Electronic Journal of Differential Equations, Vol. 2021 (2021), No. 68, pp. 1–17.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

INFINITELY MANY SOLUTIONS FOR A SINGULAR SEMILINEAR PROBLEM ON EXTERIOR DOMAINS

MAGEED ALI, JOSEPH IAIA

Abstract. In this article we prove the existence of an infinite number of radial solutions to $\Delta U + K(x)f(U) = 0$ on the exterior of the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ with $U = 0$ on ∂B_R, and $\lim_{|x|\to\infty} U(x) = 0$ where $N > 2$, $f(U) \sim \frac{-1}{|U|^{q-1}U}$ for small $U \neq 0$ with $0 < q < 1$, and $f(U) \sim |U|^{p-1}U$ for large $|U|$ with $p > 1$. Also, $K(x) \sim |x|^{-\alpha}$ with $\alpha > 2(N-1)$ for large $|x|$.

1. Introduction

In this article we consider the problem

$$\Delta U + K(|x|)f(U) = 0, \quad x \in \mathbb{R}^N \backslash B_R, \tag{1.1}$$

$$U = 0 \quad \text{on } \partial(\mathbb{R}^N \backslash B_R), \tag{1.2}$$

$$U \to 0 \quad \text{as } |x| \to \infty \tag{1.3}$$

where $U : \mathbb{R}^N \to \mathbb{R}$ with $N > 2$, B_R is the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ and $K(x) > 0$.

We use the following assumptions:

- (H1) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ and f odd, locally Lipschitz, and there exists $\beta > 0$ such that $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞).
- (H2) $f(U) = \frac{-1}{|U|^{q-1}U} + g_1(U)$ for small $U \neq 0$, $0 < q < 1$, g_1 is locally Lipschitz on $\mathbb{R}$, $g_1(0) = 0$.
- (H3) $f(U) = |U|^{p-1}U + g_2(U)$ for large U where $p > 1$ and $\lim_{U\to+\infty} \frac{g_2(U)}{|U|^p} = 0$.

Now let $F(U) = \int_0^U f(s)\,ds$. Since f is odd it follows that F is even, and from (H2) it follows that f is integrable near $U = 0$. Thus F is continuous and $F(0) = 0$. It also follows that F is bounded below by $-F_0$ with $F_0 > 0$.

- (H4) there exists γ with $0 < \beta < \gamma$ such that $F < 0$ on $(0, \gamma)$, $F > 0$ on (γ, ∞), $F > -F_0$ on $\mathbb{R}$.
- (H5) K and K' are continuous on $[R, \infty)$ with $K(r) > 0$, $2(N-1) + \frac{rK'}{K} < 0$, there exists α such that $\alpha > 2(N-1)$ and $\lim_{r\to\infty} \frac{rK'}{K} = -\alpha$.
- (H6) There exist $K_1 > 0$ and $K_2 > 0$ such that

$$\frac{K_1}{r^\alpha} \le K(r) \le \frac{K_2}{r^\alpha} \quad \text{on } [R, \infty).$$

2010 *Mathematics Subject Classification.* 34B40, 35B05.
Key words and phrases. Exterior domain; semilinear equation; radial solution.

Submitted November 2, 2020. Published August 10, 2021.

Since we are interested in studying radial solutions of (1.1)–(1.3), we rewrite these equations with $r = |x|$, $U(r) = U(|x|)$ and see that U satisfies:

$$U''(r) + \frac{N-1}{r}U'(r) + K(r)f(U(r)) = 0 \quad \text{on } (R,\infty), \tag{1.4}$$

$$U(R) = 0, \quad \lim_{r\to\infty} U(r) = 0. \tag{1.5}$$

Since $f(U)$ is discontinuous at $U = 0$ it follows that U'' is not continuous at any point where $U = 0$. However we will see that U, U' are continuous on $[R,\infty)$ and satisfy

$$r^{N-1}U'(r) = \int_r^\infty s^{N-1}K(s)f(U(s))\,ds. \tag{1.6}$$

In this article we prove the following result.

Theorem 1.1. *Assuming* (H1)–(H6) *hold and* $N > 2$*, there exist an infinite number of nontrivial radial solutions of* (1.5) *and* (1.6)*. In addition, for each nonnegative integer* n*, there is a solution of* (1.5) *and* (1.6) *with exactly* n *zeros on* $(0, R^{2-N})$*.*

The existence of a positive solution of (1.1) on $\mathbb{R}^N$ with $K(r) \equiv 1$ has been studied extensively [2, 3, 9, 12]. Recently the exterior domain $\mathbb{R}^N \backslash B_R(0)$ has been studied in [6, 7, 8, 10, 11, 13]. In addition, $f(U) = -|U|^{q-1}U + |U|^{p-1}U$ with $(1 < q < p)$ was studied in [11]. $f(U) = |U|^{q-1}U + g(U)$ with $(1 < p < q + 1)$ was studied in [1]. Also $f(U) = -|U|^{-q-1}U + g(U)$ with $(0 < q < 1 < p)$ was studied in [12].

2. Preliminaries

We first prove the existence of a solution of (1.4) with

$$U(R) = 0 \quad \text{and} \quad U'(R) = a > 0 \tag{2.1}$$

on some neighborhood to the right of R. We denote this solution by $U_a(r)$ to emphasize the dependence on the initial parameter a. To prove existence of (1.4), (2.1) we make the change of variables

$$U_a(r) = V_a(r^{2-N}). \tag{2.2}$$

Then

$$U_a'(r) = (2-N)r^{1-N}V_a'(r^{2-N}),$$
$$U_a''(r) = (2-N)(1-N)r^{-N}V_a'(r^{2-N}) + (2-N)^2 r^{2(1-N)}V_a''(r^{2-N}).$$

Letting $t = r^{2-N}$ and $r = t^{\frac{1}{2-N}}$ in (4), (7) we obtain

$$V_a''(t) + h(t)f(V_a(t)) = 0 \quad \text{on } (0, R^{2-N}), \tag{2.3}$$

$$V_a(R^{2-N}) = 0, \quad V_a'(R^{2-N}) = \frac{-aR^{N-1}}{N-2} < 0, \tag{2.4}$$

where from (H5) and (H6),

$$h(t) = \frac{1}{(N-2)^2}t^{\frac{2(N-1)}{2-N}}K(t^{\frac{1}{2-N}}) \sim \frac{t^{\tilde\alpha}}{(N-2)^2}, \tilde\alpha = \frac{\alpha - 2(N-1)}{N-2} > 0 \tag{2.5}$$

on $(0, R^{2-N})$. Also from (H5) and (H6) it follows that there are constants h_1, h_2 with $0 < h_1 \le h_2$ such that

$$h'(t) > 0, \quad h_1 t^{\tilde\alpha} \le h(t) \le h_2 t^{\tilde\alpha} \quad \text{on } (0, R^{2-N}). \tag{2.6}$$

For the existence of a solution of (2.3) on $(R^{2-N}-\epsilon, R^{2-N})$ with (2.4) for some $\epsilon > 0$ we proceed as follows. First, integrate (2.3) on (t, R^{2-N}) and use (2.4). This gives

$$-V_a'(t) = \frac{aR^{N-1}}{N-2} - \int_t^{R^{2-N}} h(s) f(V_a(s))\, ds. \tag{2.7}$$

Integrating again over (t, R^{2-N}) and using (2.4) gives

$$V_a(t) = \frac{aR^{N-1}}{N-2}(R^{2-N}-t) - \int_t^{R^{2-N}} \int_s^{R^{2-N}} h(x) f(V_a(x))\, dx\, ds. \tag{2.8}$$

Now let $W(t) = \frac{V_a(t)}{R^{2-N}-t}$ so $V_a(t) = (R^{2-N}-t)W(t)$ and

$$W(R^{2-N}) \equiv \lim_{t\to (R^{2-N})^-} \frac{V_a(t)}{R^{2-N}-t} = -V_a'(R^{2-N}) = \frac{aR^{N-1}}{N-2}.$$

Rewriting (2.8) we have

$$W(t) = \frac{aR^{N-1}}{N-2} - \frac{1}{R^{2-N}-t}\int_t^{R^{2-N}} \int_s^{R^{2-N}} h(x) f\left((R^{2-N}-x)W(x)\right)\, dx\, ds. \tag{2.9}$$

