SMALLEST EIGENVALUES FOR BOUNDARY VALUE PROBLEMS OF TWO TERM FRACTIONAL DIFFERENTIAL OPERATORS DEPENDING ON FRACTIONAL BOUNDARY CONDITIONS

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ABSTRACT. Let $n \geq 2$ be an integer, and let $n - 1 < \alpha \leq n$. We consider
eigenvalue problems for two point $n - 1, 1$ boundary value problems
$$D_{0+}^{\alpha} u + a(t)u + \lambda p(t)u = 0, \quad 0 < t < 1,$$
$$u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, n - 2, \quad D_{0+}^{\beta} u(1) = 0,$$
where $0 \leq \beta \leq n - 1$ and D_{0+}^{α} and D_{0+}^{β} denote standard Riemann-Liouville
differential operators. We prove the existence of smallest positive eigenvalues
and then obtain comparisons of these smallest eigenvalues as functions of both
p and β.

1. INTRODUCTION

Let $n \in \mathbb{N}$, $n \geq 2$, and $n - 1 < \alpha \leq n$. Assume $a \in C[0, 1]$. In this paper, we
consider the following boundary value problems:
$$D_{0+}^{\alpha} u + a(t)u + \lambda_1 p(t)u = 0, \quad 0 < t < 1, \tag{1.1}$$
satisfying the boundary conditions
$$u^{(i)}(0) = 0, \ i = 0, 1, \ldots, n - 2, \quad D_{0+}^{\beta_1} u(1) = 0, \tag{1.2}$$
or the problem
$$D_{0+}^{\alpha} u + a(t)u + \lambda_2 q(t)u = 0, \quad 0 < t < 1, \tag{1.3}$$
satisfying the boundary conditions
$$u^{(i)}(0) = 0, \ i = 0, 1, \ldots, n - 2, \quad D_{0+}^{\beta_2} u(1) = 0, \tag{1.4}$$
where $0 < \beta_1 \leq \beta_2 \leq n - 1$, or the problem
$$D_{0+}^{\alpha} u + a(t)u + \lambda_3 r(t)u = 0, \quad 0 < t < 1, \tag{1.5}$$
satisfying the boundary conditions
$$u^{(i)}(0) = 0, \ i = 0, 1, \ldots, n - 2, \quad u(1) = 0, \tag{1.6}$$
where D_{0+}^{α}, and $D_{0+}^{\beta_i}$, $i = 1, 2$, are the standard Riemann-Liouville deriva-
tives. Here p, q, and r are continuous nonnegative functions on $[0, 1]$ that do not

vanish identically on any nondegenerate compact subinterval of $[0, 1]$ and throughout this paper, we assume $a(t) \geq 0$, $0 \leq t \leq 1$.

The purpose of this work is to apply Krein-Rutman theory [16] to first, show the existence of smallest eigenvalues of each of the boundary value problems (1.1), (1.2), or (1.3), (1.4), or (1.5), (1.6) and second, to compare these eigenvalues when $0 \leq r(t) \leq p(t) \leq q(t)$ and $0 < \beta_1 \leq \beta_2 \leq n - 1$.

There is a long tradition to apply Krein-Rutman theory to obtain smallest or principal eigenvalues for boundary value problems for ordinary differential equations and we cite for example, [7, 13, 17, 20, 21]. These methods have been applied to and similar results have been developed for boundary value problems for finite difference equations and dynamic equations on time scales; see, for example, [1, 9, 12].

With the recent rapid advancements in the study of fractional calculus and fractional differential equations, these methods have applied to boundary value problems for fractional differential equations (both of Riemann-Liouville and of Caputo type) and analogous result have been obtained; see [5, 6, 10, 11, 14, 18].

Concerning the first purpose of this work, comparison theorems of Green's functions of families of boundary value problems have played a key role in the development of comparison of principal eigenvalues. For example, in [3], a partial order was defined on the type of boundary conditions that were specified at the right, and then comparison theorems for Green's functions, obtained by Peterson and Ridenhour [19], were employed to compare principal eigenvalues as a function of the partial order on the boundary conditions. For the purpose of this article, this is analogous to comparing principal eigenvalues of (1.1), (1.2) in the case $0 < \beta_1 \leq \beta_2 \leq n - 1$. Comparison theorems for Green's functions of two-point boundary value problems related to (1.1), (1.2), as a function of β have been obtained [4]; the application to the comparison of principal eigenvalues is made for the first time in this paper.

Concerning the second purpose of this work, to date, comparisons of principal eigenvalues for fractional equations have been restricted to the fractional operator D_{0+}^α. The comparison of principle eigenvalues for a fractional operator $(D_{0+}^\alpha + a\mathcal{I})$ is new. On the surface, it appears that the analogous comparison theory for ordinary differential equations applies to a general nth order linear ordinary differential operator. But in the references cited above, the operators are assumed to be disconjugate or right disfocal on the given domains and so, with the Frobenius factorization of disconjugate operators [2], the operator behaves as a one–term operator. Following the lead provided in [8], we obtain a Neumann series representation for a Green's function for the boundary value problem associated with a two–term operator, $(D_{0+}^\alpha + a\mathcal{I})$, with boundary conditions (1.2) or (1.6). With this approach, we obtain the necessary comparison theorems for the associated Green's functions and then obtain the comparisons of the eigenvalues.

