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BLOW UP AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR A $p(x)$-LAPLACIAN EQUATION WITH DELAY TERM AND VARIABLE EXPONENTS

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Abstract. In this article, we consider a nonlinear $p(x)$-Laplacian equation with time delay and variable exponents. Firstly, we prove the blow up of solutions. Then, by applying an integral inequality due to Komornik, we obtain the decay result.

1. Introduction

In this work, we study the wave equation

$$\begin{gathered}
u_{tt}-\Delta_{p(x)}u+\mu_1 u_t(x,t)|u_t|^{m(x)-2}(x,t) \\
+\mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau) \\
= bu|u|^{q(x)-2} \quad \text{in } \Omega\times(0,\infty) \\
u(x,t)=0 \quad \text{in } \partial\Omega\times[0,\infty), \\
u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x) \quad \text{in } \Omega, \\
u_t(x,t-\tau)=f_0(x,t-\tau) \quad \text{in } \Omega\times(0,\tau),
\end{gathered} \tag{1.1}$$

with delay term. Here, $\Omega\subset R^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. $\tau>0$ is a time delay term, μ_1 is a positive constant, μ_2 is a real number and $b\geq 0$ is a constant. The term $\Delta_{p(\cdot)}u=(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(\cdot)$-Laplacian. The functions u_0, u_1, f_0 are the initial data that will be specified later.

$p(\cdot)$, $q(\cdot)$ and $m(\cdot)$ are the variable exponents; these are given as measurable functions on $\overline{\Omega}$ such that:

$$\begin{gathered}
2\leq p^-\leq p(x)\leq p^+\leq p^*, \\
2\leq q^-\leq q(x)\leq q^+\leq q^*, \\
2\leq m^-\leq m(x)\leq m^+\leq m^*,
\end{gathered} \tag{1.2}$$

where

$$p^-=\operatorname{ess\,inf}_{x\in\Omega}p(x), \quad p^+=\operatorname{ess\,sup}_{x\in\Omega}p(x),$$

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$$q^- = \operatorname{ess\,inf}_{x\in\Omega}, \quad q^+ = \operatorname{ess\,sup}_{x\in\Omega} q(x),$$
$$m^- = \operatorname{ess\,inf}_{x\in\Omega} m(x), \quad m^+ = \operatorname{ess\,sup}_{x\in\Omega} m(x)$$

and

$$p^* = \begin{cases} \operatorname{ess\,inf}_{x\in\Omega} np(x)/(n-p(x)) & \text{if } p^+ < n, \\ +\infty & \text{if } p^+ > n. \end{cases} \tag{1.3}$$

Many phenomena in engineering and physics lead up to problems that deal with evolution equations, which are modeled by partial differential equations. Up to now, there are many results about partial differential equations with time delay effects. Our main goal in this work is to study the equation with $p(\cdot)$-Laplacian and the delay term $\mu_2 u_t(x,t-\tau)$ which make the problem more interesting than those studied in the literature. Equation (1.1) is a very general equation.

Time delay appears in many practical problems such as thermal, biological, economic, chemical, physical phenomena and it can be a source of instability [11]. Mathematically, these properties have practical and theoretical importance. On the other hand, the delay term is a source that may destabilize the asymptotic stability of solutions for an evolutionary system. This result is well justified in mathematical analysis and physics examples, such as non-instant transmission phenomena and biological models [32].

The problems with variable exponents arise in many branches of sciences such as nonlinear elasticity theory, electrorheological fluids and image processing [4, 5, 29]. Many works about wave equation with constant delay or delay effects with time-varying have been published.

Equations with delay. Feng and Li [7], studied the equation

$$\begin{aligned} &u_{tt} + \Delta^2 u - \operatorname{div} F(\nabla u) - \sigma(t)\int_0^t g(t-s)\Delta^2 u(s)ds + \mu_1 |u_t|^{m-1}u_t \\ &+ \mu_2 |u_t(x,t-\tau)|^{m-1}u_t(x,t-\tau) = 0, \end{aligned} \tag{1.4}$$

where $\Omega \subseteq R^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. They proved the general rates of energy decay of the initial value problem and the boundary value problem by using the energy perturbation method.

Messaoudi and Kafini [11] considered the equation

$$u_{tt} - \operatorname{div}(|\nabla u|^{m-2}\nabla u) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = b|u|^{p-2}u. \tag{1.5}$$

Under suitable conditions, they proved the blow-up of solutions of (1.5) in a finite time.

Nicaise and Pignotti [20] considered the wave equation with time delay,

$$u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0, \tag{1.6}$$

and they established stability results under the assumption $0 < \mu_2 < \mu_1$.

Park[22] treated the Kirchhoff models with time delay and perturbation of p-Laplacian type

$$u_{tt} + \Delta^2 u - \Delta_p u - a_0 \Delta u_t + a_1 u_t(x,t-\tau) + f(u) = g(x), \tag{1.7}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplacian operator and $a_0 > 0$, $a_1 \in R$, $\tau > 0$ is time delay. He established the existence of global attractors and the finite dimensionality of the attractors by establishing some functionals.

Equations without delay. Pişkin [23] studied the quasilinear hyperbolic equation

$$u_{tt} - \operatorname{div}(|\nabla u|^m \nabla u) - \Delta u_t + |u_t|^{q-1} u_t = |u|^{p-1} u, \tag{1.8}$$

where $m > 0$, p, $q \geq 1$. He investigated the global existence, decay and blow up of solutions. He proved the decay estimates of the energy function by using Nakao's inequality and obtained the blow up of solutions and lifespan estimates in three different ranges of the initial energy.

Wu and Xue [31] considered the quasi-linear wave equation

$$u_{tt} - \Delta u_t - \operatorname{div}(|\nabla u|^m \nabla u) + a|u_t|^\alpha u_t = b|u|^{p-1} u, \tag{1.9}$$

where a, b, α, m, $p \geq 0$. By using multiplier methods, they gave the precise uniform estimation of the decay rate, when the initial data are in a potential well.

Variable exponent nonlinearity. Recently, much attention has been paid to the study of nonlinear hyperbolic, parabolic and elliptic equations with variable exponents as nonlinearities. The so-called equations with non-standard growth conditions. Actually, only few work regarding hyperbolic problems with nonlinearities of variable-exponent type have appeared [15].

Equations with delay. Messaoudi and Kafini [14] studied the equation

$$\begin{aligned} &u_{tt} - \Delta u + \mu_1 u_t(x,t)|u_t|^{m(x)-2}(x,t) + \mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau) \\ &= bu|u|^{p(x)-2}. \end{aligned} \tag{1.10}$$

They obtained decay estimates and global nonexistence of solutions.

Equations without Delay. Antontsev [1, 2] considered the equation

$$\partial_{tt} u - \operatorname{div}(a(x,t)|\nabla u|^{p(x,t)} \nabla u) - \alpha \Delta u_t = b(x,t) u|u|^{\sigma(x,t)-2}, \tag{1.11}$$

in Ω, a bounded domain of R^n, where $\alpha > 0$ is a constant and a, b, p, σ are given by functions. For certain solutions with non-positive initial energy, he proved the blow-up results. Antontsev [1, 3] studied equation (1.11) and proved the local and the global existence of weak solutions.

