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LOGARITHMICALLY IMPROVED REGULARITY CRITERIA FOR THE NAVIER-STOKES EQUATIONS IN HOMOGENEOUS BESOV SPACES

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Abstract. We investigate a logarithmically improved regularity criteria in terms of the velocity, or the vorticity, for the Navier-Stokes equations in homogeneous Besov spaces. More precisely, we prove that if the weak solution u satisfies either

$$\int_0^T \frac{\|u(t)\|^{\frac{2}{1-\alpha}}_{\dot{B}^{-\alpha}_{\infty,\infty}}}{1+\log^+\|u(t)\|_{\dot{H}^{s_0}}}\,dt<\infty, \quad \text{or} \quad \int_0^T \frac{\|w(t)\|^{\frac{2}{2-\alpha}}_{\dot{B}^{-\alpha}_{\infty,\infty}}}{1+\log^+\|w(t)\|_{\dot{H}^{s_0}}}\,dt<\infty\,,$$

where $w=\operatorname{rot} u$, then u is regular on $(0,T]$. Our conclusions improve some results by Fan et al. [5].

1. Introduction

Our main purpose is to investigate a logarithmically improved regularity criteria of solutions to the Navier-Stokes equations in $\mathbb{R}^n$, $n\geq 3$:

$$\begin{gathered}\partial_t u-\Delta u+u\cdot\nabla u+\nabla p=0, \quad x\in\mathbb{R}^n,\ t\in(0,T),\\ \operatorname{div} u=0,\\ u(x,0)=u_0(x),\end{gathered}\tag{1.1}$$

where $u(x,t)=(u_1(x,t),\dots,u_n(x,t))$, and p denote the velocity vector and pressure, respectively, of the fluid at the point $(x,t)\in\mathbb{R}^n\times(0,T)$ and u_0 is a given initial velocity.

Since the pioneering works by Leray [17] and Hopf [8], the existence of global weak solutions for an arbitrary initial data $u_0\in L^2(\mathbb{R}^n)$ was well-known. However, the uniqueness and regularity of weak solutions are still open. Notice that the studying of blow-up of solutions to (1.1) plays a crucial role not only in nonlinear analysis, but also in the study of the regularity of weak solutions. Also, it is known that for each regular u_0, there exists $t_0>0$ such that u is regular for $0\leq t\leq t_0$ (see e.g. [16]). Different regularity criteria for the weak solutions have been proposed. For example, the Prodi-Serrin conditions [22, 23] states that if the weak solution u satisfies

$$u\in L^r(0,T;L^p(\mathbb{R}^n)) \quad \text{with } \frac{2}{r}+\frac{n}{p}\leq 1,\ n<p<\infty,\ 2<r<\infty,$$

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then u is smooth on $\mathbb{R}^n \times (0,T)$. The limiting case, $p=n$ and $r=\infty$, was obtained in Escauriaza et al. [4]. A logarithmically improved regularity criterion was introduced by Montgomery-Smith [18]. That is, if

$$\int_0^T \frac{\|u(t)\|_{L^p}^r}{1+\log^+\|u(t)\|_{L^p}}dt<\infty, \quad \frac{2}{r}+\frac{3}{p}=1,\ 3<p<\infty,$$

then u is smooth (in the sense that u is at least in the Sobolev spaces $W^{n,q}$ for some $q\in[2,+\infty)$ and all positive integers n).

On the other hand, in 1995, Beirão da Veiga [2] established a Serrin's type regularity criterion on the gradient of velocity field: $\nabla u \in L^r(0,T;L^p(\mathbb{R}^n))$ with $\frac{2}{r}+\frac{n}{p}\le 2$. Beale–Kato–Majda [1] and Kato–Ponce [11] showed that the L^∞-norm of the vorticity, denoted by $w=\operatorname{rot} u$, controls the breakdown of smooth solutions to the Euler and Navier-Stokes equations. To be more precise, if

$$\int_0^T \|w(\tau)\|_{L^\infty}\,d\tau<\infty\,,$$

then the smooth solution u, in $\mathcal{C}([0,T);W^{s,p}(\mathbb{R}^n))$, with $s>n/p+1$, can be continued beyond $t=T$. That was improved by Kozono–Taniuchi [14, 15] in $\mathrm{BMO}(\mathbb{R}^n)$.

