SENSITIVITY OF A NONLINEAR ORDINARY BVP WITH FRACTIONAL DIRICHLET-LAPLACE OPERATOR

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ABSTRACT. In this article, we derive a sensitivity result for a nonlinear fractional ordinary elliptic system on a bounded interval with Dirichlet boundary conditions. More precisely, using a global implicit function theorem, we show that for each functional parameter there exists a unique solution, and that its dependence on the functional parameters is continuously differentiable.

1. INTRODUCTION

In this article, we study a nonlinear ordinary boundary value problem on the interval $(0, \pi)$, involving a Dirichlet-Laplace operator $(-\Delta)^\beta$ of order $\beta > 1/2$,

$$(-\Delta)^\beta x(t) = f(t, x(t), u(t)), \quad \text{a.e. } t \in (0, \pi), \tag{1.1}$$

where $(-\Delta) : H_0^1 \cap H^2 \to L^2$ is the Dirichlet-Laplace operator, $H_0^1 = H_0^1((0,\pi), \mathbb{R}^m)$ and $H^2 = H^2((0,\pi), \mathbb{R}^m)$ are classical Sobolev spaces, $L^2 = L^2((0,\pi), \mathbb{R}^m)$ is the classical Lebesgue space, $f : (0,\pi) \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^m$ $(m, \ n \in \mathbb{N})$, $x : (0,\pi) \to \mathbb{R}^m$ is an unknown function and $u : (0,\pi) \to \mathbb{R}^r$ is a functional parameter.

Problems involving fractional Laplacians are extensively investigated in resent years because of their numerous applications, among others in probability, fluid mechanics, hydrodynamics; see, for example, [3, 4, 8, 9, 15] and references therein.

The definition of the fractional Laplacian adopted in our paper comes from the Stone-von Neumann operator calculus and is based on the spectral integral representation theorem for a self-adjoint operator in Hilbert space. It reduces to a *series* form which is taken by other authors as a definition [3, 6, 8]. Our more general approach allows us to obtain useful properties of this fractional operator in a smart way. This approach has also been used in [12].

In the first part of this paper, we recall some facts from the theory of spectral integral and Stone-von Neumann operator calculus. Next, we derive some properties of positive powers of the ordinary Dirichlet-Laplace operator and their domains (among others some embedding theorems). In the second part, we use a global implicit function theorem [10, 11] to prove existence and uniqueness of a solution to problem (1.1) as well as its sensitivity. By sensitivity we mean continuous differentiability of the mapping

$$u \mapsto x_u$$

where x_u is a unique solution to the problem, corresponding to a parameter u. This property can be used to study optimal control problems associated with system (1.1).

Similar method but based on a global diffeomorphism theorem [13] and applied to a nonlinear integral Hammerstein equation is presented in [5]. An application of the obtained results to the problem

$$\lambda(-\Delta)^{\sigma/2}x(t) + h(t, x(t)) = (-\Delta)^{\sigma/2}u(t), \quad t \in (0, 1),$$

where $\lambda \in \mathbb{R}$, $\sigma \in (1, 2]$, $h : [-1, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ $(n \in \mathbb{N})$, with the exterior Dirichlet boundary condition

$$x(t) = 0, \quad t \in (-\infty, -1] \cup [1, \infty).$$

In [6], a problem of type (1.1) on a bounded Lipschitzian domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ and with an exterior Dirichlet boundary condition, is studied. Continuous dependence of solutions on parameters (stability) is investigated therein.

In [12], using a variational method, we derive an existence result for the so-called bipolynomial fractional Dirichlet-Laplace problem

$$\sum_{i,j=0}^{k} \alpha_i \alpha_j [(-\Delta)_\omega]^{\beta_i+\beta_j} u(x) = D_u F(x, u(x)), \quad \text{a.e. } x \in \Omega,$$

where $\alpha_i > 0$ for $i = 0, \ldots, k$ $(k \in \mathbb{N} \cup \{0\})$ and $0 \leq \beta_0 < \beta_1 < \cdots < \beta_k$, $(-\Delta)_\omega : D((-\Delta)_\omega) \subset L^2 \to L^2$ is a weak Dirichlet-Laplace operator, $\Omega \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ is a bounded open set, $F : \Omega \times \mathbb{R} \to \mathbb{R}$, $D_u F$ is the partial derivative of F with respect to u.

2. Integral representation of a self-adjoint operator

Results presented in this section can be found, in the case of complex Hilbert space, for example, in [1, 14]. Their proofs can be moved without any or with small changes to the case of real Hilbert space. We will continue to deal only with real Hilbert spaces. Such a preliminary section has also been included in [12].

Let H be a real Hilbert space with a scalar product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$. Let us denote by $\Pi(H)$ the set of all projections of H on closed linear subspaces, and by $\mathcal{B}$ the σ-algebra of Borel subsets of $\mathbb{R}$. By the spectral measure in $\mathbb{R}$ we mean a set function $E : \mathcal{B} \to \Pi(H)$ that satisfies the following conditions:

- for each $x \in H$, the function

$$\mathcal{B} \ni P \mapsto E(P)x \in H \tag{2.1}$$

  is a vector measure
- $E(\mathbb{R}) = I$
- $E(P \cap Q) = E(P) \circ E(Q)$ for $P, Q \in \mathcal{B}$.

By a support of a spectral measure E we mean the complement of the sum of all open subsets of $\mathbb{R}$ with zero spectral measure.

If $b : \mathbb{R} \to \mathbb{R}$ is a bounded Borel measurable function, defined a.e. in E, then the integral $\int_{-\infty}^{\infty} b(\lambda)E(d\lambda)$ is defined by

$$\left(\int_{-\infty}^{\infty} b(\lambda)E(d\lambda) \right)x = \int_{-\infty}^{\infty} b(\lambda)E(d\lambda)x$$

for each $x \in H$ where the integral $\int_{-\infty}^{\infty} b(\lambda)E(d\lambda)x$ (with respect to the vector measure) is defined in a standard way, namely, with the aid of the sequence of simple functions converging a.e. in $E(d\lambda)x$ to b (see [1]).

If $b : \mathbb{R} \to \mathbb{R}$ is an unbounded Borel measurable function defined a.e. in E, then, for each $x \in H$ such that

$$\int_{-\infty}^{\infty} |b(\lambda)|^2 \|E(d\lambda)x\|^2 < \infty \tag{2.2}$$

(the above integral is taken with respect to the nonnegative measure $\mathcal{B} \ni P \mapsto \|E(P)x\|^2 \in \mathbb{R}_0^+$), there exists the limit

$$\lim \int_{-\infty}^{\infty} b_n(\lambda)E(d\lambda)x$$

of integrals (with respect to the vector measure (2.1)) where

$$b_n : \mathbb{R} \ni \lambda \mapsto \begin{cases} b(\lambda) & \text{if } |b(\lambda)| \le n, \\ 0 & \text{if } |b(\lambda)| > n. \end{cases}$$

Let us denote the set of all points x with property (2.2) by D. One proves that D is dense linear subspace of H, and by $\int_{-\infty}^{\infty} b(\lambda)E(d\lambda)$ one denotes the operator

$$\int_{-\infty}^{\infty} b(\lambda)E(d\lambda) : D \subset H \to H$$

given by

$$\Big(\int_{-\infty}^{\infty} b(\lambda)E(d\lambda) \Big) x = \lim \int_{-\infty}^{\infty} b_n(\lambda)E(d\lambda)x.$$

Of course, $D = H$ and

$$\lim \int_{-\infty}^{\infty} b_n(\lambda)E(d\lambda)x = \int_{-\infty}^{\infty} b(\lambda)E(d\lambda)x$$

when $b : \mathbb{R} \to \mathbb{R}$ is a bounded Borel measurable function, defined a.e. in E.

