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COUPLED POROSITY-FLUID CONCENTRATION FLUX-TEMPERATURE WAVES IN ISOTROPIC POROUS MEDIA

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Abstract. In this article, we study a problem on propagation of coupled porosity, fluid concentration flux, and temperature waves. We use a model formulated in previous papers for porous media saturated by a fluid flow, in the framework of non-equilibrium thermodynamics. We derive three modes of propagation in the one-dimensional and perfect isotropic case, and then we test the validity of the model. The waves propagation velocities are represented in diagrams as functions of the wave number. The derived results have applications in technological sectors such as seismology, medical sciences, geology and nanotechnology.

1. Introduction

In a previous article [5] a problem of propagation of coupled porosity and fluid concentration waves was studied, using a theory developed in [6, 23, 24, 25, 26]. These papers use a theory describing porous media filled by a fluid flow formulated using the procedures of extended thermodynamics with internal variables; see [1, 3, 11, 12, 13, 15, 17, 18, 19, 22].

In this article, we focus our attention on a problem of coupled porosity, fluid concentration flux, and temperature waves, in perfect isotropic porous media. This problem has applications in technological sectors such as seismic waves, medical sciences, biology, geology, and nanotechnology. In nanostructures the volume element size L along a direction is comparable or smaller than the free mean path of the heat carriers l, i.e. $\frac{l}{L} \gg 1$. Furthermore there are situations of propagation of high-frequency waves and the rate variation of properties of these porous media are faster than the time scale of the relaxation times of the fluxes to their equilibrium values.

In Section 2 the temperature equation and the rate equations for the porosity field, its flux, the heat flux and the fluid concentration flux for the considered media are presented in the anisotropic case (see [25, 26]). In Section 3 we particularize the above equations in a special case and when the geometric, transport and thermal properties of the media are invariant for all rotations and inversions of the frame axes. In Section 4, assuming that the porous medium filled by a fluid flow occupies the whole space, we derive the propagation velocities of the coupled waves of the

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porosity, fluid concentration flux, and thermal fields in the one-dimesional case. For a given numerical set of the several coefficients present in the equations, the waves propagation speeds are represented in diagrams. Then the validity of the model is tested. The appendices present a detailed derivation of the rate and temperature equations in the case of perfect isotropic media. Monographs [4, 27] present a study of the porous media filled by a fluid flux. While the authors in [7] study a thermodynamic model for erosion and/or deposition in elastic porous media.

2. Model equations

We consider a porous structure presenting a network of very thin tubes saturated by a fluid flow, whose mechanical, thermal and transport properties are analyzed using a model formulated in the framework of the extended thermodynamics (see [23, 24, 25, 26]). There the porosity field is described by an internal variable, the *structural permeability tensor* by r_{ij} as in Kubik [16], its gradient by $r_{ij,k}$ and its flux by $\mathcal{V}_{ijk}$. (a comma in the lower indices indicates the spatial derivation.)

Also, we assume that medium is elastic, and the inside the mechanical phenomena are described by the symmetric stress tensor τ_{ij} and the small-strain tensor ε_{ij}. The thermal processes are described by the temperature T, its gradient $T_{,i}$, and the heat flux q_i. The fluid flux through the porous channels is described by the fluid concentration c, its gradient $c_{,i}$, and its flux j_i^c.

Thus, we choose the thermodynamic state vector

$$C = \{\varepsilon_{ij}, c, T, r_{ij}, j_i^c, q_i, c_{,i}, T_{,i}, r_{ij,k}, \mathcal{V}_{ijk}\},$$

where $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, with u_i the displacement field. We refer to the configuration at time t, K_t, and use the standard Cartesian tensorial notation in rectangular coordinate systems. Furthermore, we assume that the porous skeleton filled by a fluid flow is a mixture of two components, so that we have

$$\rho = \rho_1 + \rho_2, \tag{2.1}$$

with ρ the density of the medium as a whole, ρ_1 the density of the fluid, and ρ_2 the density of the elastic skeleton.

We consider *the continuity equation*, where the source term has been neglected (see [3, 25])

$$\rho\dot{c} + j_{i,i}^c = 0, \tag{2.2}$$

where a superimposed dot indicates the material derivative (i.e. $\frac{d}{dt} = \frac{\partial}{\partial t} + x_\gamma \frac{\partial}{\partial x_\gamma}$, where Einstein convention for repeated indices is used). The concentration of the fluid is defined by $c = \rho_1/\rho$ and its flux j_i^c by $j_i^c = \rho_1(v_{1i} - v_i)$, with v_{1i} the fluid velocity and v_i the barycentric velocity of the mixture. These velocities satisfy $\rho v_i = \rho_1 v_{1i} + \rho_2 v_{2i}$, where v_{2i} is the porous structure velocity.

