.

The value u^ε of the game is obtained by considering that Player I has the objective of minimize (4.2) and Player II wants to maximize it.

We extend now the set up of the game to any maturity t and we formalize the definition of u^ε. Take $t \in [0,T]$ and consider the partition
$$[0,T] = \{0\} \cup \Big(\cup_{k=1}^m \left((k-1)\varepsilon^2, k\varepsilon^2\right]\Big). \tag{4.3}$$
Then if $t \neq 0$, there is a unique $k_t \in \{1, 2, \dots, m\}$ such that $t \in ((k_t-1)\varepsilon^2, k_t\varepsilon^2]$.

If $t = 0$, we define
$$u^\varepsilon(t,p) := \psi(p).$$
When $t \neq 0$, we have
$$\begin{aligned} u^\varepsilon(t,p) := \inf_{\eta_0, \mathcal{X}_0} \sup_{q_0} \dots \inf_{\eta_{k_t-1}, \mathcal{X}_{k_t-1}} \sup_{q_{k_t-1}} \Big\{ \Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{k_t} \psi(p_{k_t}) \\ + \sum_{i=0}^{k_t-1} \Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{i+1} R^\varepsilon(t - i\varepsilon^2, p_i, q_i, \eta_i, \mathcal{X}_i)\Big\}, \end{aligned} \tag{4.4}$$
where
$$R^\varepsilon(t, p_j, q_j, \eta_j, \mathcal{X}_j) := -\varepsilon\langle \eta_j, \nu_j\rangle - \frac{\varepsilon^2}{2}\langle \mathcal{X}_j \nu_j, \nu_j\rangle - \varepsilon^2 \mathcal{F}\left(t, p_j, \eta_j, \mathcal{X}_j\right). \tag{4.5}$$
We introduce a rigorous definition for the values of the game based on the Dynamic Programming Principle.

Definition 4.2. For $\psi \in BC(\mathbb{G})$, we define inductively:
$$u^\varepsilon(0,p) = \psi(p)$$
and for $t \in ((k_t-1)\varepsilon^2, k_t\varepsilon^2]$:
$$u^\varepsilon(t,p) = \frac{1}{1+\mu\varepsilon^2} \inf_{\eta, \mathcal{X}} \sup_q \left\{ u^\varepsilon(t-\varepsilon^2, p \cdot \delta_\varepsilon(q)) + R^\varepsilon(t, p, q, \eta, \mathcal{X})\right\}.$$
When $k_t = 1$, $u^\varepsilon(t - \varepsilon^2, p \cdot \delta_\varepsilon(q))$ is understood as $u^\varepsilon(0,p)$.

We start with two technical lemmas regarding some Lipschitz regularity of the value functions for smooth initial datum ψ.

Lemma 4.3. *Let ψ be smooth and such that the right and left horizontal derivatives of first and second order are bounded in $\mathbb{G}$. Then, there is a constant $C = C[\psi] > 0$ such that for all $p, \hat{p} \in \mathbb{G}$ and $t \in [0, T]$:*

$$|u^\varepsilon(t,p) - u^\varepsilon(t,\hat{p})| \le C|p \cdot \hat{p}^{-1}|_{\mathbb{G}} + k_t \varepsilon^2 \omega\left(|p \cdot \hat{p}^{-1}|_{\mathbb{G}}\right),$$

where ω is the modulus of continuity from $(\mathcal{F}3)$.

Proof. We proceed by induction. When $t = 0$ ($k_t = 0$), we have by the boundedness of the horizontal derivatives and the stratified mean value theorem [17, Theorem 1.41] (modified for right-invariant vector fields, see Remark after [17, Theorem 1.37]) that there is $C = C[\psi] > 0$ so that

$$|u^\varepsilon(0,p) - u^\varepsilon(0,\hat{p})| = |\psi(p) - \psi(\hat{p})| \le C|p \cdot \hat{p}^{-1}|_{\mathbb{G}}.$$

Assuming for $t \in ((k_t - 1)\varepsilon^2$ and $k_t\varepsilon^2]$, $k_t \in \{1, \dots, m-1\}$ that

$$|u^\varepsilon(t,p) - u^\varepsilon(t,\hat{p})| \le C|p \cdot \hat{p}^{-1}|_{\mathbb{G}} + k_t \varepsilon^2 \omega\left(|p \cdot \hat{p}^{-1}|_{\mathbb{G}}\right),$$

for $\tilde{t} = t + \varepsilon^2$ we have

$$\begin{aligned}
&u^\varepsilon(\tilde{t} - \varepsilon^2, p \cdot \delta_\varepsilon(q)) - u^\varepsilon(\tilde{t} - \varepsilon^2, \hat{p} \cdot \delta_\varepsilon(q)) + R^\varepsilon(\tilde{t}, p, q, \eta, \mathcal{X}) - R^\varepsilon(\tilde{t}, \hat{p}, q, \eta, \mathcal{X}) \\
&\le C|p \cdot \hat{p}^{-1}|_{\mathbb{G}} + k_t \varepsilon^2 \omega\left(|p \cdot \hat{p}^{-1}|_{\mathbb{G}}\right) + \varepsilon^2 \omega\left(|p \cdot \hat{p}^{-1}|_{\mathbb{G}}\right) \\
&\le C|p \cdot \hat{p}^{-1}|_{\mathbb{G}} + k_{\tilde{t}} \varepsilon^2 \omega\left(|p \cdot \hat{p}^{-1}|_{\mathbb{G}}\right),
\end{aligned}$$

where $k_{\tilde{t}} := k_t + 1$. Thus

$$\begin{aligned}
&u^\varepsilon(\tilde{t} - \varepsilon^2, p \cdot \delta_\varepsilon(q)) + R^\varepsilon(\tilde{t}, p, q, \eta, \mathcal{X}) \\
&\le C|p \cdot \hat{p}^{-1}|_{\mathbb{G}} + k_{\tilde{t}} \varepsilon^2 \omega\left(|p \cdot \hat{p}^{-1}|_{\mathbb{G}}\right) + u^\varepsilon(\tilde{t} - \varepsilon^2, \hat{p} \cdot \delta_\varepsilon(q)) + R^\varepsilon(\tilde{t}, \hat{p}, q, \eta, \mathcal{X}).
\end{aligned}$$

Taking $\sup_q$ and then $\inf_{\eta,\mathcal{X}}$ we derive

$$|u^\varepsilon(\tilde{t},p) - u^\varepsilon(\tilde{t},\hat{p})| \le C|p \cdot \hat{p}^{-1}|_{\mathbb{G}} + k_{\tilde{t}} \varepsilon^2 \omega\left(|p \cdot \hat{p}^{-1}|_{\mathbb{G}}\right).$$

□

Lemma 4.4. *Let ψ be smooth and such that the right and left horizontal derivatives of first and second order are bounded in $\mathbb{G}$. Then, there is $C = C(\psi, \lambda_0, \lambda_1) > 0$ such that for all $p \in \overline{\Omega}$ and t,*

$$|u^\varepsilon(t,p) - u^\varepsilon(t - \varepsilon^2, p)| \le C\Big(\frac{1}{1 + \mu\varepsilon^2}\Big)^{k_t} \varepsilon^2 + (k_t - 1)\varepsilon^2 \omega(\varepsilon^2),$$

where ω is the modulus of continuity from (A4).