In what follows, we provide preliminary definitions and results related to the application of Krein-Rutman theory in Section 2. In Section 3, we construct the Green's function for the fractional operator $(D_{0+}^\alpha + a\mathcal{I})$ with the boundary conditions (1.2), for $\beta_1 = \beta$ and $0 < \beta \leq n - 1$ and for the boundary conditions (1.6) (with $\beta = 0$). We obtain the comparisons of the Green's function as a function of β, analogous to the comparison theorems obtained in [4]. In Section 4, we define the appropriate linear operators associated with each of the boundary value problems (1.1), (1.2) or (1.3), (1.4) or (1.5), (1.6). We first show the compactness of

the operators. Then we apply the methods outlined in Section 2 and obtain and compare smallest eigenvalues.

2. Preliminary definitions and theorems

We first give the definitions of the Riemann-Liouville fractional integral and fractional derivative.

Definition 2.1. Let $\nu > 0$. The Riemann-Liouville fractional integral of a function u of order ν, denoted $I_{0+}^{\nu} u$, is defined as

$$I_{0+}^{\nu} u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s) ds,$$

provided the right-hand side exists. Moreover, let n denote a positive integer and assume $n - 1 < \alpha \leq n$. The Riemann-Liouville fractional derivative of order α of the function $u : [0, 1] \to \mathbb{R}$, denoted $D_{0+}^{\alpha} u$, is defined as

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds = D^n I_{0+}^{n-\alpha} u(t),$$

provided the right-hand side exists.

Definition 2.2. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided

    (i) $\alpha u + \beta v \in \mathcal{P}$, for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and

    (ii) $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ implies $u = 0$.

Definition 2.3. A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^{\circ}$, of $\mathcal{P}$ is nonempty. A cone $\mathcal{P}$ is reproducing if $\mathcal{B} = \mathcal{P} - \mathcal{P}$; i.e., given $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$.

Krasnosel'skiĭ [15] showed that every solid cone is reproducing.

Definition 2.4. Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u, v \in \mathcal{B}$, $u \leq v$ with respect to $\mathcal{P}$ if $v - u \in \mathcal{P}$. If both $M, N : \mathcal{B} \to \mathcal{B}$ are bounded linear operators, $M \leq N$ with respect to $\mathcal{P}$ if $Mu \leq Nu$ for all $u \in \mathcal{P}$.

Definition 2.5. A bounded linear operator $M : \mathcal{B} \to \mathcal{B}$ is u_0-positive with respect to $\mathcal{P}$ if there exists $u_0 \in \mathcal{P}$, $u_0 \neq 0$ such that for each $u \in \mathcal{P}$, $u \neq 0$, there exist $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \leq Mu \leq k_2 u_0$ with respect to $\mathcal{P}$.

The following three results are fundamental to our comparison results and are attributed to Krasnosel'skiĭ [15]. The proof of Theorem 2.7 can be found in Krasnosel'skiĭ's book [15]. Theorem 2.8 is provided by Keener and Travis [13] as an extension of Krasonel'skiĭ's results; a slightly more general result was recently proved by Webb [22].

Lemma 2.6. *Let $\mathcal{B}$ be a Banach space over the reals, and let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $M : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $M : \mathcal{P} \backslash \{0\} \to \mathcal{P}^{\circ}$, then M is u_0-positive with respect to $\mathcal{P}$.*

Theorem 2.7. *Let $\mathcal{B}$ be a real Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $L : \mathcal{B} \to \mathcal{B}$ be a compact, u_0-positive, linear operator. Then L has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 2.8. *Let $\mathcal{B}$ be a real Banach space and $\mathcal{P} \subset \mathcal{B}$ be a cone. Let both $M, N : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators and assume that at least one of the operators is u_0-positive. If $M \leq N$, $Mu_1 \geq \lambda_1 u_1$ for some $u_1 \in \mathcal{P}$ and some $\lambda_1 > 0$, and $Nu_2 \leq \lambda_2 u_2$ for some $u_2 \in \mathcal{P}$ and some $\lambda_2 > 0$, then $\lambda_1 \leq \lambda_2$. Furthermore, $\lambda_1 = \lambda_2$ implies u_1 is a scalar multiple of u_2.*

3. Two term differential operator

To develop the appropriate compact operators, we introduce the appropriate Banach spaces. Define the Banach Space

$$\mathcal{B} = \{u : u = t^{\alpha-1}v, \ v \in C[0,1]\},$$

with the norm

$$\|u\| = |v|_0,$$

where $|v|_0 = \sup_{t \in [0,1]} |v(t)|$ denotes the usual supremum norm. Notice that for $u \in \mathcal{B}$,

$$|u|_0 = |t^{\alpha-1}v|_0 \leq t^{\alpha-1}\|u\|,$$

implying $|u|_0 \leq \|u\|$. We also define the Banach space

$$\mathcal{B}_1 = \{u : u = t^{\alpha-1}v, v \in C^1[0,1], v(1) = 0\},$$

with the norm given by $\|u\|_1 = |v'|_0$.

