Electronic Journal of Differential Equations, Vol. 2021 (2021), No. 86, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

SINGULAR MONGE-AMPÈRE EQUATIONS OVER CONVEX DOMAINS

MENGNI LI

Abstract. In this article we are interested in the Dirichlet problem for a class of singular Monge-Ampère equations over convex domains being either bounded or unbounded. By constructing a family of sub-solutions, we prove the existence and global Hölder estimates of convex solutions to the problem over convex domains. The global regularity provided essentially depends on the convexity of the domain.

1. Introduction

Let us consider the Dirichlet problem of Monge-Ampère equation

$$\begin{aligned} \det D^2 u &= |u|^{-\alpha} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\alpha > 0$ is a constant, $\Omega \subseteq \mathbb{R}^n (n \geqslant 2)$ is a convex domain and $u : \Omega \to \mathbb{R}$ is a convex function. We note that the problem (1.1) is invariant under translation and rotation transformations. In this paper, our main purpose is to settle the issue of the existence and global regularity of the solution u to problem (1.1) over convex domains, including both bounded convex domains and unbounded convex domains.

This type of equations is one of the most important fully nonlinear partial differential equations and plays a fundamental role in a profusion of geometric applications. It is known that the equation in (1.1) arises from the L_p-Minkowski problem when we denote $\alpha = 1 - p$. As a generalization of the classical Minkowski problem [21], the L_p-Minkowski problem was first proposed to explore convex bodies in $\mathbb{R}^{n+1}$ with given p-area measures by Lutwak [20], and was furthermore associated with self-similar solutions to Gauss curvature flows by Andrews [1] and Urbas [23]. In particular, the L_p-Minkowski problem with $p = -n - 1$ (namely $\alpha = n + 2$), corresponding to the critical exponent case, is interpreted as the centroaffine Minkowski problem [6, 9]. To elaborate a little bit on a solution u to problem (1.1) with $\alpha = n + 2$, $(-1/u) \sum u_{x_i x_j} dx_i dx_j$ gives the Hilbert metric in convex domain [19], and the Legendre transform of u defines a complete hyperbolic affine sphere [4, 5]. The readers may consult [3, 7, 10, 15, 22] and the references therein for more related topics.

2010 *Mathematics Subject Classification.* 35J96, 52A20, 35B65.
Key words and phrases. Dirichlet problem; Hölder estimate; bounded convex domain; unbounded convex domain.

Submitted November 26, 2020. Published October 18, 2021.

In the past four decades, a great deal of mathematical effort has been devoted to developing the global regularity theory of the problem (1.1); see [2, 4, 8, 12, 13, 14, 16, 17, 18, 22] for example. A pivotal observation has been that the equation in (1.1) becomes singular on the boundary $\partial\Omega$ by virtue of $u=0$ there. This singularity will inevitably lead to the phenomena that the gradient Du may blow up at the boundary and hence the optimal global regularity of the solution u should be Hölder continuous. Specifically, based on the (a,η) type domain introduced by Jian and Li [12], the corresponding Hölder exponent for a class of Monge-Ampère type equations can be independent of the smoothness of domain but only essentially depends on the convexity of domain [12, 13, 18]. However, only the bounded domains were addressed, except for [11] where the existence of solution to (1.1) with $\alpha=n+2$ was obtainable on a class of unbounded domains. The main motivation of this paper is to extend such result to (1.1) with more general α, and moreover prove the existence and global regularity results on convex domains being either bounded or unbounded.

Before stating the main results, we first review the concept of (a,η) type domain in [12] to describe the convexity of domain. Roughly speaking, the less is the parameter a, the more convex is the domain. We also refer the readers to [18] for a careful understanding of the geometry at one point as the parameter a varies.

Definition 1.1. Suppose that Ω is a bounded convex domain in $\mathbb{R}^n$ and $x_0\in\partial\Omega$. We say x_0 is (a,η) type if there exist numbers $a\in[1,+\infty]$ and $\eta\geqslant 0$ such that after translation and rotation transforms, we have

$$x_0=0 \quad\text{and}\quad \Omega\subseteq\{x=(x',x_n)\in\mathbb{R}^n : x_n\geqslant\eta|x'|^a\}.$$

The domain Ω is called (a,η) type domain if its every boundary point is (a,η) type.

We note that for the case $a\in[1,2)$, there exists no (a,η) type domain though some boundary points might be (a,η) type. Thus we need only consider the (a,η) type domain with $a\in[2,+\infty]$ from now on.

Based on the existence of the solution to (1.1) over bounded convex domains [4, 13], we can derive the following global regularity result for (1.1), which can be regarded as a direct consequence of setting $F(x,u,\nabla u)=|u|^{-\alpha}$ with $\alpha>0$ in [18]. We point out that we merely pay attention to the singular case $\alpha>0$ in this paper, despite the fact that the assumption $\alpha>0$ can be trivially relaxed to $\alpha\geqslant 0$ in what follows.

Theorem 1.2. *Suppose $\Omega\subset\mathbb{R}^n$ is an (a,η) type domain with $a\in[2,+\infty]$. If u is a convex generalized solution to the problem* (1.1)*, then $u\in C^{\frac{2(a+n-1)}{a(n+\alpha)}}(\overline{\Omega})$ and*

$$|u|_{C^{\frac{2(a+n-1)}{a(n+\alpha)}}(\overline{\Omega})}\leqslant C(a,\eta,\alpha,n,\operatorname{diam}(\Omega)).$$

As our first goal is to show that this global regularity result for problem (1.1) over bounded convex domains reflects the relation of the Hölder exponent with the convexity of the domain. In particular, when we take $a=2$, Ω corresponds to a bounded convex domain satisfying exterior sphere condition (see [12, Definition 2.1 and Lemma 2.1] for details), and moreover, when we take $a=+\infty$, Ω represents a general bounded convex domain (see [18, Remark 2.3] for details). We would like to individually present this global regularity result for the two extreme cases $a=2$ and $a=+\infty$ as the following two corollaries in light of their geometric significance.

Corollary 1.3. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded convex domain satisfying exterior sphere condition. If u is a convex generalized solution to the problem* (1.1)*, then $u \in C^{\frac{n+1}{n+\alpha}}(\overline{\Omega})$ and*

$$|u|_{C^{\frac{n+1}{n+\alpha}}(\overline{\Omega})} \leqslant C(a, \eta, \alpha, n, \operatorname{diam}(\Omega)).$$

Corollary 1.4. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded convex domain. If u is a convex generalized solution to the problem* (1.1)*, then $u \in C^{\frac{2}{n+\alpha}}(\overline{\Omega})$ and*

$$|u|_{C^{\frac{2}{n+\alpha}}(\overline{\Omega})} \leqslant C(\alpha, n, \operatorname{diam}(\Omega)).$$

We remark here that $\frac{2(a+n-1)}{a(n+\alpha)} \in [\frac{2}{n+\alpha}, \frac{n+1}{n+\alpha}]$ for any $a \in [2, +\infty]$, and it equals $\frac{n+1}{n+\alpha}$ when $a = 2$ and equals $\frac{2}{n+\alpha}$ when $a = +\infty$. Similar to [18], the proof of Theorem 1.2 relies on carefully constructing sub-solutions and can be divided into two parts $2 \leqslant a < +\infty$ and $a = +\infty$. In addition, the proof of Corollary 1.4, i.e., the case $a = +\infty$, will provide great convenience for the subsequent study on the unbounded domains.

The second goal of this paper concerns the existence and global regularity of the solution to (1.1) over unbounded convex domains. On the one hand, we are inspired by the corresponding result over bounded convex domains [4, 13], for which we refer the readers to Theorem 5.1 in Section 5 with its proof based on Corollary 1.4. On the other hand, it is not surprising to generalize the existence result in [11] to the problem (1.1) over unbounded domains. The key ingredient still lies in a delicate construction of sub-solutions. Precisely, our second main result is stated as follows.