Proof. We proceed again by induction. Since ψ is smooth, and has bounded derivatives, we derive from the stratified Taylor formula (3.2) and the fact that δ_ε is horizontal that there is $C[\psi] > 0$ so that

$$\begin{aligned}
&\left|\psi(p \cdot \delta_\varepsilon(q)) - \psi(p) - \langle \nabla_{\mathcal{G},0}\psi(p), [\delta_\varepsilon(q)]_1 \rangle - \frac{1}{2} \langle \nabla^{2,*}_{\mathcal{G},0}\psi(p)[\delta_\varepsilon(q)]_1, [\delta_\varepsilon(q)]_1 \rangle\right| \\
&\le C[\psi]\varepsilon^{3/2} \cdot \sup_{|z|_{\mathbb{G}} \le b\varepsilon^{3/4}, d(I)=2} |X^I \psi(p \cdot z) - X^I \psi(p)| \quad \text{for some universal } b > 0 \\
&\le C[\psi]\varepsilon^2,
\end{aligned} \tag{4.6}$$

where we have used the stratified mean value theorem [17, Theorem 1.41] for the second order derivatives. Hence for $t \in (0, \varepsilon^2]$:

$$
\begin{aligned}
&u^\varepsilon(t,p) - \psi(p) \\
&\le \Big(\frac{1}{1+\mu\varepsilon^2}\Big) \inf_{\eta,\mathcal{X}} \sup_q \Big\{ \varepsilon\langle(\nabla_{\mathcal{G},0}\psi(p) - \eta), \nu\rangle \\
&\quad + \frac{\varepsilon^2}{2}\langle\big(\nabla^{2,*}_{\mathcal{G},0}\psi(p) - \mathcal{X}\big)\nu, \nu\rangle - \varepsilon^2\mathcal{F}(t,p,\eta,\mathcal{X}) + C[\psi]\varepsilon^2\Big\} + \frac{\mu\varepsilon^2}{1+\mu\varepsilon^2}\psi(p) \\
&\le \Big(\frac{1}{1+\mu\varepsilon^2}\Big) \inf_{\eta,\mathcal{X}} \Big\{ \varepsilon^{3/4}\|\nabla_{\mathcal{G},0}\psi(p) - \eta\| \\
&\quad + \frac{\varepsilon^{3/2}}{2}\mathcal{E}^+\left(C_0[\psi]I - \mathcal{X}\right) + C\varepsilon^2(1 + \|\mathcal{X}\|) + C[\psi]\varepsilon^2\Big\} + C_0[\psi]\varepsilon^2,
\end{aligned}
$$

where

$$
C_0[\psi] := \max\big[\|\psi\|_\infty, \|\nabla_{\mathcal{G},0}\psi\|_\infty, \|\nabla^{2,*}_{\mathcal{G},0}\psi\|_\infty\big] \tag{4.7}
$$

and $I \in S^N$ denotes the identity matrix. Let $\varepsilon > 0$ be small enough so that $C_0[\psi] \le \varepsilon^{-1/4}$, then we can choose $(\eta, \mathcal{X}) = (\nabla_{\mathcal{G},0}\psi(p), C_0[\psi]I)$ (in the case $\nabla_{\mathcal{G},0}\psi(p) = 0$, take an approximating sequence $0 \ne \eta_n \to 0$) to obtain

$$
u^\varepsilon(t,p) - \psi(p) \le [C(1 + C_0[\psi]) + C[\psi]]\Big(\frac{1}{1+\mu\varepsilon^2}\Big)\varepsilon^2. \tag{4.8}
$$

Next, we show the lower bound. Similarly to the above arguments, for any q with $|q|_{\mathbb{G}} \le \varepsilon^{-1/4}$ we have

$$
\begin{aligned}
&(1+\mu\varepsilon^2)(u^\varepsilon(t,p) - \psi(p)) \\
&\ge \inf_{\eta,\mathcal{X}} \Big\{ \varepsilon\langle(\nabla_{\mathcal{G},0}\psi(p) - \eta), \nu\rangle + \frac{\varepsilon^2}{2}\langle(-C_0[\psi]I - \mathcal{X})\nu, \nu\rangle \\
&\quad - \varepsilon^2\mathcal{F}(t,p,\eta,\mathcal{X}) - C[\psi]\varepsilon^2\Big\} - C_0[\psi]\varepsilon^2.
\end{aligned}
$$

Now, we apply Lemma 5.1 from the Appendix with $\hat{\eta} = \nabla_{\mathcal{G},0}\psi(p)$, $\hat{\mathcal{X}} = -C_0[\psi]I$ and $R_0 := C_0[\psi]$ and $K = 1$. Then. choosing an appropriate $q = \overline{q}(\epsilon, \eta, \hat{\eta}, \mathcal{X}, \hat{\mathcal{X}})$, we have in the case $\|\nabla_{\mathcal{G},0}\psi(p)\| \ge 1$, that

$$
\begin{aligned}
&(1+\mu\varepsilon^2)(u^\varepsilon(t,p) - \psi(p)) \\
&\ge -\varepsilon^2\mathcal{F}^*(t,p,\nabla_{\mathcal{G},0}\psi(p), -C_0[\psi]I) - h_1(\varepsilon^{1/4})\varepsilon^2 - C[\psi]\varepsilon^2 - C_0[\psi]\varepsilon^2,
\end{aligned}
$$

and if $\|\nabla_{\mathcal{G},0}\psi(p)\| \le 1$,

$$
(1+\mu\varepsilon^2)(u^\varepsilon(t,p) - \psi(p)) \ge -\varepsilon^2\mathcal{F}^*(t,p,0,-C_0[\psi]I) - C[\psi]\varepsilon^2 - C_0[\psi]\varepsilon^2,
$$

for all sufficiently small $\varepsilon \le \min[\varepsilon_1, \varepsilon_2]$. As in (4.8), we have

$$
u^\varepsilon(t,p) - \psi(p) \ge -\big[C(1 + C_0[\psi]) + h_1(\varepsilon^{1/4}) + C[\psi]\big]\Big(\frac{1}{1+\mu\varepsilon^2}\Big)\varepsilon^2. \tag{4.9}
$$

Combining (4.8) and (4.9), we obtain

$$
|u^\varepsilon(t,p) - \psi(p)| \le C\Big(\frac{1}{1+\mu\varepsilon^2}\Big)\varepsilon^2. \tag{4.10}
$$

Suppose now that

$$
|u^\varepsilon(t-\varepsilon^2, p) - u^\varepsilon(t,p)| \le C\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{k_t}\varepsilon^2 + (k_t - 1)\varepsilon^2\omega(\varepsilon^2).
$$

Now, taking $\tilde{t} = t + \varepsilon$,

$$\begin{aligned} &u^\varepsilon(t, p\cdot\delta_\varepsilon(q)) - u^\varepsilon(t-\varepsilon^2, p\cdot\delta_\varepsilon(q)) + R^\varepsilon(\tilde{t}, p, \eta, \mathcal{X}) - R^\varepsilon(\tilde{t}-\varepsilon^2, p, \eta, \mathcal{X}) \\ &\le C\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{k_t}\varepsilon^2 + (k_t-1)\varepsilon^2\omega(\varepsilon^2) + \varepsilon^2\omega(\varepsilon^2). \end{aligned}$$

where we have used (A4) in the latter inequality. Taking $\sup_q$ and $\inf_{\eta,\mathcal{X}}$, we derive

$$\begin{aligned} u^\varepsilon(\tilde{t}-\varepsilon^2, p) - u^\varepsilon(\tilde{t}, p) &\le C\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{k_t+1}\varepsilon^2 + k_t\varepsilon^2\omega(\varepsilon^2) \\ &= C\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{k_{\tilde{t}}}\varepsilon^2 + (k_{\tilde{t}}-1)\varepsilon^2\omega(\varepsilon^2). \end{aligned}$$

A similar argument is applied to $u^\varepsilon(\tilde{t}, p) - u^\varepsilon(\tilde{t}-\varepsilon^2, p)$. □

In the next results we shall appeal to the following constant. For a positive integer K, let us set

$$C^\varepsilon[\psi, K] := C(1 + C_0[\psi]) + h_K(\varepsilon^{1/4}) + C[\psi],$$

where C is the constant from Remark 4.1, C_0 is given by (4.7), $C[\psi]$ by (4.6) and h_K is the modulus in Lemma 5.1 from the Appendix.

The next proposition establishes the convergence of the value functions.