Note that for $v \in C^1[0,1]$ and since $v(1) = 0$, then for $0 \leq t \leq 1$,

$$|v(t)| = |v(t) - v(1)| = \left| \int_1^t v'(s)ds \right| \leq (1-t)|v'|_0 \leq \|u\|_1.$$

Therefore, $|v|_0 \leq \|u\|_1 = |v'|_0$ and

$$|u|_0 = |t^{\alpha-1}v|_0 \leq t^{\alpha-1}\|u\|_1,$$

implies

$$|u|_0 \leq \|u\|_1. \tag{3.1}$$

Let $n \in \mathbb{N}$, $n \geq 2$, and $n - 1 < \alpha \leq n$. Assume $a \in C[0,1]$ $a(t) \geq 0$, $0 \leq t \leq 1$, and consider a boundary value problem for a nonhomogeneous two–term fractional differential equation

$$D_{0+}^{\alpha}u + a(t)u(t) + h(t) = 0, \quad 0 < t < 1, \tag{3.2}$$

$$u^{(i)}(0) = 0, \ i = 0, 1, \ldots, n - 2, \quad D_{0+}^{\beta}u(1) = 0, \tag{3.3}$$

where $0 \leq \beta \leq n - 1$, and D_{0+}^{α} and D_{0+}^{β} are the standard Riemann-Liouville derivatives.

The following construction of a Neumann series representation of a Green's function can be found in [8]. We provide some details because of our choice of Banach spaces.

Let $0 \leq \beta \leq n - 1$. Let $G_0(\beta; t, s)$ denote the Green's function for $-D_{0+}^{\alpha}u = 0$, satisfying the boundary conditions $u^{(i)}(0) = 0$, $i = 0, 1, \ldots, n - 2$, $D_{0+}^{\beta}u(1) = 0$, which is given by

$$G_0(\beta; t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s < t \leq 1. \end{cases} \tag{3.4}$$

We define

$$v_0(\beta; t, s) = \begin{cases} \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \\ \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \frac{(1-\frac{s}{t})^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s < t \le 1. \end{cases} \tag{3.5}$$

Note that $G_0(\beta; t, s) = t^{\alpha-1} v_0(\beta; t, s)$.

If $0 < \beta \le n-1$, let $h \in \mathcal{B}$. It has been shown in [11] that $u \in \mathcal{B}$ is a solution of (3.2), (3.3) if, and only if, $u \in \mathcal{B}$ and u satisfies

$$\begin{aligned} u(t) &= \int_0^1 G_0(\beta; t, s)(a(s)u(s) + h(s)u(s))ds \\ &= \int_0^1 G_0(\beta; t, s)a(s)u(s)ds + \int_0^1 G_0(\beta; t, s)h(s)ds \\ &= \mathcal{A}_1 u(t) + \mathcal{A}h(t), \end{aligned} \tag{3.6}$$

where $\mathcal{A}_1$ and $\mathcal{A}$ have now been respectively defined as

$$\mathcal{A}_1 u(t) = \int_0^1 G_0(\beta; t, s)a(s)u(s)ds, \quad \mathcal{A}u(t) = \int_0^1 G_0(\beta; t, s)u(s)ds, \quad 0 \le t \le 1. \tag{3.7}$$

If $0 = \beta$, let $h \in \mathcal{B}_1$. It has been shown in [5] that $u \in \mathcal{B}_1$ is a solution of (3.2), (1.6) if, and only if, $u \in \mathcal{B}_1$ and u satisfies

$$\begin{aligned} u(t) &= \int_0^1 G_0(0; t, s)(a(s)u(s) + h(s)u(s))ds \\ &= \int_0^1 G_0(0; t, s)a(s)u(s)ds + \int_0^1 G_0(0; t, s)h(s)ds \\ &= \mathcal{A}_1 u(t) + \mathcal{A}h(t). \end{aligned} \tag{3.8}$$

Remark 3.1. We will suppress dependence on β in the operators $\mathcal{A}_1$ and $\mathcal{A}$ with the understanding that if $0 < \beta \le n-1$, the supporting Banach space is $\mathcal{B}$ and if $0 = \beta$, the Banach space is $\mathcal{B}_1$.

Solving (3.6) for u to obtain $(\mathcal{I} - \mathcal{A}_1)u = \mathcal{A}h$, or, formally

$$u = \Big(\sum_{n=0}^{\infty} \mathcal{A}_1^n \Big) \mathcal{A}h.$$

Before stating and outlining a proof of Theorem 3.3, we state a lemma (see [23, p. 795]).

Lemma 3.2. *Let $\mathcal{B}$ denote a Banach space, and assume $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ is a linear operator with operator norm $\|\mathcal{A}\|$. Let $r(\mathcal{A})$ denote the spectral radius of $\mathcal{A}$. Then*

    (i) *$r(\mathcal{A}) \le \|\mathcal{A}\|$;*
    (ii) *if $r(\mathcal{A}) < 1$, then $(\mathcal{I} - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} \mathcal{A}^n$, where $\mathcal{I}$ denotes the identity operator.*

Theorem 3.3. *Assume $a \in C[0, 1]$, and assume $|a|_0 < \Gamma(\alpha)$. If $0 < \beta \le n-1$, then a function $u \in \mathcal{B}$ is a solution of the boundary value problem (3.2), (3.3) if, and only if, $u \in \mathcal{B}$ and*

$$u(t) = \int_0^1 G(\beta; t, s)h(s)ds,$$

where

$$G(\beta; t, s) = \sum_{n=0}^{\infty} G_n(\beta; t, s), \tag{3.9}$$

and for $n \geq 1$, n an integer,

$$G_n(\beta; t, s) = \int_0^1 a(\tau) G_0(\beta; t, \tau) G_{n-1}(\beta; \tau, s) d\tau. \tag{3.10}$$