In the following the mass density ρ will be assumed constant. In [25, 26] the constitutive equations and rate equations were obtained (to close the systems of balance equations, see [25]) obeying the objectivity and frame indifference principles [9, 20, 21]. In particular *the rate equations* for r_{ij}, and the fluxes j_i^c, q_i and $\mathcal{V}_{ijk}$ have the form

$$\begin{aligned} \dot{r}_{ij} + \mathcal{V}_{ijk,k} &= \beta^1_{ijkl}\varepsilon_{kl} + \beta^2_{ijkl}r_{kl} + \beta^3_{ijk}j_k^c + \beta^4_{ijk}q_k \\ &\quad + \beta^5_{ijklm}r_{kl,m} + \beta^6_{ijk}c_{,k} + \beta^7_{ijk}T_{,k}, \end{aligned} \tag{2.3}$$

$$\tau^q\dot{q}_i = \chi^1_{ij}j_j^c - q_i + \chi^3_{ijkl}r_{jk,l} + \chi^4_{ij}c_{,j} - \chi^5_{ij}T_{,j}, \tag{2.4}$$

$$\tau^{j^c}\dot{j}_i^c = -j_i^c + \xi_{ij}^2 q_j + \xi_{ijkl}^3 r_{jk,l} - \xi_{ij}^4 c_{,j} + \xi_{ij}^5 T_{,j}, \tag{2.5}$$

$$\dot{\mathcal{V}}_{ijk} = \gamma_{ijkl}^1 j_l^c + \gamma_{ijkl}^2 q_l + \gamma_{ijklmn}^3 \mathcal{V}_{lmn} + \gamma_{ijkl}^4 c_{,l} + \gamma_{ijkl}^5 T_{,l} + \gamma_{ijklmn}^6 r_{lm,n}, \tag{2.6}$$

where the phenomenological tensors are assumed constant.

In this article for the sake of simplicity we choose

$$\mathcal{V}_{ijk} = -D_\nu r_{ij,k}, \tag{2.7}$$

with D_ν a diffusive coefficient, and thus equation (2.3) keeps the form

$$\begin{aligned}\dot{r}_{ij} - D_\nu r_{ij,kk} &= \beta_{ijkl}^1 \varepsilon_{kl} + \beta_{ijkl}^2 r_{kl} + \beta_{ijk}^3 j_k^c + \beta_{ijk}^4 q_k + \beta_{ijklm}^5 r_{kl,m} \\ &\quad + \beta_{ijk}^6 c_{,k} + \beta_{ijk}^7 T_{,k}.\end{aligned} \tag{2.8}$$

In [26] the *generalized telegraph temperature equation* was deduced as

$$\begin{aligned}\tau^q \ddot{T} + \dot{T} &= -\gamma_{ij}(\tau^q \ddot{\varepsilon}_{ij} + \dot{\varepsilon}_{ij}) + \varphi(\tau^q \ddot{c} + \dot{c}) + \eta_{ij}(\tau^q \ddot{r}_{ij} + \dot{r}_{ij}) \\ &\quad + \mathcal{K}_{ij} T_{,ji} - \nu_{ij}^1 j_{j,i}^c + D_\nu \nu_{ijkl}^3 r_{jk,li} - \nu_{ij}^4 c_{,ji},\end{aligned} \tag{2.9}$$

where the phenomenological coefficients are assumed constant, $\mathcal{K}_{ij}$ is the thermal diffusivity tensor, and (2.7) has taken into account.

Equations (2.3)–(2.6), (2.8), (2.9) describe disturbances having finite velocity of propagation and their own relaxation times to reach their respective thermodynamic equilibrium values.

In (2.4) the phenomenological tensors χ_{ij}^1, χ_{ij}^4, and χ_{ij}^3 are the thermodiffusive kinetic tensor, thermodiffusive tensor, and phenomenological tensor. These tensors describe the influences of the fluid concentration flux, the concentration gradient, and the porosity field gradient on the heat flux, respectively. The phenomenological tensor χ_{ij}^5 is the thermal conductibility.