Proposition 4.5. *There exist a subsequence $\{\varepsilon_j\}_j$ converging to 0 and a continuous function u so that $u^{\varepsilon_j} \to u$ locally uniformly as $j \to \infty$. Moreover, $u \in BUC([0,T]\times\mathbb{G})$ and $u(0,p) = \psi(p)$ for all p.*

Proof. For $\delta > 0$ consider the regularizations $\psi_\delta^\pm \in \mathcal{C}^2$ of ψ from assumption (A1),

$$\psi - \delta \le \psi_\delta^- \le \psi \le \psi_\delta^+ \le \psi + \delta. \tag{4.11}$$

Lemma 4.3 gives the next estimate for $t \in (0, \varepsilon^2]$,

$$|u^\varepsilon_{\psi_\delta^\pm}(t,p) - u^\varepsilon_{\psi_\delta^\pm}(t,\hat{p})| \le C_\delta|p\cdot\hat{p}^{-1}|_{\mathbb{G}} + \varepsilon^2\omega(|p\cdot\hat{p}^{-1}|_{\mathbb{G}}), \tag{4.12}$$

for all $p, \hat{p}$ and all sufficiently small ε, and where $u^\varepsilon_{\psi_\delta^\pm}$ denotes the value function with $\psi = \psi_\delta^\pm$. Combining (4.12) and (4.11), we conclude that

$$|u^\varepsilon(t,p) - u^\varepsilon(t,\hat{p})| \le C_\delta|p\cdot\hat{p}^{-1}|_{\mathbb{G}} + \varepsilon^2\omega(|p\cdot\hat{p}^{-1}|_{\mathbb{G}}) + \delta,$$

for $p, \hat{p} \in \overline{\Omega}$ and all $\varepsilon \le \varepsilon'$. Here $\varepsilon' = \varepsilon'(\psi_\delta^\pm, \lambda_0, \lambda_1)$ is sufficiently small. Inductively, we derive for $t \in ((k_t-1)\varepsilon^2, k_t\varepsilon^2]$,

$$|u^\varepsilon(t,p) - u^\varepsilon(t,\hat{p})| \le C_\delta|p\cdot\hat{p}^{-1}|_{\mathbb{G}} + k_t\varepsilon^2\omega(|p\cdot\hat{p}^{-1}|_{\mathbb{G}}) + \delta. \tag{4.13}$$

Now, by Lemma 4.4, the estimate

$$|u^\varepsilon_{\psi_\delta^\pm}(t,p) - u^\varepsilon_{\psi_\delta^\pm}(t-\varepsilon^2,p)| \le C^\varepsilon[\psi_\delta^\pm]\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{k_t}\varepsilon^2 + (k_t-1)\varepsilon^2\omega(\varepsilon^2),$$

holds for each t and all sufficiently small ε where

$$C^\varepsilon[\psi_\delta^\pm] := \max[C^\varepsilon[\psi_\delta^+, 1], C^\varepsilon[\psi_\delta^-, 1]].$$

If $0 \le i \le j \le m$, $t \in ((i-1)\varepsilon^2, i\varepsilon^2]$, and $s \in ((j-1)\varepsilon^2, j\varepsilon^2]$, then

$$\begin{aligned} &u^\varepsilon_{\psi_\delta^\pm}(t-\varepsilon^2, p\cdot\delta_\varepsilon(q)) - u^\varepsilon_{\psi_\delta^\pm}(s-\varepsilon^2, p\cdot\delta_\varepsilon(q)) + R^\varepsilon(t,p,q,\eta,\mathcal{X}) - R^\varepsilon(s,p,q,\eta,\mathcal{X}) \\ &\le u^\varepsilon_{\psi_\delta^\pm}(t-\varepsilon^2, p\cdot\delta_\varepsilon(q)) \pm u^\varepsilon_{\psi_\delta^\pm}(t-\varepsilon^2+\varepsilon^2, p\cdot\delta_\varepsilon(q)) \pm \dots \\ &\quad \pm u^\varepsilon_{\psi_\delta^\pm}\big(t-\varepsilon^2+(j-i)\varepsilon^2, p\cdot\delta_\varepsilon(q)\big) - u^\varepsilon_{\psi_\delta^\pm}(s-\varepsilon^2, p\cdot\delta_\varepsilon(q)) + \varepsilon^2\omega(|s-t|) \end{aligned}$$

$$\leq C^\varepsilon[\psi_\delta^\pm](j-i)\varepsilon^2 + j(j-i)\varepsilon^2 w(\varepsilon^2) + \varepsilon^2\omega(|s-t|).$$

Hence, taking $\sup_q$ and $\inf_{\eta,\mathcal{X}}$, for $t \in ((i-1)\varepsilon^2, i\varepsilon^2]$, $s \in ((j-1)\varepsilon^2, j\varepsilon^2]$ we derive that

$$|u^\varepsilon_{\psi_\delta^\pm}(t,p) - u^\varepsilon_{\psi_\delta^\pm}(s,p)| \leq C^\varepsilon[\psi_\delta^\pm](j-i)\varepsilon^2 + j(j-i)\varepsilon^2 w(\varepsilon^2) + \varepsilon^2\omega(|s-t|).$$

Therefore,

$$|u^\varepsilon(t,p) - u^\varepsilon(s,p)| \leq C^\varepsilon[\psi_\delta^\pm](j-i)\varepsilon^2 + m(j-i)\varepsilon^2\omega(\varepsilon^2) + \varepsilon^2\omega(|s-t|) + \delta. \quad (4.14)$$

Now we prove the proposition. For any $t, s \in [0,T]$ with $t \leq s$, there exist i, j such that $0 \leq i \leq j \leq m$ and

$$j\varepsilon^2 \leq s < (j+1)\varepsilon^2, \quad i\varepsilon^2 \leq t < (i+1)\varepsilon^2.$$

From (4.13) and (4.14), it follows that

$$\begin{aligned} |u^\varepsilon(t,p) - u^\varepsilon(s,\hat{p})| &\leq C^\varepsilon[\psi_\delta^\pm](j-i)\varepsilon^2 + (j-i)T\omega(\varepsilon^2) + \varepsilon^2\omega(|s-t|) \\ &\quad + C_\delta|p\cdot\hat{p}^{-1}|_{\mathbb{G}} + T\omega(|p\cdot\hat{p}^{-1}|_{\mathbb{G}}) + 2\delta. \end{aligned}$$

Set

$$C_0[\psi_\delta^\pm] := \max[C_0[\psi_\delta^+], C_0[\psi_\delta^-]].$$

Since $i\varepsilon^2 > t - \varepsilon^2$ and $j\varepsilon^2 \leq s$, we have

$$\begin{aligned} |u^\varepsilon(t,p) - u^\varepsilon(s,\hat{p})| &\leq C^\varepsilon[\psi_\delta^\pm](s-t) + m(s-t)\omega(\varepsilon^2) + \varepsilon^2\omega(|s-t|) \\ &\quad + C_\delta|p\cdot\hat{p}^{-1}|_{\mathbb{G}} + T\omega(|p\cdot\hat{p}^{-1}|_{\mathbb{G}}) + 2\delta. \end{aligned}$$

Interchanging s and t, we conclude that

$$\begin{aligned} |u^\varepsilon(t,p) - u^\varepsilon(s,\hat{p})| &\leq C^\varepsilon[\psi_\delta^\pm]|s-t| + m|s-t|\omega(\varepsilon^2) + \varepsilon^2\omega(|s-t|) \\ &\quad + C_\delta|p\cdot\hat{p}^{-1}|_{\mathbb{G}} + T\omega(|p\cdot\hat{p}^{-1}|_{\mathbb{G}}) + 2\delta. \end{aligned} \quad (4.15)$$

for all $s, t \in [0,T]$. Using (A4) on ω there is a constant C such that $m\omega(\varepsilon^2) \leq C$. Also, observe that

$$C[\psi_\delta^\pm] := \lim_{\varepsilon\to 0} C^\varepsilon[\psi_\delta^\pm] = C(1 + C_0[\psi_\delta^\pm]).$$

Next, we take $\eta \in (0,1)$ and fix $\delta < \eta/6$. Moreover, we consider $\varepsilon_0 > 0$ so that $\varepsilon_0 < \eta/4$ and for all $\varepsilon < \varepsilon_0$,

$$C^\varepsilon[\psi_\delta^\pm] \leq C(1 + C_0[\psi_\delta^\pm]) + \delta.$$

Finally, we take $r_0 > 0$ so that $|\hat{p}^{-1}\cdot p|_{\mathbb{G}} + |s-t| < r_0$. Then

$$[\frac{\eta}{4} + C(1 + C_0[\psi_\delta^\pm])]r_0 + \varepsilon_0^2\omega(r_0) + C_\delta r_0 + T\omega(r_0) < \frac{\eta}{2}.$$