Messaoudi et al. [8] studied the equation

$$u_{tt} - \Delta u + u_t|u_t|^{m(\cdot)-2} = u|u|^{p(\cdot)-2}. \tag{1.12}$$

They proved a global result and obtained the stability result by applying an integral inequality due to Komornik.

In [16], the authors considered the equation

$$u_{tt} - \operatorname{div}(|\nabla u|^{r(\cdot)-2} \nabla u) + |u_t|^{m(\cdot)-2} u_t = 0, \tag{1.13}$$

where the exponents $m(\cdot)$ and $r(\cdot)$ are given by measurable functions on Ω. They proved the decay results for the solution under suitable assumptions. Also, the authors gave two numerical applications to illustrate the theoretical results.

In the presence of the strong damping term $-\Delta u_t$, equation (1.13) takes the form

$$u_{tt} - \operatorname{div}(|\nabla u|^{r(\cdot)-2} \nabla u) - \Delta u_t + |u_t|^{m(\cdot)-2} u_t = 0, \tag{1.14}$$

where Ω is a bounded domain. Messaoudi [17] studied the nonlinear wave equation (1.14) with variable exponents. He established several decay results depending of the range of the variable exponents m and r. In recent years, some other authors investigated hyperbolic type equations with variable exponents; see [10, 21, 24, 25, 26, 27, 30].

Our purpose is to study the blow up of solutions with negative initial energy and the decay results for the nonlinear wave equation (1.1) with time-dependent delay and variable exponents. Our result extends the equation (1.5), from constant-exponent nonlinearities to variable-exponent nonlinearities.

This article is organized as follows: In Section 2, the definitions of variable exponent in Sobolev and Lebesgue spaces are introduced. In Section 3, we prove the blow up of solutions. Finally, in Section 4, the decay results will be obtained.

2. Preliminaries

In this part, we begin by introducing some preliminary facts about Lebesgue $L^{p(\cdot)}(\Omega)$ and Sobolev $W^{1,p(\cdot)}(\Omega)$ spaces with variable exponents; see [1, 5, 6, 9, 13, 14, 28].

Let $p : \Omega \to [1, \infty)$ be a measurable function. We define the variable-exponent in Lebesgue space with a variable exponent $p(\cdot)$ by

$$L^{p(\cdot)}(\Omega) = \big\{u : \Omega \to R : \text{ measurable in } \Omega, \int_\Omega |u|^{p(\cdot)} dx < \infty\big\},$$

with a Luxemburg-type norm

$$\|u\|_{p(\cdot)} = \inf\big\{\lambda > 0 : \int_\Omega |\frac{u}{\lambda}|^{p(x)} dx \le 1\big\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space. (see [5])

Now, we define the variable-exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \big\{u \in L^{p(\cdot)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega)\big\}.$$

The variable exponent Sobolev space with respect to the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

is a Banach space. The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)}.$$

The dual space of $W_0^{1,p(\cdot)}(\Omega)$ is defined as $W_0^{-1,p'(\cdot)}(\Omega)$, in the same way that the usual Sobolev spaces, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

We also suppose that $p(\cdot)$, $q(\cdot)$ and $m(\cdot)$ satisfy the log-Hölder continuity condition:

$$|q(x) - q(y)| \le -\frac{A}{\log|x-y|}, \quad \text{for a.e. } x, y \in \Omega, \text{ with } |x-y| < \delta, \tag{2.1}$$

where $A > 0$ and $0 < \delta < 1$.

Lemma 2.1 (Poincaré inequality [1]). *Assume that $q(\cdot)$ satisfies (2.1) and let Ω be a bounded domain of R^n. Then*

$$\|u\|_{p(\cdot)} \le c\|\nabla u\|_{p(\cdot)} \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where $c = c(p^-, p^+, |\Omega|) > 0$.

Lemma 2.2 ([5]). *If $p(\cdot) \in C(\overline{\Omega})$ and $q : \Omega \to [1, \infty)$ is a measurable function such that $\operatorname{ess\,inf} f_{x\in\Omega}(p^*(x) - q(x)) > 0$, with p^* defined in (1.3), is satisfied, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.*

Lemma 2.3 ([1]). *If $p^+ < \infty$ and $p : \Omega \to [1, \infty)$ is a measurable function, then $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.*

Lemma 2.4 (Hölder's inequality [1]). *Let $p, q, s \geq 1$ be measurable functions defined on Ω and*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \quad \text{for a.e. } y \in \Omega.$$

that is satisfied. If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then $fg \in L^{s(\cdot)}(\Omega)$ and

$$\|fg\|_{s(\cdot)} \leq 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

Lemma 2.5 (Unit ball property [1]). *Let $p \geq 1$ be a measurable function on Ω. Then*

$$\|f\|_{p(\cdot)} \leq 1 \quad \text{if and only if} \quad \varrho_{p(\cdot)}(f) \leq 1,$$

where

$$\varrho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx.$$

Lemma 2.6 ([1]). *If $p \geq 1$ is a measurable function on Ω. Then*

$$\min\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\} \leq \varrho_{p(\cdot)}(u) \leq \max\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\}$$

for any $u \in L^{p(\cdot)}(\Omega)$ and for a.e. $x \in \Omega$.

3. Blow up

In this part, we deal with the blow up of the solution for problem (1.1) with negative initial energy, when $b > 0$. Now, we introduce, as in [20], the new variable

$$z(x, \rho, t) = u_t(x, t - \tau\rho), \quad x \in \Omega,\ \rho \in (0, 1),\ t > 0,$$

which implies that

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \quad x \in \Omega,\ \rho \in (0, 1),\ t > 0.$$

Using the above transformation, problem (1.1) can be written in the equivalent form

$$\begin{gathered}
u_{tt} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \mu_1 u_t(x,t)|u_t(x,t)|^{m(x)-2} \\
+ \mu_2 z(x,1,t)|z(x,1,t)|^{m(x)-2} = bu|u|^{q(x)-2}, \quad \text{in } \Omega \times (0,\infty) \\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \quad \Omega \times (0,1) \times (0,\infty) \\
z(x,\rho,0) = f_0(x,-\rho\tau) \quad \Omega \times (0,1) \\
u(x,t) = 0 \quad \partial\Omega \times [0,\infty) \\
u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega.
\end{gathered} \tag{3.1}$$

Similar to [14] we can write the following definition.