Equation (2.4) is a generalization of the anisotropic transport equation Maxwell-Vernotte-Cattaneo for the heat flux $\tau^q \dot{q}_i = -q_i - \chi_{ij}^5 T_{,j}$, where τ^q is the relaxation time of the field q_i, having finite propagation velocity. When the relaxation time τ^q is null this equation reduces to the anisotropic Fourier law $q_i = -\chi_{ij}^5 T_{,j}$ describing thermal signals having infinite velocities of propagation (see [2, 8]).

In equation (2.5) the phenomenological tensors ξ_{ij}^2, ξ_{ij}^3, and ξ_{ijkl}^5 describe the influences of the heat flux, porosity field gradient, and temperature gradient on the fluid concentration flux field, respectively. Furthermore, ξ_{ik}^4 is the diffusion tensor.

Equation (2.5) generalizes the anisotropic transport equation Fick-Nonnenmacher for the fluid concentration flux $\tau^{j^c}\dot{j}_i^c = -j_i^c - \xi_{ij}^4 c_{,j}$, where τ^{j^c} is the relaxation time of the field j_i^c, having finite propagation velocity. When the relaxation time τ^{j^c} is vanishing, this equation reduces to the anisotropic Fick law $j_i^c = -\xi_{ij}^4 c_{,j}$, where the fluid concentration flux has infinite propagation velocity.

Equations (2.3) and (2.6) describe the evolution of the porosity field and its flux, and in their right-hand sides the sources terms represent contributions of several fields. Also, from the evolution equation (2.9) of the thermal field, it is seen that several fields influence the propagation of the field T.

3. Equations governing the evolution of porosity, fluid concentration flux, and temperature fields in a special case

For the treatment of the problem of coupled porosity, fluid concentration flux and temperature waves, we take into account the system of differential equations (2.8), (2.5) and (2.9), and we assume the following:

(i) the considered porous medium is at rest,
(ii) in equation (2.8) the influence of the field ε_{ij} can be neglected,
(iii) in the rate equation (2.5) the influence of the fields q_i and $c_{,i}$ can be disregarded,
(iv) in equation (2.9) the influence of the first time and second time derivatives of the small deformations field ε_{ij} and the concentration field c, the second time derivative of the porous field r_{ij}, the fluid concentration flux gradient and the gradient of the concentration gradient can be neglected.

Thus, we obtain

$$\begin{aligned}\frac{\partial r_{ij}}{\partial t} - D_\nu r_{ij,kk} = \beta^2_{ijkl} r_{kl} + \beta^3_{ijk} j^c_k + \beta^4_{ijk} q_k + \beta^5_{ijklm} r_{kl,m} \\ + \beta^6_{ijk} c_{,k} + \beta^7_{ijk} T_{,k},\end{aligned} \tag{3.1}$$

$$\tau^{j^c} \frac{\partial j^c_i}{\partial t} = -j^c_i + \xi^3_{ijkl} r_{jk,l} + \xi^5_{ij} T_{,j}, \tag{3.2}$$

$$\tau^q \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \eta_{ij} \frac{\partial r_{ij}}{\partial t} + \mathcal{K}_{ij} T_{,ji} + D_\nu \nu^3_{ijkl} r_{jk,li}. \tag{3.3}$$

In the rate equation (3.1), because of the symmetry of $r_{ij} = r_{ji}$, the phenomenological coefficients $\boldsymbol{\beta}^s$ $(s = 2, \dots, 7)$ present some symmetries. For the fourth tensor β^2_{ijkl}, present in equation (3.1), we have

$$\beta^2_{ijkl} = \beta^2_{jikl} \text{ and } \beta^2_{ijkl} = \beta^2_{ijlk}, \tag{3.4}$$

which are equivalent to

$$\beta^2_{ijkl} = \beta^2_{jikl} = \beta^2_{ijlk} = \beta^2_{jilk}.$$

Furthermore,

$$\begin{gathered}\beta^p_{ijk} = \beta^p_{jik} \quad (p = 3, 4, 6, 7), \\ \beta^5_{ijklm} = \beta^5_{jiklm} = \beta^5_{ijlkm} = \beta^5_{jilkm}.\end{gathered} \tag{3.5}$$

Also, from the symmetry properties of r_{ij} and $r_{jk,l}$, $r_{jk,li}$ (in the indexes $\{j,k\}$ and $\{l,i\}$ respectively) and of $T_{,ji}$ (in the indexes $\{j,i\}$), in the rate equations (3.2) and (3.3) we have for the following phenomenological symmetries:

$$\xi^3_{ijkl} = \xi^3_{ikjl}, \quad \eta_{ij} = \eta_{ji}, \quad \mathcal{K}_{ij} = \mathcal{K}_{ji}, \quad \nu^3_{ijkl} = \nu^3_{ikjl}, \quad \nu^3_{ijkl} = \nu^3_{ljki}\,. \tag{3.6}$$

Relations $(3.6)_4$ and $(3.5)_5$ are equivalent to

$$\nu^3_{ijkl} = \nu^3_{ikjl} = \nu^3_{ljki} = \nu^3_{lkji}. \tag{3.7}$$

The symmetry relations (3.4)-(3.7) reduce the number of the significant components of the above phenomenological tensors in equations (3.1)-(3.3). The number of these significant components can have a further reduction if we suppose the considered media are perfect isotropic.