For $x \in D$, we have

$$\Big\| \Big(\int_{-\infty}^{\infty} b(\lambda)E(d\lambda) \Big) x \Big\|^2 = \int_{-\infty}^{\infty} |b(\lambda)|^2 \|E(d\lambda)x\|^2.$$

Moreover,

$$\Big(\int_{-\infty}^{\infty} b(\lambda)E(d\lambda) \Big)^* = \int_{-\infty}^{\infty} b(\lambda)E(d\lambda), \tag{2.3}$$

i.e., the operator $\int_{-\infty}^{\infty} b(\lambda)E(d\lambda)$ is self-adjoint.

Remark 2.1. To integrate a Borel measurable function $b : B \to \mathbb{R}$ where B is a Borel set containing the support of the measure E, it is sufficient to extend b on $\mathbb{R}$ to a whichever Borel measurable function (putting, for example, $b(\lambda) = 0$ for $\lambda \notin B$).

If $b : \mathbb{R} \to \mathbb{R}$ is Borel measurable and $\sigma \in \mathcal{B}$, then by the integral

$$\int_{\sigma} b(\lambda)E(d\lambda)$$

we mean the integral

$$\int_{-\infty}^{\infty} \chi_{\sigma}(\lambda)b(\lambda)E(d\lambda),$$

where χ_σ is the characteristic function of the set σ. The integral $\int_\sigma b(\lambda)E(d\lambda)$ can be also defined with the aid of the restriction of E to the set σ. The next theorem plays the fundamental role in the spectral theory of self-adjoint operators.

Theorem 2.2. *If $A : D(A) \subset H \to H$ is self-adjoint and the resolvent set $\rho(A)$ is non-empty, then there exists a unique spectral measure E with the closed support $\Lambda = \sigma(A)$, such that*

$$A = \int_{-\infty}^{\infty} \lambda E(d\lambda) = \int_{\sigma(A)} \lambda E(d\lambda).$$

The basic notion in the Stone-von Neumann operator calculus is a function of a self-adjoint operator. Namely, if $A : D(A) \subset H \to H$ is self-adjoint and E is the spectral measure determined according to the above theorem, then, for each Borel measurable function $b : \mathbb{R} \to \mathbb{R}$, one defines the operator $b(A)$ by

$$b(A) = \int_{-\infty}^{\infty} b(\lambda)E(d\lambda) = \int_{\sigma(A)} b(\lambda)E(d\lambda).$$

It is known that the spectrum $\sigma(b(A))$ of $b(A)$ is given by

$$\sigma(b(A)) = \overline{b(\sigma(A))} \tag{2.4}$$

provided that b is continuous (it is sufficient to assume that b is continuous on $\sigma(A)$). We have the following general result.

Proposition 2.3. *If $b, d : \mathbb{R} \to \mathbb{R}$ are Borel measurable functions and E is the spectral measure for a self-adjoint operator $A : D(A) \subset H \to H$ with non-empty resolvent set, then*

$$(b \cdot d)(A) \supset b(A) \circ d(A)$$

and

$$(b \cdot d)(A) = b(A) \circ d(A) \tag{2.5}$$

if and only if

$$D((b \cdot d)(A)) \subset D(d(A)).$$

Using the above proposition one can deduce that for each $n \in \mathbb{N}$ with $n \geq 2$, and a Borel measurable function $b : \mathbb{R} \to \mathbb{R}$,

$$(b(A))^n = b^n(A). \tag{2.6}$$

When $b(\lambda) = \lambda$, equality (2.6) gives

$$A^n = \int_{-\infty}^{\infty} \lambda^n E(d\lambda). \tag{2.7}$$

If $n = 1$, then (2.7) follows from Theorem 2.2. Since $E(\mathbb{R}) = I$, therefore the identity operator I can be written as

$$I = \int_{-\infty}^{\infty} 1 E(d\lambda).$$

If $\beta > 0$, then formula (2.6) with

$$b : \mathbb{R} \ni \lambda \to \begin{cases} 0, & \lambda < 0 \\ \lambda^{\beta/2}, & \lambda \geq 0 \end{cases}$$

and $n = 2$ implies the following proposition (cf. Remark 2.1).

Proposition 2.4. *If $\sigma(A) \subset [0, \infty)$, then*

$$A^{\beta/2} \circ A^{\beta/2} = A^\beta. \tag{2.8}$$

3. Fractional Dirichlet-Laplace operator

Consider the one-dimensional Dirichlet-Laplace operator on the interval $(0, \pi)$,

$$-\Delta : H_0^1 \cap H^2 \subset L^2 \to L^2$$

given by

$$-\Delta x(t) = -x''(t).$$

In an elementary way, one can check that this operator is self-adjoint,

$$\sigma(-\Delta) = \sigma_p(-\Delta) = \{j^2 : j \in \mathbb{N}\}$$

($\sigma_p(-\Delta)$ is the pointwise spectrum of $(-\Delta)$) and the eigenspace $N(j^2)$ corresponding to the eigenvalue $\lambda_j = j^2$ is the set $\{c \sin jt : c \in \mathbb{R}^m\}$. The system of functions

$$e_{j,i} = (0, \dots, 0, \underbrace{\sqrt{\frac{2}{\pi}} \sin jt}_{\text{i-th entry}}, 0, \dots, 0), \quad j = 1, 2, \dots, \ i = 1, \dots, m,$$

is the Hilbertian basis (complete orthonormal system) in L^2.

Now, let us fix any $\beta > 0$ and consider the operator

$$(-\Delta)^\beta : D((-\Delta)^\beta) \subset L^2 \to L^2$$

where

$$D((-\Delta)^\beta) = \{x(t) \in L^2 : \int_{\sigma(-\Delta)} |\lambda^\beta|^2 \|E(d\lambda)x\|^2 = \sum_{j=1}^\infty ((j^2)^\beta)^2 |a_j|^2 < \infty, \quad (3.1)$$

where

$$x(t) = \Big(\int_{\sigma(-\Delta)} 1 E(d\lambda)x \Big)(t) = \sum_{j=1}^\infty a_j \sqrt{\frac{2}{\pi}} \sin jt \,.$$

Here E is the spectral measure given by Theorem 2.2 for the operator $(-\Delta)$, $a_j \sqrt{\frac{2}{\pi}} \sin jt$ is the projection of x on the m-dimensional eigenspace $N(j^2)$ of the operator $(-\Delta)$, and

$$(-\Delta)^\beta x(t) = \Big(\Big(\int_{\sigma(-\Delta)} \lambda^\beta E(d\lambda) \Big) x \Big)(t)$$

$$= \Big(\lim \int_{\sigma(-\Delta)} (\lambda^\beta)_n E(d\lambda)x \Big)(t) = \sum_{j=1}^\infty (j^2)^\beta a_j \sqrt{\frac{2}{\pi}} \sin jt$$

for

$$x(t) = \sum_{j=1}^\infty a_j \sqrt{\frac{2}{\pi}} \sin jt \in D((-\Delta)^\beta) \,.$$

The series is meant in L^2 but from the Carleson theorem it follows that $x(t) = \sum_{j=1}^\infty a_j \sqrt{\frac{2}{\pi}} \sin jt$ a.e. on $(0, \pi)$ (cf. [7, Theorem 5.17]).

Equality (2.4) and the fact that isolated points of the spectrum of a self-adjoint operator are the eigenvalues imply that

$$\sigma((-\Delta)^\beta) = \sigma_p((-\Delta)^\beta) = \{(j^2)^\beta : j \in \mathbb{N}\}.$$

The corresponding eigenspaces for $(-\Delta)$ and $(-\Delta)^\beta$ are the same (it follows from a general result concerning the power of any self-adjoint operator).

The operator $(-\Delta)^\beta$ will be called *the Dirichlet-Laplace operator of order β*, and the function $(-\Delta)^\beta x$ - *the Dirichlet-Laplacian of order β of x*.