We now solve this equation on $[R^{2-N}-\epsilon, R^{2-N}]$ by a fixed point method. Let $a > 0$, $0 < \epsilon < 1$, and let us define

$$\begin{gathered} S = \Big\{ W \in C[R^{2-N}-\epsilon, R^{2-N}] : W(R^{2-N}) = \frac{aR^{N-1}}{N-2}, \\ |W(t) - \frac{aR^{N-1}}{N-2}| \le \frac{aR^{N-1}}{2(N-2)} \text{ on } [R^{2-N}-\epsilon, R^{2-N}] \Big\} \end{gathered}$$

where $C[R^{2-N}-\epsilon, R^{2-N}]$ is the set of real-valued continuous functions on $[R^{2-N}-\epsilon, R^{2-N}]$. Let

$$\|W\| = \sup_{x\in[R^{2-N}-\epsilon, R^{2-N}]} |W(x)|.$$

Then $(S, \|\cdot\|)$ is a Banach space. Now let us define a map T on S by $TW(R^{2-N}) = \frac{aR^{N-1}}{N-2}$ and

$$\begin{aligned} &TW(t) \\ &= \frac{aR^{N-1}}{N-2} - \frac{1}{R^{2-N}-t}\int_t^{R^{2-N}} \int_s^{R^{2-N}} h(x) f\left((R^{2-N}-x)W(x)\right)\, dx\, ds \end{aligned} \tag{2.10}$$

on $(R^{2-N}-\epsilon, R^{2-N})$. Since $W(x) \in S$ and $0 < \epsilon < 1$ we have

$$0 < \frac{aR^{N-1}}{2(N-2)} \le W(x) \le \frac{3aR^{N-1}}{2(N-2)} \quad \text{on } [R^{2-N}-\epsilon, R^{2-N}]. \tag{2.11}$$

From (H2) we see $g_1(x)$ is locally Lipschitz and $g_1(0) = 0$ therefore it follows that

$$|g_1((R^{2-N}-x)W(x))| \le L|R^{2-N}-x||W(x)| \tag{2.12}$$

where L is the Lipschitz constant for g_1 on $[0, \frac{3aR^{N-1}}{2(N-2)}]$. It follows from (2.11) that

$$\Big|\frac{-1}{(R^{2-N}-x)^q W^q(x)}\Big| \le \frac{2^q(N-2)^q(R^{2-N}-x)^{-q}}{a^q(R^{N-1})^q} \tag{2.13}$$

and using (2.6), (2.12), and (2.13) we see that

$$\begin{aligned} &|h(x)f((R^{2-N}-x)W(x))| \\ &= \left|h(x)\Big(\frac{-1}{(R^{2-N}-x)^q W^q(x)} + g_1((R^{2-N}-x)W(x))\Big)\right| \\ &\le h(R^{2-N})\Big[\Big|\frac{2^q(N-2)^q(R^{2-N}-x)^{-q}}{a^q(R^{N-1})^q}\Big| + L\Big|(R^{2-N}-x)\frac{3aR^{N-1}}{2(N-2)}\Big|\Big]. \end{aligned} \tag{2.14}$$

Integrating once we obtain

$$\begin{aligned} &\int_t^{R^{2-N}} |h(x)f\left((R^{2-N}-x)W(x)\right)|\,dx \\ &\le h(R^{2-N})\big[\frac{C_1}{a^q}(R^{2-N}-t)^{1-q} + C_2 a(R^{2-N}-t)^2\big] \end{aligned} \tag{2.15}$$

where

$$C_1 = \frac{2^q(N-2)^q}{(R^{N-1})^q(1-q)}, \quad C_2 = \frac{3LR^{N-1}}{4(N-2)}.$$

Thus from (2.15) we have

$$\int_t^{R^{2-N}} |h(x)f\left((R^{2-N}-x)W(x)\right)|\,dx \to 0 \quad \text{as } t \to (R^{2-N})^-. \tag{2.16}$$

Next integrating (2.15) on (t, R^{2-N}) and dividing by $(R^{2-N}-t)$ we obtain

$$\begin{aligned} &\frac{1}{R^{2-N}-t}\int_t^{R^{2-N}}\int_s^{R^{2-N}} |h(x)f\left((R^{2-N}-x)W(x)\right)|\,dx\,ds \\ &\le h(R^{2-N})\big[\frac{C_3(R^{2-N}-t)^{1-q}}{a^q} + aC_4(R^{2-N}-t)^2\big] \end{aligned} \tag{2.17}$$

where $C_3 = \frac{C_1}{2-q}$ and $C_4 = \frac{C_2}{3}$. Thus from (2.17) we see that

$$\lim_{t\to(R^{2-N})^-} \frac{1}{R^{2-N}-t}\int_t^{R^{2-N}}\int_s^{R^{2-N}} |h(x)f\left((R^{2-N}-x)W(x)\right)|\,dx\,ds = 0. \tag{2.18}$$

Now we show that $T : S \to S$ is a contraction mapping with $T(W) \in S$ for each $W \in S$ if $\epsilon > 0$ is sufficiently small. First, let $W \in S$ and so it follows from (2.17) and (2.18) that

$$\frac{1}{R^{2-N}-t}\int_t^{R^{2-N}}\int_s^{R^{2-N}} h(x)f\left((R^{2-N}-x)W(x)\right)\,dx\,ds$$

is continuous on $[R^{2-N}-\epsilon, R^{2-N}]$. Then from (2.10), (2.17), and (2.18) we see that $\lim_{t\to(R^{2-N})^-} TW(t) = \frac{aR^{N-1}}{N-2}$,

$$|TW(t) - \frac{aR^{N-1}}{N-2}| \le \frac{aR^{N-1}}{2(N-2)} \quad \text{on } [R^{2-N}-\epsilon, R^{2-N}]$$

and TW is continuous if $\epsilon > 0$ is sufficiently small. Thus $T : S \to S$ if ϵ is sufficiently small. We next prove that T is a contraction mapping if ϵ is sufficiently small. Let $W_1, W_2 \in S$. Then

$$\begin{aligned} TW_1(t) - TW_2(t) = -\frac{1}{R^{2-N}-t}\int_t^{R^{2-N}}\int_s^{R^{2-N}} h(x)\big[&f((R^{2-N}-x)W_1(x)) \\ &- f((R^{2-N}-x)W_2(x))\big]\,dx\,ds. \end{aligned} \tag{2.19}$$

By (H2) we have $f((R^{2-N}-x)W(x)) = -(R^{2-N}-x)^{-q}W^{-q}(x) + g_1((R^{2-N}-x)W(x))$ where $0<q<1$. Then by (2.12) and (2.13) we first estimate

$$\begin{aligned} &|f((R^{2-N}-x)W_1) - f((R^{2-N}-x)W_2)| \\ &= \Big|\frac{-1}{(R^{2-N}-x)^q}\big[\frac{1}{W_1^q} - \frac{1}{W_2^q}\big] + g_1((R^{2-N}-x)W_1) - g_1((R^{2-N}-x)W_2)\Big| \\ &\le \frac{1}{(R^{2-N}-x)^q}\Big|\frac{1}{{W_1}^q} - \frac{1}{{W_2}^q}\Big| + L(R^{2-N}-x)|W_1-W_2| \end{aligned} \tag{2.20}$$

where L is again the Lipschitz constant for g_1 on $[0, \frac{3aR^{N-1}}{2(N-2)}]$. Next applying the mean value theorem we see that the right-hand side of (2.20) is equal to

$$\frac{1}{(R^{2-N}-x)^q}\big[\frac{q}{W_3^{q+1}}|W_1-W_2|\big] + L(R^{2-N}-x)|W_1-W_2|,$$

where W_3 is between W_1 and W_2. Since $W_i \in S$ for $i=1,2,3$, and $|W_i - \frac{aR^{N-1}}{N-2}| \le \frac{aR^{N-1}}{2(N-2)}$ then $\frac{aR^{N-1}}{2(N-2)} \le W_i \le \frac{3aR^{N-1}}{2(N-2)}$ on $[R^{2-N}-\epsilon, R^{2-N}]$. Therefore ${W_3}^{q+1} \ge \Big(\frac{aR^{N-1}}{2(N-2)}\Big)^{q+1}$, and so on $[R^{2-N}-\epsilon, R^{2-N}]$ we have

$$\begin{aligned} &|f((R^{2-N}-x)W_1) - f((R^{2-N}-x)W_2)| \\ &\le |W_1-W_2|\big[\frac{q}{(R^{2-N}-x)^q}\Big(\frac{2(N-2)}{aR^{N-1}}\Big)^{q+1} + L(R^{2-N}-x)\big]. \end{aligned} \tag{2.21}$$

Recalling from (2.5) that $h(t)$ is positive, continuous and increasing on $(0, R^{2-N}]$, with $\alpha > 2(N-1)$ we see that

$$\begin{aligned} &|TW_1 - TW_2| \\ &\le \frac{h(R^{2-N})}{R^{2-N}-t}\int_t^{R^{2-N}}\int_s^{R^{2-N}} |W_1-W_2|\Big[\frac{q}{(R^{2-N}-x)^q}\Big(\frac{2(N-2)}{aR^{N-1}}\Big)^{q+1} \\ &\quad + L(R^{2-N}-x)\Big]\,dx\,ds \\ &\le \frac{h(R^{2-N})}{R^{2-N}-t}\|W_1-W_2\|\int_t^{R^{2-N}}\int_s^{R^{2-N}} \Big[\frac{q}{(R^{2-N}-x)^q}\Big(\frac{2(N-2)}{aR^{N-1}}\Big)^{q+1} \\ &\quad + L(R^{2-N}-x)\Big]\,dx\,ds \\ &\le h(R^{2-N})\|W_1-W_2\|\big[\frac{C_5\epsilon^{1-q}}{a^{q+1}} + C_6\epsilon^2\big] \\ &= C_{7,\epsilon}\|W_1-W_2\|. \end{aligned} \tag{2.22}$$

where

$$C_5 = \frac{q}{(2-q)(1-q)}\Big(\frac{2(N-2)}{R^{N-1}}\Big)^{q+1}, \quad C_6 = \frac{L}{6}, \quad C_{7,\epsilon} = h(R^{2-N})\big[\frac{C_5\epsilon^{1-q}}{a^{q+1}} + C_6\epsilon^2\big].$$