Theorem 1.1. *Let $s>\frac{n}{2}-1$ and let $u_0\in H^s(\mathbb{R}^n)$ with $\operatorname{div} u_0=0$. Suppose that u is a strong solution of (1.1) in the class*

$$S_T:=\mathcal{C}\left((0,T);H^s(\mathbb{R}^n)\right)\cap\mathcal{C}^1\left((0,T);H^s(\mathbb{R}^n)\right)\cap\mathcal{C}\left((0,T);H^{s+2}(\mathbb{R}^n)\right)\,.$$

If

$$\int_{\delta_0}^T \|w(\tau)\|_{\mathrm{BMO}}\,d\tau<\infty\,, \tag{1.2}$$

for some $\delta_0\in(0,T)$, then u can be continued as solution in the class $S_{T'}$ for some $T'>T$.

In addition, Kozono et al. [12] improved Theorem 1.1 in the homogeneneous Besov space (for the definition of this and other spaces we will mention in this Introduction we send, for instance, to the exposition made in the monographs [24] and [16]):

$$\int_0^T \|w(\tau)\|_{\dot{B}^0_{\infty,\infty}}\,d\tau<\infty\,. \tag{1.3}$$

A version of this, as a logarithmically improved regularity criterion of (1.3), was given in Fan et al. [6]:

$$\int_0^T \frac{\|w(\tau)\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log^+\|w(\tau)\|_{\dot{\mathrm{B}}^0_{\infty,\infty}}}}\,d\tau<\infty\,. \tag{1.4}$$

Recently, Nakao-Taniuchi [19] proved a different logarithmically improved regularity criterion as follows:

$$\int_0^T \frac{\|w(\tau)\|_{\mathrm{BMO}}}{1+\log^+\|u(\tau)\|_{\mathcal{C}^{1+\alpha}}}\,d\tau<\infty \tag{1.5}$$

for some $\alpha\in(0,1)$. We point out that these authors obtained (1.4) by using the Brézis–Gallouët-Wainger type inequality.

Concerning the logarithmically improved regularity criterion on the homogeneous Besov space $\dot{B}^{-\alpha}_{\infty,\infty}$, Fan et al. [5] proved the following results.

Theorem 1.2 ([5]). *Let $u_0 \in L^{2n}(\mathbb{R}^n)$ with $\operatorname{div} u_0 = 0$. Let $u \in L^\infty\big(0,T;L^2(\mathbb{R}^n)\big) \cap L^2\big(0,T;H^1(\mathbb{R}^n)\big)$ be a weak solution of (1.1). Assume that one of the following conditions is satisfied:*

$$\int_0^T \frac{\|u(t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}}{1+\log^+\|u(t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}}\,dt < \infty, \quad \text{with } 0<\alpha<1\,, \tag{1.6}$$

$$\int_0^T \frac{\|w(t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}}}{1+\log^+\|w(t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}}\,dt < \infty, \quad \text{with } n=3, \text{ and } 0<\alpha<1\,. \tag{1.7}$$

Then u is smooth on $(0,T]$.

The main goal of this paper is to improve (1.6) and (1.7). Our main results read as follows.

Theorem 1.3. *Let $u_0 \in L^{2n}(\mathbb{R}^n)$ with $\operatorname{div} u_0 = 0$. Let $u \in L^\infty\big(0,T;L^2(\mathbb{R}^n)\big) \cap L^2\big(0,T;H^1(\mathbb{R}^n)\big)$ be a weak solution of (1.1). Suppose that*

$$\int_0^T \frac{\|u(t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}}{1+\log^+\|u(t)\|_{\dot{H}^{s_0}}}\,dt < \infty \tag{1.8}$$

holds for some $\alpha \in (0,1)$, with $s_0 = \frac{n}{2}-\alpha$. Then u is smooth on $(0,T]$.

As a consequence of the above theorem and the Sobolev embedding, we have the following corollary.

Corollary 1.4. *Let $u_0 \in L^{2n}(\mathbb{R}^n)$ with $\operatorname{div} u_0 = 0$. Let $u \in L^\infty\big(0,T;L^2(\mathbb{R}^n)\big) \cap L^2\big(0,T;H^1(\mathbb{R}^n)\big)$ be a weak solution of (1.1). If in addition*

$$\int_0^T \frac{\|u(t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}}{1+\log^+\|u(t)\|_{L^{n/\alpha}}}\,dt < \infty\,, \tag{1.9}$$

for some $\alpha \in (0,1)$, then u is smooth on $(0,T]$.

Remark 1.5. It is clear that (1.6) is weaker than (1.9) (see Proposition 2.4 below).

Our last result in this paper improves condition (1.7).