Lemma 3.1. $D((-\Delta)^\beta)$ *with the scalar product*

$$\langle x, y \rangle_\beta = \langle x, y \rangle_{L^2} + \langle (-\Delta)^\beta x, (-\Delta)^\beta y \rangle_{L^2}$$

is a Hilbert space.

Proof. The assertion follows from the operator $(-\Delta)^\beta$ being self-adjoint is closed (cf. (2.3)). $\square$

The scalar product $\langle \cdot, \cdot \rangle_\beta$ and the scalar product

$$\langle x, y \rangle_{\sim\beta} = \langle (-\Delta)^\beta x, (-\Delta)^\beta y \rangle_{L^2}$$

generate equivalent norms in $D((-\Delta)^\beta)$. Indeed, it is sufficient to observe that the following Poincare inequality holds:

$$\|x\|_{L^2}^2 = \sum_{j=1}^{\infty} a_j^2 \leq \sum_{j=1}^{\infty} ((j^2)^\beta)^2 a_j^2 = \|(-\Delta)^\beta x\|_{L^2}^2 = \|x\|_{\sim\beta}^2 \tag{3.2}$$

for each

$$x(t) = \sum_{j=1}^{\infty} a_j \sqrt{\frac{2}{\pi}} \sin jt \in D((-\Delta)^\beta).$$

Next, we shall consider $D((-\Delta)^\beta)$ with the norm $\|\cdot\|_{\sim\beta}$.

3.1. Embeddings. From the description of the domain $D((-\Delta)^\beta)$ it follows that

$$D((-\Delta)^{\beta_2}) \subset D((-\Delta)^{\beta_1}) \tag{3.3}$$

for each $0 < \beta_1 < \beta_2$. Using this relation and equality (2.7) with $A = (-\Delta)$ we assert that

$$C_c^\infty \subset D((-\Delta)^\beta)$$

for each $\beta > 0$ ($C_c^\infty = C_c^\infty((0,\pi), \mathbb{R}^m)$ is the set of smooth functions with the supports contained in $(0, \pi)$).

Lemma 3.2. *If $\beta > 1/4$, then*

$$D((-\Delta)^\beta) \subset L_m^\infty = L^\infty((0,\pi), \mathbb{R}^m)$$

and this embedding is continuous, more precisely,

$$\|x\|_{L_m^\infty} \leq \sqrt{\frac{2}{\pi}\zeta(4\beta)}\|x\|_{\sim\beta}$$

for $x \in D((-\Delta)^\beta)$, where $\zeta(4\beta)$ is the value of the Riemann zeta function $\zeta(\gamma) = \sum_{j=1}^{\infty} 1/j^\gamma$ at $\gamma = 4\beta$.

Proof. Let

$$x(t) = \sum_{j=1}^{\infty} a_j \sqrt{\frac{2}{\pi}} \sin jt \in D((-\Delta)^\beta).$$

Since $\sum_{j=1}^{\infty}((j^2)^{\beta})^2 a_j^2 < \infty$ and $\beta > 1/4$, for $t \in (0, \pi)$ a.e., we have

$$|x(t)|^2 = |\sum_{j=1}^{\infty} a_j \sqrt{\frac{2}{\pi}} \sin jt|^2 \leq \frac{2}{\pi} \Big(\sum_{j=1}^{\infty} |a_j| \Big)^2$$

$$= \frac{2}{\pi} \Big(\sum_{j=1}^{\infty} \frac{(j^2)^{\beta}|a_j|}{(j^2)^{\beta}} \Big)^2$$

$$\leq \frac{2}{\pi} \Big(\sum_{j=1}^{\infty}((j^2)^{\beta})^2|a_j|^2 \Big) \Big(\sum_{j=1}^{\infty} \frac{1}{\big((j^2)^{\beta}\big)^2} \Big)$$

$$= \frac{2}{\pi} \|x\|_{\sim_\beta}^2 \zeta(4\beta) < \infty$$

and the proof is complete.   □

Lemma 3.3. *If $\beta \geq 1/2$, then $D((-\Delta)^{\beta}) \subset H_0^1$, and consequently*

$$D((-\Delta)^{\beta}) \subset C = C([0, \pi], \mathbb{R}^m).$$

Proof. Of course it is sufficient to show that $D((-\Delta)^{1/2}) \subset H_0^1$ (cf. (3.3)). Indeed, let $x(t) = \sum_{j=1}^{\infty} a_j \sqrt{\frac{2}{\pi}} \sin jt \in D((-\Delta)^{1/2})$ and consider this series on the interval $[0, \pi]$. The sequence (S_n) of partial sums converges in L^2 to x. From the convergence of the series $\sum_{j=1}^{\infty} j^2 a_j^2$ it follows that the sequence (S_n') of derivatives converges in L^2 to a function. So (cf. [7]), one can choose a subsequence (S_{n_k}') convergent a.e. on $[0, \pi]$ to this function and bounded pointwise a.e. on $[0, \pi]$ by a function $g \in L^2$. Consequently, the sequence (S_{n_k}') is equiabsolutely integrable on $[0, \pi]$. So, the sequence (S_{n_k}) is equiabsolutely continuous on $[0, \pi]$. Of course, $S_{n_k}(0) = 0$, thus

$$|S_{n_k}(t)| = \big|S_{n_k}(0) + \int_0^t S_{n_k}'(s)ds\big| \leq \int_0^\pi g(s)ds < \infty$$

for $t \in [0, \pi]$. It means that elements of the sequence (S_{n_k}) satisfy the assumptions of the Ascoli-Arzela theorem for absolutely continuous functions and, in consequence, there exists a subsequence $(S_{n_{k_i}})$ converging uniformly on $[0, \pi]$ to an absolutely continuous function $\overline{x}$. Clearly, $(S_{n_{k_i}})$ converges to $\overline{x}$ in L^2. The uniqueness of the limit in L^2 means that $x = \overline{x}$ a.e. on $(0, \pi)$. So, x has a representative which is absolutely continuous on $[0, \pi]$ and satisfies Dirichlet boundary conditions, i.e. $x \in W_0^{1,1}((0, \pi), \mathbb{R}^m)$ (the classical Sobolev space). Consequently, there exists a function $g \in L^1$ such that

$$\int_0^\pi x(t)\varphi'(t)\, dt = -\int_0^\pi g(t)\varphi(t)\, dt$$

for each $\varphi \in C_c^{\infty}$. But

$$\int_0^\pi x(t)\varphi'(t)\, dt = \int_0^\pi \Big(\sum_{j=1}^{\infty} a_j \sqrt{\frac{2}{\pi}} \sin jt \Big) \varphi'(t)\, dt$$

$$= \int_0^\pi \lim_{n \to \infty} S_n(t)\varphi'(t)\, dt$$

$$= \sum_{j=1}^{\infty} \int_0^\pi a_j \sqrt{\frac{2}{\pi}} \sin jt\varphi'(t)\, dt$$

$$= \sum_{j=1}^{\infty} - \int_0^\pi j a_j \sqrt{\frac{2}{\pi}} \cos jt \varphi(t)\, dt$$

$$= - \int_0^\pi \sum_{j=1}^{\infty} j a_j \sqrt{\frac{2}{\pi}} \cos jt \varphi(t)\, dt$$

for $\varphi \in C_c^\infty$. The last equality follows from $\sum_{j=1}^{\infty} j^2 a_j^2 < \infty$, and consequently, $\sum_{n=1}^{\infty} j a_j \sqrt{\frac{2}{\pi}} \cos jt \in L^2$. Thus,

$$g(t) = \sum_{n=1}^{\infty} j a_j \sqrt{\frac{2}{\pi}} \cos jt \in L^2$$

and, finally, $x \in H_0^1$.