Since

$$\lim_{\epsilon\to 0^+} C_{7,\epsilon} = \lim_{\epsilon\to 0^+} h(R^{2-N})\big[\frac{C_5\epsilon^{1-q}}{a^{q+1}} + C_6\epsilon^2\big] = 0,$$

for ϵ sufficiently small we see that $0 < C_{7,\epsilon} < 1$, and therefore it follows from (2.22) that T is a contraction. Then by the contraction mapping principle on S [4] we see there exists a unique solution $W \in S$ to $TW = W$ on $[R^{2-N}-\epsilon, R^{2-N}]$ for some

$\epsilon > 0$. Then $V_a(t) = (R^{2-N} - t)W(t)$ is a solution of (2.3) and satisfies (2.4) for some $\epsilon > 0$.

Now define the energy of solutions to (2.3) and (2.4) as

$$E_a(t) = \frac{1}{2}\frac{V_a'^2(t)}{h(t)} + F(V_a(t)). \tag{2.23}$$

Differentiating E_a, using (2.3), and using that from (2.6) that $h'(t) > 0$, we have

$$E_a'(t) = -\frac{V_a'^2(t)h'(t)}{2h^2(t)} \le 0. \tag{2.24}$$

Thus E_a is non-increasing where it is defined. Therefore for these t with $t < R^{2-N}$ we have

$$0 < \frac{1}{2}\frac{a^2 R^{2(N-1)}}{(N-2)^2 h(R^{2-N})} = E_a(R^{2-N}) \le E_a(t) = \frac{1}{2}\frac{V_a'^2(t)}{h(t)} + F(V_a(t)). \tag{2.25}$$

Remark 2.1. It follows from (2.3) that if $V_a(t_0) \neq 0$ then $V_a''(t_0)$ is defined and V_a'' is continuous in a neighborhood of t_0. We also note if V_a is a solution of (2.7) and there exists a $Z_a \in (0, R^{2-N}]$ such that $V_a(Z_a) = 0$, then from (2.25) we see $0 < E_a(Z_a) = \frac{1}{2}\frac{V_a'^2(Z_a)}{h(t)}$ and so $V_a'(Z_a) \neq 0$. We also observe that if $V_a(Z_0) = 0$ then it follows from (2.3) and (H2) that $V_a''(Z_0)$ is undefined and that $\lim_{t\to Z_0^+} |V_a''(t)| = \infty$. Therefore due to these considerations for the rest of this paper we will seek functions V_a that are continuously differentiable on $[0, R^{2-N}]$ and satisfy (2.7).

Lemma 2.2. *Assume* -(H1)-(H6) *hold, $N > 2$, and $a > 0$. Let $V_a(t)$ be the solution of* (2.7) *on $(R^{2-N} - \epsilon, R^{2-N})$ whose existence we have just proved. Then V_a and V_a' are defined and continuous on* $[0, R^{2-N}]$. *Also $|V_a'(t)| \le \frac{aR^{N-1}}{N-2} + \sqrt{2F_0 h(R^{2-N})}$ on* $[0, R^{2-N}]$, *$|V_a(t)| \le \frac{aR}{N-2} + R^{2-N}\sqrt{2F_0 h(R^{2-N})}$ on* $[0, R^{2-N}]$, *and $V_a(t)$ satisfies* (2.7) *on* $[0, R^{2-N}]$.

Proof. It follows from (2.3) that

$$\Big(\frac{1}{2}V_a'^2(t) + h(t)F(V_a(t))\Big)' = h'(t)F(V_a(t)). \tag{2.26}$$

Integrating from t to R^{2-N} and using (2.4) yields

$$-\frac{1}{2}V_a'^2(t) - h(t)F(V_a(t)) = -\frac{1}{2}\frac{a^2 R^{2(N-1)}}{(N-2)^2} + \int_t^{R^{2-N}} h'(s)F(V_a(s))\,ds.$$

Since $-F_0 < F$ by (H4) and $h > 0, h' > 0$ by (2.6) then $hF_0 \ge -hF$ thus

$$\begin{aligned}
-\frac{1}{2}V_a'^2(t) + h(t)F_0 &\ge -\frac{1}{2}V_a'^2(t) - h(t)F(V_a(t)) \\
&= -\frac{1}{2}\frac{a^2 R^{2(N-1)}}{(N-2)^2} + \int_t^{R^{2-N}} h'(s)F(V_a(s))\,ds \\
&\ge -\frac{1}{2}\frac{a^2 R^{2(N-1)}}{(N-2)^2} - F_0\int_t^{R^{2-N}} h'(s)\,ds \\
&= -\frac{1}{2}\frac{a^2 R^{2(N-1)}}{(N-2)^2} - F_0\left(h(R^{2-N}) - h(t)\right).
\end{aligned}$$

Therefore,

$$V_a'^2(t) \le \frac{a^2 R^{2(N-1)}}{(N-2)^2} + 2F_0 h(R^{2-N}).$$

Finally since $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$ for $x \ge 0$ and $y \ge 0$ we see that

$$|V_a'(t)| \le \frac{aR^{N-1}}{N-2} + \sqrt{2F_0 h(R^{2-N})}. \tag{2.27}$$

Integrating on (t, R^{2-N}) and using (2.3), (2.4) we obtain

$$\begin{aligned}
|V_a(t)| &= \Big| \int_t^{R^{2-N}} V_a'(s)\,ds \Big| \\
&\le \int_t^{R^{2-N}} |V_a'(s)|\,ds \\
&\le \int_t^{R^{2-N}} \Big(\frac{aR^{N-1}}{N-2} + \sqrt{2F_0 h(R^{2-N})} \Big) ds \\
&= (R^{2-N} - t) \Big(\frac{aR^{N-1}}{N-2} + \sqrt{2F_0 h(R^{2-N})} \Big) \\
&\le \frac{aR}{N-2} + R^{2-N} \sqrt{2F_0 h(R^{2-N})}.
\end{aligned} \tag{2.28}$$

From (2.27) and (2.28) it follows that V_a and V_a' are bounded where they are defined and hence V_a, V_a' exist on $[0, R^{2-N}]$ and V_a' satisfies (2.7) on $[0, R^{2-N}]$. This completes the proof of Lemma 2.2. □

Lemma 2.3. *Assume* (H1)–(H6) *hold, $N > 2$, $a > 0$, and $V_a(t)$ solves* (13). *Then the solutions $V_a(t)$ depend continuously on the parameter $a > 0$ on* $[0, R^{2-N}]$.

Proof. First, let $0 < a_1 < a_2$. It follows from (2.27) and (2.28) that V_a' and V_a are bounded on $[0, R^{2-N}]$ and these upper bounds can be chosen to be independent of a for $0 < a_1 \le a \le a_2$. Then from (2.27) and (2.28) we have

$$|V_a'(t)| \le C_8 a_2 + C_9 \quad \text{on } [0, R^{2-N}]\ \forall a \text{ with } 0 < a_1 \le a \le a_2 \tag{2.29}$$

where $C_8 = \frac{R^{2-N}}{N-2}$, $C_9 = \sqrt{2F_0 h(R^{2-N})}$, and

$$|V_a(t)| \le C_{10} a_2 + C_{11} \quad \text{on } [0, R^{2-N}]\ \forall a \text{ with } 0 < a_1 \le a \le a_2 \tag{2.30}$$

where $C_{10} = \frac{R}{N-2}$ and $C_{11} = R^{2-N} C_9$. Thus we see that $|V_a'|$ and $|V_a|$ are uniformly bounded on $[0, R^{2-N}]$ for all a with $0 < a_1 \le a \le a_2$. Next, we suppose there exists $a^* > 0$, and we want to show that $V_a \to V_{a^*}$ uniformly on $[0, R^{2-N}]$ as $a \to a^*$. By way of contradiction suppose not. Then there exist a_j such that $a_j \to a^*$ as $j \to \infty$, $t_j \in [0, R^{2-N}]$ and there is an $\epsilon_0 > 0$ such that

$$|V_{a_j}(t_j) - V_{a^*}(t_j)| \ge \epsilon_0 \quad \forall j. \tag{2.31}$$

Since $a_j \to a^*$ as $j \to \infty$ then if j is sufficiently large we have $|a_j| \le a^* + 1$ and by (2.29), (2.30) we see that V_a and V_a' are uniformly bounded and therefore equicontinuous on $[0, R^{2-N}]$. Then by the Arzela-Ascoli theorem there is a subsequence a_{j_l}, of V_{a_j} such that $V_{a_{j_l}} \to V_a^*$ uniformly on $[0, R^{2-N}]$. So as $l \to \infty$,

$$0 \leftarrow |V_{a_{j_l}}(t_{j_l}) - V_{a^*}(t_{j_l})| \ge \epsilon_0$$

which is impossible. Thus V_a varies continuously with a on $[0, R^{2-N}]$ for all a with $0 < a_1 \le a \le a_2$. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Assume* (H1)–(H6)*,* $N > 2$*, and let* $V_a(t)$ *be the solution of* (2.7)*. If* a *is sufficiently large then* $V_a(t)$ *has a local maximum,* M_a*, and a zero,* Z_a*, with* $0 < Z_a < M_a < R^{2-N}$*. Further* $V_a(M_a) \to \infty$*,* $M_a \to R^{2-N}$*,* $Z_a \to R^{2-N}$*, and* $|V_a'(Z_a)| \to \infty$ *as* $a \to \infty$*.*