Theorem 1.6. *Let $u_0 \in L^6(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. Let $u \in L^\infty\big(0,T;L^2(\mathbb{R}^3)\big) \cap L^2\big(0,T;H^1(\mathbb{R}^3)\big)$ be a weak solution of (1.1). If*

$$\int_0^T \frac{\|w(t)\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}}}{1+\log^+\|w(t)\|_{\dot{H}^{s_0}}}\,dt < \infty \tag{1.10}$$

for $0<\alpha<1$, $s_0 = \frac{3}{2}-\alpha$, then u is smooth on $(0,T]$.

Notation. Through this paper, we use the following general abbreviation $X = X(\mathbb{R}^n)$. So, for instance, $L^p \equiv L^p(\mathbb{R}^n)$, and $H^s \equiv H^s(\mathbb{R}^n)$. Moreover, we denote by C a positive constant which can change from line to line.

2. Definitions and preliminary results

Let us first define a weak solution, introduced by Leray [17].

Definition 2.1. Let $u_0 \in L^2$ with $\operatorname{div} u_0 = 0$ in $\mathbb{R}^n$. Then u is called a weak solution of (1.1) if $u \in L^\infty\big(0,T;L^2(\mathbb{R}^n)\big) \cap L^2\big(0,T;H^1(\mathbb{R}^n)\big)$ satisfies the equation in distributional sense and the following inequality

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\tau)\|_{L^2}^2\,d\tau \le \|u_0\|_{L^2}^2 \tag{2.1}$$

for all $t \in (0,T)$.

The following results (see e.g. [10, Theorem 4], [7]) will be repeatedly used.

Proposition 2.2. (i) *Suppose that $u_0 \in L^\gamma$, for some $\gamma \ge n$ with $\operatorname{div} u_0 = 0$ in $\mathbb{R}^n$. Then, there exists a time $T_0 > 0$ and a unique solution of (1.1) on $[0,T_0)$ such that*

$$u \in \mathcal{BC}\left([0,T_0);L^\gamma\right) \cap L^s\left(0,T_0;L^r\right), \quad t^{1/s}u \in \mathcal{BC}\left([0,T_0);L^r\right), \tag{2.2}$$

with $\frac{2}{s} + \frac{n}{r} = \frac{n}{\gamma}$, $s,r > n$, and $\mathcal{BC}$ denotes the space of bounded and continuous functions.

(ii) *Let $(0,T^*)$ be the maximal interval such that u solves (1.1) in $\mathcal{C}\left((0,T^*),L^\gamma\right)$, with $\gamma > n$. Then*

$$\|u(t)\|_{L^\gamma} \ge C(T^* - t)^{\frac{n-\gamma}{2\gamma}}, \tag{2.3}$$

where constant $C > 0$ is independent of T^* and t.

(iii) *Let u be a solution of (1.1) on $(0,T_0)$ in the functions class (2.2). Suppose that $u_0 \in L^2$. Then u is also a weak solution in Definition 2.1.*

(iv) *Let u be a weak solution of (1.1) satisfying $u \in L^s\left(0,T;L^r(\mathbb{R}^n)\right)$, for some $r > n$, with $\frac{2}{s} + \frac{n}{r} \le 1$. Then $u \in \mathcal{C}^\infty\left(\mathbb{R}^n \times (0,T)\right)$.*

To define the homogeneous Besov spaces, we recall the Littlewood-Paley decomposition (see, e.g., [24]). Let $\phi_j(x)$ be the inverse Fourier transform of the j-th component of the dyadic decomposition i.e.,

$$\sum_{j\in\mathbb{Z}} \hat\phi(2^{-j}\xi) = 1$$

except $\xi = 0$, where $\operatorname{supp}(\hat\phi) \subset \{\xi : 1/2 < |\xi| < 2\}$. Let

$$\mathcal{Z}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}(\mathbb{R}^n), D^\alpha \hat f(0) = 0, \forall \alpha \in \mathbb{N}^n, \text{ multi-index} \right\}.$$

Definition 2.3. For every $s \in \mathbb{R}$, and for every $1 \le q,\ r \le \infty$, the homogeneous Besov space is denoted by

$$\dot B^s_{q,r} = \left\{ f \in \mathcal{Z}'(\mathbb{R}^n) : \|f\|_{\dot B^s_{q,r}} < \infty \right\},$$

with

$$\|f\|_{\dot B^s_{q,r}} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsr}\|\phi_j * f\|_{L^q}^r\right)^{1/r}, & \text{if } 1 \le r < \infty, \\ \sup_{j\in\mathbb{Z}}\{2^{js}\|\phi_j * f\|_{L^q}\}, & \text{if } r = \infty, \end{cases}$$

where $\phi_j(x) = 2^{jn}\phi(2^j x)$.