The second part of the theorem follows from a known property of Sobolev space $W^{1,1}((0,\pi), \mathbb{R}^m)$. $\qquad\square$

Lemma 3.4. *If $\beta > 3/4$, then any bounded the set $B \subset D((-\Delta)^\beta)$ is equicontinuous on $[0, \pi]$.*

Proof. Similarly as in the proof of Lemma 3.2 we obtain

$$|x(t_1) - x(t_2)|^2 = \Big| \sum_{j=1}^{\infty} a_j \sqrt{\frac{2}{\pi}} (\sin jt_1 - \sin jt_2) \Big|^2$$

$$\leq \Big(\sum_{j=1}^{\infty} |a_j| \sqrt{\frac{2}{\pi}} 2 |\sin \frac{j(t_1 - t_2)}{2}| \Big)^2$$

$$\leq \frac{2}{\pi} |t_1 - t_2|^2 \Big(\sum_{j=1}^{\infty} |a_j| j \Big)^2$$

$$= \frac{2}{\pi} |t_1 - t_2|^2 \Big(\sum_{j=1}^{\infty} \frac{(j^2)^\beta |a_j|}{(j^2)^\beta j^{-1}} \Big)^2$$

$$\leq \frac{2}{\pi} |t_1 - t_2|^2 \Big(\sum_{j=1}^{\infty} ((j^2)^\beta)^2 |a_j|^2 \Big) \Big(\sum_{j=1}^{\infty} \frac{1}{(j^{2\beta-1})^2} \Big)$$

$$= \frac{2}{\pi} |t_1 - t_2|^2 \|x\|_{\sim\beta}^2 \zeta(4\beta - 2) < \infty$$

for $t_1, t_2 \in (0, \pi)$ a.e., where $x(t) = \sum_{j=1}^{\infty} a_j \sqrt{\frac{2}{\pi}} \sin jt \in D((-\Delta)^\beta)$. Identifying x with its absolutely continuous representative on $[0, \pi]$ we assert that the above estimation holds for all $t_1, t_2 \in [0, \pi]$. $\qquad\square$

Using Lemmas 3.2, 3.3, 3.4 we obtain the following result.

Corollary 3.5. *If $\beta > 3/4$, then the embedding $D((-\Delta)^\beta) \subset C$ is compact.*

3.2. **Equivalence of equations.** Fact that the operator $(-\Delta)^\beta$ $(\beta > 0)$ is self-adjoint means that its domain satisfies the equality

$$D((-\Delta)^\beta)$$
$$= \Big\{ x \in L^2 : \text{there exists } z \in L^2 \text{ such that} \qquad (3.4)$$
$$\int_0^\pi x(t)(-\Delta)^\beta y(t)\, dt = \int_0^\pi z(t)y(t)\, dt \text{ for each } y \in D((-\Delta)^\beta) \Big\}$$

and

$$(-\Delta)^\beta x = z \qquad (3.5)$$

for $x \in D((-\Delta)^\beta)$.

From (2.8) it follows that $x \in D((-\Delta)^\beta)$ if and only if $x \in D((-\Delta)^{\beta/2})$, $(-\Delta)^{\beta/2} x \in D((-\Delta)^{\beta/2})$, and this case

$$(-\Delta)^{\beta/2}((-\Delta)^{\beta/2}x) = (-\Delta)^\beta x. \qquad (3.6)$$

Using this fact and (3.4), (3.5), we obtain the following lemma.

Lemma 3.6. *If $\beta > 0$ and $g \in L^2$, then $x \in D((-\Delta)^\beta)$ and $(-\Delta)^\beta x = g$ if and only if $x \in D((-\Delta)^{\beta/2})$ and*

$$\int_0^\pi (-\Delta)^{\beta/2} x(t)(-\Delta)^{\beta/2} y(t)\, dt = \int_0^\pi g(t)y(t)\, dt$$

for each $y \in D((-\Delta)^{\beta/2})$.

4. Global implicit function theorem

Let X be a real Banach space and $I : X \to \mathbb{R}$ be a functional of class C^1. We say that I satisfies *Palais-Smale (PS) condition* if any sequence (x_k) such that

- $|I(x_k)| \le M$ for all $k \in \mathbb{N}$ and some $M > 0$,
- $I'(x_k) \to 0$,

admits a convergent subsequence. Here $I'(x_k)$ denotes the Frechet differential of I at x_k. A sequence (x_k) satisfying the above conditions is called the (PS) sequence for I.

From [10, 11] we have the following result.

Theorem 4.1. *Let X, U be real Banach spaces, H be a real Hilbert space. If $F : X \times U \to H$ is continuously differentiable with respect to $(x, u) \in X \times U$ and*

- *for each $u \in U$, the functional*

$$\varphi : X \ni x \mapsto \frac{1}{2} \|F(x, u)\|^2 \in \mathbb{R}$$

  satisfies (PS) condition
- *$F_x'(x, u) : X \to H$ is bijective for each $(x, u) \in X \times U$,*

then there exists a unique function $\lambda : U \to X$ such that $F(\lambda(u), u) = 0$ for each $u \in U$ and this function is of class C^1 with differential $\lambda'(u)$ at u given by

$$\lambda'(u) = -[F_x'(\lambda(u), u)]^{-1} \circ F_u'(\lambda(u), u). \qquad (4.1)$$

5. A boundary value problem

Let us consider boundary value problem (1.1). Using the global implicit function theorem, we shall show (under suitable assumptions) that, for each fixed $u \in L_r^\infty = L^\infty((0,\pi), \mathbb{R}^r)$, problem (1.1) has a unique solution $x_u \in D((-\Delta)^\beta)$ and the mapping

$$L_r^\infty \ni u \mapsto x_u \in D((-\Delta)^\beta)$$

is continuously differentiable.

Consider the mapping

$$F : D((-\Delta)^\beta) \times L_r^\infty \ni (x,u) \mapsto (-\Delta)^\beta x(t) - f(t, x(t), u(t)) \in L^2.$$

We shall formulate conditions guaranteeing that

- F is of class C^1,
- differential $F_x(x,u) : D((-\Delta)^\beta) \to L^2$ is bijective for each (x,u) in $D((-\Delta)^\beta) \times L_r^\infty$,
- for each $u \in L_r^\infty$, functional

$$\varphi : D((-\Delta)^\beta) \ni x \mapsto \frac{1}{2}\|F(x,u)\|_{L^2}^2 \in \mathbb{R}$$

satisfies the (PS) condition.

5.1. **Smoothness of F.** Assume that function f is measurable in $t \in (0,\pi)$, continuously differentiable in $(x,u) \in \mathbb{R}^m \times \mathbb{R}^r$ and

$$|f(t,x,u)|, |f_x(t,x,u)|, |f_u(t,x,u)| \le a(t)\gamma(|x|) + b(t)\delta(|u|) \qquad (5.1)$$

for $(t,x,u) \in (0,\pi) \times \mathbb{R}^m \times \mathbb{R}^r$, where $a, b \in L^2$ and $\gamma, \delta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous functions.

Proposition 5.1. *If $\beta > 1/4$, then F is of class C^1 and the differential $F'(x,u) : D((-\Delta)^\beta) \times L_r^\infty \to L^2$ of F at (x,u) is given by*

$$F'(x,u)(h,v) = (-\Delta)^\beta h(t) - f_x(t, x(t), u(t))h(t) - f_u(t, x(t), u(t))v(t)$$

for $(h,v) \in D((-\Delta)^\beta) \times L_r^\infty$.