Therefore, $|\hat{p}^{-1}\cdot p|_{\mathbb{G}} + |s-t| < r_0$ and $\varepsilon < \varepsilon_0$ imply

$$|u^\varepsilon(t,p) - u^\varepsilon(s,\hat{p})| < \eta.$$

Moreover, the functions u^ε are uniformly bounded. Indeed, for $t \in (0, \varepsilon^2]$, we have the upper bound

$$\begin{aligned} (1+\mu\varepsilon^2)u^\varepsilon(t,p) &\leq \|\psi\|_\infty + \inf_{\eta,\mathcal{X}}\sup_q R^\varepsilon(t,p,q,\eta,\mathcal{X}) \\ &\leq \|\psi\|_\infty + \inf_\eta \sup_q R^\varepsilon(t,p,q,\eta,\mathcal{O}) \\ &= \|\psi\|_\infty + \inf_\eta \sup_q \left(-\varepsilon\langle\eta,\nu\rangle - \varepsilon^2\mathcal{F}_*(t,p,\eta,\mathcal{O})\right). \end{aligned}$$

Taking a sequence $\eta_k \searrow 0$ with $\|\eta_k\| \le \varepsilon^{-1/4}$ and using the lower semicontinuity of $\mathcal{F}_*$ together with ($\mathcal{F}5$), we obtain

$$u^\varepsilon(t,p) \le \|\psi\|_\infty$$

for all $\varepsilon > 0$. Now, taking $\hat{\eta} = 0$, $\hat{\mathcal{X}} = O$ and $R_0 = 1$ in Lemma 5.1, there is $\overline{q}$ (depending on ε, η and $\mathcal{X}$) so that

$$\begin{aligned}
(1+\mu\varepsilon^2)u^\varepsilon(t,p) &\ge -\|\psi\|_\infty + \inf_{\eta,\mathcal{X}} \sup_q R^\varepsilon(t,p,q,\eta,\mathcal{X}) \\
&\ge -\|\psi\|_\infty + \inf_{\eta,\mathcal{X}} R^{*,\varepsilon}(t,p,\overline{q},0,\mathcal{O}) \\
&= -\|\psi\|_\infty.
\end{aligned}$$

Therefore, $|u^\varepsilon(t,p)| \le \|\psi\|_\infty$. By induction we deduce that for all (t,p),

$$|u^\varepsilon(t,p)| \le \|\psi\|_\infty.$$

In this way, we may apply [23, Lemma 4.2] to obtain the convergence (up to a subsequence) of u^ε to some continuous u, locally uniformly in $[0,T]\times\mathbb{G}$.

We now prove the final statement. Taking $\varepsilon_j \to 0$ in (4.15), we obtain

$$|u(t,p) - u(s,\hat{p})| \le C(1 + C_0[\psi_\delta^\pm])|s-t| + C_\delta|p\cdot\hat{p}^{-1}|_{\mathbb{G}} + T\omega(|p\cdot\hat{p}^{-1}|_{\mathbb{G}}) + 2\delta.$$

Hence $u \in BUC([0,T]\times\mathbb{G})$. Applying (4.15) to $s=0$ and $p=\hat{p}$, and taking the limit $\varepsilon_j \to 0$ we obtain

$$|u(t,p) - \psi(p)| \le C(1 + C_0[\psi_\delta^\pm])|t| + 2\delta.$$

Letting $t \to 0$ and then $\delta \to 0$ it follows that $u(0,p) = \psi(p)$. □

Proposition 4.6. *The function u is a viscosity subsolution of* (2.1).

Proof. We argue by contradiction. Then there exist a positive constant θ_0 and a smooth function φ, such that the following holds in a neighbourhood $\overline{\mathcal{B}}_0 := [t_0-\delta, t_0+\delta]\times\overline{B}_{\mathbb{G}}(p_0,r_0)$ of a strict local maximal point $P_0 = (t_0,p_0) \in (0,T)\times\mathbb{G}$ of $u-\varphi$,

$$\partial_t\varphi + \mu u + \mathcal{F}_*(t,p,\nabla_{\mathcal{G},0}\varphi, \nabla^{2,*}_{\mathcal{G},0}\varphi) \ge \theta_0 > 0, \tag{4.16}$$

where δ and r_0 are sufficiently small, with

$$4\delta < T. \tag{4.17}$$

We also assume that $\max_{\overline{\mathcal{B}}_0}(u-\varphi) = 0$.

Let $P = (t,p) \in \overline{\mathcal{B}}_0$. Reasoning as in (4.6), we have

$$\begin{aligned}
u^\varepsilon(P) = \Big(\frac{1}{1+\mu\varepsilon^2}\Big) \inf_{\eta,\mathcal{X}} \sup_q \Big\{ &(u^\varepsilon - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(q)) + \varphi(P) - \varepsilon^2\partial_t\varphi(P) \\
&+ \varepsilon\langle \nabla_{\mathcal{G},0}\varphi(P) - \eta, \nu\rangle + \frac{\varepsilon^2}{2}\big\langle\big(\nabla^{2,*}_{\mathcal{G},0}\varphi(P) - \mathcal{X}\big)\nu,\nu\big\rangle \\
&- \varepsilon^2\mathcal{F}\big(t,p,\eta,\mathcal{X}\big) + o(\varepsilon^2)\Big\}.
\end{aligned}$$

In the sequel, ε is small enough so that

$$\|\nabla_{\mathcal{G},0}\varphi\|_{\infty,\overline{\mathcal{B}}_0} \le \varepsilon^{-1/4}, \quad \|\nabla^{2,*}_{\mathcal{G},0}\varphi\|_{\infty,\overline{\mathcal{B}}_0} \le \varepsilon^{-1/2} \text{ and } o(\varepsilon^2) - \varepsilon^2\theta_0 \le 0.$$

Using $-\varphi \le -u$ in $\overline{\mathcal{B}}_0$ we obtain

$$(u^\varepsilon - \varphi)(P) \le \Big(\frac{1}{1+\mu\varepsilon^2}\Big) \inf_{\eta,\mathcal{X}} \sup_q \Big\{(u^\varepsilon - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(q))$$

$$
\begin{aligned}
&- \varepsilon^2[\partial_t\varphi(P) + \mu u(P) + \mathcal{F}_*\big(t, p, \eta, \mathcal{X}\big)] \\
&+ \varepsilon\langle\nabla_{\mathcal{G},0}\varphi(P) - \eta, \nu\rangle + \frac{\varepsilon^2}{2}\big\langle\big(\nabla^{2,*}_{\mathcal{G},0}\varphi(P) - \mathcal{X}\big)\nu, \nu\big\rangle + o(\varepsilon^2)\Big\}.
\end{aligned}
$$

Taking the special choices $\eta = \nabla_{\mathcal{G},0}\varphi(P)$, $\mathcal{X} = \nabla^{2,*}_{\mathcal{G},0}\varphi(P)$ (if $\nabla_{\mathcal{G},0}\varphi(P) = 0$, we choose an approximating sequence $0 \neq \eta_k \to 0$) and appealing to (4.16) we deduce that

$$
\begin{aligned}
(u^\varepsilon - \varphi)(P) &\leq \Big(\frac{1}{1+\mu\varepsilon^2}\Big)\sup_q\big\{(u^\varepsilon - \varphi)(t - \varepsilon^2, p\cdot\delta_\varepsilon(q)) + o(\varepsilon^2) - \varepsilon^2\theta_0\big\} \\
&\leq \Big(\frac{1}{1+\mu\varepsilon^2}\Big)\sup_q\big\{(u^\varepsilon - \varphi)(t - \varepsilon^2, p\cdot\delta_\varepsilon(q))\big\} \\
&\leq \Big(\frac{1}{1+\mu\varepsilon^2}\Big)\sup_q\big\{((u^\varepsilon)^* - \varphi)(t - \varepsilon^2, p\cdot\delta_\varepsilon(q))\big\}.
\end{aligned}
$$

Taking a sequence of points $P_n = (t_n, p_n) \in \overline{\mathcal{B}}_0$ converging to P so that

$$
(u^\epsilon)^*(P) = \lim_{n\to\infty} u^\varepsilon(P_n)
$$

we obtain

$$
\begin{aligned}
&((u^\varepsilon)^* - \varphi)(P) \\
&= \lim_{n\to\infty}(u^\varepsilon - \varphi)(P_n) \\
&\leq \Big(\frac{1}{1+\mu\varepsilon^2}\Big)\lim_{n\to\infty}\sup_q\big\{((u^\varepsilon)^* - \varphi)(t_n - \varepsilon^2, p_n\cdot\delta_\varepsilon(q))\big\} \\
&= \Big(\frac{1}{1+\mu\varepsilon^2}\Big)\lim_{n\to\infty}\big\{((u^\varepsilon)^* - \varphi)(t_n - \varepsilon^2, p_n\cdot\delta_\varepsilon(q_n))\big\} \\
&\leq \Big(\frac{1}{1+\mu\varepsilon^2}\Big)((u^\varepsilon)^* - \varphi)(t - \varepsilon^2, p\cdot\delta_\varepsilon(q_0^\varepsilon))
\end{aligned}
\tag{4.18}
$$

for some $|q_n|_{\mathbb{G}} \leq \varepsilon^{-1/4}$, where $q_n \to q_0^\varepsilon$ (up to a subsequence that we do not re label) and where we have used the upper semicontinuity of $(u^\varepsilon)^* - \varphi$.