3.1. **Perfect isotropic media.** In this subsection we consider perfect isotropic systems, having invariant symmetry properties with respect all rotations and the inversion of the frame axes. These properties simplify the form of the temperature and rate equations (3.1)–(3.3) in such a way that the number of the significant Cartesian components of the phenomenological tensors have a further reduction (see [6, 10, 14]). In fact, in this case the phenomenological tensors of order two ξ^5_{ij}, $\mathcal{K}_{ij}$, and η_{ij} take the form

$$\xi^5_{ij} = \xi^5 \delta_{ij}, \quad \mathcal{K}_{ij} = \mathcal{K} \delta_{ij}, \quad \eta_{ij} = \eta \delta_{ij}; \tag{3.8}$$

the tensors of order three and five vanish,

$$\beta^3_{ijk} = \beta^4_{ijk} = \beta^5_{ijklm} = \beta^6_{ijk} = \beta^7_{ijk} = 0; \tag{3.9}$$

the tensors of order four have the form

$$L_{ijkl} = L_1\delta_{ij}\delta_{kl} + L_2\delta_{ik}\delta_{jl} + L_3\delta_{il}\delta_{jk}, \tag{3.10}$$

where L_s $(s = 1, 2, 3)$ are the 3 significant components of L_{ijkl}, so that β^2_{ijkl}, ξ^3_{ijkl} and ν^3_{ijkl} have three significant components. Because of these tensors satisfy also the symmetry properties $(3.4)_3$, $(3.6)_1$, and (3.7), in the appendix we show that the tensors β^2_{ijkl}, ξ^3_{ijkl}, and ν^3_{ijkl} have only *two significant components.*

Furthermore, in the following we will consider only the *scalar (or spherical) part* $r\delta_{ij}$ of r_{ij}, with r defined by

$$r = \frac{1}{3}r_{kk}, \tag{3.11}$$

having split r_{ij} in its deviatoric part $\tilde{r}_{ij} = r_{ij} - r\delta_{ij}$, and its spherical part $r\delta_{ij}$.

By this assumption we have

$$\begin{gathered} r_{ij} = r\delta_{ij}, \quad r_{ij,kk} = r_{,kk}\delta_{ij}, \quad r_{kl,m} = r_{,m}\delta_{kl}, \\ r_{jk,l} = r_{,l}\delta_{jk}, \quad r_{jk,li} = r_{,li}\delta_{jk}. \end{gathered} \tag{3.12}$$

From equations (3.1)-(3.3), taking into consideration the results (3.12) and relations (3.8), (3.9), (4.2), (4.4), and (4.5) for the phenomenological tensors (see detailed calculations in Appendices 4–7), we derive the following simplified system of equations governing the evolution of porosity, fluid concentration flux and temperature fields

$$\frac{\partial r}{\partial t} - D_\nu r_{,kk} = -\alpha_r r, \tag{3.13}$$

$$\tau^{j^c}\frac{\partial j^c_i}{\partial t} = -j^c_i + \alpha_c r_{,i} + \beta_c T_{,i}, \tag{3.14}$$

$$\tau^q\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \mathcal{K}T_{,kk} + \alpha_T r_{,kk} - 3\eta\alpha_r r, \tag{3.15}$$

where

$$\alpha_r = (\tau^r)^{-1} > 0 \tag{3.16}$$

is the inverse of the relaxation time of the field r, given by relation (5.3) of Appendix 5, α_c, β_c, and α_T are coupling coefficients reflecting some cross-kinetic effects of the porosity gradient field and the temperature gradient on the the fluid concentration flux, and the effect of the field $r_{,kk}$ on the temperature field, respectively (see relations (6.3) and (7.4) of Appendices 6 and 7, respectively). A detailed derivation of equations (3.13)-(3.15) has been obtained in Appendix 5–7.