Definition 3.1. Fix $T > 0$. We call (u, z) a strong solution of (3.1) if

$$\begin{gathered}
u \in W^{2,\infty}([0,T]; L^2(\Omega)) \cap W^{1,\infty}([0,T]; H_0^1(\Omega)) \cap L^\infty([0,T]; H^2(\Omega) \cap H_0^1(\Omega)), \\
u_t \in L^{m(\cdot)}(\Omega \times (0,T)), \\
z \in W^{1,\infty}([0,1] \times [0,T]; L^2(\Omega)) \cap L^\infty([0,1]; L^{m(\cdot)}(\Omega) \cap [0,T])
\end{gathered}$$

and (u,z) satisfies the initial data and (3.1) in the sense that

$$\begin{aligned}
&\int_\Omega u_{tt}(\cdot,t)v dx-\int_\Omega \operatorname{div}(|\nabla u(\cdot,t)|^{p(\cdot)-2}\nabla u(\cdot,t))v dx\\
&+\mu_1\int_\Omega |u_t(\cdot,t)|^{m(\cdot)-2}u_t(\cdot,t)v dx+\mu_2\int_\Omega |z(\cdot,1,t)|^{m(\cdot)-2}z(\cdot,1,t)v dx\\
&=b\int_\Omega |u(\cdot,t)|^{q(\cdot)-2}u(\cdot,t)v dx
\end{aligned}\tag{3.2}$$

and

$$\tau\int_\Omega z_t(\cdot,\rho,t)w dx+\int_\Omega z_\rho(\cdot,\rho,t)w dx=0,\tag{3.3}$$

for a.e. $t\in[0,T)$ and for $(v,w)\in H_0^1(\Omega)\cap L^2(\Omega)$.

The energy functional associated with problem (3.1) is defined as

$$\begin{aligned}
E(t)=\frac12\|u_t\|^2&+\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\\
&+\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho-b\int_\Omega \frac{|u|^{q(x)}}{q(x)}dx
\end{aligned}\tag{3.4}$$

for $t\ge 0$, where ξ is a continuous function that satisfies

$$\tau|\mu_2|(m(x)-1)<\xi(x)<\tau(\mu_1 m(x)-|\mu_2|),\quad x\in\overline{\Omega}.\tag{3.5}$$

The following lemma gives that, under the condition $\mu_1>|\mu_2|$, $E(t)$ is non-increasing.

Lemma 3.2. *Let (u,z) be a solution of* (3.1). *Then there exists $C_0>0$ such that*

$$E'(t)\le -C_0\int_\Omega(|u_t|^{m(x)}+|z(x,1,t)|^{m(x)})dx\le 0.\tag{3.6}$$

Proof. Multiplying the first equation in (3.1) by u_t, integrating over Ω, then multiplying the second equation of (3.1) by $\frac1\tau\xi(x)|z|^{m(x)-2}z$, and integrating over $\Omega\times(0,1)$, we obtain

$$\begin{aligned}
&\frac{d}{dt}[\frac12\int_\Omega u_t^2dx+\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx+\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho\\
&-b\int_\Omega\frac{|u|^{q(x)}}{q(x)}dx]\\
&=-\mu_1\int_\Omega|u_t|^{m(x)}dx-\frac1\tau\int_\Omega\int_0^1\xi(x)|z(x,\rho,t)|^{m(x)-2}zz_\rho(x,\rho,t)d\rho dx\\
&\quad-\mu_2\int_\Omega u_tz(x,1,t)|z(x,1,t)|^{m(x)-2}dx.
\end{aligned}\tag{3.7}$$

The last two terms of the right-hand side of (3.7) can be estimated as follows,

$$\begin{aligned}
&-\frac1\tau\int_\Omega\int_0^1\xi(x)|z(x,\rho,t)|^{m(x)-2}zz_\rho(x,\rho,t)d\rho dx\\
&=-\frac1\tau\int_\Omega\int_0^1\frac{\partial}{\partial\rho}\Big(\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\Big)d\rho dx\\
&=\frac1\tau\int_\Omega\frac{\xi(x)}{m(x)}\Big(|z(x,0,t)|^{m(x)}-|z(x,1,t)|^{m(x)}\Big)dx
\end{aligned}$$

$$= \int_\Omega \frac{\xi(x)}{\tau m(x)} |u_t|^{m(x)} dx - \int_\Omega \frac{\xi(x)}{\tau m(x)} |z(x,1,t)|^{m(x)}.$$

We use Young's inequality, $q = \frac{m(x)}{m(x)-1}$ and $q' = m(x)$ for the last term, and then we obtain

$$\|u_t\| |z(x,1,t)|^{m(x)-1} \le \frac{1}{m(x)} |u_t|^{m(x)} + \frac{m(x)-1}{m(x)} |z(x,1,t)|^{m(x)}.$$

As a result,

$$\begin{aligned} & -\mu_2 \int_\Omega u_t z |z(x,1,t)|^{m(x)-2} dx \\ & \le |\mu_2| \Big(\int_\Omega \frac{1}{m(x)} |u_t(t)|^{m(x)} dx + \int_\Omega \frac{m(x)-1}{m(x)} |z(x,1,t)|^{m(x)} dx \Big). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{dE(t)}{dt} \le & -\int_\Omega [\mu_1 - (\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)})] |u_t(t)|^{m(x)} dx \\ & - \int_\Omega (\frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)}) |z(x,1,t)|^{m(x)} dx. \end{aligned}$$

Consequently, for all $x \in \overline{\Omega}$, relation (3.5) gives

$$\begin{gathered} f_1(x) = \mu_1 - (\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)}) > 0, \\ f_2(x) = \frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} > 0. \end{gathered}$$

Since that $m(x)$, and hence, $\xi(x)$ is bounded, we infer that $f_1(x)$ and $f_2(x)$ are also bounded. We define

$$C_0(x) = \min\{f_1(x), f_2(x)\} > 0 \quad \text{for any } x \in \overline{\Omega},$$

and take $C_0 = \inf_{x\in\overline{\Omega}} C_0(x)$, then $C_0(x) \ge C_0 > 0$. Therefore,

$$E'(t) \le -C_0 \Big[\int_\Omega |u_t(t)|^{m(x)} dx + \int_\Omega |z(x,1,t)|^{m(x)} dx \Big] \le 0.$$

□

To establish the blow up, we suppose that $E(0) < 0$ in addition to (1.3). Setting

$$H(t) = -E(t); \tag{3.8}$$

we have

$$\begin{gathered} H'(t) = -E'(t) \ge 0, \\ 0 < H(0) \le H(t) \le b \int_\Omega \frac{|u|^{q(x)}}{q(x)} dx \le \frac{b}{p^-} \varrho(u), \end{gathered} \tag{3.9}$$

where

$$\varrho(u) = \varrho_{q(\cdot)}(u) = \int_\Omega |u|^{q(x)} dx.$$

Lemma 3.3 ([15]). *Assume that the conditions of Lemma 2.2 hold. Then, exists a constant $C > 1$, depending only of Ω, such that*

$$\varrho^{s/q^-}(u) \le C\big(\|\nabla u\|_{p(\cdot)}^{p^-} + \varrho(u)\big). \tag{3.10}$$

Then, we have the following inequalities:

$$\|u\|_{q^-}^s \le C(\|\nabla u\|_{p(\cdot)}^{p^-} + \|u\|_{q^-}^{q^-}), \tag{3.11}$$

$$\varrho^{s/q^-}(u) \le C\Big(|H(t)| + \|u_t\|_2^2 + \varrho(u) + \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho\Big), \tag{3.12}$$

$$\|u\|_{q^-}^s \le C\Big(|H(t)| + \|u_t\|_2^2 + \|u\|_{q^-}^{q^-} + \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho\Big), \tag{3.13}$$

for any $u \in W_0^{1,p(\cdot)}(\Omega)$ and $p^- \le s \le q^-$. Let (u,z) be a solution of (3.1). Then

$$\varrho(u) \ge C\|u\|_{q^-}^{q^-}, \tag{3.14}$$

$$\int_\Omega |u|^{m(x)}dx \le C\Big(\varrho^{m^-/q^-}(u) + \varrho^{m^+/q^-}(u)\Big). \tag{3.15}$$

The blow up of problem (3.1) is given by the following theorem.