Define $P_0^\varepsilon = P_0$, and, for $k \geq 1$, $P_k^\varepsilon = (t_k^\varepsilon, p_k^\varepsilon)$ as follows:

$$
P_k^\varepsilon = (t_{k-1}^\varepsilon - \varepsilon^2, p_{k-1}^\varepsilon\cdot\delta_\varepsilon(q_0^\varepsilon(P_{k-1}^\varepsilon))), \quad 1 \leq k \leq m.
$$

If $P_1^\varepsilon, P_2^\varepsilon, \dots, P_k^\varepsilon \in \overline{\mathcal{B}}_0$, from (4.18), we obtain

$$
((u^\varepsilon)^* - \varphi)(P_{k-1}^\varepsilon) \leq \Big(\frac{1}{1+\mu\varepsilon^2}\Big)\left[(u^\varepsilon)^* - \varphi\right](P_k^\varepsilon),
$$

and so

$$
((u^\varepsilon)^* - \varphi)(P_0^\varepsilon) \leq \Big(\frac{1}{1+\mu\varepsilon^2}\Big)^k[(u^\varepsilon)^* - \varphi](P_k^\varepsilon). \tag{4.19}
$$

Taking $n = n^\varepsilon$ so that $n\varepsilon^2 \in (\delta, 4\delta)$ it follows $t_n^\varepsilon \notin [t_0 - \delta, t_0 + \delta]$ (i.e., $P_n^\varepsilon \notin \overline{\mathcal{B}}_0$). In addition, $n \leq m$, by the choice (4.17). Hence, there exists a minimal number $K_\varepsilon \leq m$ such that $P_{K_\varepsilon}^\varepsilon \in \overline{\mathcal{B}}_0$ but $P_{K_\varepsilon+1}^\varepsilon \notin \overline{\mathcal{B}}_0$. By compactness, $P_{K_\varepsilon}^\varepsilon \to P' = (t', p') \in \overline{\mathcal{B}}_0\backslash\{P_0\}$ as $\varepsilon \to 0$ (or equivalently $m \to \infty$). Note that

$$
0 < e^{-\mu T} = e^{-\mu m\varepsilon^2} \leq \big(1 + \mu\varepsilon^2\big)^{-m} \leq \big(1 + \mu\varepsilon^2\big)^{-K_\varepsilon} \leq 1.
$$

Thus,

$$
\lim_{\substack{\varepsilon\to 0\\ (m\to\infty)}}\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{K_\varepsilon} := \alpha \in (0, 1].
$$

Consequently,

$$\begin{aligned}0 = (u-\varphi)(P_0) &= \lim_{\varepsilon\to 0}(u^\varepsilon - \varphi)(P_0^\varepsilon)\\ &\le \lim_{\varepsilon\to 0}\Big(\frac{1}{1+\mu\varepsilon^2}\Big)^{K_\varepsilon}((u^\varepsilon)^* - \varphi)(P_{K_\varepsilon}^\varepsilon)\\ &\le \alpha(u-\varphi)(P'),\ \text{with } P' \in \overline{\mathcal{B}}_0\backslash\{P_0\}.\end{aligned}$$

We have a contradiction, since P_0 is a strict maximum in $\overline{\mathcal{B}}_0$. □

Proposition 4.7. *The function u is a viscosity supersolution of* (2.1).

Proof. Reasoning by contradiction again, there exist $\theta_0 > 0$ and a smooth function φ, such that the following holds in a neighbourhood $\overline{\mathcal{B}}_0$ of a strict local minimal point $P_0 = (t_0, p_0) \in (0,T)\times\Omega$ of $u-\varphi$,

$$\partial_t\varphi + \mu u + \mathcal{F}^*(t,p,\nabla_{\mathcal{G},0}\varphi, \nabla^{2,*}_{\mathcal{G},0}\varphi) \le -\theta_0 < 0. \tag{4.20}$$

We assume that the value of $u-\varphi$ at P_0 is 0.

Let $R_0 > 0$ be such that

$$\|\nabla_{\mathcal{G},0}\varphi\|_{\infty,\overline{\mathcal{B}}_0},\ \|\nabla^{2,*}_{\mathcal{G},0}\varphi\|_{\infty,\overline{\mathcal{B}}_0} \le R_0.$$

Suppose first that $\nabla_{\mathcal{G},0}\varphi(P_0) \neq 0$. We may assume that there is $\gamma_0 > 0$ such that

$$\|\nabla_{\mathcal{G},0}\varphi\|_{\infty,\overline{\mathcal{B}}_0} \ge \gamma_0 > 0$$

Hence there exists $j_0 \in \mathbb{N}$ such that $\gamma_0 > 1/j_0$. By Lemma 5.1 and performing a Taylor expansion as in (4.6), we have for $P = (t,p) \in \overline{\mathcal{B}}_0$,

$$\begin{aligned}u^\varepsilon(P) \ge \frac{1}{1+\mu\varepsilon^2}\inf_{\eta,\mathcal{X}}\Big\{&(u^\varepsilon-\varphi)((t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q})) + \varphi(P) - \varepsilon^2\partial_t\varphi(P)\\ &-\varepsilon^2\mathcal{F}^*\Big(t,p,\nabla_{\mathcal{G},0}\varphi(P), \nabla^{2,*}_{\mathcal{G},0}\varphi(P)\Big) - \varepsilon^2 h_{j_0}(\varepsilon^{1/4}) + o(\varepsilon^2)\Big\},\end{aligned}$$

for some $\overline{q} \in \mathbb{G}$, with $|\overline{q}|_{\mathbb{G}} \le \varepsilon^{-1/4}$. So we obtain

$$\begin{aligned}&(u^\varepsilon-\varphi)(P)\\ &\ge \frac{1}{1+\mu\varepsilon^2}\inf_{\eta,\mathcal{X}}\Big\{(u^\varepsilon-\varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q})) - \varepsilon^2\big[\partial_t\varphi(P) + \mu u(P)\\ &\quad + \mathcal{F}^*\Big(t,p,\nabla_{\mathcal{G},0}\varphi(P), \nabla^{2,*}_{\mathcal{G},0}\varphi(P)\Big)\big] - \varepsilon^2 h_{j_0}(\varepsilon^{1/4}) + o(\varepsilon^2)\Big\}\\ &\ge \frac{1}{1+\mu\varepsilon^2}\inf_{\eta,\mathcal{X}}\{(u^\varepsilon-\varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q})) + \varepsilon^2[\theta_0 + o(1) - h_{j_0}(\varepsilon^{1/4})]\}\\ &\ge \frac{1}{1+\mu\varepsilon^2}\inf_{\eta,\mathcal{X}}\{(u^\varepsilon-\varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q}))\}\\ &\ge \frac{1}{1+\mu\varepsilon^2}\inf_{\eta,\mathcal{X}}\{((u^\varepsilon)_*-\varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q}))\},\end{aligned} \tag{4.21}$$

where we used $-\varphi \ge -u$ in $\overline{\mathcal{B}}_0$ and (4.20). By a similar argument as in (4.18), we derive from (4.21) that

$$((u^\varepsilon)_*-\varphi)(P) \ge \frac{1}{1+\mu\varepsilon^2}\inf_{\eta,\mathcal{X}}\{((u^\varepsilon)_*-\varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q}))\}. \tag{4.22}$$