If $0 = \beta$, then a function $u \in \mathcal{B}_1$ is a solution of the boundary value problem (3.2), (1.6) if, and only if, $u \in \mathcal{B}_1$ and

$$u(t) = \int_0^1 G(0; t, s) h(s) ds,$$

where

$$G(0; t, s) = \sum_{n=0}^{\infty} G_n(0; t, s), \tag{3.11}$$

and for $n \geq 1$, n an integer,

$$G_n(0; t, s) = \int_0^1 a(\tau) G_0(0; t, \tau) G_{n-1}(0; \tau, s) d\tau. \tag{3.12}$$

Proof. To obtain (3.9) inductively from (3.10) (or respectively (3.11) from (3.12)), compute each $\mathcal{A}_1^n \mathcal{A} h$ inductively. If

$$\mathcal{A}_1^n \mathcal{A} h = \int_0^1 G_n(\beta; t, s) h(s) ds$$

then

$$\begin{aligned}
\mathcal{A}_1^{n+1} \mathcal{A} h &= \mathcal{A}_1 \mathcal{A}_1^n \mathcal{A} h \\
&= \int_0^1 G_0(\beta; t, s) a(s) \int_0^1 G_n(\beta; s, r) h(r) dr ds \\
&= \int_0^1 \left(\int_0^1 a(\tau) G_0(\beta; t, \tau) G_n(\beta; \tau, s) d\tau \right) h(s) ds \\
&= \int_0^1 G_{n+1}(\beta; t, s) h(s) ds.
\end{aligned}$$

To address the convergence in (3.9), it is shown in [4] that for $0 < \beta \leq n - 1$,

$$G_0(\beta; t, s) \geq 0, \quad (t, s) \in [0, 1] \times [0, 1],$$

and so it follows from (3.4) that

$$0 \leq G_0(\beta; t, s) \leq \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} \bigg|_{(t=1, s=0)} = \frac{1}{\Gamma(\alpha)}.$$

To see this, it is clear that for $s \in [0, 1]$,

$$G_0(\beta; t, s) \leq \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} \bigg|_{t=1} = \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}.$$

So now maximize the function of s at $s = 0$. Assume inductively that for $n \geq 1$,

$$|G_n(\beta_1; t, s)| \leq \frac{|a|_0^n}{\Gamma^{n+1}(\alpha)}. \tag{3.13}$$

Then

$$|G_{n+1}(\beta_1; t, s)| \leq \int_0^1 |a(\tau)||G_0(\beta_1; t, \tau)||G_n(\beta_1; \tau, s)|d\tau \leq \frac{|a|_0^{n+1}}{\Gamma^{n+2}(\alpha)}.$$

So, (3.13) is valid for each $n \geq 1$. Straightforward applications of the Weierstrass M-test and the ratio test imply the uniform and absolute convergence of (3.9) on $[0, 1] \times [0, 1]$.

For $\beta = 0$, to address convergence in (3.11), first define, for $n \geq 1$,

$$v_n(0; t, s) = \int_0^1 a(\tau)v_0(0; t, \tau)G_{n-1}(0; \tau, s)d\tau, \tag{3.14}$$

where $v_0(\beta; t, s)$ has been defined in (3.5). Then $G_n(0; t, s) = t^{\alpha-1}v_n(0; t, s)$ and

$$G(0; t, s) = t^{\alpha-1}\sum_{n=0}^{\infty} v_n(0; t, s) = t^{\alpha-1}V(0; t, s), \tag{3.15}$$

where $V(0; t, s) = \sum_{n=0}^{\infty} v_n(0; t, s)$.

It is shown in [5] that $\int_0^1 v_0(0; t, s)ds \in C^1[0, 1]$. Moreover, if $h \in \mathcal{B}_1$ and $\|h\|_1 = 1$, which implies by (3.1) that $|h|_0 \leq 1$, then

$$\left|\frac{d}{dt}\int_0^t \frac{(1-\frac{s}{t})^{\alpha-1}}{\Gamma(\alpha)}a(s)h(s)ds\right| \leq \int_0^t \frac{(\alpha-1)(1-\frac{s}{t})^{\alpha-2}}{\Gamma(\alpha)}\frac{s}{t^2}ds|a|_0 = \frac{|a|_0}{\Gamma(\alpha)}. \tag{3.16}$$

Thus,

$$\|v_n(0; t, s)\|_1 \leq \frac{|a|_0^n}{\Gamma^{n+1}(\alpha)},$$

the analogue of (3.13).

To apply Lemma 3.2 and complete the proof, for $\beta = 0$ or $0 < \beta \leq n - 1$, calculate

$$\|\mathcal{A}_1\| = \sup_{h \in \mathcal{B}, \|h\|=1} \|\mathcal{A}_1 h\|$$

$$= \sup_{h \in \mathcal{B}, \|h\|=1} \left\|\int_0^1 G_0(\beta; t, s)a(s)h(s)ds\right\| \leq \frac{|a|_0}{\Gamma(\alpha)} < 1. \qquad \square$$

The following inequalities are known for G_0 and v_0.