Theorem 3.4. *Let $u_0 \in W_0^{1,p(\cdot)}(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that the condition (2.1) holds and*

$$2 \le \max\{m^+, p^+\} < q^- \le q(x) \le q^+ \le p_*(x),$$

where

$$p_*(x) = \begin{cases} \frac{np(x)}{\operatorname{ess\,inf}_{x\in\Omega}(n-m(x))} & \text{if } p^+ < n,\\ +\infty & \text{if } p^+ > n. \end{cases}$$

Moreover, suppose that $E(0) < 0$. Then, the solution of (3.1) blows up in finite time.

Proof. We define

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t(x,t)dx \tag{3.16}$$

for a small ε to be chosen later, and

$$0 < \alpha \le \min\big\{\frac{q^- - 2}{2q^-}, \frac{q^- - m^+}{q^-(m^+ - 1)}\big\}. \tag{3.17}$$

Differentiation of (3.16), using the first equation in (3.1), gives

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega u_t^2 dx - \varepsilon \int_\Omega |\nabla u|^{p(x)}dx \\ &\quad + \varepsilon b \int_\Omega |u|^{q(x)}dx - \varepsilon\mu_1 \int_\Omega u u_t(x,t)|u_t(x,t)|^{m(x)-2}dx \\ &\quad - \varepsilon\mu_2 \int_\Omega u z(x,1,t)|z(x,1,t)|^{m(x)-2}dx. \end{aligned}$$

From the definition of $H(t)$ and for $0 < a < 1$, we obtain

$$\begin{aligned} L'(t) &\ge C_0(1-\alpha)H^{-\alpha}(t)[\int_\Omega |u_t(t)|^{m(x)}dx + \int_\Omega |z(x,1,t)|^{m(x)}dx] \\ &\quad + \varepsilon\Big((1-a)q^- H(t) + \frac{(1-a)q^-}{2}\|u_t\|^2\Big) \\ &\quad + \varepsilon(1-a)q^- \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \\ &\quad + \varepsilon(1-a)q^- \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\int_\Omega u_t^2dx-\varepsilon\int_\Omega|\nabla u|^{p(x)}+\varepsilon ab\int_\Omega|u|^{q(x)}dx\\
&-\varepsilon\mu_1\int_\Omega uu_t(x,t)|u_t(x,t)|^{m(x)-2}dx\\
&-\varepsilon\mu_2\int_\Omega uz(x,1,t)|z(x,1,t)|^{m(x)-2}dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
L'(t)&\geq C_0(1-\alpha)H^{-\alpha}(t)[\int_\Omega|u_t(t)|^{m(x)}dx+\int_\Omega|z(x,1,t)|^{m(x)}dx]\\
&\quad+\varepsilon(1-a)q^-H(t)+\varepsilon\frac{(1-a)q^-+2}{2}\|u_t\|^2\\
&\quad+\varepsilon\Big(\frac{(1-a)q^-}{p^+}-1\Big)\int_\Omega|\nabla u|^{p(x)}dx\\
&\quad+\varepsilon(1-a)q^-\int_0^1\int_\Omega\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho+\varepsilon ab\varrho(u)\\
&\quad-\varepsilon\mu_1\int_\Omega uu_t(x,t)|u_t(x,t)|^{m(x)-2}dx\\
&\quad-\varepsilon\mu_2\int_\Omega uz(x,1,t)|z(x,1,t)|^{m(x)-2}dx.
\end{aligned}\tag{3.18}
$$

From Young's inequality, we obtain

$$
\begin{aligned}
&\int_\Omega|u_t|^{m(x)-1}|u|dx\\
&\leq\frac{1}{m^-}\int_\Omega\delta^{m(x)}|u|^{m(x)}dx+\frac{m^+-1}{m^+}\int_\Omega\delta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}dx
\end{aligned}\tag{3.19}
$$

and

$$
\begin{aligned}
&\int_\Omega|z(x,1,t)|^{m(x)-1}|u|dx\\
&\leq\frac{1}{m^-}\int_\Omega\delta^{m(x)}|u|^{m(x)}dx+\frac{m^+-1}{m^+}\int_\Omega\delta^{-\frac{m(x)}{m(x)-1}}|z(x,1,t)|^{m(x)}dx.
\end{aligned}\tag{3.20}
$$

As in [19], the estimates (3.19) and (3.20) remain valid if δ is time-dependent. Thus, taking δ such that

$$
\delta^{-\frac{m(x)}{m(x)-1}}=kH^{-\alpha}(t),
$$

where $k\geq1$ is specified later, we obtain

$$
\int_\Omega\delta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}dx=kH^{-\alpha}(t)\int_\Omega|u_t|^{m(x)}dx,\tag{3.21}
$$

$$
\int_\Omega\delta^{-\frac{m(x)}{m(x)-1}}|z(x,1,t)|^{m(x)}dx=kH^{-\alpha}(t)|z(x,1,t),|^{m(x)}dx\tag{3.22}
$$

$$
\begin{aligned}
\int_\Omega\delta^{m(x)}|u|^{m(x)}dx&=\int_\Omega k^{1-m(x)}H^{\alpha(m(x)-1)}(t)|u|^{m(x)}dx\\
&\leq\int_\Omega k^{1-m^-}H^{\alpha(m^+-1)}(t)\int_\Omega|u|^{m(x)}dx.
\end{aligned}\tag{3.23}
$$

From (3.14) and (3.15), we have

$$\begin{aligned} &H^{\alpha(m^+-1)}(t)\int_\Omega |u|^{m(x)}dx \\ &\le C[(\varrho(u))^{m^-/q^-+\alpha(m^+-1)} + (\varrho(u))^{m^+/q^-+\alpha(m^+-1)}]. \end{aligned} \tag{3.24}$$

From (3.17), we conclude that

$$s = m^- + \alpha q^-(m^+-1) \le q^-, \quad s = m^+ + \alpha q^-(m^+-1) \le q^-.$$