Theorem 1.5. *Suppose $\Omega \subset \mathbb{R}^n$ is an unbounded convex domain such that $\partial\Omega$ is strictly convex at some point $x_0 \in \partial\Omega$. Then problem* (1.1) *admits a convex solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$. Moreover, for any $r > 0$, $u \in C^{\frac{2}{n+\alpha}}(\overline{\Omega \cap B_r(0)})$ and*

$$|u|_{C^{\frac{2}{n+\alpha}}(\overline{\Omega \cap B_r(0)})} \leqslant C(\alpha, n, \operatorname{diam}(\Omega \cap B_r(0))).$$

Here $B_r(0)$ denotes the ball in $\mathbb{R}^n$ centered at the origin with radius r.

An outline of this paper is as follows. In Section 2, we revisit necessary results on the convexity and establish the general setup for choosing sub-solutions. To better understand the geometry of (a, η) domain, we deal with two particular cases $a = 2$ and $a = +\infty$ of Theorem 1.2 in Section 3 and Section 5 respectively. Moreover, Section 4 and Section 5 provide a complete proof for Theorem 1.2. Finally, Section 6 is devoted to the construction of sub-solutions and the existence of solutions to (1.1) on unbounded convex domains, which completes the proof of Theorem 1.5.

2. Preliminaries

2.1. **Useful observations on convexity.** Firstly, we give a brief review on convex bodies in $\mathbb{R}^n$ as follows.

Remark 2.1. *The definitions of convex bodies and strictly convex bodies in $\mathbb{R}^n$ are well known:*

(i) *A subset $\Omega \subset \mathbb{R}^n$ is said to be* convex *if for every two points $x, y \in \Omega$, the line segment joining x to y is contained in Ω, that is, for any $t \in [0, 1]$, we have $tx + (1 - t)y \in \Omega$.*

(ii) *A subset* $\Omega \subset \mathbb{R}^n$ *is said to be* strictly convex *if for every two points* $x, y \in \Omega$, *the line segment joining* x *to* y *is strictly contained in* Ω, *that is, for any* $t \in (0,1)$, *we have* $tx + (1-t)y \in \Omega^\circ$, *where* Ω° *denotes the interior of* Ω.

From the geometric intuition, the curvature at every boundary point of smooth convex domains is nonnegative, while the curvature at every boundary point of smooth strictly convex domains is positive. For instance, balls are not only convex but also strictly convex, while cubes are merely convex rather than strictly convex. In addition, the concept of (a,η) *type as in Definition 1.1 gives a more precise description of the convexity of domains, where we refer the readers to* [18, Remarks 2.2 and 2.3] *for more details.*

Later on, we present the following fact based on rotation and translation transforms of convex domains.

Lemma 2.2. *Given a general convex domain* Ω, *an invertible n-order matrix* $A \in M_n$ *and a point* $x_0 \in \mathbb{R}^n$, *we denote*

$$\widetilde{\Omega} := A\Omega + x_0 = \{y : \text{ there exists } x \in \Omega \text{ such that } y = Ax + x_0\}.$$

If $u(x)$ *is a convex solution to* (1.1) *on* Ω, *then*

$$\widetilde{u}(Ax + x_0) := |\det A|^{\frac{2}{n+\alpha}} u(x)$$

is a convex solution to (1.1) *on* $\widetilde{\Omega}$.

Proof. Let $\widetilde{x} = Ax + x_0$. Then there holds

$$\det D_x^2 u = (\det A)^2 \det D_{\widetilde{x}}^2 u = |\det A|^2 \det D_{\widetilde{x}}^2 u.$$

According to the equation in (1.1), i.e., $|u|^\alpha \det D_x^2 u = 1$, we infer

$$|u|^\alpha |\det A|^2 \det D_{\widetilde{x}}^2 u = 1.$$

Let $\widetilde{u} = |\det A|^{\frac{2}{n+\alpha}} u$. Then we have

$$|u|^\alpha = |\det A|^{-\frac{2\alpha}{n+\alpha}} |\widetilde{u}|^\alpha$$

and

$$\det D_{\widetilde{x}}^2 u = \det D_{\widetilde{x}}^2 \big(|\det A|^{-\frac{2}{n+\alpha}} \widetilde{u}\big) = |\det A|^{-\frac{2n}{n+\alpha}} \det D_{\widetilde{x}}^2 \widetilde{u}.$$

Combining the previous three formulas, we derive

$$|\widetilde{u}|^\alpha \det D_{\widetilde{x}}^2 \widetilde{u} = 1.$$

The proof of the lemma is now complete. □

We turn to review an interesting lemma concerning the boundary Hölder regularity of convex functions over convex domains, for which we refer the readers to [12, Lemma 2.3] for the proof.

Lemma 2.3. *Let* Ω *be a bounded convex domain and* $u \in C(\overline{\Omega})$ *be a convex function in* Ω *with* $u|_{\partial\Omega} = 0$. *If there exist* $\lambda \in (0,1]$ *and* $M > 0$ *such that*

$$|u(x)| \leqslant M d_x^\lambda, \quad \forall x \in \Omega,$$

where $d_x = \operatorname{dist}(x, \partial\Omega)$, *then* $u \in C^\lambda(\overline{\Omega})$ *and*

$$|u|_{C^\lambda(\overline{\Omega})} \leqslant M\big((\operatorname{diam}(\Omega))^\lambda + 1\big).$$

2.2. **Equivalent conditions of sub-solution.** We present the comparison principle and define sub-solutions of (1.1). For simplicity of expression, we denote

$$H[W] := \det D^2W \cdot |W|^\alpha.$$

We have an application of the comparison principle for fully nonlinear equations, i.e. [8, Theorem 17.1].

Theorem 2.4 (comparison principle)**.** *Let $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $F[u] \geqslant F[v]$ in Ω and $u \leqslant v$ on $\partial\Omega$. It then follows that $u \leqslant v$ in Ω.*

Definition 2.5. A non-positive function W is called a sub-solution of (1.1) if

$$\det D^2W \geqslant |W|^{-\alpha} \quad \text{in } \Omega,$$

i.e. $H[W] \geqslant 1$ in Ω.

For convenience of constructing sub-solutions in the next sections, we give two equivalent conditions for which W is a sub-solution to the problem (1.1). In fact, the only difference between these two equivalent conditions lies in what the variable r represents.

Lemma 2.6. *Consider $W(x) = W(r)$ and write for $i, j \in \{1, 2, \dots, n\}$,*

$$W_r = \frac{\partial W}{\partial r}, \quad W_i = \frac{\partial W}{\partial x_i}, \quad W_{ij} = \frac{\partial^2 W}{\partial x_i \partial x_j}.$$

Then a non-positive function W is a sub-solution to the problem (1.1) *if and only if*

$$H[W] = \Big(\frac{W_r}{r}\Big)^{n-1} W_{rr}|W|^\alpha \geqslant 1 \quad \textit{in } \Omega.$$

Proof. For $i, j \in \{1, 2, \dots, n\}$, by direct computation, we have

$$W_i = W_r \frac{x_i}{r},$$
$$W_{ij} = \frac{W_r}{r}\delta_{ij} + \Big(W_{rr} - \frac{W_r}{r}\Big)\frac{x_i}{r}\frac{x_j}{r}.$$