Proposition 2.4. *For any $0 < \sigma \le n$, we have*

$$L^{\frac{n}{\sigma}}(\mathbb{R}^n) \hookrightarrow \dot B^{-\sigma}_{\infty,\infty}(\mathbb{R}^n).$$

Proof. From Young's inequality, for any $j \in \mathbb{Z}$, we have

$$2^{-j\sigma}\|\phi_j * f\|_{L^\infty} \le C2^{-j\sigma}\|\phi_j\|_{L^{\frac{n}{n-\sigma}}}\|f\|_{L^{\frac{n}{\sigma}}} \le C\|f\|_{L^{\frac{n}{\sigma}}}\,.$$

This completes the proof. $\square$

3. Proof of main results

Proof of Theorem 1.3. Since $u \in L^{2n}$, by applying Proposition 2.2, we obtain a weak solution u which is smooth in $(0, T_0)$. Therefore, for any $T > 0$, we can assume that u is smooth on $(0, T)$.

For $s \in (s_0, \frac{n}{2})$, applying $(-\Delta)^{s/2}$ to (1.1), and using $(-\Delta)^{s/2}u$ as a test function to the resulting equation we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |(-\Delta)^{s/2}u(t)|^2\,dx + \int |\nabla(-\Delta)^{s/2}u|^2\,dx\\
&= -\int (-\Delta)^{s/2}(u\cdot\nabla u)\cdot(-\Delta)^{s/2}u\,dx\\
&= -\int (-\Delta)^{s/2}\operatorname{div}(u\otimes u)\cdot(-\Delta)^{s/2}u\,dx\\
&= -\int (-\Delta)^{\frac{s-\alpha}{2}}\operatorname{div}(u\otimes u)\cdot(-\Delta)^{\frac{s+\alpha}{2}}u\,dx\\
&\le \|u\otimes u\|_{H^{1+s-\alpha}}\|(-\Delta)^{\frac{s+\alpha}{2}}u\|_{L^2}\\
&\le \|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|(-\Delta)^{\frac{s+1}{2}}u\|_{L^2}\|(-\Delta)^{s/2}u\|_{L^2}^{1-\alpha}\|(-\Delta)^{\frac{s+1}{2}}u\|_{L^2}^{\alpha}\\
&\le \delta\|(-\Delta)^{\frac{s+1}{2}}u\|_{L^2}^2 + C_\delta\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}\|(-\Delta)^{s/2}u\|_{L^2}^2\,,
\end{aligned}$$

where $\delta > 0$ is small enough.

Notice that we have used the inequality ([13])

$$\|u\otimes u\|_{H^{1+s-\alpha}} \le C\|u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|(-\Delta)^{\frac{s+1}{2}}u\|_{L^2}\,,$$

the Gagliardo–Nirenberg inequality [3], and the Young inequality. Therefore,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|(-\Delta)^{s/2}u(t)\|_{L^2}^2 &\le C\|u(t)\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}\|(-\Delta)^{s/2}u(t)\|_{L^2}^2\\
&\le C\frac{\|u(t)\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{1-\alpha}}}{1+\log^+\|(-\Delta)^{\frac{s_0}{2}}u(t)\|_{L^2}}\|(-\Delta)^{s/2}u(t)\|_{L^2}^2\\
&\quad\times\big(1+\log^+\|(-\Delta)^{s_0/2}u(t)\|_{L^2}\big)\,.
\end{aligned}\tag{3.1}$$

Thanks to the Gagliardo–Nirenberg inequality, we obtain

$$\|(-\Delta)^{\frac{s_0}{2}}u(t)\|_{L^2} \le \|u(t)\|_{L^2}^{1-\frac{s_0}{s}}\|(-\Delta)^{s/2}u(t)\|_{L^2}^{\frac{s_0}{s}} \le \|u\|_{L^\infty(0,T;L^2)}^{1-\frac{s_0}{s}}\|(-\Delta)^{s/2}u(t)\|_{L^2}^{\frac{s_0}{s}}$$

for $t \in (0, T)$. Since $u \in L^\infty\big(0, T; L^2(\mathbb{R}^n)\big)$, it follows from the above inequality that

$$1+\log^+\|(-\Delta)^{\frac{s_0}{2}}u(t)\|_{L^2} \le C\log\Big(e+\|(-\Delta)^{s/2}u(t)\|_{L^2}^2\Big)\,. \tag{3.2}$$