Proof. We first show that if a is sufficiently large then there exists $t_{a,\gamma} > 0$ such that $V_a(t_{a,\gamma}) = \gamma$ and $0 < V_a < \gamma$ on $(t_{a,\gamma}, R^{2-N})$. Suppose not. Then $0 < V_a(t) < \gamma$ on $(0, R^{2-N})$ and all sufficiently large a. Since E_a is non-increasing on $0 < t < R^{2-N}$ and $|V_a| < \gamma$ then $F(V_a) < 0$ and from (2.25) it follows that

$$\frac{1}{2}\frac{V_a'^2(t)}{h(t)} \geq \frac{1}{2}\frac{V_a'^2(t)}{h(t)} + F(V_a(t)) \geq \frac{1}{2}\frac{a^2R^{2(N-1)}}{(N-2)^2h(R^{2-N})} > 0. \tag{2.32}$$

Thus $V_a' < 0$ on (t, R^{2-N}) and we obtain

$$-V_a'(t) \geq \frac{aR^{N-1}}{(N-2)\sqrt{h(R^{2-N})}}\sqrt{h(t)}. \tag{2.33}$$

Integrating (2.33) from t to R^{2-N} gives

$$V_a(t) = \int_t^{R^{2-N}} -V_a'(s)\,ds \geq \int_t^{R^{2-N}} \frac{aR^{N-1}}{(N-2)\sqrt{h(R^{2-N})}}\sqrt{h(s)}\,ds. \tag{2.34}$$

Evaluating this expression at $t = 0$ we obtain

$$\gamma \geq V_a(0) \geq \frac{aR^{N-1}}{(N-2)\sqrt{h(R^{2-N})}}\int_0^{R^{2-N}} \sqrt{h(s)}\,ds. \tag{2.35}$$

The right-hand side approaches infinity as a goes to infinity which contradicts the assumption that the left-hand side is bounded by γ. Thus V_a gets larger than γ as $a \to \infty$ and so there exists $t_{a,\gamma}$ with $0 < t_{a,\gamma} < R^{2-N}$ such that $V_a(t_{a,\gamma}) = \gamma$ and $0 < V_a(t) < \gamma$ on $(t_{a,\gamma}, R^{2-N})$. In addition, evaluating (2.34) at $t = t_{a,\gamma}$ we obtain

$$\gamma = V_a(t_{a,\gamma}) \geq \frac{aR^{N-1}}{(N-2)\sqrt{h(R^{2-N})}}\int_{t_{a,\gamma}}^{R^{2-N}} \sqrt{h(s)}\,ds. \tag{2.36}$$

Thus we see that

$$t_{a,\gamma} \to R^{2-N} \quad \text{as } a \to \infty. \tag{2.37}$$

It then follows immediately that there is $t_{a,\beta}$ such that $t_{a,\gamma} < t_{a,\beta} < R^{2-N}$ and $V_a(t_{a,\beta}) = \beta$. Since $t_{a,\gamma} \to R^{2-N}$ as $a \to \infty$ then it follows that

$$t_{a,\beta} \to R^{2-N} \quad \text{as } a \to \infty. \tag{2.38}$$

Next we show that if V_a is decreasing for all $t \in [\frac{1}{2}R^{2-N}, R^{2-N}]$ then we have $\lim_{a\to\infty} V_a\left(\frac{1}{2}R^{2-N}\right) = \infty$. We suppose by the way of contradiction that $V_a\left(\frac{1}{2}R^{2-N}\right) \leq A$ where $A > 0$ does not depend on a for a large. For $\frac{1}{2}R^{2-N} \leq t \leq R^{2-N}$ it follows that there exists $B > 0$ such that $F(V_a) < B$ on $[\frac{1}{2}R^{2-N}, R^{2-N}]$ and all large a. Since E_a is non-increasing,

$$\frac{1}{2}\frac{V_a'^2(t)}{h(t)} + B \geq \frac{1}{2}\frac{V_a'^2(t)}{h(t)} + F(V_a(t)) = E_a(t) \geq E_a(R^{2-N}) = \frac{1}{2}\frac{a^2R^{2(N-1)}}{(N-2)^2h(R^{2-N})}$$

on $[\frac{R^{2-N}}{2}, R^{2-N}]$. Rewriting the above expression we have

$$-V_a'(t) \geq \sqrt{\frac{a^2R^{2(N-1)}}{(N-2)^2h(R^{2-N})} - 2B}\sqrt{h(t)} \quad \text{on } [\frac{R^{2-N}}{2}, R^{2-N}].$$

Integrating this on (t, R^{2-N}) we obtain:

$$V_a(t) \geq \sqrt{\frac{a^2 R^{2(N-1)}}{(N-2)^2 h(R^{2-N})} - 2B} \int_t^{R^{2-N}} \sqrt{h(s)}\, ds. \tag{2.39}$$

Now evaluating (2.39) at $t = \frac{R^{2-N}}{2}$ we have

$$A \geq V_a\Big(\frac{R^{2-N}}{2}\Big) \geq \sqrt{\frac{a^2 R^{2(N-1)}}{(N-2)^2 h(R^{2-N})} - 2B} \int_{\frac{R^{2-N}}{2}}^{R^{2-N}} \sqrt{h(s)}\, ds. \tag{2.40}$$

As $a \to \infty$, the right-hand side aappraoches infinity, which is a contradiction since we were assuming A is finite. Thus

$$\lim_{a\to\infty} V_a\Big(\frac{1}{2}R^{2-N}\Big) = \infty \quad \text{if } V_a \text{ is decreasing on } [\frac{R^{2-N}}{2}, R^{2-N}]. \tag{2.41}$$

We next show that if V_a is decreasing on $[\frac{R^{2-N}}{2}, R^{2-N}]$ then $V_a\big(\frac{3R^{2-N}}{4}\big) \to \infty$ as $a \to \infty$. From (2.38) we know $t_{a,\beta} \to R^{2-N}$ as $a \to \infty$ so for a sufficiently large we have $\frac{R^{2-N}}{2} \leq t_{a,\beta}$ and $V_a(t) > \beta$ on $[\frac{R^{2-N}}{2}, t_{a,\beta})$. From (2.3) and (H3) we see that $V_a''(t) < 0$ on $[\frac{R^{2-N}}{2}, t_{a,\beta})$ for sufficiently large a. Thus $V_a(t)$ is concave down here so we have for $0 \leq \lambda \leq 1$,

$$\begin{aligned}
V_a\Big(\lambda\frac{R^{2-N}}{2} + (1-\lambda)t_{a,\beta}\Big) &\geq \lambda V_a\Big(\frac{R^{2-N}}{2}\Big) + (1-\lambda)V_a(t_{a,\beta}) \\
&= \lambda V_a\Big(\frac{R^{2-N}}{2}\Big) + (1-\lambda)\beta \\
&\geq \lambda V_a\Big(\frac{R^{2-N}}{2}\Big).
\end{aligned}$$

Now for $t \in [\frac{R^{2-N}}{2}, t_{a,\beta}]$ we can write $t = \frac{\lambda\ R^{2-N}}{2} + (1-\lambda)t_{a,\beta}$, i.e.

$$\lambda = \frac{t_{a,\beta} - t}{t_{a,\beta} - \frac{R^{2-N}}{2}}$$

and thus $0 \leq \lambda \leq 1$, and we obtain

$$V_a(t) \geq \frac{t_{a,\beta} - t}{t_{a,\beta} - \frac{R^{2-N}}{2}} V_a\Big(\frac{R^{2-N}}{2}\Big) \quad \text{on } [\frac{R^{2-N}}{2}, t_{a,\beta}]. \tag{2.42}$$

Evaluating at $t = \frac{3R^{2-N}}{4}$ gives

$$V_a\Big(\frac{3R^{2-N}}{4}\Big) \geq \frac{t_{a,\beta} - \frac{3R^{2-N}}{4}}{t_{a,\beta} - \frac{R^{2-N}}{2}} V_a\Big(\frac{R^{2-N}}{2}\Big). \tag{2.43}$$

From (2.38) we saw that $t_{a,\beta} \to R^{2-N}$ as $a \to \infty$ thus for sufficiently large a we have $\frac{t_{a,\beta} - \frac{3R^{2-N}}{4}}{t_{a,\beta} - \frac{R^{2-N}}{2}} \geq \frac{1}{3}$ and therefore (50) along with (2.41) gives

$$V_a\Big(\frac{3R^{2-N}}{4}\Big) \geq \frac{1}{3} V_a\Big(\frac{R^{2-N}}{2}\Big) \to \infty \quad \text{as } a \to \infty. \tag{2.44}$$

Now let us show that $V_a(t)$ has a local maximum M_a on $[\frac{R^{2-N}}{2}, R^{2-N}]$ if a is sufficiently large. Suppose not. Then $V_a(t)$ is decreasing on $[\frac{R^{2-N}}{2}, R^{2-N}]$.