Consequently,

$$\det D^2W = \frac{W_r}{r} I + \Big(W_{rr} - \frac{W_r}{r}\Big)\theta\theta^T,$$

where I is the unit matrix and $\theta^T = \big(\frac{x_1}{r}, \dots, \frac{x_n}{r}\big)$. We notice that all the n eigenvalues of matrix $\theta\theta^T$ are $1, 0, \dots, 0$ and thus all eigenvalues of matrix D^2W are $W_{rr}, \frac{W_r}{r}, \dots, \frac{W_r}{r}$. Then we have the following explicit formula for $\det D^2W$:

$$\det D^2W = \Big(\frac{W_r}{r}\Big)^{n-1} W_{rr}.$$

As a result, we obtain

$$H[W] = \det D^2W \cdot |W|^\alpha = \Big(\frac{W_r}{r}\Big)^{n-1} W_{rr}|W|^\alpha.$$

With $W \leqslant 0$ on $\partial\Omega$, the lemma follows immediately. □

Lemma 2.7. *Consider $W(x) = W(r, x_n)$ and write for $i, j \in \{1, 2, \dots, n\}$,*

$$W_r = \frac{\partial W}{\partial r}, \quad W_i = \frac{\partial W}{\partial x_i}, \quad W_{ij} = \frac{\partial^2 W}{\partial x_i \partial x_j}.$$

Then a non-positive function W is a sub-solution to the problem (1.1) *if and only if*

$$H[W] = \Big(\frac{W_r}{r}\Big)^{n-2}\big(W_{rr}W_{nn} - |W_{rn}|^2\big)|W|^\alpha \geqslant 1 \quad \text{in } \Omega.$$

Proof. Let

$$D^2W := \begin{pmatrix} A & \alpha \\ \alpha^T & W_{nn} \end{pmatrix},$$

where $\alpha^T = (W_{n1}, \dots, W_{n(n-1)})$ and A is the $(n-1)$-order matrix. We infer that

$$\det D^2W = \det A(W_{nn} - \alpha^T A^{-1}\alpha).$$

For $k, l \in \{1, 2, \dots, n-1\}$, a direct computation gives

$$W_k = W_r \frac{x_k}{r},$$
$$W_{kl} = \frac{W_r}{r}\delta_{kl} + \Big(W_{rr} - \frac{W_r}{r}\Big)\frac{x_k}{r}\frac{x_l}{r},$$
$$W_{kn} = W_{rn}\frac{x_k}{r}.$$

Thus we obtain

$$A = \frac{W_r}{r} I + \Big(W_{rr} - \frac{W_r}{r}\Big)\theta\theta^T,$$

where I is the unit matrix and $\theta^T = \big(\frac{x_1}{r}, \dots, \frac{x_{n-1}}{r}\big)$. We note that all the $n-1$ eigenvalues of matrix $\theta\theta^T$ are $1, 0, \dots, 0$ and hence all eigenvalues of matrix A are $W_{rr}, \frac{W_r}{r}, \dots, \frac{W_r}{r}$. In particular, α is an eigenvector of A with respect to the eigenvalue W_{rr}. Then we obtain

$$\det A = \Big(\frac{W_r}{r}\Big)^{n-2} W_{rr},$$
$$\alpha^T A^{-1}\alpha = \alpha^T \frac{1}{W_{rr}}\alpha = \frac{|W_{rn}|^2}{W_{rr}}.$$

This implies the following explicit formula for $\det D^2W$:

$$\det D^2W = \Big(\frac{W_r}{r}\Big)^{n-2} W_{rr}\Big(W_{nn} - \frac{|W_{rn}|^2}{W_{rr}}\Big) = \Big(\frac{W_r}{r}\Big)^{n-2}\big(W_{rr}W_{nn} - |W_{rn}|^2\big).$$

Therefore,

$$H[W] = \det D^2W \cdot |W|^\alpha = \Big(\frac{W_r}{r}\Big)^{n-2}\big(W_{rr}W_{nn} - |W_{rn}|^2\big)|W|^\alpha.$$

Combining this with $W \leqslant 0$ on $\partial\Omega$, we have proved the lemma. $\square$

3. Bounded convex domains satisfying exterior sphere condition

In this section, we focus on a bounded convex domain Ω satisfying exterior sphere condition, or in other words, Ω is a $(2, \eta)$ type domain. We construct a sub-solution to the problem (1.1) which exploits the strength of exterior sphere condition, and then give an alternative proof of Corollary 1.3 for the case $\alpha > 1$. In fact, a complete proof of Corollary 1.3 for all $\alpha > 0$ is given in the following Section 4 since $a = 2$ is a particular case of $2 \leqslant a < +\infty$.

We begin with several simplifications of the problem:

(i) By Lemma 2.3, it suffices to show that

$$|u(y)| \leqslant C(\eta, \alpha, n, \operatorname{diam}(\Omega)) d_y^{\frac{n+1}{n+\alpha}}, \quad \forall y \in \Omega. \tag{3.1}$$

(ii) For any point $y \in \Omega$, we can find $z \in \partial\Omega$ be the nearest boundary point to y. Without loss of generality, we assume that the domain Ω satisfies the exterior sphere condition with radius R. By some translations and rotations, we can further assume $z = 0$, $0 \in \partial\Omega \cap \partial B_R(y_0)$, $\Omega \subset B_R(y_0)$, and the line yz is the x_n-axis.

We notice that the tangent plane of Ω at $z = 0$ is unique since $z = 0$ is the nearest boundary point to y. Moreover, y is on the line determined by 0 and y_0 (with the order 0, y, y_0), and hence

$$d_y = \operatorname{dist}(y, \partial\Omega) = |y - 0| = |y_0 - 0| - |y_0 - y| = R - |y_0 - y|.$$

We are now ready to construct a sub-solution to (1.1) by using Lemma 2.6. Let

$$U(x) = -K(R^2 - |y_0 - x|^2)^{\frac{n+1}{n+\alpha}} = -K(R^2 - r^2)^{\frac{n+1}{n+\alpha}},$$

where $r = |y_0 - x|$ and K is a positive constant to be determined such that U is a sub-solution to (1.1) in Ω. It is trivial to see that $U \leqslant 0$ on $\overline{\Omega}$, thus

$$U \leqslant u \quad \text{on } \partial\Omega. \tag{3.2}$$

A routine computation leads us to

$$\begin{gathered}
U_r = 2K\frac{n+1}{n+\alpha}(R^2 - r^2)^{\frac{n+1}{n+\alpha}-1} r, \\
U_{rr} = 2K\frac{n+1}{n+\alpha}(R^2 - r^2)^{\frac{n+1}{n+\alpha}-2}\big(R^2 + \frac{\alpha - n - 2}{n+\alpha} r^2\big),
\end{gathered}$$

which gives

$$H[U] = \Big(\frac{U_r}{r}\Big)^{n-1} U_{rr}|U|^\alpha = 2^n K^{n+\alpha}\frac{(n+1)^n}{(n+\alpha)^n}\big(R^2 + \frac{\alpha - n - 2}{n+\alpha} r^2\big).$$

We split $\alpha > 1$ into the following two cases:

(i) When $\alpha \geqslant n + 2$, we derive that

$$H[U] \geqslant 2^n K^{n+\alpha}\frac{(n+1)^n}{(n+\alpha)^n}(R^2 + 0r^2) = 2^n K^{n+\alpha}\frac{(n+1)^n}{(n+\alpha)^n} R^2.$$

Then we can take M sufficiently large such that $H[U] \geqslant 1$.

(ii) When $1 < \alpha < n + 2$, we infer that

$$H[U] \geqslant 2^n K^{n+\alpha}\frac{(n+1)^n}{(n+\alpha)^n}\big(R^2 + \frac{\alpha - n - 2}{n+\alpha} R^2\big) = 2^{n+1} K^{n+\alpha}\frac{(\alpha-1)(n+1)^n}{(n+\alpha)^{n+1}} R^2.$$

Then we can take M sufficiently large such that $H[U] \geqslant 1$.