Proof. Smoothness of the first term of F is obvious. So, let us consider the mapping

$$G : D((-\Delta)^\beta) \times L_r^\infty \ni (x,u) \mapsto f(t, x(t), u(t)) \in L^2.$$

We shall show that the mappings

$$G_x(x,u) : D((-\Delta)^\beta) \ni h \mapsto f_x(t, x(t), u(t))h(t) \in L^2,$$
$$G_u(x,u) : L_r^\infty \ni v \mapsto f_u(t, x(t), u(t))v(t) \in L^2$$

are partial Frechet differentials of G at (x,u) and the mappings

$$D((-\Delta)^\beta) \times L_r^\infty \ni (x,u) \mapsto G_x(x,u) \in \mathcal{L}(D((-\Delta)^\beta), L^2), \qquad (5.2)$$
$$D((-\Delta)^\beta) \times L_r^\infty \ni (x,u) \mapsto G_u(x,u) \in \mathcal{L}(L_r^\infty, L^2) \qquad (5.3)$$

are continuous. Of course, it is sufficient to check the differentiability in Gateaux sense and continuity of the above two mappings (in such a case, the Gateaux differentials are Frechet ones).

So, let us consider differentiability of G with respect to x. Linearity and continuity of the mapping $G_x(x, u)$ are obvious (in view of Lemma 3.2). To prove that $G_x(x, u)$ is Gateaux differential of G with respect to x, we shall show that

$$\left\| \frac{G(x + \lambda_k h, u) - G(x, u)}{\lambda_k} - G_x(x, u)h \right\|_{L^2}^2$$
$$= \int_0^\pi \left| \frac{f(t, x(t) + \lambda_k h(t), u(t)) - f(t, x(t), u(t))}{\lambda_k} - f_x(t, x(t), u(t))h(t) \right|^2 dt \to 0$$

for each sequence $(\lambda_k) \subset (-1, 1)$ such that $\lambda_k \to 0$. Indeed, the sequence of functions

$$t \mapsto \frac{f(t, x(t) + \lambda_k h(t), u(t)) - f(t, x(t), u(t))}{\lambda_k} - f_x(t, x(t), u(t))h(t)$$

converges pointwise a.e. on $(0, \pi)$ to the zero function (by differentiability of f in x). Moreover, from the mean value theorem it follows that this sequence is bounded by a function from L^2:

$$\left| \frac{f(t, x(t) + \lambda_k h(t), u(t)) - f(t, x(t), u(t))}{\lambda_k} - f_x(t, x(t), u(t))h(t) \right|$$
$$= |f_x(t, x(t) + s_{t,k} \lambda_k h(t), u(t))h(t) - f_x(t, x(t), u(t))h(t)|$$
$$\leq \text{const}_{x,u,h}(a(t) + b(t))|h(t)|,$$

where $s_{t,k} \in (0, 1)$ and $\text{const}_{x,u,h}$ is a constant depending on x, u, h. Thus, using the Lebesgue dominated convergence theorem we assert that $G_x(x, u)$ is Gateaux differential of G with respect to x.

In the same way, we check that $G_u(x, u)$ is Gateaux differential of G with respect to u.

To finish the proof we shall show that the mappings (5.2), (5.3) are continuous. Let $(x_k, u_k) \to (x_0, u_0)$ in $D((-\Delta)^\beta) \times L_r^\infty$. Then

$$\|(G_x(x_k, u_k) - G_x(x_0, u_0))h\|_{L^2}^2$$
$$\leq \int_0^\pi |f_x(t, x_k(t), u_k(t)) - f_x(t, x_0(t), u_0(t))|^2 |h(t)|^2 dt$$
$$\leq \|h\|_\infty^2 \int_0^\pi |f_x(t, x_k(t), u_k(t)) - f_x(t, x_0(t), u_0(t))|^2 dt$$
$$\leq \frac{2}{\pi} \zeta(4\beta) \|h\|_{\sim \beta}^2 \int_0^\pi |f_x(t, x_k(t), u_k(t)) - f_x(t, x_0(t), u_0(t))|^2 dt.$$

Consequently,

$$\|G_x(x_k, u_k) - G_x(x_0, u_0)\|_{\mathcal{L}(D((-\Delta)^\beta), L^2)}$$
$$\leq \sqrt{\frac{2}{\pi} \zeta(4\beta)} \left(\int_0^\pi |f_x(t, x_k(t), u_k(t)) - f_x(t, x_0(t), u_0(t))|^2 dt \right)^{1/2}.$$

Using Lemma 3.2, assumption (5.1) and the Lebesgue dominated convergence theorem we assert that $G_x(x_k, u_k) \to G_x(x_0, u_0)$ in $\mathcal{L}(D((-\Delta)^\beta), L^2)$.

In a similar way, we check the continuity of the mapping

$$D((-\Delta)^\beta) \times L_r^\infty \ni (x, u) \mapsto G_u(x, u) \in \mathcal{L}(L_r^\infty, L^2).$$

The proof is complete.   □

5.2. **Bijectivity of $F_x(x,u)$.** In view of the previous theorem and its proof, it is clear that if $\beta > 1/4$ and functions f, f_x satisfy growth condition (5.1), then the partial differential of F with respect to x is of the form

$$F_x(x,u) : D((-\Delta)^\beta) \ni h \mapsto (-\Delta)^\beta h(t) - f_x(t, x(t), u(t))h(t) \in L^2.$$

for each $(x,u) \in D((-\Delta)^\beta) \times L_r^\infty$.

Proposition 5.2. *Assume that functions f, f_x satisfy growth condition (5.1). If $\beta > 1/2$ and one of the following conditions is satisfied*

(a) $\|\Lambda\|_{L^1_{m\times m}} < \frac{\pi}{2\zeta(2\beta)}$,
(b) $\Lambda(t) \le 0$, *i.e. matrix $\Lambda(t)$ is nonpositive, for a.e. $t \in (0, \pi)$,*
(c) $\Lambda \in L^\infty_{m\times m}$ *and* $\|\Lambda\|_\infty < 1$,

where $\Lambda(t) := f_x(t, x(t), u(t))$, $L^p_{m\times m} = L^p((0,\pi), \mathbb{R}^{m\times m})$ for $p = 1, \infty$, then differential $F_x(x,u) : D((-\Delta)^\beta) \to L^2$ is bijective.

By the norm of a matrix $C = [c_{i,j}] \in \mathbb{R}^{m\times m}$ we mean the value $(\sum_{i,j=1}^m |c_{i,j}|^2)^{1/2}$.

Remark 5.3. In Part (c) one can assume that $\beta > 1/4$. In such a case the proof of coercivity of a (see the proof of Proposition 5.2) remains unchanged and to show its continuity one estimates

$$|a(h,y)| \le \|h\|_{\sim\beta/2}\|y\|_{\sim\beta/2} + \|\Lambda\|_\infty\|h\|_{L^2}\|y\|_{L^2}$$
$$\le (1 + \|\Lambda\|_\infty)\|h\|_{\sim\beta/2}\|y\|_{\sim\beta/2}.$$

Proof of Proposition 5.2. We shall show that, for each function $g \in L^2$, equation

$$(-\Delta)^\beta h(t) - \Lambda(t)h(t) = g(t) \tag{5.4}$$

has a unique solution in $D((-\Delta)^\beta)$. Using Lemma 3.6, we see that it is equivalent to show that there exists a unique function $h \in D((-\Delta)^{\beta/2})$ such that

$$\int_0^\pi (-\Delta)^{\beta/2}h(t)(-\Delta)^{\beta/2}y(t)\,dt = \int_0^\pi (\Lambda(t)h(t) + g(t))y(t)\,dt$$

for each $y \in D((-\Delta)^{\beta/2})$. So, let us define a bilinear form $a : D((-\Delta)^{\beta/2}) \times D((-\Delta)^{\beta/2}) \to \mathbb{R}$ by

$$a(h,y) = \int_0^\pi (-\Delta)^{\beta/2}h(t)(-\Delta)^{\beta/2}y(t)\,dt - \int_0^\pi \Lambda(t)h(t)y(t)\,dt.$$