Next let

$$I_a = \min_{[\frac{1}{2}R^{2-N}, \frac{3}{4}R^{2-N}]} \frac{h(t)f(V_a(t))}{V_a(t)}. \tag{2.45}$$

Since $h(t) > 0$ is bounded from below on $[\frac{1}{2}R^{2-N}, \frac{3}{4}R^{2-N}]$ then there is an $h_0 > 0$ such that $h(t) > h_0$ on $[\frac{1}{2}R^{2-N}, \frac{3R^{2-N}}{4}]$. Since we are assuming V_a is decreasing on $[\frac{1}{2}R^{2-N}, \frac{3R^{2-N}}{4}]$ for all $a > 0$ sufficiently large and since by (2.44) we have $V_a\big(\frac{3R^{2-N}}{4}\big) \to \infty$ as $a \to \infty$, it therefore follows that $V_a \to \infty$ uniformly on $[\frac{1}{2}R^{2-N}, \frac{3R^{2-N}}{4}]$. By (H3) it then follows for sufficiently large a that $\frac{f(V_a)}{V_a} \geq \frac{1}{2}{V_a}^{p-1}$ and therefore

$$\begin{aligned} I_a &= \min_{[\frac{1}{2}R^{2-N}, \frac{3}{4}R^{2-N}]} \frac{h(t)f(V_a)}{V_a} \\ &\geq h_0 \min_{[\frac{1}{2}R^{2-N}, \frac{3}{4}R^{2-N}]} \frac{f(V_a)}{V_a} \\ &\geq \frac{h_0}{2} \min_{[\frac{1}{2}R^{2-N}, \frac{3}{4}R^{2-N}]} V_a^{p-1} \\ &\geq \frac{h_0}{2} {V_a}^{p-1}\Big(\frac{3R^{2-N}}{4}\Big). \end{aligned}$$

By (2.44) the right-hand side goes to infinity, and thus we obtain

$$\lim_{a\to\infty} I_a = \infty. \tag{2.46}$$

Now we apply the Sturm Comparison theorem [5] on $[\frac{1}{2}R^{2-N}, \frac{3R^{2-N}}{4}]$. Consider

$$V_a'' + \big[\frac{h(t)f(V_a)}{V_a}\big]V_a = 0, \tag{2.47}$$

$$W_a'' + I_a W_a = 0 \tag{2.48}$$

where

$$\beta < V_a\Big(\frac{3}{4}R^{2-N}\Big) = W_a\Big(\frac{3}{4}R^{2-N}\Big), \tag{2.49}$$

$$V_a'\Big(\frac{3}{4}R^{2-N}\Big) = W_a'\Big(\frac{3}{4}R^{2-N}\Big) < 0. \tag{2.50}$$

Since $W_a''+I_aW_a = 0$ and $W_a \not\equiv 0$, it follows that $W_a = C_{12}\sin(\sqrt{I_a}t)+C_{13}\cos(\sqrt{I_a}t)$ where C_{12} and C_{13} are not both zero. It is well-known that any interval of length $\frac{\pi}{\sqrt{I_a}}$ has a zero of W_a and so it follows that W_a has a local maximum $\tilde{M}_a \in [\frac{3}{4}R^{2-N} - \frac{\pi}{\sqrt{I_a}}, \frac{3}{4}R^{2-N}]$ and W_a is decreasing on $[\tilde{M}_a, \frac{3}{4}R^{2-N}]$. Also for a sufficiently large then from (2.47), $\frac{3}{4}R^{2-N} - \frac{\pi}{\sqrt{I_a}} > \frac{1}{2}R^{2-N}$. Multiplying (2.47) by W_a, (2.48) by V_a, and subtracting we obtain

$$(W_aV_a' - V_aW_a')' + \Big(\frac{h(t)f(V_a)}{V_a} - I_a\Big)V_aW_a = 0. \tag{2.51}$$

Using (2.49), (2.50) and since W_a has a local maximum $\tilde{M}_a$ then integrating (2.51) on $[\tilde{M}_a, \frac{3}{4}R^{2-N}]$ we obtain

$$-W_a(\tilde{M}_a)V_a'(\tilde{M}_a) + \int_{\tilde{M}_a}^{\frac{3}{4}R^{2-N}} \Big(\frac{h(t)f(V_a)}{V_a} - I_a\Big)V_aW_a = 0. \tag{2.52}$$

Since $W_a(\tilde{M}_a) \geq W_a\left(\frac{3}{4}R^{2-N}\right) > \beta > 0$ by (2.49) and $\left(\frac{h(t)f(V_a)}{V_a} - I_a\right)V_aW_a \geq 0$ on $[\tilde{M}_a, \frac{3}{4}R^{2-N}]$ then $\int_{\tilde{M}_a}^{\frac{3}{4}R^{2-N}} \left(\frac{h(t)f(V_a)}{V_a} - I_a\right)V_aW_a > 0$ and so it follows that $V_a'(\tilde{M}_a) > 0$ which is a contradiction to the assumption that $V_a'(t) < 0$ on $[\frac{R^{2-N}}{2}, R^{2-N})$. Thus $V_a(t)$ must have a local maximum, M_a, with $\frac{1}{2}R^{2-N} < M_a < R^{2-N}$ and V_a decreasing on (M_a, R^{2-N}) if a is sufficiently large.

Now let us show that $V_a(M_a) \to \infty$ as $a \to \infty$. Suppose by the way of the contradiction that there exists a constant $C_{14} > 0$ independent of a such that $V_a(M_a) < C_{14}$ and so $V_a(t) < C_{14}$ on (M_a, R^{2-N}). Integrating (2.3) on (M_a, R^{2-N}) and using (2.4) gives

$$\int_{M_a}^{R^{2-N}} V_a''(t)\,dt + \int_{M_a}^{R^{2-N}} h(t)f(V_a(t))\,dt = 0.$$

Therefore

$$\begin{aligned}\frac{aR^{2-N}}{N-2} &= \int_{M_a}^{R^{2-N}} h(t)f(V_a(t))\,dt \\ &= \int_{M_a}^{R^{2-N}} h(t)(-{V_a}^{-q}(t))\,dt + \int_{M_a}^{R^{2-N}} h(t)g_1(V_a(t))\,dt \\ &\leq \int_{M_a}^{R^{2-N}} h(t)g_1(V_a(t))\,dt.\end{aligned} \tag{2.53}$$

Since $0 \leq V_a(t) \leq V_a(M_a) \leq C_{14}$ and g_1 is continuous, $g_1(V_a) \leq C_{15}$ for some constant $C_{15} > 0$ on $[M_a, R^{2-N}]$, and since $h(t) \leq h_2t^{\tilde{\alpha}}$ (by (2.4)), estimating (2.53) gives

$$\frac{aR^{2-N}}{N-2} \leq \frac{h_2C_{15}}{1+\tilde{\alpha}}\left[(R^{2-N})^{1+\tilde{\alpha}} - M_a^{1+\tilde{\alpha}}\right] \leq \frac{h_2C_{15}}{1+\tilde{\alpha}}(R^{2-N})^{1+\tilde{\alpha}}. \tag{2.54}$$

The left-hand side of (2.54) goes to $+\infty$ as $a \to \infty$ but the right-hand side is bounded which contradicts the assumption that $0 \leq V_a(M_a) \leq C_{14}$. Thus

$$V_a(M_a) \to \infty \text{ as } a \to \infty. \tag{2.55}$$

Now let us show that $\lim_{a\to\infty} M_a = R^{2-N}$. Since $V_a''(t) \leq 0$ on $(M_a, t_{a,\beta})$ then V_a is concave down here and so we obtain

$$V_a(\lambda M_a + (1-\lambda)t_{a,\beta}) \geq \lambda V_a(M_a) + (1-\lambda)\beta \tag{2.56}$$

where $0 \leq \lambda \leq 1$. Letting $\lambda = 1/2$ gives

$$V_a\Big(\frac{M_a + t_{a,\beta}}{2}\Big) \geq \frac{1}{2}V_a(M_a) + \frac{1}{2}\beta = \frac{V_a(M_a) + \beta}{2}. \tag{2.57}$$