To sum up, for any $\alpha > 1$, we always have

$$H[U] \geqslant 1. \tag{3.3}$$

According to (3.2) and (3.3), we obtain from Lemma 2.6 that U is a sub-solution to the problem (1.1).

By Theorem 2.4 (comparison principle), we obtain

$$0 \geqslant u(y) \geqslant U(y).$$

Taking this inequality on y_n-axis, we arrive at the conclusion that

$$\begin{aligned}
|u(y)| &\leqslant |U(y)| \\
&= K(R^2 - |y_0 - y|^2)^{\frac{n+1}{n+\alpha}}
\end{aligned}$$

$$
\begin{aligned}
&= K(R+|y_0-y|)^{\frac{n+1}{n+\alpha}}(R-|y_0-y|)^{\frac{n+1}{n+\alpha}}\\
&\leqslant K(2R)^{\frac{n+1}{n+\alpha}} d_y^{\frac{n+1}{n+\alpha}},
\end{aligned}
$$

which implies (3.1). Thus we have proved Corollary 1.3 for the case $\alpha > 1$.

4. (a,η) TYPE DOMAINS WITH $2 \leqslant a < +\infty$

In this section, we prove Theorem 1.2 for the case $2 \leqslant a < +\infty$, which can imply Corollary 1.3 immediately. We present here only the main procedure of the proof and refer the readers to [18] for a rigorous derivation since this section can be regarded as its special case $F(x,u,\nabla u) = |u|^{-\alpha}$ with $\alpha > 0$.

Firstly, we will adopt the following simplifications:

(i) Thanks to Lemma 2.3, it suffices to show that

$$
|u(y)| \leqslant C(a,\eta,\alpha,n,\mathrm{diam}(\Omega)) d_y^{\frac{2(a+n-1)}{a(n+\alpha)}}, \quad \forall y \in \Omega. \tag{4.1}
$$

(ii) For any point $y \in \Omega$, there exists $z \in \partial\Omega$ such that $\mathrm{dist}(y,z) = d_y$. Since the domain Ω is (a,η) type, without loss of generality, we can assume $z = 0$ and take the line determined by y and z as the x_n-axis such that

$$
\Omega \subseteq \{x \in \mathbb{R}^n : x_n \geqslant \eta |x'|^a\}.
$$

We construct a sub-solution to the problem (1.1) by using Lemma 2.7. From now on, we let

$$
U(r,x_n) = -\Big(\big(\frac{x_n}{\varepsilon}\big)^{\frac{2}{a}} - r^2\Big)^{1/b},
$$

where $b = \frac{n+\alpha}{a+n-1}$ and ε is positive constant to be determined such that U is a sub-solution to (1.1) in Ω. It is obvious that $U \leqslant 0$ on $\overline{\Omega}$ and hence

$$
U \leqslant u \quad \text{on } \partial\Omega. \tag{4.2}
$$

By straightforward calculation, we obtain

$$
\begin{gathered}
U_r = \frac{2}{b}|U|^{1-b} r,\\
U_n = -\frac{2}{ab}|U|^{1-b}\big(\frac{x_n}{\varepsilon}\big)^{\frac{2}{a}-1}\frac{1}{\varepsilon},\\
U_{rr} = \frac{2}{b}|U|^{1-b} - \frac{4(1-b)}{b^2}|U|^{1-2b} r^2,\\
U_{nn} = -\frac{2(2-a)}{a^2 b}|U|^{1-b}\big(\frac{x_n}{\varepsilon}\big)^{\frac{2}{a}-2}\frac{1}{\varepsilon^2} - \frac{4(1-b)}{a^2 b^2}|U|^{1-2b}\big(\frac{x_n}{\varepsilon}\big)^{\frac{4}{a}-2}\frac{1}{\varepsilon^2},\\
U_{rn} = \frac{4(1-b)}{ab^2}|U|^{1-2b} r\big(\frac{x_n}{\varepsilon}\big)^{\frac{2}{a}-1}\frac{1}{\varepsilon},
\end{gathered}
$$

which yields

$$
\begin{aligned}
U_{rr}U_{nn} - |U_{rn}|^2 = &\underbrace{\frac{8(a-2)(b-1)}{a^2 b^3}|U|^{2-3b}\big(\frac{x_n}{\varepsilon}\big)^{\frac{2}{a}-2} r^2 \frac{1}{\varepsilon^2}}_{I_1}\\
&+ \underbrace{\frac{8(b-1)}{a^2 b^3}|U|^{2-3b}\big(\frac{x_n}{\varepsilon}\big)^{\frac{4}{a}-2}\frac{1}{\varepsilon^2}}_{I_2}
\end{aligned}
$$

$$+\underbrace{\frac{4(a-2)}{a^2b^2}|U|^{2-2b}\big(\frac{x_n}{\varepsilon}\big)^{\frac{2}{a}-2}\frac{1}{\varepsilon^2}}_{I_3}.$$

To estimate $I_1+I_2+I_3$, we will choose $\delta\in(0,1)$ and $\varepsilon=\varepsilon(\delta,a,\eta)>0$ such that $\varepsilon\big(\frac{1}{\delta}\big)^{\frac{a}{2}}\leqslant\eta$. As a consequence, we obtain

$$\Omega\subseteq\{x\in\mathbb{R}^n|x_n\geqslant\eta|x'|^a\}\subseteq\{x\in\mathbb{R}^n|\delta\big(\frac{x_n}{\varepsilon}\big)^{\frac{2}{a}}\geqslant r^2\},$$

and thus

$$|U|^b=\big(\frac{x_n}{\varepsilon}\big)^{\frac{2}{a}}-r^2\in\Big[(1-\delta)\big(\frac{x_n}{\varepsilon}\big)^{\frac{2}{a}},\big(\frac{x_n}{\varepsilon}\big)^{\frac{2}{a}}\Big].\tag{4.3}$$

Step 1: Consider the case $\alpha+1>2$, i.e. $\alpha>1$. We distinguish two cases: $2\leqslant a<\alpha+1$ and $\alpha+1\leqslant a<+\infty$.

Case 1: When $2\leqslant a<\alpha+1$, we have $b=\frac{n+\alpha}{a+n-1}>1$. In such a case, we obtain I_1, I_2, $I_3\geqslant 0$ and hence

$$\begin{aligned}U_{rr}\cdot U_{nn}-|U_{rn}|^2\geqslant I_2&=\frac{8(b-1)}{a^2b^3}|U|^{2-3b}\big(\frac{x_n}{\varepsilon}\big)^{\frac{4}{a}-2}\frac{1}{\varepsilon^2}\\&\overset{(4.3)}{\geqslant}\frac{8(b-1)}{a^2b^3}|U|^{2-3b}\big((1-\delta)^{-\frac{a}{2}}|U|^{\frac{ab}{2}}\big)^{\frac{4}{a}-2}\frac{1}{\varepsilon^2}\\&=\frac{8(b-1)}{a^2b^3}(1-\delta)^{a-2}|U|^{2-b-ab}\frac{1}{\varepsilon^2}.\end{aligned}$$

This and the formula of $H[\cdot]$ in Lemma 2.7 imply

$$\begin{aligned}H[U]&=\Big(\frac{U_r}{r}\Big)^{n-2}\big(U_{rr}U_{nn}-|U_{rn}|^2\big)|U|^\alpha\\&\geqslant\big(\frac{2}{b}|U|^{1-b}\big)^{n-2}\frac{8(b-1)}{a^2b^3}(1-\delta)^{a-2}|U|^{2-b-ab}\frac{1}{\varepsilon^2}|U|^\alpha\\&=\big(\frac{2}{b}\big)^{n-2}\frac{8(b-1)}{a^2b^3}(1-\delta)^{a-2}\frac{1}{\varepsilon^2}.\end{aligned}$$

Since $b>1$, we can take $\varepsilon=C(a,\alpha,n,\delta)>0$ sufficiently small such that

$$H[U]\geqslant 1\ \text{ in }\Omega.\tag{4.4}$$

By (4.2), (4.4) and Lemma 2.7, we conclude that U is a sub-solution to the problem (1.1).