This form is continuous. Indeed (cf. Lemma 3.2),

$$|a(h,y)| \le \|h\|_{\sim\beta/2}\|y\|_{\sim\beta/2} + \|\Lambda\|_{L^1}\|h\|_\infty\|y\|_\infty$$
$$\le (1 + \|\Lambda\|_{L^1}\frac{2}{\pi}\varsigma(2\beta))\|h\|_{\sim\beta/2}\|y\|_{\sim\beta/2}$$

for $h, y \in D((-\Delta)^{\beta/2})$. We have the following three parts
Part a.

$$|a(h,h)| = \left|\int_0^\pi (-\Delta)^{\beta/2}h(t)(-\Delta)^{\beta/2}h(t)\,dt - \int_0^\pi \Lambda(t)h(t)h(t)\,dt\right|$$
$$\ge \|h\|_{\sim\beta/2}^2 - \|\Lambda\|_{L^1}\|h\|_\infty^2$$
$$\ge (1 - \|\Lambda\|_{L^1}\frac{2}{\pi}\varsigma(2\beta))\|h\|_{\sim\beta/2}^2.$$

Part b.

$$|a(h,h)| \geq \int_0^\pi (-\Delta)^{\beta/2} h(t)(-\Delta)^{\beta/2} h(t)\, dt - \int_0^\pi \Lambda(t)h(t)h(t)\, dt \geq \|h\|_{\sim\beta/2}^2.$$

Part c.

$$|a(h,h)| = \Big| \int_0^\pi (-\Delta)^{\beta/2} h(t)(-\Delta)^{\beta/2} h(t)\, dt - \int_0^\pi \Lambda(t)h(t)h(t)\, dt \Big|$$
$$\geq \|h\|_{\sim\beta/2}^2 - \|\Lambda\|_\infty \|h\|_{L^2}^2$$
$$\geq (1 - \|\Lambda\|_\infty)\|h\|_{\sim\beta/2}^2.$$

So, a is coercive. From Lax-Milgram theorem it follows that for each linear continuous functional $l : D((-\Delta)^{\beta/2}) \to \mathbb{R}$ there exists a unique $h \in D((-\Delta)^{\beta/2})$ such that

$$a(h,y) = l(y)$$

for each $y \in D((-\Delta)^{\beta/2})$. Since the functional

$$D((-\Delta)^{\beta/2}) \ni y \mapsto \int_0^\pi g(t)y(t)\, dt \in \mathbb{R}$$

is linear and continuous, therefore there exists a unique $h \in D((-\Delta)^{\beta/2})$ such that

$$\int_0^\pi (-\Delta)^{\beta/2} h(t)(-\Delta)^{\beta/2} y(t)\, dt - \int_0^\pi \Lambda(t)h(t)y(t)\, dt = \int_0^\pi g(t)y(t)\, dt$$

for each $y \in D((-\Delta)^{\beta/2})$. The proof is complete. $\qquad\square$

5.3. **(PS) condition.** As in the proof of Proposition 5.2, one can show that, for each $\beta > 0$ and any function $g \in L^2$, there exists a unique function $x_g \in D((-\Delta)^{\beta/2})$ such that

$$\int_0^\pi (-\Delta)^{\beta/2} x_g(t)(-\Delta)^{\beta/2} y(t)\, dt = \int_0^\pi g(t)y(t)\, dt$$

for each $y \in D((-\Delta)^{\beta/2})$. It means, in view of Lemma 3.6, that the following lemma holds.

Lemma 5.4. *For any $\beta > 0$ and $g \in L^2$, there exists a unique solution $x_g \in D((-\Delta)^\beta)$ of the equation*

$$(-\Delta)^\beta x = g.$$

Lemma 5.5. *If $\beta > 1/2$, then the operator*

$$[(-\Delta)^\beta]^{-1} : L^2 \ni g \mapsto x_g \in L^2$$

is compact, i.e. the image of any bounded set in L^2 is relatively compact in L^2.

Proof. Since $x_{(g_1,\dots,g_m)} = (x_{g_1}, \dots, x_{g_m})$ for each $(g_1, \dots, g_m) \in L^2$, one can assume that $m = 1$.

Let us recall the Kolmogorov-Frechet-Riesz theorem [7]: if $\mathcal{F}$ is a bounded set in $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ and

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{|h|<\delta}{\forall}\ \underset{f\in\mathcal{F}}{\forall}\ \|\tau_h f - f\|_{L^p(\mathbb{R}^n)} < \varepsilon \tag{5.5}$$

(where, $\tau_h f(x) = f(x+h)$), then $\mathcal{F}|_\Omega$ is relatively compact in $L^p(\Omega)$ for each measurable set $\Omega \subset \mathbb{R}^n$ with finite Lebesgue measure.

Let $G \subset L^2((0,\pi),\mathbb{R})$ be a set bounded by a constant C. Consider the functions

$$g(t) = \sum_{j=1}^{\infty} b_j^g \sqrt{\frac{2}{\pi}} \sin jt \in G,$$

$$x_g(t) = \sum_{j=1}^{\infty} a_j^g \sqrt{\frac{2}{\pi}} \sin jt$$

(both series are convergent in L^2 and, in view of the Carleson theorem, a.e. on $(0,\pi)$). Since $(-\Delta)^\beta x_g(t) = g(t)$, i.e.

$$\sum_{j=1}^{\infty} (j^2)^\beta a_j^g \sqrt{\frac{2}{\pi}} \sin jt = \sum_{j=1}^{\infty} b_j^g \sqrt{\frac{2}{\pi}} \sin jt,$$

it follows that

$$a_j^g = \frac{b_j^g}{(j^2)^\beta}$$

for $j \in \mathbb{N}$. Now, we shall show that the set of functions $\{\widetilde{x}_g;\ g \in G\}$, where

$$\widetilde{x}_g : \mathbb{R} \ni t \mapsto \begin{cases} x_g(t) & t \in (0,\pi), \\ 0 & \text{otherwise}, \end{cases}$$

satisfies condition (5.5) (of course, it is bounded in $L^2(\mathbb{R},\mathbb{R})$). Let us fix $0 < h < \pi$ and consider the integral

$$
\begin{aligned}
\int_{-\infty}^{\infty} & |\widetilde{x}_g(t+h) - \widetilde{x}_g(t)|^2 dt \\
&= \int_{-h}^{0} |\widetilde{x}_g(t+h)|^2 dt + \int_{0}^{\pi-h} |\widetilde{x}_g(t+h) - \widetilde{x}_g(t)|^2 dt \\
&\quad + \int_{\pi-h}^{\pi} |\widetilde{x}_g(t+h) - \widetilde{x}_g(t)|^2 dt \\
&= \int_{0}^{h} |x_g(t)|^2 dt + \int_{0}^{\pi-h} |x_g(t+h) - x_g(t)|^2 dt + \int_{\pi-h}^{\pi} |x_g(t)|^2 dt.
\end{aligned}
\tag{5.6}
$$

The first term of the above expression can be estimated as follows (to obtain third inequality we use Hölder inequality for series)

$$
\begin{aligned}
\int_{0}^{h} |x_g(t)|^2 dt &= \int_{0}^{h} \Big| \sum_{j=1}^{\infty} \frac{b_j^g}{(j^2)^\beta} \sqrt{\frac{2}{\pi}} \sin jt \Big|^2 dt \\
&\leq \frac{2}{\pi} \int_{0}^{h} \Big(\sum_{j=1}^{\infty} \frac{|b_j^g|}{(j^2)^\beta} \Big)^2 dt \\
&\leq \frac{2}{\pi} h \sum_{j=1}^{\infty} |b_j^g|^2 \sum_{j=1}^{\infty} \frac{1}{(j^2)^{2\beta}} \\
&= \frac{2}{\pi} h \|g\|_{L^2}^2 \zeta(4\beta) \leq \frac{2}{\pi} C \zeta(4\beta) h.
\end{aligned}
$$

In the same way one can estimate third term of (5.6).