Therefore,

$$H^{\alpha(m^+-1)}(t)\int_\Omega |u|^{m(x)}dx \le C(\|\nabla u\|_{p(\cdot)}^{p^-} + \varrho(u)). \tag{3.25}$$

Combining (3.19)-(3.25), we obtain

$$\begin{aligned} L'(t) \ge{}& (1-\alpha)H^{-\alpha}(t)\Big[C_0 - \varepsilon\Big(\frac{m^+-1}{m^+}\Big)ck\Big]\int_\Omega |u_t(t)|^{m(x)}dx \\ &+ (1-\alpha)H^{-\alpha}(t)\Big[C_0 - \varepsilon\Big(\frac{m^+-1}{m^+}\Big)ck\Big]\int_\Omega |z(x,1,t)|^{m(x)}dx \\ &+ \varepsilon\Big(\frac{(1-a)q^- - p^+}{p^+} - \frac{C}{m^- k^{m^- -1}}\Big)\int_\Omega |\nabla u|^{p(x)}dx \\ &+ \varepsilon(1-a)q^- H(t) + \varepsilon\frac{(1-a)q^-+2}{2}\|u_t\|^2 \\ &+ \varepsilon\Big(ab - \frac{C}{m^- k^{m^- -1}}\Big)\varrho(u) \\ &+ \varepsilon(1-a)q^-\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho. \end{aligned} \tag{3.26}$$

Let us choose a small enough such that

$$\frac{(1-a)q^-+2}{2} > 0,$$

and k so large that

$$\frac{(1-a)q^- - p^+}{p^+} - \frac{C}{m^- k^{m^- -1}} > 0 \quad \text{and} \quad ab - \frac{C}{m^- k^{m^- -1}} > 0.$$

Once k and a are fixed, we choose ε small enough such that

$$\begin{gathered} C_0 - \varepsilon\Big(\frac{m^+-1}{m^+}\Big)ck > 0, \\ L(0) = H^{1-\alpha}(0) + \varepsilon\int_\Omega u_0(x)u_1(x)dx > 0. \end{gathered}$$

Hence, (3.26) becomes

$$\begin{aligned} L'(t) \ge \varepsilon\eta\Big[&H(t) + \|u_t\|^2 + \|\nabla u\|_{p(\cdot)}^{p^-} + \varrho_{p(\cdot)}(u) \\ &+ \int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho\Big] \end{aligned} \tag{3.27}$$

for a constant $\eta > 0$. Eventually,

$$L(t) \ge L(0) > 0, \quad \forall t \ge 0.$$

Now, for some constants σ, $\Gamma > 0$, we denote

$$L'(t) \geq \Gamma L^{\sigma}(t).$$

For this reason, we estimate

$$\big|\int_\Omega uu_t(x,t)dx\big| \leq \|u\|_2\|u_t\|_2 \leq C\|u\|_{q^-}\|u_t\|_2,$$

which indicates

$$\big|\int_\Omega uu_t(x,t)dx\big|^{1/(1-\alpha)} \leq C\|u\|_{q^-}^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)}$$

and by Young's inequality

$$\big|\int_\Omega uu_t(x,t)dx\big|^{1/(1-\alpha)} \leq C[\|u\|_{q^-}^{\mu/(1-\alpha)} + \|u_t\|_2^{\Theta/(1-\alpha)}],$$

where $1/\mu + 1/\Theta = 1$. The choice of $\Theta = 2(1-\alpha)$ will make $\mu/(1-\alpha) = 2/(1-2\alpha) \leq q^-$. Thus,

$$\big|\int_\Omega uu_t(x,t)dx\big|^{1/(1-\alpha)} \leq C[\|u\|_{q^-}^{s} + \|u_t\|_2^2],$$

where $s = \mu/(1-\alpha)$. From (3.13), we obtain

$$\begin{aligned} &\big|\int_\Omega uu_t(x,t)dx\big|^{1/(1-\alpha)} \\ &\leq C\Big[|H(t)| + \|u_t(t)\|^2 + \varrho(u) + \int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho\Big]. \end{aligned} \tag{3.28}$$

Moreover, we have

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \Big[H^{(1-\alpha)}(t) + \varepsilon\int_\Omega uu_t(x,t)dx\Big]^{1/(1-\alpha)} \\ &\leq 2^{\alpha/(1-\alpha)}\Big[H(t) + \big|\int_\Omega uu_t\,dx\big|^{1/(1-\alpha)}\Big] \\ &\leq C\Big[|H(t)| + \|u_t(t)\|^2 + \varrho(u) + \int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho\Big]. \end{aligned}$$

Therefore, for some $\Psi > 0$, from (3.27), we obtain

$$L'(t) \geq \Psi L^{1/(1-\alpha)}(t).$$

A simple integration over $(0,t)$ gives

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Psi\alpha t/(1-\alpha)}$$

which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\alpha}{\Psi\alpha[L(0)]^{\alpha/(1-\alpha)}}.$$

The proof is complete. $\square$

4. Decay of solutions

In this part, we prove our decay result, when $b = 0$. Now, we introduce the variable

$$z(x,\rho,t) = u_t(x, t-\tau\rho), \quad x \in \Omega,\ \rho \in (0,1),\ t > 0;$$

thus,

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Consequently, problem (1.1) is transformed into

$$\begin{gathered}
u_{tt} - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \mu_1 u_t(x,t)|u_t(x,t)|^{m(x)-2} \\
+ \mu_2 z(x,1,t)|z(x,1,t)|^{m(x)-2} = 0 \quad \text{in } \Omega \times (0,\infty), \\
\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \quad \text{in } \Omega \times (0,1) \times (0,\infty) \\
z(x,\rho,0) = f_0(x,-\rho\tau) \quad \text{in } \Omega \times (0,1) \\
u(x,t) = 0 \quad \text{on } \partial\Omega \times [0,1) \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega.
\end{gathered} \tag{4.1}$$

We define the modified energy functional for problem (4.1) by

$$\begin{aligned}
E(t) = \frac{1}{2}\|u_t\|^2 &+ \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} dx \\
&+ \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\, dx\, d\rho,
\end{aligned} \tag{4.2}$$

where ξ is the continuous function given in (3.5) and $t \ge 0$.

Similar to Lemma 3.2, we easily establish, for $\mu_1 > |\mu_2|$ and for some $C_0 > 0$, that

$$E'(t) \le -C_0 \int_\Omega \Big(|u_t|^{m(x)} + |z(x,1,t)|^{m(x)}\Big) dx \le 0. \tag{4.3}$$

Lemma 4.1 (Komornik, [12]). *Let $E : R^+ \to R^+$ be a non-increasing function and assume that there are constants σ, $\omega > 0$ such that*

$$\int_s^\infty E^{1+\sigma}(t)dt \le \frac{1}{\Omega} E^\sigma(0)E(s) = cE(s), \quad \forall s > 0.$$

Then

$$E(t) \le \begin{cases} cE(0)/(1+t)^{1/\sigma} & \text{if } \sigma > 0, \\ cE(0)e^{-\omega t} & \text{if } \sigma = 0. \end{cases}$$

for all $t \ge 0$.