Since for each admissible η and $\mathcal{X}$ we have $|\overline{q}|_{\mathbb{G}} \le \varepsilon^{-1/4}$, and the function $((u^\varepsilon)_* - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\cdot))$ is lower semicontinuous, the infimum:

$$\inf_{|q|_{\mathbb{G}}\le\varepsilon^{-1/4}} \left\{((u^\varepsilon)_* - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q}))\right\}$$

is finite. Hence, so is:

$$\inf_{\eta,\mathcal{X}} \left\{((u^\varepsilon)_* - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q}))\right\}.$$

Take a sequence $(\eta_n, \mathcal{X}_n)$ so that if $\overline{q}_n = \overline{q}_n(\varepsilon, \varphi, P, \eta_n, \mathcal{X}_n)$, then

$$\lim_{n\to\infty} ((u^\varepsilon)_* - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q}_n)) = \inf_{\eta,\mathcal{X}} \left\{((u^\varepsilon)_* - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q}))\right\}.$$

By compactness, there is a point $\overline{q}_0^\varepsilon = \overline{q}_0^\varepsilon(P)$, $|\overline{q}_0^\varepsilon|_{\mathbb{G}} \le \varepsilon^{-1/4}$, so that

$$\overline{q}_0^\varepsilon = \lim_{n\to\infty} \overline{q}_n.$$

The lower semicontinuity of $((u^\varepsilon)_* - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\cdot))$ yields

$$\lim_{n\to\infty} (u^\varepsilon)_* - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q}_n)) \ge ((u^\varepsilon)_* - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q}_0^\varepsilon)).$$

Thus, by (4.22), we derive that

$$((u^\varepsilon)_* - \varphi)(P) \ge \Big(\frac{1}{1+\mu\varepsilon^2}\Big)((u^\varepsilon)_* - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q}_0^\varepsilon)). \tag{4.23}$$

Next, we consider the case $\nabla_{\mathcal{G},0}\varphi(P_0) = 0$. Let $\mathbf{F} : \overline{\mathcal{B}}_0 \to \mathbb{R}$ be so that

$$\mathbf{F}(\cdot) := \partial_t\varphi(\cdot) + \mu u(\cdot) + \mathcal{F}^*(\cdot, 0, \nabla^{2,*}_{\mathcal{G},0}\varphi(\cdot)).$$

We can assume $\mathbf{F}(P) \le -\theta_0$ and that $\|\nabla_{\mathcal{G},0}\varphi(P)\| \le 1/j$ for any $P \in \overline{\mathcal{B}}_0$ and some positive integer j. Applying Lemma 5.1, for any $P \in \overline{\mathcal{B}}_0$ and any $\eta, \mathcal{X}$ admissible, there exists $\overline{q}_0^\varepsilon$, with $|\overline{q}_0^\varepsilon|_{\mathbb{G}} \le \varepsilon^{-1/4}$, such that

$$\begin{aligned}
(u^\varepsilon - \varphi)(P) &\ge \frac{1}{1+\mu\varepsilon^2} \inf_{\eta,\mathcal{X}} \Big\{(u^\varepsilon - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(\overline{q}_0^\varepsilon)) - \varepsilon^2\big[\partial_t\varphi(P) + \mu u(P) \\
&\quad + \mathcal{F}^*\Big(t, p, 0, \nabla^{2,*}_{\mathcal{G},0}\varphi(P)\Big)\big] + o(\varepsilon^2)\Big\} \\
&\ge \frac{1}{1+\mu\varepsilon^2} \inf_{|q|_{\mathbb{G}}\le\varepsilon^{-1/4}} \left\{((u^\varepsilon)_* - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(q)) - \varepsilon^2\mathbf{F}(P) + o(\varepsilon^2)\right\} \\
&\ge \frac{1}{1+\mu\varepsilon^2} \inf_{|q|_{\mathbb{G}}\le\varepsilon^{-1/4}} \left\{((u^\varepsilon)_* - \varphi)(t-\varepsilon^2, p\cdot\delta_\varepsilon(q))\right\}
\end{aligned}$$

Thus, there is $q_0^\varepsilon = q_0^\varepsilon(P)$ where the latter infimum is attained. Hence (4.23) holds.

We may proceed as in the end of Proposition 4.6 (right after (4.18)) to get a contradiction with the fact that P_0 is a strict minimum. □

5. Appendix: A technical lemma for existence

We provide the proof of the next lemma which is [19, Lemma 4.6]. We give full details to show that $\bar{q}$ may be taken horizontal.

Lemma 5.1. *Let $(\hat{\eta}, \hat{\mathcal{X}}) \in \mathbb{R}^{m_1} \times \mathcal{S}^{m_1}(\mathbb{R})$ and let R_0 so that $\|\hat{\eta}\|, \|\hat{\mathcal{X}}\| \le R_0$.*

(1) *If* $\|\hat{\eta}\| \geq K^{-1}$ *($K \in \mathbb{N}$), then there exists* $\varepsilon_1 = \varepsilon_1(K, R_0, \lambda_0, \lambda_1)$ *such that for all* $(\eta, \mathcal{X}) \in (\mathbb{R}^{m_1}\backslash\{0\}) \times \mathcal{S}^{m_1}(\mathbb{R})$*, with* $\|\eta\| \leq \varepsilon^{-1/4}, \|\mathcal{X}\| \leq \varepsilon^{-1/2}$*, there exists* $\overline{q} = \overline{q}(\varepsilon, \eta, \hat{\eta}, \mathcal{X}, \hat{\mathcal{X}})$*,* $\overline{q} = (\overline{\nu}, 0)$*, with* $|\overline{q}|_{\mathbb{G}} \leq \varepsilon^{-1/4}$ *such that for all* $\varepsilon \leq \varepsilon_1$ *and all* (t, p)*,*

$$R^{\varepsilon}(t, p, \overline{q}, \eta, \mathcal{X}) \geq R^{*,\varepsilon}(t, p, \overline{q}, \hat{\eta}, \hat{\mathcal{X}}) - \varepsilon^2 h_K(\varepsilon^{1/4}), \tag{5.1}$$

where $R^{*,\varepsilon}$ *is defined as* R^{ε} *changing* $\mathcal{F}$ *by* $\mathcal{F}^*$ *and* $h_K(r) := \omega_{(1/2)K,R_0}(r)$*, with* $r \geq 0$*.*

(2) *If* $\|\hat{\eta}\| \leq K^{-1}$ *($K \in \mathbb{N}$), then there exists* $\varepsilon_2 = \varepsilon_2(K, R_0, \lambda_0, \lambda_1)$ *such that for all* $(\eta, \mathcal{X}) \in (\mathbb{R}^{m_1}\backslash\{0\}) \times \mathcal{S}^{m_1}(\mathbb{R})$*, with* $\|\eta\| \leq \varepsilon^{-1/4}, \|\mathcal{X}\| \leq \varepsilon^{-1/2}$*, there exists* $\overline{q} = \overline{q}(\varepsilon, \eta, \hat{\eta}, \mathcal{X}, \hat{\mathcal{X}})$*,* $\overline{q} = (\overline{\nu}, 0)$*, with* $|\overline{q}|_{\mathbb{G}} \leq \varepsilon^{-1/4}$ *such that for each* $\varepsilon \leq \varepsilon_2$ *and all* (t, p)*,*

$$R^{\varepsilon}(t, p, \overline{q}, \eta, \mathcal{X}) \geq R^{*,\varepsilon}(t, p, \overline{q}, 0, \hat{\mathcal{X}}). \tag{5.2}$$