Lemma 3.4. *The following hold.*

(1) $G_0(\beta; t, s) \geq 0$ *for* $(t, s) \in [0, 1] \times [0, 1)$, $0 \leq \beta \leq n - 1$;
(2) $G_0(\beta; t, s) > 0$ *for* $(t, s) \in (0, 1] \times [0, 1)$ *for* $\beta > 0$ *and* $G_0(0; t, s) > 0$ *for* $(t, s) \in (0, 1) \times (0, 1)$;
(3) $v_0(\beta; 0, s) > 0$ *for* $s \in (0, 1)$, $0 \leq \beta \leq n - 1$;
(4) *If* $0 \leq \beta_1 < \beta_2 \leq n - 1$, *then* $G(\beta_1; t, s) < G(\beta_2; t, s)$ *for* $(t, s) \in (0, 1) \times (0, 1)$;
(5) *If* $0 \leq \beta_1 < \beta_2 \leq n - 1$, *then* $v_0(\beta_1; 0, s) < v_0(\beta_2; 0, s)$ *for* $s \in (0, 1)$;
(6) $v_0(0; 1, s) = 0$ *for* $s \in (0, 1)$;
(7) $v_0'(0; 1, s) < 0$ *for* $s \in (0, 1)$.

Proof. The proofs of (1), (2), and (4) can be found in [4]. The proof of (3) can be found in [18]. For (5), notice that

$$v_0(\beta_2; 0, s) - v_0(\beta_1; 0, s) = \frac{1}{\Gamma(\alpha)}\left[(1-s)^{\alpha-1-\beta_2} - (1-s)^{\alpha-1-\beta_1}\right]$$

$$= \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-1-\beta_2} \left[1 - (1-s)^{\beta_2-\beta_1} \right] > 0.$$

So (5) holds. Property (6) can be verified directly. For property (7), notice

$$v_0(0; 1, s) = -\frac{(\alpha-1)s(1-s)^{\alpha-2}}{\Gamma(\alpha)} < 0. \qquad \square$$

Because of the construction of $G(\beta; t, s)$ through (3.10) and (3.9) for $\beta > 0$ or through (3.11) and (3.12), the following extension to Lemma 3.4 is valid. So we need the following inequalities.

Lemma 3.5. *The following hold.*

(1) $G(\beta; t, s) \geq 0$ *for* $(t, s) \in [0, 1] \times [0, 1)$, $0 \leq \beta \leq n - 1$;
(2) $G(\beta; t, s) > 0$ *for* $(t, s) \in (0, 1] \times [0, 1)$ *for* $\beta > 0$ *and* $G(0; t, s) > 0$ *for* $(t, s) \in (0, 1) \times (0, 1)$;
(3) $V(\beta; 0, s) > 0$ *for* $s \in (0, 1)$, $0 \leq \beta \leq n - 1$;
(4) *If* $0 \leq \beta_1 < \beta_2 \leq n - 1$, *then* $G(\beta_1; t, s) < G(\beta_2; t, s)$ *for* $(t, s) \in (0, 1) \times (0, 1)$;
(5) *If* $0 \leq \beta_1 < \beta_2 \leq n - 1$, *then* $V(\beta_1; 0, s) < V(\beta_2; 0, s)$ *for* $s \in (0, 1)$;
(6) $V(0; 1, s) = 0$ *for* $s \in (0, 1)$;
(7) $V'(0; 1, s) < 0$ *for* $s \in (0, 1)$.

Proof. If $a \equiv 0$, $G(\beta; t, s) = G_0(\beta; t, s)$ and so (1)-(7) hold. Suppose $a \not\equiv 0$. Let $0 \leq \beta_1 < \beta_2 \leq n - 1$. For (2) and (4), notice for $(t, s) \in (0, 1) \times (0, 1)$, $0 < G_0(\beta_1; t, s) < G_0(\beta_2; t, s)$. Now assume for $k \in \mathbb{N}$, $0 < G_k(\beta_1; t, s) < G_k(\beta_2; t, s)$. Then for $(t, s) \in (0, 1) \times (0, 1)$,

$$G_{k+1}(\beta_2; t, s) = \int_0^1 a(\tau) G_0(\beta_2; t, \tau) G_k(\beta_2; \tau, s) d\tau$$

$$> \int_0^1 a(\tau) G_0(\beta_1; t, \tau) G_k(\beta_1; \tau, s) d\tau$$

$$= G_{k+1}(\beta_1; t, s) > 0.$$

So for each $n \in \mathbb{N}$, $0 < G_n(\beta_1; t, s) < G_n(\beta_2; t, s)$ for $(t, s) \in (0, 1) \times (0, 1)$. Then

$$G(\beta_2; t, s) = \sum_{n=0}^{\infty} G_n(\beta_2; t, s) > \sum_{n=0}^{\infty} G_n(\beta_1; t, s) = G(\beta_1; t, s) > 0.$$

The proof of (1) is similar. For (3) and (5), similarly notice for $s \in (0, 1)$, $0 < v_0(\beta_1; 0, s) < v_0(\beta_2; 0, s)$. Assume for $k \in \mathbb{N}$, $0 < v_k(\beta_2; 0, s) < v_k(\beta_1; 0, s)$. For $s \in (0, 1)$,

$$v_{k+1}(\beta_2; 0, s) = \int_0^1 a(\tau) v_0(\beta_2; 0, \tau) v_k(\beta_2; \tau, s) d\tau$$

$$> \int_0^1 a(\tau) v_0(\beta_1; 0, \tau) v_k(\beta_1; \tau, s) d\tau$$

$$= v_{k+1}(\beta_1; t, s) > 0.$$

Thus, for each $n \in \mathbb{N}$, $0 < v_n(\beta_1; 0, s) < v_n(\beta_2; 0, s)$ for $s \in (0, 1)$. This implies

$$V(\beta_2; 0, s) = \sum_{n=0}^{\infty} v_n(\beta_2; 0, s) > \sum_{n=0}^{\infty} G_n(\beta_1; 0, s) = V(\beta_1; 0, s) > 0.$$