To prove our main result, we need of the following lemmas.

Lemma 4.2 ([14]). *The functional*

$$F(t) = \tau \int_0^1 \int_\Omega e^{-\rho\tau}\xi(x)|z(x,\rho,t)|^{m(x)}\, dx\, d\rho$$

satisfies

$$F'(t) \le \int_\Omega \xi(x)|u_t|^{m(x)} dx - \tau e^{-\tau} \int_0^1 \int_\Omega \xi(x)|z(x,\rho,t)|^{m(x)}\, dx\, d\rho$$

along the solution of (4.1).

Lemma 4.3 ([18])**.** *Let u be a solution of* (4.1)*. Then, for some $C > 0$,*

$$\varrho_{p(x)}(\nabla u) \geq C\|\nabla u\|_{p^-}^{p^+}. \tag{4.4}$$

Theorem 4.4. *Let $u_0 \in W_0^{1,p(\cdot)}(\Omega)$, $u_1 \in L^2(\Omega)$ be given and assume that $m(\cdot)$ and $p(\cdot)$ belong to $C(\bar{\Omega})$. Suppose that condition* (2.1) *is satisfied and*

$$2 \leq p^- \leq p(x) \leq p^+ \leq m^- \leq m(x) \leq m^+ \leq {p^-}^*, \quad \forall x \in C(\overline{\Omega}),$$

where

$${p^-}^* = \begin{cases} \operatorname{ess\,inf}_{x\in\Omega} \frac{np^-}{(n-p^-)} & \text{if } p^- < n, \\ +\infty & \text{if } p^- \geq n. \end{cases}$$

Then, there exist two constants $c, \alpha > 0$ independent of t such that for any global solution of (4.1)*, it holds*

$$E(t) \leq \begin{cases} ce^{-\alpha t} & \text{if } m(x) = 2, \\ cE(0)/(1+t)^{2/(m^+-2)} & \text{if } m^+ > 2. \end{cases}$$

Proof. We multiply the first equation of (4.1) by $uE^r(t)$, for $r > 0$ that will be specified later, and integrate over $\Omega \times (s, T)$, $s < T$. So, we obtain

$$\begin{aligned} &\int_s^T E^r(t) \int_\Omega \Big[uu_{tt} - u \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \mu_1 uu_t|u_t|^{m(x)-2} \\ &+ \mu_2 uz(x,1,t)|z(x,1,t)|^{m(x)-2}\Big]\,dx\,dt = 0, \end{aligned}$$

which implies that

$$\begin{aligned} &\int_s^T E^r(t) \int_\Omega \Big(\frac{d}{dt}(uu_t) - u_t^2 + |\nabla u|^{p(x)} + \mu_1 uu_t(x,t)|u_t(x,t)|^{m(x)-2} \\ &+ \mu_2 uz(x,1,t)|z(x,1,t)|^{m(x)-2}\Big)\,dx\,dt = 0. \end{aligned} \tag{4.5}$$

By using the definition of $E(t)$, given in (4.2), and the relation

$$\frac{d}{dt}\Big(E^r(t)\int_\Omega uu_t\,dx\Big) = rE^{r-1}(t)E'(t)\int_\Omega uu_t\,dx + E^r(t)\frac{d}{dt}\int_\Omega uu_t\,dx,$$

equation (4.5) becomes

$$\begin{aligned} &2\int_s^T E^{r+1}(t)dt \\ &\leq -\int_s^T \frac{d}{dt}\Big(E^r(t)\int_\Omega uu_t\,dx\Big)dt + r\int_s^T E^{r-1}(t)E'(t)\int_\Omega uu_t\,dx\,dt \\ &\quad + 2\int_s^T E^r(t)\int_\Omega u_t^2\,dx\,dt - \mu_1\int_s^T E^r(t)\int_\Omega uu_t|u_t|^{m(x)-2}\,dx\,dt \\ &\quad - \mu_2\int_s^T E^r(t)\int_\Omega uz(x,1,t)|z(x,1,t)|^{m(x)-2}\,dx\,dt \\ &\quad + 2\int_s^T E^r(t)\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho dt. \end{aligned} \tag{4.6}$$

Now, we estimate the right-hand side terms of equation (4.6).

The first term is estimated as follows:

$$
\begin{aligned}
&\big| -\int_s^T \frac{d}{dt}(E^r(t)\int_\Omega uu_t\,dx)dt\big| \\
&= \big|E^r(s)\int_\Omega uu_t(x,s)dx - E^r(T)\int_\Omega uu_t(x,T)dx\big| \\
&\le \frac{1}{2}E^r(s)\Big[\int_\Omega u^2(x,s)dx + \int_\Omega u_t^2(x,s)dx\Big] \\
&\quad + \frac{1}{2}E^r(T)\Big[\int_\Omega u^2(x,T)dx + \int_\Omega u_t^2(x,T)dx\Big] \\
&\le E^r(s)\Big[\frac{1}{2}c_*\int_\Omega |\nabla u(x,s)|^2dx + E(s)\Big] \\
&\quad + E^r(T)\Big[\frac{1}{2}c_*\int_\Omega |\nabla u(x,T)|^2dx + E(T)\Big]
\end{aligned}
$$

where c_* is the embedding constant. So, we obtain

$$
\begin{aligned}
&\big| -\int_s^T \frac{d}{dt}(E^r(t)\int_\Omega uu_t\,dx)dt\big| \\
&\le E^r(s)[c\|\nabla u(s)\|_{p^-}^2 + E(s)] + E^r(T)[c\|\nabla u(T)\|_{p^-}^2 + E(T)] \\
&\le E^{r+1}(s) + cE^r(s)(\|\nabla u(s)\|_{p^-}^{p^+})^{2/p^+} \\
&\quad + E^{r+1}(T) + cE^r(T)(\|\nabla u(T)\|_{p^-}^{p^+})^{\frac{2}{p^+}},
\end{aligned}
$$

where c is a generic positive constant that may change their value from a line to another. Then, we use (4.4), and recalling that $E(t)$ is non-increasing, we obtain

$$
\begin{aligned}
&\big| -\int_s^T \frac{d}{dt}(E^r(t)\int_\Omega uu_t\,dx)dt\big| \\
&\le E^{r+1}(s) + cE^r(s)(\varrho_{p(x)}(\nabla u(s)))^{2/p^+} \\
&\quad + E^{r+1}(T) + cE^r(T)(\varrho_{p(x)}(\nabla u(T)))^{\frac{2}{p^+}} \\
&\le E^{r+1}(s) + c(E(s))^{r+\frac{2}{p^+}} + E^{r+1}(T) + c(E(T))^{r+\frac{2}{p^+}} \\
&\le E^{r+1}(s) + c(E(s))^{r+\frac{2}{p^+}}.
\end{aligned}
\tag{4.7}
$$

In the above estimate, for $p^- > 2$, we applied the Hölder inequality

$$
\int_\Omega |\nabla u|^2dx \le |\Omega|^{\frac{p^--2}{p^-}}\Big(\int_\Omega |\nabla u|^{p^-}dx\Big)^{2/p^-}.
$$

Estimate (4.7) for the case $p^- = 2$ is also true.