By Theorem 2.4 (comparison principle), we obtain

$$0\geqslant u(y)\geqslant U(y).$$

Restricting this inequality onto y_n-axis, we have

$$\begin{aligned}|u(y)|\leqslant|U(y)|&\leqslant\big(\frac{y_n}{\varepsilon(a,\alpha,n,\delta)}\big)^{\frac{2}{ab}}\\&=C(a,\eta,\alpha,n,\mathrm{diam}(\Omega))y_n^{\frac{2}{ab}}\\&=C(a,\eta,\alpha,n,\mathrm{diam}(\Omega))d_y^{\frac{2(a+n-1)}{a(n+\alpha)}},\end{aligned}$$

which implies (4.1).

Case 2: When $\alpha+1 \leqslant a < +\infty$, we have $b = \frac{n+\alpha}{a+n-1} \in (0,1]$. Since $a \geqslant \alpha+1 > 2$, then $I_1 \leqslant 0$, $I_2 \leqslant 0$, $I_3 \geqslant 0$. On the one hand, we observe that

$$I_1 = (a-2)r^2\big(\frac{x_n}{\varepsilon}\big)^{-\frac{2}{a}} I_2 \overset{(4.3)}{\geqslant} \delta(a-2)I_2.$$

In view of (4.3), we obtain $x_n \geqslant \varepsilon |U|^{\frac{ab}{2}}$ and then

$$\big(\frac{x_n}{\varepsilon}\big)^{\frac{4}{a}-2} \leqslant \big(|U|^{\frac{ab}{2}}\big)^{\frac{4}{a}-2} = |U|^{2b-ab},$$

which gives rise to

$$\begin{aligned}
I_1 + I_2 &\geqslant \big(\delta(a-2)+1\big) I_2 \\
&= \big(\delta(a-2)+1\big)\frac{8(b-1)}{a^2b^3}|U|^{2-3b}\big(\frac{x_n}{\varepsilon}\big)^{\frac{4}{a}-2}\frac{1}{\varepsilon^2} \\
&\geqslant \big(\delta(a-2)+1\big)\frac{8(b-1)}{a^2b^3}|U|^{2-3b}|U|^{2b-ab}\frac{1}{\varepsilon^2} \\
&= \big(\delta(a-2)+1\big)\frac{8(b-1)}{a^2b^3}|U|^{2-b-ab}\frac{1}{\varepsilon^2}.
\end{aligned}$$

On the other hand, since (4.3) also yields

$$x_n \leqslant \varepsilon(1-\delta)^{-\frac{a}{2}}|U|^{\frac{ab}{2}}$$

and moreover

$$\big(\frac{x_n}{\varepsilon}\big)^{\frac{2}{a}-2} \geqslant \big((1-\delta)^{-\frac{a}{2}}|U|^{\frac{ab}{2}}\big)^{\frac{2}{a}-2} = (1-\delta)^{a-1}|U|^{b-ab},$$

we can infer that

$$\begin{aligned}
I_3 &= \frac{4(a-2)}{a^2b^2}|U|^{2-2b}\big(\frac{x_n}{\varepsilon}\big)^{\frac{2}{a}-2}\frac{1}{\varepsilon^2} \\
&\geqslant \frac{4(a-2)}{a^2b^2}|U|^{2-2b}(1-\delta)^{a-1}|U|^{b-ab}\frac{1}{\varepsilon^2} \\
&= \frac{4(a-2)}{a^2b^2}(1-\delta)^{a-1}|U|^{2-b-ab}\frac{1}{\varepsilon^2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&U_{rr}\cdot U_{nn} - |U_{rn}|^2 \\
&\geqslant \delta(a-2)\cdot I_2 + I_2 + I_3 \\
&\geqslant \Big(\big(\delta(a-2)+1\big)\frac{8(b-1)}{a^2b^3} + \frac{4(a-2)}{a^2b^2}(1-\delta)^{a-1}\Big)|U|^{2-b-ab}\frac{1}{\varepsilon^2}.
\end{aligned}$$

We can proceed as in Case 1 and derive that

$$H[U] \geqslant \big(\frac{2}{b}\big)^{n-2}\Big(\big(\delta(a-2)+1\big)\frac{8(b-1)}{a^2b^3} + \frac{4(a-2)}{a^2b^2}(1-\delta)^{a-1}\Big)\frac{1}{\varepsilon^2}.$$

We remark here that we require

$$\big(\delta(a-2)+1\big)\frac{8(b-1)}{a^2b^3} + \frac{4(a-2)}{a^2b^2}(1-\delta)^{a-1} > 0, \tag{4.5}$$

which is equivalent to

$$(a-2)(1-\delta)^{a-1} > (\delta(a-2)+1)\Big(\frac{2}{b}-2\Big). \tag{4.6}$$

In fact, we note that $\alpha > 1$ and $a \geqslant \alpha + 1 > 2$ lead us to

$$\begin{aligned} a - 2 &> \frac{a(n+1)}{n+\alpha} - 2 = \frac{a(n-1)+2a}{n+\alpha} - 2 \\ &> \frac{2(n-1)+2a}{n+\alpha} - 2 = \frac{2(a+n-1)}{n+\alpha} - 2 \\ &= \frac{2}{b} - 2. \end{aligned}$$

Thus, we can take $\delta = C(a, \alpha, n) > 0$ small enough such that (4.6) holds and hence (4.5) holds. As a result, we can take $\varepsilon = C(a, \alpha, n, \delta) > 0$ sufficiently small such that

$$H[U] \geqslant 1 \quad \text{in } \Omega. \tag{4.7}$$

Using (4.2), (4.7) and Lemma 2.7, we derive that U is a sub-solution to problem (1.1).

In view of Theorem 2.4 (comparison principle), we obtain $0 \geqslant u(y) \geqslant U(y)$. Restricting this inequality onto x_n-axis, we have

$$\begin{aligned} |u(y)| \leqslant |U(y)| &\leqslant \big(\frac{y_n}{\varepsilon(a,\alpha,n,\delta)}\big)^{\frac{2}{ab}} \\ &= C(a, \eta, \alpha, n, \operatorname{diam}(\Omega)) y_n^{\frac{2}{ab}} \\ &= C(a, \eta, \alpha, n, \operatorname{diam}(\Omega)) d_y^{\frac{2(a+n-1)}{a(n+\alpha)}}, \end{aligned}$$

which leads us to (4.1).

Step 2: Consider the case $\alpha + 1 \leqslant 2$, i.e. $\alpha \leqslant 1$. In such a case, we always have

$$\alpha + 1 \leqslant 2 \leqslant a < +\infty.$$

We can adopt the same procedure as in Case 2 of Step 1 to obtain (4.1).

Up to now, we have proved Theorem 1.2 for the case $2 \leqslant a < +\infty$. It remains to prove Theorem 1.2 for the case $a = +\infty$.

5. General bounded convex domains

In this section, we consider Ω as a general bounded convex domain, i.e., $(+\infty, \eta)$ type domain. We first prove Corollary 1.4 and hence the $a = +\infty$ limit case of Theorem 1.2. We next apply Corollary 1.4 to provide a proof for the existence result of solutions on bounded convex domains.