3.2. **Propagation velocities of the coupled waves.** The aim of this Subsection is to find the dispersion relation, the propagation velocities of the coupled porosity, fluid concentration flux, and temperature waves as functions of the wave number.

We assume that the porous medium occupies the whole space, and confine our study to one-dimensional waves, that propagate along the x direction, so that $\mathbf{j}^c = (j^c, 0, 0)$. Thus, we assume that the solutions of the set of equations (3.13)-(3.15) keep the form

$$r(x,t) = \widehat{r}e^{ik(x-vt)}, \quad j^c(x,t) = \widehat{j^c}e^{ik(x-vt)}, \quad T(x,t) = \widehat{T}e^{ik(x-vt)}, \tag{3.17}$$

where v is the wave velocity, k is the wave number and $\widehat{r}$, $\widehat{j^c}$ and $\widehat{T}$ are the amplitudes of the waves $r(x,t)$, $j^c(x,t)$ and $T(x,t)$ (3.17); v is defined by $v=\omega/k\,[\mathrm{m\,s^{-1}}]$, with ω the angular frequency, $\omega=2\pi f\,[\mathrm{s^{-1}}]$, being f the wave frequency, k the wave number, given by $k=2\pi/\lambda\,[\mathrm{m^{-1}}]$, and λ the wavelength.

Thus, inserting the relations (3.17) and their suitable derivatives into (3.13)-(3.15), we obtain the system of equations

$$\big(D_\nu k^2+\alpha_r-ikv\big)\,\widehat{r}=0, \tag{3.18}$$

$$\alpha_c ik\widehat{r}+\big(\tau^{j^c}ikv-1\big)\widehat{j^c}+\beta_c ik\widehat{T}=0, \tag{3.19}$$

$$(-\alpha_T k^2-3\eta\alpha_r)\widehat{r}+\big(\tau^q k^2v^2+ikv-\mathcal{K}k^2\big)\,\widehat{T}=0, \tag{3.20}$$

which has non-trivial solutions only if its determinant vanishes, i.e.

$$\mathcal{D}=\begin{vmatrix} D_\nu k^2+\alpha_r-ikv & 0 & 0\\ \alpha_c ik & \tau^{j^c}ikv-1 & \beta_c ik\\ -\alpha_T k^2-3\eta\alpha_r & 0 & \tau^q k^2v^2+ikv-\mathcal{K}k^2 \end{vmatrix}=0. \tag{3.21}$$

Developing $\mathcal{D}$ we obtain the following *dispersion relation* for the waves propagation velocities v concerning possible propagation modes:

$$\begin{aligned} &\tau^{j^c}\tau^q k^3v^4+ik^2\Big[\tau^{j^c}+\tau^q+\tau^{j^c}\tau^q\left(D_\nu k^2+\alpha_r\right)\Big]v^3\\ &-k\Big[\Big(\tau^{j^c}+\tau^q\Big)\left(D_\nu k^2+\alpha_r\right)+\tau^{j^c}\mathcal{K}k^2+1\Big]v^2\\ &-i\Big[\big(\mathcal{K}\tau^{j^c}k^2+1\big)\left(D_\nu k^2+\alpha_r\right)+\mathcal{K}k^2\Big]v+\mathcal{K}k\left(D_\nu k^2+\alpha_r\right)=0. \end{aligned} \tag{3.22}$$

From the real part of this dispersion relation, we obtain

$$\begin{aligned} &\tau^{j^c}\tau^q k^2v^4-\Big[\big(\tau^{j^c}+\tau^q\big)\left(D_\nu k^2+\alpha_r\right)+\tau^{j^c}\mathcal{K}k^2+1\Big]v^2\\ &+\mathcal{K}\left(D_\nu k^2+\alpha_r\right)=0, \end{aligned} \tag{3.23}$$

from which we derive two possible modes

$$v_{(1)}=\sqrt{\mathscr{A}+\sqrt{\mathscr{A}^2-\mathscr{B}}},\quad v_{(2)}=\sqrt{\mathscr{A}-\sqrt{\mathscr{A}^2-\mathscr{B}}}, \tag{3.24}$$

where

$$\mathscr{A}=\frac{\tau^{j^c}\mathcal{K}k^2+1+\big(\tau^{j^c}+\tau^q\big)\left(D_\nu k^2+\alpha_r\right)}{2\tau^{j^c}\tau^q k^2},\quad \text{with } \mathscr{A}>0, \tag{3.25}$$

$$\mathscr{B}=\frac{\mathcal{K}\left(D_\nu k^2+\alpha_r\right)}{\tau^{j^c}\tau^q k^2},\quad \text{with } \mathscr{B}>0. \tag{3.26}$$