The proofs of (6) and (7) are similar.　　　　　　　　　　　　　　　□

4. Comparison of smallest eigenvalues

We derive existence and comparison results. To do this, we will define integral operators whose kernels are the Green's function for $-D_{0+}^{\alpha}u - a(t)u = 0$, satisfying the boundary conditions $u^{(i)}(0) = 0$, $i = 0, 1, \ldots, n-2$, $D_{0+}^{\beta}u(1) = 0$, which are given by (3.9). So u solves (1.1), (1.2) if, and only if,

$$u(t) = \lambda_1 \int_0^1 G(\beta_1; t, s)p(s)u(s)ds.$$

Similarly, u solves (1.3), (1.4) if, and only if,

$$u(t) = \lambda_2 \int_0^1 G(\beta_2; t, s)q(s)u(s)ds,$$

and u solves (1.5), (1.6) if, and only if,

$$u(t) = \lambda_3 \int_0^1 G(0; t, s)r(s)u(s)ds.$$

We define the linear operators

$$Mu(t) = \int_0^1 G(\beta_1; t, s)p(s)u(s)ds, \tag{4.1}$$

$$Nu(t) = \int_0^1 G(\beta_2; t, s)q(s)u(s)ds, \; Lu(t) = \int_0^1 G(0; t, s)r(s)u(s)ds. \tag{4.2}$$

Theorem 4.1. *The operators $M, N, L : \mathcal{B} \to \mathcal{B}$ are compact. Also, $L : \mathcal{B}_1 \to \mathcal{B}_1$ is compact.*

Proof. Let $0 \le \beta \le n-1$. It is proved in [14] that if $0 < \beta \le n-1$, then $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ is compact, where $\mathcal{A}$ has been defined in (3.7). For $\beta = 0$, it is proved in [5] that $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ is compact.

For the sake of completeness, we remind the reader the technique of proof. Let $h \in \mathcal{B}$ so $h = t^{\alpha-1}v$. If $\beta > 0$, $v \in C[0,1]$; if $\beta = 0$, $v \in C^1[0,1]$. Write

$$\mathcal{A}h(t) = t^{\alpha-1} \int_0^1 v_0(\beta; t, s)s^{\alpha-1}v(s)ds = t^{\alpha-1}K_0(\beta)v(t),$$

where

$$K_0(\beta)v(t) = \int_0^1 v_0(\beta; t, s)s^{\alpha-1}v(s)ds,$$

and v_0 has been defined in (3.5). For $\beta > 0$, $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ is compact if, and only if, $K_0(\beta) : C[0,1] \to C[0,1]$ is compact; for $\beta = 0$, $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ is compact if, and only if, $K_0(\beta) : C^1[0,1] \to C^1[0,1]$ is compact. For $\beta > 0$, a standard application of the Arzela-Ascoli theorem then gives the compactness of $K_0(\beta)$. For $\beta = 0$, (3.16) is employed.

It is clear that the operators $\mathcal{A}$ and $\mathcal{A}_1$ commute and if $h \in \mathcal{B}$, then

$$\left(\sum_{n=0}^{\infty} (\mathcal{A}_1)^n \mathcal{A} \right) h = \mathcal{A} \sum_{n=0}^{\infty} (\mathcal{A}_1)^n h.$$

Thus, if $u \in \mathcal{B}$, then $pu \in \mathcal{B}$ and

$$
\begin{aligned}
Mu(t) &= \int_0^1 G(\beta_1; t, s)p(s)u(s)ds \\
&= \sum_{n=0}^\infty (\mathcal{A}_1)^n \mathcal{A}pu \\
&= \mathcal{A}\Big(\sum_{n=0}^\infty \mathcal{A}_1^n\Big)pu.
\end{aligned}
$$

Once we argue that $u \in \mathcal{B}$ implies $\big(\sum_{n=0}^\infty \mathcal{A}_1^n\big)pu \in \mathcal{B}$; then the compactness of M is proved by the compactness of $\mathcal{A}$.

The analysis to show the uniform and absolute convergence of (3.9) on $[0,1] \times [0,1]$ can be applied to $\big(\sum_{n=0}^\infty \mathcal{A}_1^n\big)h$. Now $\bar{G}_0(\beta; t, s) = a(s)G_0(\beta; t, s)$,

$$
\bar{G}_n(\beta; t, s) = \int_0^1 \bar{G}_0(\beta; t, \tau)\bar{G}_{n-1}(\beta; \tau, s)d\tau,
$$

for each $n \geq 1$, and

$$
\bar{G}(\beta; t, s) = \sum_{n=0}^\infty \bar{G}_n(\beta; t, s) = t^{\alpha-1}\sum_{n=0}^\infty \bar{v}_n(\beta; t, s).
$$

The assumption $|a|_0 < \Gamma(\alpha)$ implies that $\bar{G}(\beta; t, s)$ converges uniformly and absolutely on $[0,1] \times [0,1]$. Thus, $\big(\sum_{n=0}^\infty \mathcal{A}_1^n\big)pu \in \mathcal{B}$.