5.1. **Global regularity of solution on bounded convex domain.**

First of all, we adopt several simplifications:

(i) According to Lemma 2.3, we only need to show that

$$|u(y)| \leqslant C(\alpha, n, \operatorname{diam}(\Omega)) d_y^{\frac{2}{n+\alpha}}, \quad \forall y \in \Omega.$$

(ii) For any point $y \in \Omega$, letting $z \in \partial\Omega$ be the nearest boundary point to y, by some translations and rotations, we can assume that $z = 0$, $0 \in \overline{\Omega} \subset \mathbb{R}^n_+$, and the line yz is the x_n-axis with y over the plane $z = 0$. We remark here that

$$d_y = \operatorname{dist}(y, \partial\Omega) = |y - 0| = y_n.$$

We now construct a sub-solution to (1.1) based on Lemma 2.7. Denote $l = \operatorname{diam}(\Omega)$ and let

$$V(r, x_n) = -M x_n^{\frac{2}{n+\alpha}} (N^2 l^2 - r^2)^{\frac{1}{2}},$$

where $r = |x'|$ and M, N are positive constants to be determined such that V is a sub-solution to the problem (1.1) in Ω. It is clear that $V \leqslant 0$ on $\overline{\Omega}$ and thus

$$V \leqslant u \quad \text{on} \quad \partial\Omega. \tag{5.1}$$

By straightforward calculation, we obtain

$$\begin{gathered}
V_r = M x_n^{\frac{2}{n+\alpha}} (N^2 l^2 - r^2)^{-\frac{1}{2}} r, \\
V_n = -M \frac{2}{n+\alpha} x_n^{\frac{2}{n+\alpha}-1} (N^2 l^2 - r^2)^{\frac{1}{2}}, \\
V_{rr} = M N^2 l^2 x_n^{\frac{2}{n+\alpha}} (N^2 l^2 - r^2)^{-\frac{3}{2}}, \\
V_{nn} = M \frac{2}{n+\alpha} \Big(1 - \frac{2}{n+\alpha}\Big) x_n^{\frac{2}{n+\alpha}-2} (N^2 l^2 - r^2)^{\frac{1}{2}}, \\
V_{rn} = M \frac{2}{n+\alpha} x_n^{\frac{2}{n+\alpha}-1} (N^2 l^2 - r^2)^{-\frac{1}{2}} r.
\end{gathered}$$

Therefore,

$$\begin{aligned}
H[V] &= \Big(\frac{V_r}{r}\Big)^{n-2} \big(V_{rr} V_{nn} - |V_{rn}|^2\big) |V|^\alpha \\
&= M^{n+\alpha} N^2 l^2 \frac{2}{n+\alpha} \Big(1 - (1 + r^2 N^{-2} l^{-2}) \frac{2}{n+\alpha}\Big) (N^2 l^2 - r^2)^{\frac{\alpha-n}{2}}.
\end{aligned}$$

Observing that $r = |x'| \leqslant \operatorname{diam}(\Omega) = l$ in Ω, we first take $N = C(\alpha, n, l)$ sufficiently large such that

$$1 - (1 + r^2 N^{-2} l^{-2}) \frac{2}{n+\alpha} > 0.$$

Because $N^2 l^2 - r^2 \in [(N^2 - 1) l^2, N^2 l^2]$, we take $M = C(\alpha, n, N, l)$ sufficiently large such that

$$M^{n+\alpha} N^2 l^2 \frac{2}{n+\alpha} \Big(1 - (1 + r^2 N^{-2} l^{-2}) \frac{2}{n+\alpha}\Big) (N^2 l^2 - r^2)^{\frac{\alpha-n}{2}} \geqslant 1.$$

It follows that

$$H[V] \geqslant 1 \quad \text{in } \Omega. \tag{5.2}$$

This with (5.1), (5.2), and Lemma 2.7 implies that V is a sub-solution to the problem (1.1).

Using Theorem 2.4 (comparison principle), we obtain

$$0 \geqslant u(y) \geqslant V(y).$$

By taking this inequality on y_n-axis, we can summarize that

$$|u(y)| \leqslant |V(y)| \leqslant M N l y_n^{\frac{2}{n+\alpha}} = M N l d_y^{\frac{2}{n+\alpha}}.$$

This completes the proof of Corollary 1.4 as well as the $a = +\infty$ case of Theorem 1.2. Combined with Section 4, we have thus completed the proof of Theorem 1.2.

5.2. **Existence of solution on bounded convex domain.** In fact, the existence of solution to (1.1) on bounded convex domain can be seen as a particular case of [4, Theorem 5] and [13, Theorem 1.1]. We state the result as the following theorem and prove it by using Corollary 1.4. The proof below will serve as a heuristic argument to the next section dealing with unbounded convex domains.

Theorem 5.1. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded convex domain. Then problem* (1.1) *admits a convex solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$. Moreover, $u \in C^{\frac{2}{n+\alpha}}(\overline{\Omega})$ and*

$$|u|_{C^{\frac{2}{n+\alpha}}(\overline{\Omega})} \leqslant C(\alpha, n, \mathrm{diam}(\Omega)).$$

Proof. Let $\{\Omega_i\}$ be a sequence of bounded C^2 strictly convex domains such that $\Omega_i \subset \Omega_{i+1}$ and $\bigcup_{i=1}^\infty \Omega_i = \Omega$. In view of [4, Theorem 5], problem (1.1) admits a convex solution $u_i \in C^\infty(\Omega_i) \cap C(\overline{\Omega_i})$ for each Ω_i. According to Corollary 1.4, we have $u_i \in C^{\frac{2}{n+\alpha}}(\overline{\Omega_i})$ and

$$|u_i|_{C^{\frac{2}{n+\alpha}}(\overline{\Omega_i})} \leqslant C(\alpha, n, \mathrm{diam}(\Omega_i)).$$

Let us define $u_i(x) = 0$ for all $x \in \overline{\Omega} \setminus \overline{\Omega_i}$. Then we obtain $u_i \in C^{\frac{2}{n+\alpha}}(\overline{\Omega})$ and the uniform Hölder estimate

$$|u_i|_{C^{\frac{2}{n+\alpha}}(\overline{\Omega})} = |u_i|_{C^{\frac{2}{n+\alpha}}(\overline{\Omega_i})} \leqslant C(\alpha, n, \mathrm{diam}(\Omega)).$$

By Theorem 2.4 (comparison principle) and the proof of Corollary 1.4, we also obtain the decreasing property

$$0 \geqslant u_i(x) \geqslant u_{i+1}(x) \geqslant V(x), \quad \forall x \in \overline{\Omega}.$$

Using the diagonal technique of choosing subsequence, we obtain that $\{u_i\}$ is locally uniformly bounded. Due to the convexity of u_i and [7, Corollary A.23], it follows that u_i is locally uniformly Lipschitz and thus $\{u_i\}$ is locally equicontinuous. Thanks to Arzela-Ascoli theorem, a subsequence of $\{u_i\}$ (still denoted by $\{u_i\}$) locally uniformly converges to a convex function $u \in C(\overline{\Omega})$, which also satisfies

$$|u|_{C^{\frac{2}{n+\alpha}}(\overline{\Omega})} \leqslant C(\alpha, \mathrm{diam}(\Omega), n)$$

and hence $u \in C^{\frac{2}{n+\alpha}}(\overline{\Omega})$. Moreover, $u \in C(\overline{\Omega})$ is a convex generalized solution to (1.1) by [22, Lemma 2.2]. Based on Caffarelli's interior $C^{2,\alpha}$ regularity in [2, 8], we can derive further regularity by bootstrapping from the equation in (1.1). Repeating the bootstrap argument, we upgrade the regularity to $u \in C^\infty(\Omega)$. The theorem follows immediately. □

6. Unbounded convex domains

It remains to concentrate on the existence and global regularity result over an unbounded convex domain Ω. Similar to [11], we first construct sub-solutions to problem (1.1) over unbounded convex domains, and then the next step is to prove Theorem 1.5, which can be regarded as an application of Section 5 in spirit.