The velocity $v_{(1)}$ is real when

$$\mathscr{A}^2-\mathscr{B}\ge 0\quad\text{and}\quad \mathscr{A}+\sqrt{\mathscr{A}^2-\mathscr{B}}\ge 0. \tag{3.27}$$

Condition $(3.27)_1$ is satisfied when

$$\Big[\tau^{j^c}\mathcal{K}k^2+1+\big(\tau^{j^c}+\tau^q\big)\big(D_\nu k^2+\alpha_r\big)\Big]^2-4\tau^{j^c}\tau^q\mathcal{K}k^2\left(D_\nu k^2+\alpha_r\right)\ge 0, \tag{3.28}$$

whereas $(3.27)_2$ is always satisfied, if (3.28) holds, because it is a sum of two positive terms.

The velocity $v_{(2)}$ is real when: (i) expression (3.28) holds, and from $(3.24)_2$ we have

$$\mathscr{A}-\sqrt{\mathscr{A}^2-\mathscr{B}}\ge 0, \tag{3.29}$$

from which we derive $\mathscr{B} \geq 0$, that is always true.

From the imaginary part of the dispersion relation (3.22), we derive

$$\begin{aligned} &k^2\big[\tau^{j^c} + \tau^q + \tau^{j^c}\tau^q\left(D_\nu k^2 + \alpha_r\right)\big]v^3 \\ &-\Big[\Big(\mathcal{K}\tau^{j^c}k^2 + 1\Big)\left(D_\nu k^2 + \alpha_r\right) + \mathcal{K}k^2\Big]v = 0, \end{aligned} \tag{3.30}$$

from which we obtain the values

$$v_{(3)} = 0, \quad v_{(4)} = \sqrt{\frac{\mathcal{K}k^2 + (\mathcal{K}\tau^{j^c}k^2 + 1)(D_\nu k^2 + \alpha_r)}{k^2\left[\tau^{j^c} + \tau^q + \tau^{j^c}\tau^q(D_\nu k^2 + \alpha_r)\right]}}. \tag{3.31}$$

Notice that the velocity $v_{(4)}$ is real for all $k \neq 0$ because in (3.31) the radicand is always positive. Thus, we have obtained three possible modes of propagation: $v_{(1)}$, $v_{(2)}$ and $v_{(4)}$.

In Figures 1–3 the propagation speeds $v_{(1)}$, $v_{(2)}$, and $v_{(4)}$ as functions of k, solving the real part (3.23) or the imaginary part (3.30) of the dispersion relation (3.22), are represented in the case where, as an example, we have considered a given numerical set of the several coefficients present in the equations of the examined problem: $D_\nu = 10^{-2}\,\mathrm{m}^2\,\mathrm{s}^{-1}$, $\mathcal{K} = 10^{-4}\,\mathrm{m}^2\,\mathrm{s}^{-1}$, $\tau^{j^c} = 10^{-2}\,\mathrm{s}$, $\tau^q = 10^{-2}\,\mathrm{s}$, and $\alpha_r = 10^2\,\mathrm{s}^{-1}$. In this assumption condition (3.28) is satisfied for all k and furthermore the velocities $v_{(1)}$ and $v_{(2)}$ are real. We recall that the velocity $v_{(4)}$ is real for all $k \neq 0$.

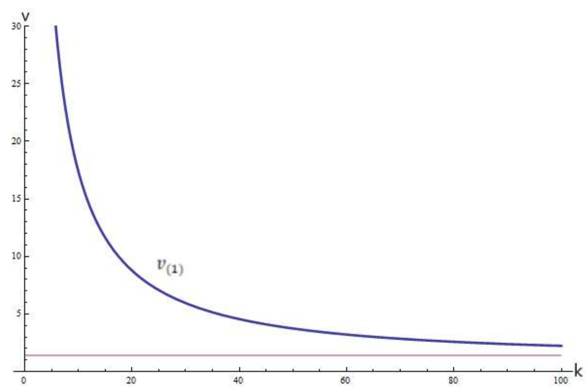

Figure 1. Representation of the wave propagation speed $v_{(1)}$ (in blue color) as function of k, for a given numerical set of the several coefficients present in the examined problem. The horizontal line in fuchsia color is its horizontal asymptote.