Similarly, we deal with the term

$$\begin{aligned}
&\big|r\int_s^T E^{r-1}(t)E'(t)\int_\Omega uu_t\,dx\,dt\big| \\
&\le -c\int_s^T E^{r-1}(t)E'(t)[E(T)+cE^{\frac{2}{p^+}}(t)]dt \\
&\le -c[\int_s^T E^{r}(t)E'(t)+\int_s^T (E(t))^{r+\frac{2}{p^+}-1}E'(t)dt] \\
&\le c[E^{r+1}(s)+(E(s))^{r+\frac{2}{p^+}}].
\end{aligned} \tag{4.8}$$

To treat the other term, we define

$$\Omega_+=\{x\in\Omega:|u_t(x,t)|\ge 1\},\quad \Omega_-=\{x\in\Omega:|u_t(x,t)|<1\},$$

and use Hölder's and Young's inequalities. Then we have

$$\begin{aligned}
&\big|2\int_s^T E^r(t)\int_\Omega u_t^2\,dx\,dt\big| \\
&=\Big|2\int_s^T E^r(t)\Big[\int_{\Omega_+}u_t^2dx+\int_{\Omega_-}u_t^2dx\Big]dt\Big| \\
&\le c\int_s^T E^r(t)\Big[\Big(\int_{\Omega_+}|u_t|^{m^-}dx\Big)^{2/m^-}+\Big(\int_{\Omega_-}|u_t|^{m^+}dx\Big)^{2/m^+}\Big]dt \\
&\le c\int_s^T E^r(t)\Big[\Big(\int_{\Omega}|u_t|^{m(x)}dx\Big)^{2/m^-}+\Big(\int_{\Omega}|u_t|^{m(x)}dx\Big)^{2/m^+}\Big]dt \\
&\le c\int_s^T E^r(t)[(-E'(t))^{2/m^-}+(-E'(t))^{2/m^+}]dt \\
&\le c\varepsilon\int_s^T (E(t))^{rm^-/(m^--2)}dt+c_\varepsilon\int_s^T(-E'(t))dt \\
&\quad+c\varepsilon\int_s^T E^{r+1}(t)dt+c_\varepsilon\int_s^T(-E'(t))^{2(r+1)/m^+}dt.
\end{aligned}$$

where $c_\varepsilon=\frac{1}{r+1}\big(\frac{\varepsilon(r+1)}{r}\big)^{-r}$.

Choose r such that $r=m^+/2-1$ will make $\frac{rm^-}{m^--2}=r+1+\frac{m^+-m^-}{m^--2}$. Now, we consider two cases, $m^->2$ and $m^-=2$.

For $m^->2$, we have

$$\begin{aligned}
&\big|2\int_s^T E^r(t)\int_\Omega u_t^2\,dx\,dt\big| \\
&\le c\varepsilon\int_s^T E^{r+1}(t)dt+c\varepsilon(E(0))^{\frac{m^+-m^-}{m^--2}}\int_s^T E^{r+1}(t)dt+c_\varepsilon E(s) \\
&\le \tilde c\varepsilon\int_s^T E^{r+1}(t)dt+c_\varepsilon E(s),
\end{aligned} \tag{4.9}$$

where $\tilde c$ is a positive constant.

For $m^-=2$, we obtain

$$\big|2\int_s^T E^r(t)\int_\Omega u_t^2\,dx\,dt\big|$$

$$
\begin{aligned}
&= \Big|2\int_s^T E^r(t)\Big[\int_{\Omega_+} u_t^2 dx + \int_{\Omega_-} u_t^2 dx\Big] dt\Big| \\
&\le c\int_s^T E^r(t)\Big[\int_{\Omega_+} |u_t|^{m(x)} dx + \Big(\int_{\Omega_-} |u_t|^{m^+} dx\Big)^{2/m^+}\Big] dt \\
&\le c\int_s^T E^r(t)\Big[\int_{\Omega} |u_t|^{m(x)} dx + (\int_{\Omega} |u_t|^{m(x)} dx)^{2/m^+}\Big] dt \\
&\le c\int_s^T E^r(t)(-E'(t)) dt + c\int_s^T E^r(t)(-E'(t))^{2/m^+} dt \\
&\le cE^{r+1}(s) + c\varepsilon\int_s^T E^{r+1}(t) dt + c_\varepsilon\int_s^T (-E'(t))^{2(r+1)/m^+} dt.
\end{aligned}
$$

Therefore, with the choice of $r = m^+/2 - 1$, we obtain

$$
\begin{aligned}
\big|2\int_s^T E^r(t)\int_\Omega u_t^2\, dx\, dt\big| &\le cE^{r+1}(s) + c\varepsilon\int_s^T E^{r+1}(t) dt + c_\varepsilon E(s) \\
&\le c\varepsilon\int_s^T E^{r+1}(t) dt + (c_\varepsilon + cE^r(0))E(s) \\
&\le c\varepsilon\int_s^T E^{r+1}(t) dt + \tilde{c}_\varepsilon E(s),
\end{aligned} \tag{4.10}
$$

where $\tilde{c}_\varepsilon = c_\varepsilon + cE^r(0)$.

Because of $m^+ \ge p^+$ and $r = \frac{m^+}{2} - 1$, we have $r + \frac{2}{p^+} - 1 \ge 0$. As a result, the estimates (4.7) and (4.8) become

$$
\begin{aligned}
\big| -\int_s^T \frac{d}{dt}\Big(E^r(t)\int_\Omega uu_t\, dx\Big) dt\big| &\le E^{r+1}(s) + c(E(s))^{r+\frac{2}{p^+}} \\
&\le [E^r(0) + c(E(0))^{r+\frac{2}{p^+}-1}]E(s) \\
&= \tilde{c}E(s),
\end{aligned} \tag{4.11}
$$

and

$$
\begin{aligned}
\big|r\int_s^T E^{r-1}(t)E'(t)\int_\Omega uu_t\, dx\, dt\big| &\le c[E^{r+1}(s) + (E(s))^{r+\frac{2}{p^+}}] \\
&\le c[E^r(0) + (E(0))^{r+\frac{2}{p^+}-1}]E(s) \\
&= \tilde{c}E(s),
\end{aligned} \tag{4.12}
$$

respectively.