From (2.55) we know that $V_a(M_a) \to \infty$ as $a \to \infty$ so then (2.57) implies

$$V_a\Big(\frac{M_a + t_{a,\beta}}{2}\Big) \to \infty \quad \text{as } a \to \infty. \tag{2.58}$$

Since V_a is decreasing on $[M_a, \frac{M_a+t_{a,\beta}}{2}]$ it follows that $V_a \to \infty$ uniformly on $[M_a, \frac{M_a+t_{a,\beta}}{2}]$ for sufficiently large a. Since $f(V_a(t)) \geq \frac{1}{2}V_a^p(t)$ for V_a large by $(H3)$, from (2.3) $-V_a''(t) \geq f(V_a(t)) \geq \frac{1}{2}h(t)V_a^p(t)$ on $[M_a, \frac{M_a+t_{a,\beta}}{2}]$. Since V_a is decreasing on (M_a, t), integrating from M_a to t where $M_a \leq t \leq \frac{M_a+t_{a,\beta}}{2}$ we obtain

$$-V_a'(t) = -V_a'(t) + V_a'(M_a)$$

$$\begin{aligned} &= \int_{M_a}^{t} -V_a''(s)\,ds \\ &\geq \frac{1}{2}\int_{M_a}^{t} h(s)V_a^p(s)ds \\ &\geq \frac{1}{2}V_a^p(t)\int_{M_a}^{t} h(s)\,ds. \end{aligned}$$

Therefore,

$$\frac{-V_a'(t)}{V_a^p(t)} \geq \frac{1}{2}\int_{M_a}^{t} h(s)\,ds. \tag{2.59}$$

Integrating on (M_a, t) gives

$$\frac{1}{(p-1)V_a^{p-1}(t)} \geq \frac{1}{p-1}[V_a^{1-p}(t) - V_a^{1-p}(M_a)] \geq \frac{1}{2}\int_{M_a}^{t}\int_{M_a}^{s} h(x)\,dx\,ds. \tag{2.60}$$

Evaluating at $t = \frac{M_a + t_{a,\beta}}{2}$ gives

$$\frac{1}{(p-1)V_a^{p-1}(\frac{M_a+t_{a,\beta}}{2})} \geq \frac{1}{2}\int_{M_a}^{\frac{M_a+t_{a,\beta}}{2}}\int_{M_a}^{x} h(x)\,dx\,ds. \tag{2.61}$$

The left-hand side goes to zero as $a \to \infty$ by (2.58). Since we saw in (2.38) $t_{a,\beta} \to R^{2-N}$ as $a \to \infty$ and $h(s)$ is continuous and positive, it follows that

$$M_a \to R^{2-N} \quad \text{as } a \to \infty. \tag{2.62}$$

Next we show there is a $Z_a \in (0, M_a)$ such that $V_a(Z_a) = 0$, $V_a(t) > 0$ on (Z_a, R^{2-N}), and $Z_a \to R^{2-N}$ as $a \to \infty$. Moreover $V_a'(Z_a) \to -\infty$ as $a \to \infty$. Again we do this by contradiction. Let us assume $V_a(t) > 0$ on $(0, M_a)$. Since $E_a(t)$ is non-increasing then we have

$$F(V_a(M_a)) \leq \frac{1}{2}\frac{V_a'^2}{h(t)} + F(V_a(t)) \quad \text{for } 0 \leq t \leq M_a. \tag{2.63}$$

Now if V_a has a positive local minimum m_a, then $V_a''(m_a) \geq 0$ so $f(V_a(m_a)) \leq 0$ so $0 < V_a(m_a) \leq \beta$ but also $0 < E_a(m_a) = F(V_a(m_a))$ so $V_a(m_a) > \gamma \geq \beta$ which is a contradiction. Thus $V_a' > 0$ on $(0, M_a)$. Rewriting, integrating (2.63) over $[\frac{M_a}{2}, M_a]$, using (2.5), and making a change of variables gives

$$\begin{aligned} \int_0^{V_a(M_a)} \frac{ds}{\sqrt{F(V_a(M_a)) - F(s)}} &\geq \int_{V_a(\frac{M_a}{2})}^{V_a(M_a)} \frac{ds}{\sqrt{F(V_a(M_a)) - F(s)}} \\ &= \int_{\frac{M_a}{2}}^{M_a} \frac{|V_a'(t)|\,dt}{\sqrt{F(V_a(M_a)) - F(V_a(t))}} \\ &\geq \int_{\frac{M_a}{2}}^{M_a} \sqrt{2h(s)}\,ds \\ &\geq \int_{\frac{M_a}{2}}^{M_a} \sqrt{2h_1}s^{\tilde{\alpha}/2}\ \mathrm{ds} \\ &= \frac{\sqrt{2h_1}(1 - \frac{1}{2^{1+\frac{\tilde{\alpha}}{2}}})}{1 + \frac{\tilde{\alpha}}{2}} M_a^{1+\frac{\tilde{\alpha}}{2}}. \end{aligned} \tag{2.64}$$

Now we estimate the left-hand side. It follows from (H3) that $f(U) \geq \frac{1}{2}U^p$ for U sufficiently large therefore for U large enough we see that $\min_{[\frac{1}{2}U,U]} f \geq \frac{1}{2^{p+1}}U^p$ and since $p > 1$, it follows that

$$\lim_{U\to\infty} \frac{U}{\min_{[\frac{1}{2}U,U]} f} = 0. \tag{2.65}$$

We now estimate the integral on the left-hand side of (2.64) when $s \in [0, \frac{V_a(M_a)}{2}]$ and a is sufficiently large. We then have $F(s) < F(\frac{V_a(M_a)}{2})$ for all $s \in (0, \frac{V_a(M_a)}{2})$ and thus $F(V_a(M_a)) - F(\frac{V_a(M_a)}{2}) < F(V_a(M_a)) - F(s)$ so

$$\begin{aligned}\int_0^{\frac{V_a(M_a)}{2}} \frac{ds}{\sqrt{F(V_a(M_a)) - F(s)}} &\leq \int_0^{\frac{V_a(M_a)}{2}} \frac{ds}{\sqrt{F(V_a(M_a)) - F(\frac{V_a(M_a)}{2})}} \\ &= \frac{\frac{V_a(M_a)}{2}}{\sqrt{F(V_a(M_a)) - F(\frac{V_a(M_a)}{2})}}.\end{aligned} \tag{2.66}$$

By the mean value theorem there is a $d_1 > 0$ such that $\frac{V_a(M_a)}{2} < d_1 < V_a(M_a)$ and

$$\begin{aligned}F(V_a(M_a)) - F(\frac{V_a(M_a)}{2}) &= f(d_1)[V_a(M_a) - \frac{V_a(M_a)}{2} \\ &= f(d_1)[\frac{V_a(M_a)}{2}] \\ &\geq \Big[\min_{[\frac{V_a(M_a)}{2}, V_a(M_a)]} f\Big] \frac{V_a(M_a)}{2}\end{aligned}$$

so

$$\begin{aligned}\frac{\frac{V_a(M_a)}{2}}{\sqrt{F(V_a(M_a)) - F(\frac{V_a(M_a)}{2})}} &\leq \frac{\sqrt{\frac{V_a(M_a)}{2}}}{\sqrt{\min_{[\frac{V_a(M_a)}{2}, V_a(M_a)]} f}} \\ &\leq \frac{1}{\sqrt{2}} \sqrt{\frac{V_a(M_a)}{\min_{[\frac{V_a(M_a)}{2}, V_a(M_a)]} f}} \to 0\end{aligned} \tag{2.67}$$

as $a \to \infty$, by (2.65). Thus by (2.66) and (2.67) we see that

$$\lim_{a\to\infty} \int_0^{\frac{V_a(M_a)}{2}} \frac{ds}{\sqrt{2}\sqrt{F(V_a(M_a)) - F(s)}} = 0. \tag{2.68}$$

Next, we estimate the integral on the left-hand side of (2.64) for $s \in [\frac{V_a(M_a)}{2}, V_a(M_a)]$. By the mean value theorem there is a $d_2 > 0$ with $\frac{V_a(M_a)}{2} < d_2 < V_a(M_a)$ such that

$$F(V_a(M_a)) - F(s) = f(d_2)[V_a(M_a) - s] \geq \Big[\min_{[\frac{V_a(M_a)}{2}, V_a(M_a)]} f\Big][V_a(M_a) - s].$$