In a similar way, $N : \mathcal{B} \to \mathcal{B}$ is compact. In [5], it was shown that $K_0(0) : C^1[0,1] \to C^1[0,1]$ is compact and $K_0(0)u(1) = 0$ for any $u \in \mathcal{B}_1$. Then $L : \mathcal{B}_1 \to \mathcal{B}_1$ is compact, which implies $L : \mathcal{B} \to \mathcal{B}$ is also compact. $\square$

We define the cone

$$
\mathcal{P} = \{u \in \mathcal{B} : u(t) \geq 0 \text{ for } t \in [0,1]\},
$$

and the set $\Omega := \{u = t^{\alpha-1}v \in \mathcal{B} : u(t) > 0 \text{ for } t \in (0,1], \; v(0) > 0\}$. We also define the cone

$$
\mathcal{P}_1 = \{u \in \mathcal{B}_1 : u(t) \geq 0 \text{ for } t \in [0,1]\},
$$

and the set $\Omega_1 := \{u = t^{\alpha-1}v \in \mathcal{B} : u(t) > 0 \text{ for } t \in (0,1), \; v(0) > 0, \; v'(1) < 0\}$. The proof of the following Lemma 4.2 can be found in [6].

Lemma 4.2. *The set $\Omega \subset \mathcal{P}^\circ$. Hence the cone $\mathcal{P}$ is solid in $\mathcal{B}$ and therefore reproducing.*

The proof of the following Lemma 4.3 can be found in [5].

Lemma 4.3. *The set $\Omega_1 \subset \mathcal{P}_1^\circ$. Hence the cone $\mathcal{P}_1$ is solid in $\mathcal{B}_1$ and therefore reproducing.*

Lemma 4.4. *The operators M, N are u_0-positive with respect to $\mathcal{P}$.*

Proof. We first show $M : \mathcal{P} \to \mathcal{P}$. Let $u \in \mathcal{P}$. Then

$$
Mu(t) = \int_0^1 G(\beta; t, s)p(s)u(s)ds \geq 0.
$$

So $Mu \in \mathcal{P}$ and $M : \mathcal{P} \to \mathcal{P}$. Next, let $u \in \mathcal{P} \setminus \{0\}$. Now, there exists a compact subinterval $[a, b] \subset [0, 1]$ such that $p(t) > 0$ and $u(t) > 0$ for $t \in [a, b]$. So for $t \in (0, 1]$,

$$Mu(t) = \int_0^1 G(\beta; t, s)p(s)u(s)ds \geq \int_a^b G(\beta; t, s)p(s)u(s)ds > 0.$$

Let $Mu(t) = t^{\alpha-1}v(t)$. Then

$$v(0) = \int_0^1 V(\beta; 0, s)p(s)u(s)ds > 0.$$

So $M : \mathcal{P} \setminus \{0\} \to \Omega \subset \mathcal{P}^\circ$. By Lemma 2.6, M is u_0-positive with respect to $\mathcal{P}$. Similarly, N is u_0-positive with respect to $\mathcal{P}$. $\qquad\square$

Lemma 4.5. *The operators L is u_0-positive with respect to $\mathcal{P}_1$.*

Proof. Following the proof of the previous theorem, $L : \mathcal{P}_1 \to \mathcal{P}_1$ and if $u \in \mathcal{P}_1 \setminus \{0\}$, then $Lu(t) > 0$ for $t \in (0, 1)$. Let $Lu(t) = t^{\alpha-1}v(t)$. Again, similar to above, $v(0) > 0$. Finally,

$$v'(1) = \int_0^1 V'(0; 1, s)r(s)u(s)ds < 0.$$

So $L : \mathcal{P}_1 \setminus \{0\} \to \Omega_1 \subset \mathcal{P}_1^\circ$. By Lemma 2.6, L is u_0-positive with respect to $\mathcal{P}_1$. $\quad\square$

The following result is a direct consequence of Theorem 2.7.

Theorem 4.6. *Let $\mathcal{B}$, $\mathcal{B}_1$, $\mathcal{P}$, $\mathcal{P}_1$ M, N, and L be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$. Similarly, L has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}_1^\circ$.*

Theorem 4.7. *Let $\mathcal{B}$, $\mathcal{B}_1$, $\mathcal{P}$, $\mathcal{P}_1$ M, N, and L be defined as earlier. Let $r(t) \leq p(t) \leq q(t)$ on $[0, 1]$. Let Λ_1, Λ_2, and Λ_3 be the eigenvalues defined in Theorem 4.6 associated with M, N, and L, respectively, with the essentially unique eigenvectors u_1, $u_2 \in \mathcal{P}^\circ$, $u_3 \in \mathcal{P}_1^\circ$. Then $\Lambda_3 < \Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(t) = q(t)$ on $[0, 1]$ and $\beta_1 = \beta_2$.*

Proof. Let $p(t) \leq q(t)$ on $[0, 1]$. So for any $u \in \mathcal{P}$ and $t \in [0, 1]$,

$$(N - M)u(t) = \int_0^1 G(\beta_2; t, s)q(s)u(s)ds - \int_0^1 G(\beta_1; t, s)p(s)u(s)ds$$

$$\geq \int_0^1 G(\beta_1; t, s)p(s)u(s)ds - \int_0^1 G(\beta_1; t, s)p(s)u(s)ds = 0.$$

So $(N - M)(u) \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \leq N$ with respect to $\mathcal{P}$. Then, by Theorem 2.8, $\Lambda_1 \leq \Lambda_2$.