6.1. **Construction of a sub-solution.**

Lemma 6.1. *Denote* $x = (x', x_n)$ *and* $r = |x'|$. *If*

$$\Omega = \Big\{(x', x_n) \in \mathbb{R}^n : x_n \geqslant \sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha - n}{n+\alpha}}\, r\Big\},$$

then

$$W(x) = -\Big(\Big(\frac{x_n}{\sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}}\Big)^2 - r^2\Big)^{\frac{n}{n+\alpha}}$$

is a solution to (1.1) *on* Ω.

Proof. It is straightforward to compute that

$$\begin{aligned}
W_r &= \frac{2n}{n+\alpha}\Big(\Big(\frac{x_n}{\sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}}\Big)^2 - r^2\Big)^{\frac{n}{n+\alpha}-1} r, \\
W_n &= -\frac{\frac{2n}{n+\alpha}}{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}\Big(\Big(\frac{x_n}{\sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}}\Big)^2 - r^2\Big)^{\frac{n}{n+\alpha}-1} x_n, \\
W_{rr} &= \frac{4n\alpha}{(n+\alpha)^2}\Big(\Big(\frac{x_n}{\sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}}\Big)^2 - r^2\Big)^{\frac{n}{n+\alpha}-2} r^2 \\
&\quad + \frac{2n}{n+\alpha}\Big(\Big(\frac{x_n}{\sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}}\Big)^2 - r^2\Big)^{\frac{n}{n+\alpha}-1}, \\
W_{nn} &= \frac{\frac{4n\alpha}{(n+\alpha)^2}}{\big(\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}\big)^2}\Big(\Big(\frac{x_n}{\sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}}\Big)^2 - r^2\Big)^{\frac{n}{n+\alpha}-2} x_n^2 \\
&\quad - \frac{\frac{2n}{n+\alpha}}{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}\Big(\Big(\frac{x_n}{\sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}}\Big)^2 - r^2\Big)^{\frac{n}{n+\alpha}-1}, \\
W_{rn} &= -\frac{\frac{4n\alpha}{(n+\alpha)^2}}{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}\Big(\Big(\frac{x_n}{\sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}}\Big)^2 - r^2\Big)^{\frac{n}{n+\alpha}-2} r x_n,
\end{aligned}$$

which gives rise to

$$W_{rr}W_{nn} - |W_{rn}|^2 = \frac{\frac{4n^2}{(n+\alpha)^2}}{\big(\frac{2n}{n+\alpha}\big)^n}\Big(\Big(\frac{x_n}{\sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}}\Big)^2 - r^2\Big)^{\frac{2n}{n+\alpha}-2}.$$

Putting this expression into the formula of $H[\cdot]$ in Lemma 2.7, we finally verify that

$$H[W] = \Big(\frac{W_r}{r}\Big)^{n-2}\big(W_{rr}W_{nn} - |W_{rn}|^2\big)|W|^\alpha = 1.$$

The proof is complete. $\square$

Using Lemma 6.1, we can construct sub-solutions over unbounded convex domains as follows.

Lemma 6.2. *Suppose Ω is an unbounded convex domain in $\mathbb{R}^n$ such that $\partial\Omega$ is strictly convex at some point $x_0 \in \partial\Omega$. Then*

$$W(x) = -\Big(\Big(\frac{x_n}{\sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}}\Big)^2 - r^2\Big)^{\frac{n}{n+\alpha}}$$

is a sub-solution to (1.1) *on* Ω.

Proof. Since $\partial\Omega$ is strictly convex at some point $x_0 \in \partial\Omega$, then there exists a tangent plane P of $\partial\Omega$ at x_0 such that $P \cap \partial\Omega = \{x_0\}$. By some translations and rotations, we can assume $x_0 = 0$ and P is given by the equation $x_n = 0$. Without loss of generality, we further assume that $\partial\Omega$ near the origin can be expressed as $x_n = \varphi(x')$ which is the graph of a function over the tangent plane $x_n = 0$.

We use D to denote the largest domain such that φ is well-defined. Under these assumptions, we have $\varphi(0) = 0$ and the origin is the lowest point of Ω, and thus $\varphi(x') \geqslant 0$ for all $x' \in \partial D$. Moreover, for any small number $\delta > 0$, we have

$$\{x \in \mathbb{R}^{n-1} : |x'| < \delta\} \Subset D,$$
$$h(\delta) := \min\{\varphi(x') : |x'| = \delta\} > 0.$$

We are now in a position to show that

$$\Omega \subset \Sigma := \big\{(x', x_n) : x_n \geqslant \frac{h(\delta)}{\delta}|x'| - h(\delta)\big\}. \tag{6.1}$$

For any $(x', x_n) \in \partial\Omega$, we can distinguish the following two cases to prove this claim:

(i) When $|x'| \leqslant \delta$, we can derive $x_n = \varphi(x') \geqslant 0 \geqslant \frac{h(\delta)}{\delta}|x'| - h(\delta)$ and thus $(x', x_n) \in \Sigma$.

(ii) When $|x'| > \delta$, the convexity of Ω gives rise to

$$\frac{\varphi(x') - \varphi(\frac{\delta}{|x'|}x')}{|x' - \frac{\delta}{|x'|}x'|} \geqslant \frac{\varphi(\frac{\delta}{|x'|}x') - \varphi(0)}{|\frac{\delta}{|x'|}x' - 0|},$$

which implies that

$$\begin{aligned}
x_n &= \varphi(x') \\
&\geqslant \varphi(\frac{\delta}{|x'|}x') + \frac{\varphi(\frac{\delta}{|x'|}x')}{\delta}(|x'| - \delta) \\
&\geqslant h(\delta) + \frac{h(\delta)}{\delta}(|x'| - \delta) \\
&= \frac{h(\delta)}{\delta}|x'| \\
&\geqslant \frac{h(\delta)}{\delta}|x'| - h(\delta).
\end{aligned}$$

Thus we obtain $(x', x_n) \in \Sigma$ in such a case.

Summing up, for any $(x', x_n) \in \partial\Omega$, we always have $(x', x_n) \in \Sigma$, namely $\partial\Omega \subset \Sigma$. We note that the lowest point of Ω, i.e., the origin, is in Σ. Thus by the convexity of Ω, we obtain that $\Omega \subset \Sigma$.

Now we take a linear transformation $T : (x', x_n) \to (\widetilde{x'}, \widetilde{x_n})$ as follows:

$$\widetilde{x'} = x',$$

$$\widetilde{x_n} = \frac{\delta}{h(\delta)} \sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha - n}{n+\alpha}}\, (x_n + h(\delta))\,.$$

It follows that

$$T\Sigma = \Big\{ (\widetilde{x'}, \widetilde{x_n}) \in \mathbb{R}^n : \widetilde{x_n} \geqslant \sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha - n}{n+\alpha}} |\widetilde{x'}| \Big\}.$$

We note that (6.1) gives $T\Omega \subset T\Sigma$. Without loss of generality, by Lemma 2.2, we can assume $\Omega \subset T\Sigma$. According to Lemma 6.1, we obtain that

$$W(x) = -\Big(\Big(\frac{x_n}{\sqrt{\big(\frac{2n}{n+\alpha}\big)^n \frac{\alpha-n}{n+\alpha}}} \Big)^2 - r^2 \Big)^{\frac{n}{n+\alpha}}$$

is a solution to (1.1) on $T\Sigma$. Therefore it is a sub-solution to (1.1) on Ω as a result of $\Omega \subset T\Sigma$. The proof is complete. □