Therefore,

$$\begin{aligned} &\int_{\frac{V_a(M_a)}{2}}^{V_a(M_a)} \frac{ds}{\sqrt{F(V_a(M_a)) - F(s)}} \\ &\le \int_{\frac{V_a(M_a)}{2}}^{V_a(M_a)} \frac{ds}{\sqrt{[\min_{[\frac{V_a(M_a)}{2}, V_a(M_a)]} f][V_a(M_a) - s]}} \\ &= \sqrt{2}\sqrt{\frac{V_a(M_a)}{\min_{[\frac{V_a(M_a)}{2}, V_a(M_a)]} f}}. \end{aligned} \tag{2.69}$$

Thus by (2.65) we see that

$$\lim_{a\to\infty} \int_{\frac{V_a(M_a)}{2}}^{V_a(M_a)} \frac{dt}{\sqrt{2}\sqrt{F(V_a(M_a)) - F(s)}} = 0. \tag{2.70}$$

Combining (2.67) and (2.70) we have

$$\lim_{a\to\infty} \int_0^{V_a(M_a)} \frac{ds}{\sqrt{2}\sqrt{F(V_a(M_a)) - F(s)}} = 0. \tag{2.71}$$

Thus the left-hand side of (2.64) goes to 0 as $a \to \infty$ but the right-hand side of (2.64) does not because by (2.62) we know $M_a \to R^{2-N}$ as $a \to \infty$ and so we get a contradiction. Thus for a sufficiently large $V_a(t)$ has a first zero, Z_a, with $V_a(Z_a) = 0$ and $V_a(t) > 0$ on (Z_a, R^{2-N}). Similarly rewriting (2.63) and integrating on (Z_a, M_a) we obtain

$$\int_0^{V_a(M_a)} \frac{ds}{\sqrt{2}\sqrt{F(V_a(M_a)) - F(s)}} \ge \sqrt{h_1}\Big(\frac{{M_a}^{1+\frac{\tilde\alpha}{2}} - {Z_a}^{1+\frac{\tilde\alpha}{2}}}{1 + \frac{\tilde\alpha}{2}}\Big). \tag{2.72}$$

Since the left-hand side approaches 0 as $a \to \infty$ (by(2.71)), we see ${M_a}^{1+\frac{\tilde\alpha}{2}} - {Z_a}^{1+\frac{\tilde\alpha}{2}} \to 0$ as $a \to \infty$. Also since we know from (2.62) that $M_a \to R^{2-N}$ as $a \to \infty$ this then implies that $Z_a \to R^{2-N}$ as $a \to \infty$.

Finally we show that $V_a'(Z_a) \to +\infty$ as $a \to \infty$. Since $Z_a \to R^{2-N}$ as $a \to \infty$ and $E_a(t)$ is non-increasing, since $0 < Z_a \le M_a$ we have

$$0 < F(V_a(M_a)) = E_a(M_a) \le E_a(Z_a) = \frac{1}{2}\frac{{V_a'}^2(Z_a)}{h(Z_a)}$$

and so rewriting this inequality gives

$$2h(Z_a)F(V_a(M_a)) \le {V_a'}^2(Z_a). \tag{2.73}$$

As $a \to \infty$ the left-hand side appraoches ∞ because $\lim_{a\to\infty} h(Z_a) = h(R^{2-N}) > 0$ and $\lim_{a\to\infty} F(V_a(M_a)) = \infty$ by (2.55). Thus ${V_a'}^2(Z_a) \to \infty$ as $a \to \infty$ and thus it follows that $V_a'(Z_a) \to +\infty$ as $a \to \infty$. In similar way if $a > 0$ is sufficiently large then $V_a(t)$ has a second zero $Z_{a,2}$ on $(0, R^{2-N})$ with $Z_{a,2} \to R^{2-N}$ as $a \to \infty$ and $V_a'(Z_{a,2}) \to -\infty$. More generally $V_a(t)$ has n zeros on $(0, R^{2-N})$ if $a > 0$ is sufficiently large. This completes the proof. □

Lemma 2.5. *Let $V_a(t)$ be the solution of* (2.7), (H1)–(H6) *hold, and* $N > 2$. *If R is sufficiently large then $V_a(t) > 0$ for all $t \in (0, R^{2-N})$ if a sufficiently small.*

Proof. To reach a contradiction, suppose there is $Z_a \in (0, R^{2-N})$ such that $V_a(Z_a) = 0$ for all a sufficiently small. Then there exists $0 < M_a < R^{2-N}$ such that $V_a'(M_a) = 0$ and $V_a'(t) < 0$ on (M_a, R^{2-N}). Also $0 < E_a(M_a) = F(V_a(M_a))$ so $V_a(M_a) > \gamma$. Then by Lemma 2.2 we see that $|V_a'(t)| \le \frac{aR^{2-N}}{N-2} + \sqrt{2F_0 h(R^{2-N})}$, and since $V_a(t)$ is decreasing on (M_a, R^{2-N}) this gives

$$-V_a'(t) \le \frac{aR^{2-N}}{N-2} + \sqrt{2F_0 h(R^{2-N})} \quad \text{on } (M_a, R^{2-N}). \tag{2.74}$$

Integrating from t to R^{2-N} and using (2.4) we obtain:

$$V_a(t) \le \Big(\frac{aR^{2-N}}{N-2} + \sqrt{2F_0 h(R^{2-N})}\Big)(R^{2-N} - t) \le \Big(\frac{aR^{2-N}}{N-2} + \sqrt{2F_0 h(R^{2-N})}\Big)R^{2-N}.$$

Substituting $t = M_a$ gives

$$\gamma \le \Big(\frac{aR^{2-N}}{N-2} + \sqrt{2F_0 h(R^{2-N})}\Big)R^{2-N}.$$

Taking the limit as $a \to 0^+$ we obtain

$$\gamma \le \sqrt{2F_0 h(R^{2-N})}R^{2-N} = \sqrt{2F_0 h_2}(R^{2-N})^{\tilde{\alpha}/2}R^{2-N}. \tag{2.75}$$

Then using (2.6) we obtain

$$\gamma \le \sqrt{2F_0 h_2}R^{1-\frac{\alpha}{2}} \text{ where } \alpha > 2(N-1). \tag{2.76}$$

Thus we see that the right-hand side of (2.76) is larger than γ for R sufficiently large but since $\alpha > 2$ we see the right-hand side goes to 0 as $R \to \infty$ contradicting (2.76). Thus if R is sufficiently large then $0 < V_a(t) < \gamma$ if a is sufficiently small. This completes the proof. □

3. Proof of the main Theorem 1.1

Lemma 3.1. *Assume $N > 2$ and (H1)–(H6) hold. For $a > 0$ Let $V_a(t)$ be the solution of (2.7). Then $V_a(t)$ has at most a finite numbers of zeros on $(0, R^{2-N})$.*

Proof. Suppose by way of contradiction that there are distinct zero's $Z_n \in (0, R^{2-N})$ such that $V_a(Z_n) = 0$. Then either there is a decreasing subsequence (still labeled Z_n) or an increasing subsequence and a $Z^* \in [0, R^{2-N}]$ such that $Z_n \to Z^*$ as $n \to \infty$. By continuity $V_a(Z^*) = 0$. Also since $V_a'(R^{2-N}) < 0$ there exists $\epsilon > 0$ such that V_a is not zero on $(R^{2-N} - \epsilon, R^{2-N})$ and thus $Z^* \ne R^{2-N}$. Therefore $0 \le Z^* < R^{2-N}$. Without loss of generality assume Z_n is decreasing. Then there is a local maximum or local minimum M_n of V_a with $Z_{n+1} < M_n < Z_n$ so $M_n \to Z^*$ as $n \to \infty$ and notice also that since $E_a(t) > 0$ on $[0, R^{2-N}]$ by (2.25) then $E_a(M_n) = F(V_a(M_n)) > 0$ which implies that $|V_a(M_n)| > \gamma$. Now by the mean value theorem,

$$\gamma < |V_a(M_n)| = |V_a(M_n) - V_a(Z_n)| = |V_a'(c_n)||M_n - Z_n|, \tag{3.1}$$

where $c_n \ne 0$ and $M_n < c_n < Z_n$. Since $M_n \to Z^*$ and $Z_n \to Z^*$ it follows that $|M_n - Z_n| \to 0$ as $a \to \infty$. Also by (2.27) we see $|V_a'(c_n)| < \frac{aR^{2-N}}{N-2} + \sqrt{2F_0 h(R^{2-N})} < \infty$. This implies that the right-hand side of (84) goes to zero which contradicts the fact that $\gamma > 0$. Thus V_a has at most a finite numbers of zeros on $(0, R^{2-N})$. This completes the proof. □

Let

$$S_n = \{a > 0 : V_a(t) \text{ has exactly } n \text{ zeros on } (0, R^{2-N})\}.$$

By Lemma 3.1 we know that S_n is nonempty for some n. Let $n_0 \geq 0$ be the smallest non-negative integer n such that $S_n \neq \emptyset$ (so $S_{n_0} \neq \emptyset$ and $S_0, S_1, S_2, \dots, S_{n_0-1}$ are all empty). By Lemma 2.3 it follows that S_{n_0} is bounded above. Therefore the supremum of S_{n_0} exists, and so we let

$$a_{n_0} = \sup S_{n_0}.$$

If in addition R is sufficiently small then $S_0 \neq \emptyset$ by Lemma 2.4 and so $n_0 = 0$.