Conclusions

In this article we worked out for a perfect isotropic porous media filled by a fluid flow, a system of rate equations for the porosity, a fluid concentration flux, and temperature fields to study the propagation of coupled waves of these fields. We used a model formulated in previous papers, in the framework of rational extended irreversible thermodynamics with internal variables. A structural permeability tensor r_{ij}, its gradient $r_{ij,k}$, and its flux $\mathcal{V}_{ijk}$ were introduced in the thermodynamic state vector and the mass density of the mixture consisting of the porous skeleton

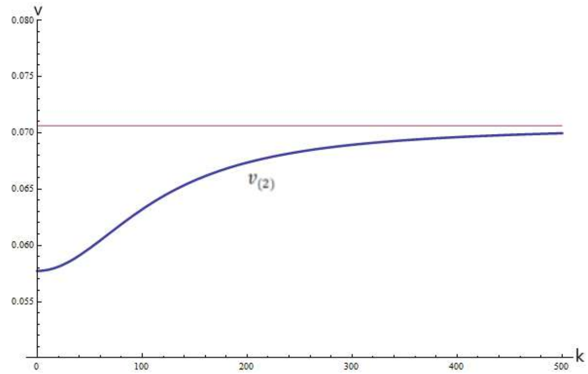

FIGURE 2. Representation of the wave propagation speed $v_{(2)}$ (in blue color) as function of k, for a given numerical set of the several coefficients present in the studied problem.The horizontal line in fuchsia color is its horizontal asymptote.

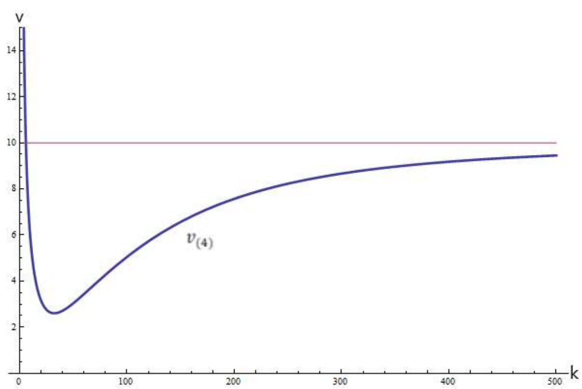

FIGURE 3. Representation of the wave propagation speed $v_{(4)}$ (in blue color) as function of k, for a given numerical set of the several coefficients present in the examined problem. The horizontal line in fuchsia color is its horizontal asymptote.

and the fluid flowing inside of it was assumed constant. The body was supposed occupying the whole space. The dispersion relation was derived, three possible propagation modes were obtained, and the corresponding wave propagation velocities as functions of the wave number k, were represented in diagrams, for a given set of the several phenomenological coefficients present in the studied problem.

4. Appendix A: Perfect isotropic tensors with special symmetry properties

Here we consider perfect isotropic tensors of fourth order, having special symmetry properties, and thus a reduced number of significant components. In particular, we demonstrate that the tensors β^2_{ijkl}, ξ^3_{ijkl} and ν^3_{ijkl} can be expressed only by two significant independent components.

Case (a) Let us consider the fourth order perfect isotropic tensor β^2_{ijkl}, present in equation (3.1) and having the symmetries $\beta^2_{ijkl} = \beta^2_{jikl} = \beta^2_{ijlk} = \beta^2_{jilk}$. Using relation (3.10) we obtain

$$\beta^2_{jikl} = \beta^2_a \delta_{ji}\delta_{kl} + \beta^2_b \delta_{jk}\delta_{il} + \beta^2_c \delta_{jl}\delta_{ik} \tag{4.1}$$

and an analogous expression for β^2_{ijkl}. Matching the two relations obtained by the help of (3.10), the tensor β^2_{ijkl} can be written as

$$\beta^2_{ijkl} = \beta^2_1 \delta_{ij}\delta_{kl} + \beta^2_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad \text{with } \beta^2_1 = \beta^2_a,\ \beta^2_2 = (\beta^2_b + \beta^2_c)/2. \tag{4.2}$$

Case (b) Let us consider the fourth order perfect isotropic tensor ξ^3_{ijkl}, present in equation (3.2) and having the symmetry

$$\xi^3_{ijkl} = \xi^3_{ikjl}.$$

From relation (3.10) we have

$$\xi^3_{ikjl} = \xi^3_a \delta_{ik}\delta_{jl} + \xi^3_b \delta_{ij}\delta_{kl} + \xi^3_c \delta_{il}\delta_{kj} \tag{4.3}$$

and an analogous result for ξ^3_{ikjl}. Matching the two results we have

$$\xi^3_{ijkl} = \xi^3_1 \delta_{il}\delta_{jk} + \xi^3_2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}). \tag{4.4}$$