For the second term, we have

$$\int_0^{\pi-h} |x_g(t+h) - x_g(t)|^2 dt$$

$$= \int_0^{\pi-h} \Big| \sum_{j=1}^{\infty} \frac{b_j^g}{(j^2)^\beta} \sqrt{\frac{2}{\pi}} (\sin j(t+h) - \sin jt) \Big|^2 dt$$

$$\leq \int_0^{\pi-h} \Big(\sum_{j=1}^{\infty} \frac{|b_j^g|}{(j^2)^\beta} \sqrt{\frac{2}{\pi}} |2 \sin \frac{jh}{2} \cos(jt + \frac{jh}{2})| \Big)^2 dt$$

$$\leq \frac{8}{\pi} \int_0^{\pi-h} \Big(\sum_{j=1}^{\infty} \frac{|b_j^g|}{(j^2)^\beta} |\sin \frac{jh}{2}| \Big)^2 dt \leq \frac{8}{\pi}(\pi - h) \sum_{j=1}^{\infty} |b_j^g|^2 \sum_{j=1}^{\infty} \frac{\sin^2 \frac{jh}{2}}{(j^2)^{2\beta}}$$

$$\leq \frac{8}{\pi}(\pi - h) C \sum_{j=1}^{\infty} \frac{jh}{2(j^2)^{2\beta}}$$

$$\leq 4Ch \sum_{j=1}^{\infty} \frac{1}{j^{4\beta-1}} = 4C\zeta(4\beta - 1)h.$$

If $-\pi < h < 0$, we proceed in the same way. Finally,

$$\|\tau_h f - f\|_{L^p(\mathbb{R}^n)} \leq \text{const}|h|$$

for $|h| < \pi$. So, the set $\{\widetilde{x}_g\big|_{(0,\pi)} : g \in G\} = \{x_g : g \in G\}$ is relatively compact in L^2. The proof is complete.   □

Using the above lemma we obtain the following result.

Lemma 5.6. *If $\beta > 1/2$ and $x_k \rightharpoonup x_0$ weakly in $D((-\Delta)^\beta)$, then $x_k \to x_0$ strongly in L^2 and $(-\Delta)^\beta x_k \rightharpoonup (-\Delta)^\beta x_0$ weakly in L^2.*

Proof. From the continuity of the linear operators

$$D((-\Delta)^\beta) \ni x \mapsto x \in L^2,$$
$$D((-\Delta)^\beta) \ni x \mapsto (-\Delta)^\beta x \in L^2,$$

it follows that $x_k \rightharpoonup x_0$ weakly in L^2 and $(-\Delta)^\beta x_k \rightharpoonup (-\Delta)^\beta x_0$ weakly in L^2. Lemma 5.5 implies that the sequence (x_k) contains a subsequence (x_{k_i}) converging strongly in L^2 to a limit. Of course, this limit is the function x_0, i.e. $x_{k_i} \to x_0$ strongly in L^2. Supposing contrary and repeating the above argumentation we assert that $x_k \to x_0$ strongly in L^2.   □

Remark 5.7. Lemmas 5.5 and Lemma 5.6 are valid for each $\beta > 0$. The proofs of such stronger results, in the case of bounded open set $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, can be found in [12]. We give here weaker theorems for two reasons. First, to prove more general results (in fact, a counterpart of Lemma 5.5 because the proof of Lemma 5.6 remains unchanged) some additional considerations, concerning the spectral representation of the inverse operator, are needed. Second, due to the other assumptions (cf. Proposition 5.2) assumption $\beta > 1/2$ in Theorem 6.1 can not be omitted.

The main tool for proving that φ satisfies the (PS) condition is the following lemma.

Lemma 5.8. *If $\beta > 1/4$, f satisfies the growth condition*

$$|f(t, x, u)| \leq a(t)|x| + b(t)\delta(|u|)$$

for $(t, x, u) \in (0, \pi) \times \mathbb{R}^m \times \mathbb{R}^r$, where $a, b \in L^2$, $\delta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function and

$$\sqrt{\frac{2}{\pi}\zeta(4\beta)}\|a\|_{L^2} < 1, \tag{5.7}$$

then, for each $u \in L_r^\infty$, the functional

$$\varphi : D((-\Delta)^\beta) \ni x \mapsto \frac{1}{2}\|F(x, u)\|_{L^2}^2 \in \mathbb{R}$$

is coercive, i.e. $\|x\|_{\sim\beta} \to \infty$ implies $\varphi(x) \to \infty$.

Proof. We have

$$\|F(x, u)\|_{L^2} = \|(-\Delta)^\beta x(t) - f(t, x(t), u(t))\|_{L^2}$$
$$\geq \|(-\Delta)^\beta x(t)\|_{L^2} - \|f(t, x(t), u(t))\|_{L^2}.$$

But

$$\|f(t, x(t), u(t))\|_{L^2} \leq \Big(\int_0^\pi (a(t)|x(t)| + b(t)\delta(|u(t)|))^2 dt\Big)^{1/2}$$
$$\leq \Big(\int_0^\pi |a(t)|^2|x(t)|^2 dt\Big)^{1/2} + D$$
$$\leq \|x\|_\infty \|a\|_{L^2} + D$$
$$\leq \sqrt{\frac{2}{\pi}\zeta(4\beta)}\|a\|_{L^2}\|(-\Delta)^\beta x\|_{L^2} + D$$

where $D = \big(\int_0^\pi |b(t)|^2(\delta(|u(t)|))^2 dt\big)^{1/2}$. Thus,

$$\|F(x, u)\|_{L^2} \geq \|(-\Delta)^\beta x\|_{L^2} - \sqrt{\frac{2}{\pi}\zeta(4\beta)}\|a\|_{L^2}\|(-\Delta)^\beta x\|_{L^2} - D$$
$$= (1 - \sqrt{\frac{2}{\pi}\zeta(4\beta)}\|a\|_{L^2})\|x\|_{\sim\beta} - D.$$

It means that φ is coercive. $\square$

Now, we are in a position to prove that φ satisfies the (PS) condition.

Proposition 5.9. *If $\beta > 1/2$, f and f_x satisfy the growth conditions*

$$|f(t, x, u)| \leq a(t)|x| + b(t)\delta(|u|),$$
$$|f_x(t, x, u)| \leq a(t)\gamma(|x|) + b(t)\delta(|u|)$$

for $(t, x, u) \in (0, \pi) \times \mathbb{R}^m \times \mathbb{R}^r$, where $a, b \in L^2$ and $\gamma, \delta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous functions, and (5.7) holds true, then φ (with any fixed $u \in L_r^\infty$) satisfies the (PS) condition.