Lemma 3.2. *$V_{a_n}(t)$ has exactly n zeros on $(0, R^{2-N})$ and $V_{a_n}(0) = 0$ for all $n \geq n_0$.*

Proof. Since S_{n_0} is the smallest value of n such that $S_n \neq \emptyset$ this implies that $V_{a_{n_0}}(t)$ has at least n_0 zeros on $(0, R^{2-N})$. Next we show that $V_{a_{n_0}}(t)$ has at most n_0 zeros on $(0, R^{2-N})$. By way of contradiction, suppose there exists an (n_0+1)st zero Z^* with $Z^* \in (0, R^{2-N})$ such that $V_{a_{n_0}}(Z^*) = 0$ and $0 < Z^* < Z_{n_0} < \cdots < Z_1 < R^{2-N}$ and suppose without loss of generality that $V_{a_{n_0}} > 0$ on $(0, Z^*)$. Since E_a is non-increasing then $0 < E_a(Z^*) = \frac{1}{2}\frac{V'^2_{a_{n_0}}(Z^*)}{h(Z^*)}$ which implies that $V'^2_{a_{n_0}}(Z^*) > 0$. Since $V'_{a_{n_0}} > 0$ on $(0, Z^*)$ it follows that $V'_{a_{n_0}}(Z^*) < 0$. So $V_{a_{n_0}}(Z^* - \delta) > 0$ for $\delta > 0$ sufficiently small. By continuity with respect to a it follows that if $a < a_{n_0}$ then V_a also has a (n_0+1)st zero on $(0, R^{2-N})$ which is a contradiction to the definition of a_{n_0}. Therefore we see that $V_{a_{n_0}}(t)$ has exactly n_0 zeros on $(0, R^{2-N})$. Now we denote $Z_{a_{n_0}}$ as the $n_0{}^{th}$ zero of $V_{a_{n_0}}(t)$. Then $V_{a_{n_0}}(t) \neq 0$ if $0 < t < Z_{a_{n_0}}$. So without loss of generality we assume that $V_{a_{n_0}} < 0$ on $(0, Z_{a_{n_0}})$. It follows by continuity of $V_{a_{n_0}}$ that $V_{a_{n_0}}(0) = \lim_{t\to 0^+} V_{a_{n_0}}(t) \leq 0$. Thus $V_{a_{n_0}}(0) \leq 0$. Next we show that $V_{a_{n_0}}(0) = 0$. So suppose not. Then $V_{a_{n_0}} < 0$ on $[0, Z_{a_{n_0}})$. From the remark before Lemma 2.2 we saw that $V'_{a_{n_0}}(Z) \neq 0$ if $V_{a_{n_0}}(Z) = 0$. For $a_{n_0+1} > a > a_{n_0}$ we see that $|V'_a| \leq |a_{n_0+1}|\frac{R^{N-1}}{N-2} + \sqrt{2F_0 h(R^{2-N})}$ by Lemma 2.2. It follows then that V_a will also have n_0 zeros on $(0, R^{2-N})$ if $a_{n_0+1} > a > a_{n_0}$. On the other hand, if $a > a_{n_0}$ then by the definition of a_{n_0} we see that V_a has at least (n_0+1) zeros on $(0, R^{2-N})$ which is a contradiction. Thus the assumption that $V_{a_{n_0}}(0) < 0$ is false and since $V_{a_{n_0}}(0) \leq 0$ then it follows that $V_{a_{n_0}}(0) = 0$.

Next let

$$S_{n_0+1} = \{a > 0 : V_a(t) \text{ has exactly } n_0 + 1 \text{ zeros on } (0, R^{2-N})\}.$$

For a slightly larger than a_{n_0} than V_a has at least n_0+1 zeros on $(0, R^{2-N})$ by definition of a_{n_0}. Next we show that $V_a(t)$ has at most n_0+1 zeros on $(0, R^{2-N})$ if a is close to a_{n_0} and $a > a_{n_0}$. So suppose not and suppose that V_a has an (n_0+2)nd zero on $(0, R^{2-N})$. Then V_a has a local maximum or a local minimum at some M_a where $0 < Z_{a_{n_0+2}} < M_a < Z_{a_{n_0+1}}$ and for a slightly larger than a_{n_0}. Also $\lim_{a\to a_{n_0}} V_a = V_{a_{n_0}}$ uniformly on $(0, R^{2-N})$ and $Z_{a_{n_0+1}} \to 0$, hence $M_a \to 0$ as $a \to a_{n_0}$. Since $0 < E_a(M_a) = F(V_a(M_a))$ it follows that $|V_a(M_a)| > \gamma > \beta$ so $\beta \leq |V_a(M_a)| \to |V_{a_{n_0}}(0)| = 0$ which is false. Thus if $a > a_{n_0}$ and a is close to a_{n_0} then V_a has at most n_0+1 zeros on $(0, R^{2-N})$ and since we showed earlier V_a has at least n_0+1 zeros on $(0, R^{2-N})$ then it follows that $S_{n_0+1} \neq \emptyset$. By Lemma 2.2 it follows that S_{n_0+1} is bounded from above.

Let

$$a_{n_0+1} = \sup\ S_{n_0+1}.$$

In a similar fashion way we can show that $V_{a_{n_0+1}}(t)$ has exactly $n_0 + 1$ zeros on $(0, R^{2-N})$ and $V_{a_{n_0+1}}(0) = 0$. Proceeding inductively we can show that for each $n \in \mathbb{N}$ there exists a solution $V_{a_{n_0+n}}(t)$ of (2.7) which has exactly $n_0 + n$ zeros on $(0, R^{2-N})$ and $V_{a_{n_0+n}}(0) = 0$. This completes the proof of Lemma 3.2 and the proof of the main theorem. □

References

[1] B. Azeroual, A. Zertiti; On multiplicity of radial solutions to Dirichlet problem involving the $p-$Laplacian on exterior domains, *International Journal of Applied Mathematics*, Volume 31 (2018), Number 1, 121-147.

[2] H. Berestycki, P.L. Lions; Non-linear scalar field equations I, *Arch. Rational Mech. Anal.*, Volume 82 (1983), 313-347.

[3] H. Berestycki, P.L. Lions; Non-linear scalar field equations II, *Arch. Rational Mech. Anal.*, Volume 82 (1983), 347-375.

[4] M. Berger; *Nonlinearity and functional analysis,* Academic Free Press, New York, 1977.

[5] G. Birkhoff, G. C. Rota; *Ordinary Differential Equations,* John Wiley and Sons, 1962.

[6] A. Castro, L. Sankar, R. Shivaji; Uniqueness of nonnegative solutions for semipositone problems on exterior domains, *Journal of Mathematical Analysis and Applications*, Volume 394(2012), Issue 1, 432-437.

[7] J. Iaia; Loitering at the hilltop on exterior domains, *Electronic Journal of the Qualitative Theory of Differential Equations*, Volume 2015, 82 (2015), 1-11.

[8] J. Iaia; Existence for semilinear equations on exterior domains, *Electronic Journal of the Qualitative Theory of Differential Equations*, 2016 (2016) No. 108 , 1-12.

[9] C.K.R.T. Jones, T. Kupper; On the infinitely many solutions of a semilinear equation, *SIAM J. Math. Anal.*, Volume 17 (1986), 803-835.

[10] J. Joshi; Existence and nonexistence of solutions of sublinear problems with prescribed number of zeros on exterior domains, *Electronic Journal of Differential Equations*, Volume 2017 (2017) No. 133, 1-10.

[11] E.K. Lee, R. Shivaji, B. Son; Positive radial solutions to classes of singular problems on the exterior of a ball, *Journal of Mathematical Analysis and Applications,* 434 (2016), No. 2, 1597-1611, 2016.

[12] K. McLeod, W. C. Troy, F. B. Weissler; Radial solutions of $\Delta u + f(u) = 0$ with prescribed numbers of zeros, *Journal of Differential Equations*, Volume 83 (1990), Issue 2, 368-373.

[13] L. Sankar, S. Sasi, R. Shivaji; Semipositone problems with falling zeros on exterior domains, *Journal of Mathematical Analysis and Applications*, Volume 401 (2013), Issue 1, 146-153.

Mageed Ali
Department of Mathematics, University of Kirkuk, Kirkuk, Iraq
Email address: mageedali@uokirkuk.edu.iq

Joseph Iaia
Department of Mathematics, University of North Texas, Denton, TX, USA
Email address: iaia@unt.edu