6.2. **Proof of Theorem 1.5.** Let $\{\Omega_i\}$ be a sequence of bounded convex domains such that $\Omega_i \subset \Omega_{i+1}$ and $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$. According to Theorem 5.1, the problem (1.1) admits a convex solution $u_i \in C^\infty(\Omega_i) \cap C(\overline{\Omega_i})$ for each Ω_i. Corollary 1.4 also gives rise to $u_i \in C^{\frac{2}{n+\alpha}}(\overline{\Omega_i})$ as well as

$$|u_i|_{C^{\frac{2}{n+\alpha}}(\overline{\Omega_i})} \leqslant C(\alpha, n, \mathrm{diam}(\Omega_i)).$$

Define $u_i(x) = 0$ for all $x \in \mathbb{R}^n \setminus \overline{\Omega_i}$. For any $r > 0$, we further obtain $u_i \in C^{\frac{2}{n+\alpha}}(\overline{\Omega \cap B_r(0)})$ and the uniform Hölder estimate

$$|u_i|_{C^{\frac{2}{n+\alpha}}(\overline{\Omega \cap B_r(0)})} = |u_i|_{C^{\frac{2}{n+\alpha}}(\overline{\Omega_i \cap B_r(0)})} \leqslant C(\alpha, n, \mathrm{diam}(\Omega \cap B_r(0))).$$

Thanks to Theorem 2.4 (comparison principle) and Lemma 6.2, we also derive the decreasing property

$$0 \geqslant u_i(x) \geqslant u_{i+1}(x) \geqslant W(x), \quad \forall x \in \Omega.$$

The diagonal technique of choosing subsequence leads us to the conclusion that $\{u_i\}$ is locally uniformly bounded. By the convexity of u_i and [7, Corollary A.23], we infer that all u_i are locally uniformly Lipschitz and thus $\{u_i\}$ is locally equicontinuous. By Arzela-Ascoli theorem, a subsequence of $\{u_i\}$ (still denoted by $\{u_i\}$) locally uniformly converges to a convex function $u \in C(\overline{\Omega \cap B_r(0)})$, which also satisfies

$$|u|_{C^{\frac{2}{n+\alpha}}(\overline{\Omega \cap B_r(0)})} \leqslant C(\alpha, n, \mathrm{diam}(\Omega \cap B_r(0)))$$

and therefore $u \in C^{\frac{2}{n+\alpha}}(\overline{\Omega \cap B_r(0)})$. Since $r > 0$ is arbitrary, we can derive $u \in C(\overline{\Omega})$. In fact, if there exists a point $y \in \overline{\Omega}$ such that u is not continuous at y, then we will always find a sufficiently large $r' > 0$ such that $y \in B_{r'}(0)$ and hence $y \in \overline{\Omega \cap B_{r'}(0)}$, which contradicts $u \in C(\overline{\Omega \cap B_{r'}(0)})$. Moreover, $u \in C(\overline{\Omega})$ is a convex generalized solution to (1.1) by [22, Lemma 2.2]. Using Caffarelli's interior $C^{2,\alpha}$ regularity in [2, 8], we can obtain further regularity by a bootstrap argument from the equation in (1.1). Repeating the bootstrap argument, we can upgrade the regularity to $u \in C^\infty(\Omega)$. This completes the proof.

Acknowledgments. The author would like to thank Profs. Huaiyu Jian and Pin Yu for their helpful suggestions and encouragement along the way. Sincere thanks also go to Drs. You Li and Haoyu Wang for providing useful comments and long-term help. The author is also grateful to the anonymous referees for offering helpful suggestions to improve the paper.

The major part of this paper was carried out while the author was participating in summer social practice at Institute for Electronics and Information Technology in Tianjin, Tsinghua University. The author acknowledges IEIT for the warm and peaceful atmosphere that allows her to calm down in spare time and helps provoke her thinking.

References

[1] B. Andrews; Gauss curvature flow: the fate of the rolling stones, *Invent. Math.*, **138** (1999), 151–161.

[2] L. A. Caffarelli; Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation, *Ann. of Math. (2)*, **131** (1990), 135–150.

[3] S. B. Chen, Q.-R. Li, G.X. Zhu; On the L_p Monge-Ampère equation, *J. Differential Equations*, **263** (2017), 4997–5011.

[4] S.-Y. Cheng, S.-T. Yau; On the regularity of the Monge-Ampère equation det $\frac{\partial^2 u}{\partial x_i \partial x_j} = F(x,u)$, *Comm. Pure Appl. Math.*, **30** (1977), 41–68.

[5] S.-Y. Cheng, S.-T. Yau; Complete affine hypersurfaces I: The completeness of affine metrics, *Comm. Pure Appl. Math.*, **39** (1986), 839–866.

[6] K.-S. Chou, X.-J. Wang; The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry, *Adv. Math.*, **205** (2006), 33–83.

[7] A. Figalli; *The Monge-Ampère equation and its applications*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2017.

[8] D. Gilbarg, N. S. Trudinger; *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 2001.

[9] Y. He, Q.-R. Li, X.-J. Wang; Multiple solutions of the L_p-Minkowski problem, *Calc. Var. Partial Differential Equations*, **55** (2016), Art. 117.

[10] Y. Huang, J. K. Liu, L. Xu; On the uniqueness of L_p-Minkowski problems: the constant p-curvature case in $\mathbb{R}^3$, *Adv. Math.*, **281** (2015), 906–927.

[11] H. Y. Jian, Y. Li; A singular Monge-Ampère equation on unbounded domains, *Sci. China Math.*, **61** (2018), 1473–1480.

[12] H. Y. Jian, Y. Li; Optimal boundary regularity for a singular Monge-Ampère equation, *J. Differential Equations*, **264** (2018), 6873–6890.

[13] H. Y. Jian, Y. Li, X. S. Tu; On a class of degenerate and singular Monge-Ampère equations, arXiv:1908.06396.

[14] H. Y. Jian, X.-J. Wang, Y. W. Zhao; Global smoothness for a singular Monge-Ampère equation, *J. Differential Equations*, **263** (2017), 7250–7262.

[15] M.-Y. Jiang; Remarks on the 2-dimensional L_p-Minkowski problem, *Adv. Nonlinear Stud.*, **10** (2010), 297–313.

[16] F. D. Jiang, N. S. Trudinger, X.-P. Yang; On the Dirichlet problem for Monge-Ampère type equations, *Calc. Var. Partial Differential Equations*, **49** (2014), 1223–1236.

[17] N. Q. Le, O. Savin; Schauder estimates for degenerate Monge-Ampère equations and smoothness of the eigenfunctions, *Invent. Math.*, **207** (2017), 389–423.

[18] M. N. Li, Y. Li; Global regularity for a class of Monge-Ampère type equations, *Sci. China Math.*, (2020).

[19] C. Loewner, L. Nirenberg; *Partial differential equations invariant under conformal or projective transformations*, in: "Contributions to analysis (a collection of papers dedicated to Lipman Bers)", pp. 245-272, Academic Press, New York, 1974.

[20] E. Lutwak; The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem, *J. Differential Geom.*, **38** (1993), 131–150.

[21] H. Minkowski; Volumen und Oberfläche. (German), *Math. Ann.*, **57** (1903), 447–495.

[22] N. S. Trudinger, X.-J. Wang; *The Monge-Ampère equation and its geometric applications*, in: "Handbook of geometric analysis", pp. 467-524, Adv. Lect. Math, Int. Press, Somerville, MA, 2008.

[23] J. Urbas; *Self-similar solutions of Gauss curvature flows. Monge Ampère equation: applications to geometry and optimization (Deerfield Beach, FL, 1997)*, pp. 157–172, Contemp. Math., 226, Amer. Math. Soc., Providence, RI, 1999.

Mengni Li

Department of Mathematical Sciences and Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China

Email address: `krisymengni@163.com`