If $p(t) = q(t)$ on $[0, 1]$ and $\beta_1 = \beta_2$, then $\Lambda_1 = \Lambda_2$. Next, suppose $p(t) \neq q(t)$ or $\beta_1 \neq \beta_2$. If $p(t) \neq q(t)$, then $p(t) < q(t)$ on some subinterval $[a, b] \subset [0, 1]$, which implies $(N - M)u_1(t) > 0$ for $t \in (0, 1]$. Let $(N - M)u_1(t) = t^{\alpha-1}v(t)$. So

$$v(0) = \int_0^1 V(\beta_2; 0, s)q(s)u(s)ds - \int_0^1 V(\beta_1; 0, s)p(s)u(s)ds$$

$$\geq \int_0^1 V(\beta_1; 0, s)q(s)u(s)ds - \int_0^1 V(\beta_1; 0, s)p(s)u(s)ds$$

$$= \int_0^1 V(\beta_1; 0, s)(q(s) - p(s))u(s)ds.$$

Since $p(t) < q(t)$ on $[a, b] \subset [0, 1]$, then $v(0) > 0$. So, $(N-M)u_1 \in \Omega \subset \mathcal{P}^\circ$. So there exists $\epsilon > 0$ such that $(N - M)u_1 - \epsilon u_1 \in \mathcal{P}$. So $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \leq Nu_1$, implying $Nu_1 \geq (\Lambda_1 + \epsilon)u_1$. Since $M \leq N$ and $Nu_2 = \Lambda_2 u_2$, Theorem 2.8 implies $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. Next, suppose $\beta_1 \neq \beta_2$ and $p(t) = q(t)$ on $[0, 1]$. Then $\beta_1 < \beta_2$, and by Lemma 3.4 (4), $(N - M)u_1(t) > 0$ for $t \in (0, 1]$. Let $(N - M)u_1(t) = t^{\alpha-1}v(t)$. Then

$$v(0) = \int_0^1 V(\beta_2; 0, s)q(s)u(s)ds - \int_0^1 V(\beta_1; 0, s)p(s)u(s)ds$$

$$> \int_0^1 V(\beta_1; 0, s)q(s)u(s)ds - \int_0^1 V(\beta_1; 0, s)p(s)u(s)ds = 0.$$

So, $(N - M)u_1 \in \Omega \subset \mathcal{P}^\circ$. A similar argument gives that $\Lambda_1 < \Lambda_2$.

Finally, let $p(t) \geq r(t)$ on $[0, 1]$. For $u \in \mathcal{P}$ and $t \in [0, 1]$,

$$(M - L)u(t) = \int_0^1 G(\beta_1; t, s)p(s)u(s)ds - \int_0^1 G(0; t, s)r(s)u(s)ds$$

$$\geq \int_0^1 G(0; t, s)r(s)u(s)ds - \int_0^1 G(0; t, s)r(s)u(s)ds = 0.$$

So $(M - L)(u) \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $L \leq M$ with respect to $\mathcal{P}$. Notice Theorem 2.8 only requires M be u_0-positive with respect to $\mathcal{P}$. Consequently, by Theorem 2.8, $\Lambda_3 \leq \Lambda_1$. Since $u_3 \in \mathcal{P}_1$, $u_3 \in \mathcal{P}$. By Lemma 3.4, $(M - L)u_3(t) > 0$ for $t \in (0, 1]$. Let $(M - L)u_3(t) = t^{\alpha-1}v(t)$. Then

$$v(0) = \int_0^1 V(\beta_1; 0, s)q(s)u(s)ds - \int_0^1 V(0; 0, s)r(s)u(s)ds$$

$$> \int_0^1 V(0; 0, s)q(s)u(s)ds - \int_0^1 V(0; 0, s)r(s)u(s)ds = 0.$$

So, $(M - L)u_3 \in \Omega \subset \mathcal{P}^\circ$. So there exists $\epsilon > 0$ such that $(M - L)u_3 - \epsilon u_3 \in \mathcal{P}$. So $\Lambda_3 u_3 + \epsilon u_3 = Lu_3 + \epsilon u_3 \leq Mu_3$, implying $Mu_3 \geq (\Lambda_3 + \epsilon)u_3$. Since $M \leq M$ and $Mu_1 = \Lambda_1 u_1$, by Theorem 2.8, $\Lambda_3 + \epsilon \leq \Lambda_1$, or $\Lambda_3 < \Lambda_1$. □

Lemma 4.8. *The eigenvalues of* (1.1), (1.2) *are reciprocals of eigenvalues of* M, *and conversely. Similarly, eigenvalues of* (1.3), (1.4) *are reciprocals of eigenvalues of* N, *and conversely, and eigenvalues of* (1.5), (1.6) *are reciprocals of eigenvalues of* N, *and conversely.*

The main result is a direct consequence of Theorem 4.7 and Lemma 4.8.

Theorem 4.9. *Assume the hypotheses of Theorem 4.6. Then there exists smallest positive eigenvalues* λ_1 *and* λ_2 *of* (1.1), (1.2) *and* (1.3), (1.4), *and* λ_3 *of* (1.5), (1.6), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to* λ_1 *and* λ_2 *may be chosen to belong to* $\mathcal{P}^\circ$, *and eigenfunctions corresponding to* λ_3 *can be chosen to belong to* $\mathcal{P}_1^\circ$. *Finally,* $\lambda_3 > \lambda_1 \geq \lambda_2$, *and* $\lambda_1 = \lambda_2$ *if and only if* $p(t) = q(t)$ *for all* $t \in [0, 1]$ *and* $\beta_1 = \beta_2$.

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