Proof. Assume that $\hat{\eta} \neq \eta$ and $\hat{\mathcal{X}} \neq \mathcal{X}$. Using orthonormal eigenvectors $\xi_0, \xi_1, \dots, \xi_{m_1-1} \in \mathbb{R}^{m_1}$ of $\mathcal{X} - \hat{\mathcal{X}}$, we can represent ν with $\|\nu\| \leq \varepsilon^{-1/4}$ by

$$\nu = \sum_{i=0}^{m_1-1} s_i \xi_i,$$

where $s_i \in \mathbb{R}$ ($i = 0, 1, \dots, m_1 - 1$) with $s_0^2 + \dots + s_{m_1-1}^2 \leq \varepsilon^{-1/4}$. In particular, let ξ_0 be the unit eigenvector which gives the maximum eigenvalue of $\hat{\mathcal{X}} - \mathcal{X}$. Thus $\varepsilon^{-2}[R^{\varepsilon}(t, p, \overline{q}, \eta, \mathcal{X}) - R^{*,\varepsilon}(t, p, \overline{q}, \hat{\eta}, \hat{\mathcal{X}})]$ is bounded from below by

$$\begin{aligned} &\varepsilon^{-1} s_0 \langle \hat{\eta} - \eta, \xi_0 \rangle + \varepsilon^{-1} \sum_{i=1}^{m_1-1} s_i \langle \hat{\eta} - \eta, \xi_i \rangle + \frac{1}{2} s_0^2 \mathcal{E}\big(\hat{\mathcal{X}} - \mathcal{X}\big) \\ &+ \frac{1}{2} \sum_{i=1}^{m_1-1} s_i^2 \langle (\hat{\mathcal{X}} - \mathcal{X}) \xi_i, \xi_i \rangle + \big[\mathcal{F}\big(t, p, \hat{\eta}, \hat{\mathcal{X}}\big) - \mathcal{F}\big(t, p, \eta, \mathcal{X}\big)\big]. \end{aligned} \tag{5.3}$$

Case 1: Assume $\|\hat{\eta}\| \geq K^{-1}$ for some $K \in \mathbb{N}$.

(a) If $\|\hat{\eta} - \eta\| \leq \varepsilon^{1/4}$, then $\|\eta\| \geq 1/2K$ for all sufficiently small ε. In the case $\mathcal{E}\left(\hat{\mathcal{X}} - \mathcal{X}\right) > 0$, we take $|s_0| = \lambda_1$, $s_i = 0$ for $i = 1, \dots, m_1 - 1$ in (5.3). Then (5.3) is rewritten as

$$\begin{aligned} &\varepsilon^{-1} \lambda_1 |\langle \hat{\eta} - \eta, \xi_0 \rangle| + \frac{\lambda_1^2}{2} \mathcal{E}^+\big(\hat{\mathcal{X}} - \mathcal{X}\big) \\ &+ \big[\mathcal{F}\big(t, p, \hat{\eta}, \hat{\mathcal{X}}\big) - \mathcal{F}\big(t, p, \hat{\eta}, \mathcal{X}\big)\big] + \big[\mathcal{F}\big(t, p, \hat{\eta}, \mathcal{X}\big) - \mathcal{F}\big(t, p, \eta, \mathcal{X}\big)\big], \end{aligned} \tag{5.4}$$

where we choose an appropriate sign of s_0 so that $s_0 \langle \hat{\eta} - \eta, \xi_0 \rangle$ is non-negative. From (A4), for any $\eta \in \mathbb{R}^{m_1}\backslash\{0\}$,

$$\frac{\lambda_1^2}{2} \mathcal{E}^+\big(\mathcal{X} - \hat{\mathcal{X}}\big) + \big[\mathcal{F}\big(t, p, \hat{\eta}, \hat{\mathcal{X}}\big) - \mathcal{F}\big(t, p, \hat{\eta}, \mathcal{X}\big)\big] \geq 0. \tag{5.5}$$

From (A5) and $\|\eta\| \geq 1/2K$ it follows

$$\mathcal{F}(t, p, \hat{\eta}, \mathcal{X}) - \mathcal{F}(t, p, \eta, \mathcal{X}) \geq -\omega_{1/2K,R_0}(\varepsilon^{1/4}) \tag{5.6}$$

where $\omega_{1/2K,R_0}$ is a modulus of continuity depending only on K and R_0. Thus, from (5.5), (5.6) and (5.4) we obtain

$$\varepsilon^{-2}[R^{\varepsilon}(t, p, \overline{q}, \eta, \mathcal{X}) - R^{*,\varepsilon}(t, p, \overline{q}, \hat{\eta}, \hat{\mathcal{X}})] \geq -h_K(\varepsilon^{1/4}), \tag{5.7}$$

where $h_K(s) = \omega_{1/2K,R_0}(s)$.

Now if $\mathcal{E}(\hat{\mathcal{X}} - \mathcal{X}) \le 0$, we take $s_i = 0$ for $i = 0, 1, \dots, m_1 - 1$ in the formula (5.3). Then $\mathcal{F}(t, p, \hat{\eta}, \hat{\mathcal{X}}) \ge \mathcal{F}(t, p, \hat{\eta}, \mathcal{X})$ for any $\eta \in \mathbb{R}^{m_1} \backslash \{0\}$ holds, since $\mathcal{F}$ is degenerate elliptic (see Remark 4.1). From (5.6), the inequality (5.7) is derived (with the same modulus h_K).

(b) If $\|\hat{\eta} - \eta\| \ge \varepsilon^{1/4}$, then

$$\frac{\hat{\eta} - \eta}{\|\hat{\eta} - \eta\|} = \sum_{i=0}^{m_1-1} r_i \xi_i, \tag{5.8}$$

where $r_i \in \mathbb{R}$ with $r_0^2 + r_1^2 + \dots + r_{m_1-1}^2 = 1$. Let us divide this case into two parts.

Suppose first that $|\langle \hat{\eta} - \eta, \xi_0 \rangle| \ge \frac{\varepsilon^{1/2}}{\lambda_1}$. Then if $\mathcal{E}(\hat{\mathcal{X}} - \mathcal{X}) > 0$, we choose s_i so that $|s_0| = \lambda_1$, $s_i = 0$ $(i = 1, \dots, m_1 - 1)$ and obtain a formula similar to (5.4):

$$\begin{aligned}
&\varepsilon^{-1}\lambda_1 |\langle \hat{\eta} - \eta, \xi_0 \rangle| + \frac{\lambda_1^2}{2} \mathcal{E}^+\big(\hat{\mathcal{X}} - \mathcal{X}\big) \\
&+ \big[\mathcal{F}\big(t, p, \eta, \hat{\mathcal{X}}\big) - \mathcal{F}\big(t, p, \eta, \mathcal{X}\big)\big] + \big[\mathcal{F}\big(t, p, \hat{\eta}, \hat{\mathcal{X}}\big) - \mathcal{F}\big(t, p, \eta, \hat{\mathcal{X}}\big)\big].
\end{aligned}$$

Similarly, by (A4) and Remark 4.1, we have for all $\varepsilon \le \varepsilon(R_0, \lambda_0, \lambda_1)$,

$$\begin{aligned}
\varepsilon^{-2}[R^{\varepsilon}(t, p, \overline{q}, \eta, \mathcal{X}) - R^{*,\varepsilon}(t, p, \overline{q}, \hat{\eta}, \hat{\mathcal{X}})] &\ge \varepsilon^{-1/2} - 2C(1 + \|\hat{\mathcal{X}}\|) \\
&\ge \varepsilon^{-1/4} - 2C(1 + R_0) \ge 0.
\end{aligned}$$

If $\mathcal{E}(\hat{\mathcal{X}} - \mathcal{X}) \le 0$, we choose s_i so that $|s_0| = \varepsilon^{1/4}\lambda_1$, $s_i = 0$ $(i = 1, \dots, m_1 - 1)$, and obtain a similar formula to (5.4). Hence, there is $\varepsilon_1(R_0, K, \lambda_0, \lambda_1)$ so that

$$\begin{aligned}
&\varepsilon^{-2}[R^{\varepsilon}(t, p, \overline{q}, \eta, \mathcal{X}) - R^{*,\varepsilon}(t, p, \overline{q}, \hat{\eta}, \hat{\mathcal{X}})] \\
&\ge \varepsilon^{-1/2} + \frac{\lambda_1^2}{2} \varepsilon^{1/2} \mathcal{E}(\hat{\mathcal{X}} - \mathcal{X}) - 2C(1 + \|\hat{\mathcal{X}}\|) \\
&\ge \varepsilon^{-1/4} - \frac{\lambda_1^2}{2}\Big(\varepsilon^{1/2} R_0 + 1\Big) - 2C(1 + R_0) \\
&\ge \varepsilon^{-1/4} - \lambda_1^2 - 2C(1 + R_0) \ge 0,
\end{aligned}$$