For the next term, by using Young's inequality, we have

$$
\begin{aligned}
&\big| -\mu_1\int_s^T E^r(t)\int_\Omega u|u_t|^{m(x)-1}\, dx\, dt\big| \\
&\le \varepsilon\int_s^T E^r(t)\int_\Omega |u|^{m(x)}\, dx\, dt + c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}\, dx\, dt \\
&\le \varepsilon\int_s^T E^r(t)\Big[\int_{\Omega_+} |u|^{m^-} dx + \int_{\Omega_-} |u|^{m^+} dx\Big] dt \\
&\quad + c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}\, dx\, dt,
\end{aligned}
$$

where we used Young's inequality with

$$p(x) = \frac{m(x)}{m(x)-1} \quad \text{and} \quad p'(x) = m(x),$$

and thus

$$c_\varepsilon(x) = (m(x)-1)m(x)^{m(x)/(1-m(x))}\varepsilon^{1/(1-m(x))}.$$

Using the embedding, we obtain

$$\begin{aligned}
&\big| -\mu_1 \int_s^T E^r(t) \int_\Omega u|u_t|^{m(x)-1}\,dx\,dt\big| \\
&\le \varepsilon \int_s^T E^r(t)[c_1\|\nabla u\|_{p^-}^{m^-} + c_2\|\nabla u\|_{p^-}^{m^+}] + \int_s^T E^r(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}dx,
\end{aligned}$$

where c_1 and c_2 are positive constants independent of ε.

From (4.2) and (4.4), we have

$$\begin{aligned}
&\big| -\mu_1 \int_s^T E^r(t) \int_\Omega u|u_t|^{m(x)-1}\,dx\,dt\big| \\
&\le \varepsilon \int_s^T E^r(t)\big[c_1(\varrho_{p(x)}(\nabla u))^{\frac{m^-}{p^+}} + c_2(\varrho_{p(x)}(\nabla u))^{\frac{m^+}{p^+}}\big]dt \\
&\quad + c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}\,dx\,dt \\
&\le \varepsilon c_1' \int_s^T E^{r+1}(t)(E(t))^{\frac{m^-}{p^+}-1}dt + \varepsilon c_2' \int_s^T E^{r+1}(t)(E(t))^{\frac{m^+}{p^+}-1}dt \\
&\quad + c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}\,dx\,dt \\
&\le c'\varepsilon\Big((E(0))^{\frac{m^-}{p^+}-1} + (E(0))^{\frac{m^+}{p^+}-1}\Big)\int_s^T E^{r+1}(t)dt \\
&\quad + \int_s^T E^r(t)\int_\Omega c_\varepsilon(x)|u_t|^{m(x)}\,dx\,dt,
\end{aligned} \tag{4.13}$$

where c_1', c_2' and c' are positive constants independent of ε.

The next term of (4.6) can be estimated in a similar way to obtain

$$\begin{aligned}
&\big| -\mu_2 \int_s^T E^r(t) \int_\Omega u|z(x,1,t)|^{m(x)-1}\,dx\,dt\big| \\
&\le \varepsilon \int_s^T E^r(t)\big[c_1(\varrho_{p(x)}(\nabla u))^{\frac{m^-}{p^+}} + c_2(\varrho_{p(x)}(\nabla u))^{\frac{m^+}{p^+}}\big]dt \\
&\quad + c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)|z(x,1,t)|^{m(x)}\,dx\,dt \\
&\le c'\varepsilon\Big((E(0))^{\frac{m^-}{p^+}-1} + (E(0))^{\frac{m^+}{p^+}-1}\Big)\int_s^T E^{r+1}(t)dt \\
&\quad + c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)|z(x,1,t)|^{m(x)}\,dx\,dt.
\end{aligned} \tag{4.14}$$

For the last term of (4.6), by using Lemma 4.2, we have

$$
\begin{aligned}
&2\int_s^T E^r(t)\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho dt\\
&\le \frac{2}{m^-}\int_s^T E^r(t)\int_0^1\int_\Omega \xi(x)|z(x,\rho,t)|^{m(x)}\,dx\,d\rho dt\\
&\le -\frac{2\tau}{m^-}\int_s^T E^r(t)\frac{d}{dt}\Big(\int_0^1\int_\Omega e^{-\rho\tau}\xi(x)|z|^{m(x)}\,dx\,d\rho\Big)dt\\
&\quad+\frac{2}{m^-}\int_s^T E^r(t)\int_\Omega \xi(x)|u_t|^{m(x)}\,dx\,dt\\
&\le -\frac{2\tau}{m^-}[E^r(t)\int_0^1\int_\Omega e^{-\rho\tau}\xi(x)|z|^{m(x)}\,dx\,d\rho]_{t=s}^{t=T}\\
&\quad+\frac{2}{m^-}\int_s^T E^r(t)\int_\Omega \xi(x)|u_t|^{m(x)}\,dx\,dt.
\end{aligned}
$$

As $\xi(x)$ is bounded, by using (4.2), we obtain

$$
\begin{aligned}
&2\int_s^T E^r(t)\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\,dx\,d\rho dt\\
&\le \frac{2\tau e^{-\tau}}{m^-}E^r(s)E(s)+\frac{2c}{m^-}E^{r+1}(T)\\
&\le \frac{2\tau e^{-\tau}}{m^-}E^r(0)E(s)+\frac{2c}{m^-}E^r(T)E(s)\le c^*E(s),
\end{aligned}
\tag{4.15}
$$

for some $c^*>0$.

By combining (4.6)-(4.15), we have

$$
\begin{aligned}
\int_s^T E^{r+1}(t)dt &\le c\varepsilon\Big(1+(E(0))^{\frac{m^-}{p^+}-1}+(E(0))^{\frac{m^+}{p^+}-1}\Big)\int_s^T E^{r+1}(t)dt\\
&\quad+cE(s)+c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)|z(x,1,t)|^{m(x)}\,dx\,dt.
\end{aligned}
\tag{4.16}
$$

We choose $\varepsilon>0$ small such that

$$
c\varepsilon\Big(1+(E(0))^{\frac{m^-}{p^+}-1}+(E(0))^{\frac{m^+}{p^+}-1}\Big)<1.
$$

Then, we have

$$
\int_s^T E^{r+1}(t)dt\le cE(s)+c\int_s^T E^r(t)\int_\Omega c_\varepsilon(x)|z(x,1,t)|^{m(x)}\,dx\,dt.
$$

Once ε is fixed, $c_\varepsilon(x)\le M$, since $m(x)$ is bounded. So, we obtain

$$
\begin{aligned}
\int_s^T E^{r+1}(t)dt &\le cE(s)+cM\int_s^T E^r(t)\int_\Omega |z(x,1,t)|^{m(x)}\,dx\,dt\\
&\le cE(s)-C_0M\int_s^T E^r(t)E'(t)dt\\
&\le cE(s)+\frac{C_0M}{r+1}(E^{r+1}(s)-E^{r+1}(T))\\
&\le cE(s).
\end{aligned}
\tag{4.17}
$$

Thus, by taking $T \to \infty$, we obtain

$$\int_s^\infty E^{\frac{m^+}{2}}(t)dt \le cE(s).$$

Therefore, Komornik's Lemma (with $\sigma = r = m^+/2 - 1$) provides the desired result. $\square$

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