Case (c) The perfect isotropic fourth tensor ν^3_{ijkl}, present in equation (3.3), has the symmetries $\nu^3_{ijkl} = \nu^3_{ikjl} = \nu^3_{ljki} = \nu^3_{lkji}$. Thus, by an analogous method used in the cases (a) and (b) the tensor ν^3_{ijkl} can be written as

$$\nu^3_{ijkl} = \nu^3_1 \delta_{il}\delta_{jk} + \nu^3_2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}). \tag{4.5}$$

5. Appendix B: Derivation of the rate equation for porosity field

To obtain equation (3.13), we use (3.9), $(3.12)_1$, $(3.12)_2$, and the special form (4.2) assumed by the fourth order tensor β^2_{ijkl}, so that (3.1) takes the form

$$\frac{\partial r}{\partial t}\delta_{ij} - D_\nu r_{,kk}\delta_{ij} = [\beta^2_1 \delta_{ij}\delta_{kl} + \beta^2_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})] r \delta_{kl}, \tag{5.1}$$

where β^2_1, β^2_2 are the 2 significant independent components of the fourth tensor β^2_{ijkl}. Then, from (5.1) we obtain

$$\frac{\partial r}{\partial t}\delta_{ij} - D_\nu r_{,kk}\delta_{ij} = \left(3\beta^2_1 + 2\beta^2_2\right) r \delta_{ij}, \tag{5.2}$$

i.e. equation (3.13), when $i = j$ and we define

$$\left(3\beta^2_1 + 2\beta^2_2\right) = -\alpha_r = -\left(\tau^r\right)^{-1}, \tag{5.3}$$

where τ^r the relaxation time of the porosity field.

6. Appendix C: Derivation of the rate equation for the fluid concentration flux

To derive (3.14), we use (3.2) and the special forms $(3.8)_1$ and (4.4) for the tensors ξ^5_{ij} and ξ^3_{ijkl}, respectively, so that we obtain

$$\tau^{j^c} \frac{\partial j^c_i}{\partial t} = -j^c_i + [\xi^3_1 \delta_{il}\delta_{jk} + \xi^3_2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})] r_{,l}\delta_{jk} + \xi^5 T_{,i}, \tag{6.1}$$

where ξ_1^3, ξ_2^3 are the 2 significant independent components of the fourth tensor ξ_{ijkl}^3 and ξ^5 is the only significant component of the second order tensor ξ_{ij}^5. Thus, equation (6.1) keeps the form

$$\tau^{j^c}\frac{\partial j_i^c}{\partial t} = -j_i^c + (3\xi_1^3 + 2\xi_2^3)r_{,i} + \xi^5 T_{,i}, \tag{6.2}$$

i.e. equation (3.14), when we define

$$\beta_c = \xi^5, \quad \alpha_c = 3\xi_1^3 + 2\xi_2^3. \tag{6.3}$$

7. Appendix D: Derivation of temperature equation

To deduce (3.15), we use (3.3) , (3.11), $(3.12)_1$, and $(3.12)_5$, and the special forms $(3.8)_2$, $(3.8)_3$, and (4.5) of the tensors $\mathcal{K}_{ij}$, η_{ij} and ν_{ijkl}^3, so that we obtain

$$\tau^q\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = 3\eta\frac{\partial r}{\partial t} + \mathcal{K}T_{,ii} + D_\nu\left[\nu_1^3\delta_{il}\delta_{jk} + \nu_2^3(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})\right]r_{,li}\delta_{jk}, \tag{7.1}$$

where ν_1^3, ν_2^3 are the 2 significant independent components of the fourth tensor ν_{ijkl}^3 and $\mathcal{K}$, η are the only significant components of the second order tensors $\mathcal{K}_{ij}$ and η_{ij}. Then equation (7.1) reads

$$\tau^q\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = 3\eta\frac{\partial r}{\partial t} + \mathcal{K}T_{,ii} + D_\nu\left(3\nu_1^3 + 2\nu_2^3\right)r_{,ii}. \tag{7.2}$$

Using (3.13), equation (7.2) assumes the form

$$\tau^q\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \mathcal{K}T_{,ii} + D_\nu\left(3\nu_1^3 + 2\nu_2^3 + 3\eta\right)r_{,ii} - 3\eta\alpha_r r, \tag{7.3}$$

i.e. equation (3.15), when we define

$$\alpha_T = D_\nu\left(3\nu_1^3 + 2\nu_2^3 + 3\eta\right). \tag{7.4}$$

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