Proof. From Proposition 5.1 it follows that φ is of class C^1 and its differential $\varphi'(x) : D((-\Delta)^\beta) \to \mathbb{R}$ is

$$\varphi'(x)h = \int_0^\pi ((-\Delta)^\beta x(t) - f(t, x(t), u(t)))((-\Delta)^\beta h(t) - f_x(t, x(t), u(t))h(t)) \, dt$$

for $h \in D((-\Delta)^\beta)$. Consequently, for $x_k, x_0 \in D((-\Delta)^\beta)$, we have

$$(\varphi'(x_k) - \varphi'(x_0))(x_k - x_0) = \|x_k - x_0\|^2_{\sim \beta/2} + \sum_{i=1}^{5} \psi_i(x_k)$$

where

$$\psi_1(x_k) = \int_0^\pi (-\Delta)^\beta x_k(t) f_x(t, x_k(t), u(t))(x_0(t) - x_k(t))\, dt,$$

$$\psi_2(x_k) = \int_0^\pi (-\Delta)^\beta x_0(t) f_x(t, x_0(t), u(t))(x_k(t) - x_0(t))\, dt,$$

$$\psi_3(x_k) = \int_0^\pi f(t, x_k(t), u(t)) f_x(t, x_k(t), u(t))(x_k(t) - x_0(t))\, dt,$$

$$\psi_4(x_k) = \int_0^\pi f(t, x_0(t), u(t)) f_x(t, x_0(t), u(t))(x_0(t) - x_k(t))\, dt,$$

$$\psi_5(x_k) = \int_0^\pi (f(t, x_0(t), u(t)) - f(t, x_k(t), u(t)))((-\Delta)^\beta x_k(t) - (-\Delta)^\beta x_0(t))\, dt.$$

Now, let (x_k) be a (PS) sequence for φ. Since φ is coercive, therefore (x_k) is bounded in $D((-\Delta)^\beta)$. So, one can choose a subsequence (x_{k_i}) weakly converging in $D((-\Delta)^\beta)$ to some x_0. From Lemma 5.6 it follows that $x_{k_i} \to x_0$ strongly in L^2 and $(-\Delta)^\beta x_{k_i}(t) \rightharpoonup (-\Delta)^\beta x_0(t)$ weakly in L^2. Since the sequence (x_{k_i}) is bounded in $D((-\Delta)^\beta)$, therefore it is bounded in L^∞_m and, consequently $(\beta > 1/2)$, in C. Moreover, there exists a subsequence of the sequence (x_{k_i}) (let us denote it by (x_{k_i})) converging to x_0 pointwise a.e. on $(0, \pi)$.

Term $\psi_1(x_{k_i})$ tends to zero. Indeed, $f_x(t, x_{k_i}(t), u(t))$, $k \in \mathbb{N}$, are equibounded on $(0, \pi)$ by a square integrable function. Functions $f_x(t, x_{k_i}(t), u(t))(x_0(t) - x_{m_k}(t))$ belong to L^2 and converge pointwise (a.e. on $(0, \pi)$) to zero function. Moreover, they are equibounded on $(0, \pi)$ by a square integrable function. So, from the Lebesgue dominated convergence theorem it follows that the sequence

$$(f_x(t, x_{k_i}(t), u(t))(x_0(t) - x_{m_k}(t)))$$

converges in L^2 to the zero function. Thus, in view of the weak convergence of the sequence $((-\Delta)^\beta x_k)$ to $(-\Delta)^\beta x_0$ in L^2, $\psi_1(x_{k_i}) \to 0$.

Similarly, $\psi_l(x_{k_i}) \to 0$ for remaining l. Finally, since $\varphi'(x_{k_i})(x_{k_i} - x_0) \to 0$ and $\varphi'(x_0)(x_{k_i} - x_0) \to 0$, it follows that

$$\|x_{k_i} - x_0\|^2_{\sim \beta/2} \to 0,$$

i.e. φ satisfies the (PS) condition. □

6. Final result

Thus, we have proved the following result.

Theorem 6.1. *Assume that $\beta > 1/2$, function f is measurable in $t \in (0, \pi)$, continuously differentiable in $(x, u) \in \mathbb{R}^m \times \mathbb{R}^r$ and*

$$|f(t, x, u)| \le a(t)|x| + b(t)\delta(|u|),$$
$$|f_x(t, x, u)|, |f_u(t, x, u)| \le a(t)\gamma(|x|) + b(t)\delta(|u|)$$

for $(t, x, u) \in (0, \pi) \times \mathbb{R}^m \times \mathbb{R}^r$, where $a, b \in L^2$, $\gamma, \delta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous functions and

$$\sqrt{\frac{2}{\pi} \zeta(4\beta)} \|a\|_{L^2} < 1.$$

If, for each pair $(x, u) \in D((-\Delta)^\beta) \times L_r^\infty$ one of the following assumptions is satisfied

(a) $\|f_x(t, x(t), u(t))\|_{L^1_{m \times m}} < \frac{\pi}{2\zeta(2\beta)}$,
(b) $f_x(t, x(t), u(t)) \le 0$ *for a.e.* $t \in (0, \pi)$,
(c) $f_x(t, x(t), u(t)) \in L^\infty_{m \times m}$ *and* $\|f_x(t, x(t), u(t))\|_\infty < 1$,

then, for each $u \in L_r^\infty$, there exists a unique solution $x_u \in D((-\Delta)^\beta)$ of problem (1.1) and the mapping

$$\lambda : L_r^\infty \ni u \mapsto x_u \in D((-\Delta)^\beta)$$

is continuously differentiable with the differential $\lambda'(u)$ at $u \in L_r^\infty$ such that, for each $v \in L_r^\infty$,

$$(-\Delta)^\beta (\lambda'(u)v)(t) - f_x(t, x_u(t), u(t))(\lambda'(u)v)(t) = f_u(t, x_u(t), u(t))v(t)$$

for $t \in (0, \pi)$ a.e.

Remark 6.2. Thus, for each $u \in L_r^\infty$, $v \in D((-\Delta)^\beta)$ the function $\lambda'(u)v \in D((-\Delta)^\beta)$ is a solution to the equation

$$(-\Delta)^\beta y(t) - f_x(t, x_u(t), u(t))y(t) = f_u(t, x_u(t), u(t))v(t), \quad \text{a.e.} t \in (0, \pi).$$

Example 6.3. Let $\beta > 1/2,, m = 2$, and $r = 2$. It is easy to see that the function

$$f(t, x, u) = \left(f^1(t, x_1, x_2, u_1, u_2), f^2(t, x_1, x_2, u_1, u_2) \right)$$

$$= (a \sin(x_2) + t^{-1/3} e^{u_1}, b \cos(x_1) + t u_2)$$

satisfies assumptions of Theorem 6.1 with

$$a(t) = \sqrt{a^2 + b^2}, \quad \gamma(s) = \sqrt{2}, \quad b(t) = t^{-1/3} + t + |b|, \quad \delta(s) = e^s,$$

where $a, b \in \mathbb{R}$ are such that

$$\sqrt{a^2 + b^2} \le \frac{1}{2\sqrt{2}\zeta(2\beta)}.$$

Consequently, for each $u = (u_1, u_2) \in L_2^\infty$, there exists a unique solution $x_u \in D((-\Delta)^\beta)$ of the problem

$$(-\Delta)^\beta x_1(t) = a \sin(x_2(t)) + t^{-1/3} e^{u_1(t)}$$

$$(-\Delta)^\beta x_2(t) = b \cos(x_1(t)) + t u_2(t)$$

for $t \in (0, \pi)$ a.e., and the mapping $\lambda(u) = (x_u^1, x_u^2)$ is continuously differentiable with the differential $\lambda'(u) : L_2^\infty \to D((-\Delta)^\beta)$ such that

$$(-\Delta)^\beta (\lambda'(u)v)(t) - \begin{bmatrix} 0 & a \cos((x_u)_2(t)) \\ -b \sin((x_u)_1(t)) & 0 \end{bmatrix} (\lambda'(u)v)(t)$$

$$= \begin{bmatrix} t^{-1/3} e^{u_1(t)} & 0 \\ 0 & t u_2(t) \end{bmatrix} v(t), \quad \text{a.e. } t \in (0, \pi),$$

for each $v \in L_2^\infty$, i.e.

$$(-\Delta)^\beta ((\lambda'(u)v)_1)(t) = a \cos((x_u)_2(t)) (\lambda'(u)v)_2(t) + t^{-\frac{1}{3}} e^{u_1(t)} v_1(t)$$

$$(-\Delta)^{\beta}((\lambda'(u)v)_1)(t) = -b\sin((x_u)_1(t))\,(\lambda'(u)v)_1\,(t) + tu_2(t)v_2(t)$$

for a.e. $t \in (0, \pi)$ and every $v = (v_1, v_2) \in L_2^\infty$.

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