Now we consider the case $|\langle \hat{\eta} - \eta, \xi_0 \rangle| \le \frac{\varepsilon^{1/2}}{\lambda_1}$. Then from (5.8):

$$|r_0| = \Big|\frac{\langle \hat{\eta} - \eta, \xi_0 \rangle}{\|\hat{\eta} - \eta\|}\Big| \le \frac{\varepsilon^{1/4}}{\lambda_1} =: c_0 \varepsilon^{1/4}. \tag{5.9}$$

Since $r_0^2 + r_1^2 + \dots + r_{m_1-1}^2 = 1$, we have the inequality

$$1 - c_0^2 \varepsilon^{1/2} \le 1 - r_0^2 = r_1^2 + r_2^2 + \dots + r_{m_1-1}^2 \le |r_1| + |r_2| + \dots + |r_{m_1-1}|,$$

where we take ε so that $c_0^2 \varepsilon^{1/2} < 1/2$. This inequality implies that there exists at least one number j_0 such that

$$|r_{j_0}| \ge \frac{1 - c_0^2 \varepsilon^{1/2}}{m_1 - 1} > \frac{1}{2(m_1 - 1)}.$$

Now we take s_i so that $s_i = 0$ $(i \ne 0, j_0)$ in (5.3) to obtain

$$\begin{aligned}
&\varepsilon^{-1} s_0 \langle \hat{\eta} - \eta, \xi_0 \rangle + \varepsilon^{-1} s_{j_0} \langle \hat{\eta} - \eta, \xi_{j_0} \rangle + \frac{s_0^2}{2} \mathcal{E}\big(\hat{\mathcal{X}} - \mathcal{X}\big) + \frac{s_{j_0}^2}{2} \big\langle \big(\hat{\mathcal{X}} - \mathcal{X}\big) \xi_{j_0}, \xi_{j_0} \big\rangle \\
&+ \big[\mathcal{F}\big(t, p, \eta, \hat{\mathcal{X}}\big) - \mathcal{F}\big(t, p, \eta, \mathcal{X}\big)\big] + \big[\mathcal{F}\big(t, p, \hat{\eta}, \hat{\mathcal{X}}\big) - \mathcal{F}\big(t, p, \eta, \hat{\mathcal{X}}\big)\big].
\end{aligned} \tag{5.10}$$

If $\mathcal{E}(\hat{\mathcal{X}}-\mathcal{X})>0$, we chose $|s_0|=\lambda_1$ with $s_0\langle\hat\eta-\eta,\xi_0\rangle\geq 0$. In addition, take $|s_{j_0}|=\lambda_1\varepsilon^{1/4}$ so that $s_{j_0}\langle\hat\eta-\eta,\xi_{j_0}\rangle\geq 0$. Then, using also (5.9), (5.10) is rewritten as

$$\varepsilon^{-1}\lambda_1|r_0|\|\hat\eta-\eta\|+\varepsilon^{-3/4}\lambda_1|r_{j_0}|\|\hat\eta-\eta\|+\frac{\lambda_1^2}{2}\mathcal{E}\big(\hat{\mathcal{X}}-\mathcal{X}\big)+\frac{\lambda_1^2}{2}\varepsilon^{1/2}\langle\big(\hat{\mathcal{X}}-\mathcal{X}\big)\xi_{j_0},\xi_{j_0}\rangle$$
$$+\big[\mathcal{F}\big(t,p,\eta,\hat{\mathcal{X}}\big)-\mathcal{F}\big(t,p,\eta,\mathcal{X}\big)\big]+\big[\mathcal{F}\big(t,p,\hat\eta,\hat{\mathcal{X}}\big)-\mathcal{F}\big(t,p,\eta,\hat{\mathcal{X}}\big)\big].$$

From (A4) and Remark 4.1, there exists $\epsilon_1(R_0,K,\lambda_0,\lambda_1)$ so that

$$\begin{aligned}&\varepsilon^{-2}[R^{\varepsilon}(\overline{q},\eta,\mathcal{X})-R^{*,\varepsilon}(\overline{q},\hat\eta,\hat{\mathcal{X}})]\\ &\geq\varepsilon^{-3/4}\lambda_1|r_{j_0}|\|\hat\eta-\eta\|+\frac{\lambda_1^2}{2}\varepsilon^{1/2}\langle\big(\hat{\mathcal{X}}-\mathcal{X}\big)\xi_{j_0},\xi_{j_0}\rangle+\big[\mathcal{F}\big(p,t,\hat\eta,\hat{\mathcal{X}}\big)-\mathcal{F}\big(t,p,\eta,\hat{\mathcal{X}}\big)\big]\\ &\geq\frac{\lambda_1\varepsilon^{-1/2}}{2(m_1-1)}-\frac{\lambda_1^2}{2}\varepsilon^{1/2}\|\hat{\mathcal{X}}-\mathcal{X}\|-2C(1+\|\hat{\mathcal{X}}\|)\geq 0.\end{aligned}$$

In the case $\mathcal{E}(\hat{\mathcal{X}}-\mathcal{X})\leq 0$, we take $s_0=0$ and $|s_{j_0}|=\lambda_1\varepsilon^{1/4}$ so that $s_{j_0}\langle\hat\eta-\eta,\xi_{j_0}\rangle\geq 0$. Then, as in the previous case we have

$$\begin{aligned}&\varepsilon^{-3/4}\lambda_1|r_{j_0}|\|\hat\eta-\eta\|+\frac{\lambda_1^2}{2}\varepsilon^{1/2}\langle\big(\hat{\mathcal{X}}-\mathcal{X}\big)\xi_{j_0},\xi_{j_0}\rangle\\ &\quad+\big[\mathcal{F}\big(t,p,\eta,\hat{\mathcal{X}}\big)-\mathcal{F}\big(t,p,\eta,\mathcal{X}\big)\big]+\big[\mathcal{F}\big(t,p,\hat\eta,\hat{\mathcal{X}}\big)-\mathcal{F}\big(t,p,\eta,\hat{\mathcal{X}}\big)\big]\geq 0.\end{aligned}$$

In particular, since $\|\hat\eta\|\geq K^{-1}$, we see $\mathcal{F}(t,p,\hat\eta,\mathcal{X})=\mathcal{F}^*(t,p,\hat\eta,\mathcal{X})$. Consequently if we set $\varepsilon_1=\varepsilon_1(K,R_0,\lambda_0,\lambda_1)$, then the formula (5.1) holds with $h_K(s)=\omega_{R_0,1/K}(s)$.

Case 2: For $\|\hat\eta\|\leq K^{-1}$ with $K\in\mathbb{N}$, we argue as in case 1 to derive the estimate (5.2).

Finally, we consider the case of $\eta=\hat\eta$ or $\mathcal{X}=\hat{\mathcal{X}}$ for $\hat\eta\in\mathbb{R}^{m_1}\backslash\{0\}$. We can choose sequences $\{\eta_k\}\subset\mathbb{R}^{m_1}\backslash\{0\}$ and $\{\mathcal{X}_n\}\subset\mathcal{S}^{m_1}(\mathbb{R})$ such that $\eta_k\neq\eta$, $\eta_k\to\hat\eta$, $\mathcal{X}_k\neq\mathcal{X}$, $\mathcal{X}_n\to\hat{\mathcal{X}}$ as $k,n\to\infty$, respectively. Now let us set $\overline{q}_{k,n}:=\overline{q}(\varepsilon,\eta_k,\hat\eta,\mathcal{X}_n,\hat{\mathcal{X}})$ where $\overline{q}(\varepsilon,\eta_k,\hat\eta,\mathcal{X}_n,\hat{\mathcal{X}})$ satisfies the inequality (5.1) or (5.2). As $|\overline{q}_{k,n}|_{\mathbb{G}}\leq\varepsilon^{-1/4}$, by compactness, the conclusion follows by taking as $k\to\infty$ and then $n\to\infty$ in (5.1) and (5.2)). □

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