Combining (3.1) and (3.2) we obtain

$$\frac{d}{dt}\|(-\Delta)^{s/2}u(t)\|_{L^2}^2$$

$$\leq C\frac{\|u(t)\|^{\frac{2}{1-\alpha}}_{\dot B^{-\alpha}_{\infty,\infty}}}{1+\log^+\|(-\Delta)^{\frac{s_0}{2}}u(t)\|_{L^2}}\|(-\Delta)^{s/2}u(t)\|_{L^2}\log\big(e+\|(-\Delta)^{s/2}u(t)\|^2_{L^2}\big)\,,$$

which implies

$$\|u\|_{L^\infty(0,T;H^s)}\leq C\,.$$

Hence, from the Sobolev embedding, we deduce that

$$u\in L^\infty(0,T;L^{\frac{2n}{n-2s}}(\mathbb{R}^n))\,.$$

This and Proposition 2.2 imply that u is smooth on $[0,T]$. The proof of Theorem 1.3 is complete. □

Proof of Theorem 1.6. It is not difficult to verify that w satisfies the equation

$$w_t+u\cdot\nabla w-\Delta w=w\cdot\nabla u\,. \tag{3.3}$$

Testing (3.3) with $-\Delta w$, and using that $\operatorname{div}u=0$ we obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\|\nabla w(t)\|^2_{L^2}+\|\Delta w(t)\|^2_{L^2}\\
&=\int(u\cdot\nabla)w\cdot\Delta w\,dx-\int(w\cdot\nabla)u\cdot\Delta w\,dx\\
&=\sum_{i,j}\int u_i\partial_i w\cdot\partial_j^2 w\,dx+\int(-\Delta)^{\frac{1-\alpha}{2}}(w\cdot\nabla u)\cdot(-\Delta)^{\frac{1+\alpha}{2}}w\,dx\\
&=-\sum_{i,j}\int\partial_j u_i\partial_i w\cdot\partial_j w\,dx+\int(-\Delta)^{\frac{1-\alpha}{2}}(w\cdot\nabla u)\cdot(-\Delta)^{\frac{1+\alpha}{2}}w\,dx\\
&=-\sum_{i,j}\int\partial_i(\partial_j u_i w)\cdot\partial_j w\,dx+\int(-\Delta)^{\frac{1-\alpha}{2}}(w\cdot\nabla u)\cdot(-\Delta)^{\frac{1+\alpha}{2}}w\,dx\\
&=-\sum_{i,j}\int(-\Delta)^{\frac{-\alpha}{2}}\partial_i(\partial_j u_i w)\cdot(-\Delta)^{\frac{\alpha}{2}}\partial_j w\,dx\\
&\quad+\int(-\Delta)^{\frac{1-\alpha}{2}}(w\cdot\nabla u)\cdot(-\Delta)^{\frac{1+\alpha}{2}}w\,dx\,.
\end{aligned}\tag{3.4}$$

By Hölder's inequality and the Plancherel theorem, we obtain

$$\begin{aligned}
&\Big|\int(-\Delta)^{\frac{-\alpha}{2}}\partial_i(\partial_j u_i w)\cdot(-\Delta)^{\frac{\alpha}{2}}\partial_j w\,dx+\int(-\Delta)^{\frac{1-\alpha}{2}}(w\cdot\nabla u)\cdot(-\Delta)^{\frac{1+\alpha}{2}}w\,dx\Big|\\
&\leq C\|(-\Delta)^{\frac{1-\alpha}{2}}(w\cdot\nabla u)\|_{L^2}\|(-\Delta)^{\frac{1+\alpha}{2}}(w)\|_{L^2}\,.
\end{aligned}\tag{3.5}$$

On the other hand, we recall the following two inequalities obtained in [13] and [9]:

$$\begin{gathered}
\|fg\|_{\dot H^s}\leq C\big(\|f\|_{\dot B^{-\alpha}_{\infty,\infty}}\|g\|_{\dot H^{s+\alpha}}+\|g\|_{\dot B^{-\alpha}_{\infty,\infty}}\|f\|_{\dot H^{s+\alpha}}\big)\,,\\
\|\nabla u\|_{\dot B^{-\alpha}_{\infty,\infty}}\leq C\|w\|_{\dot B^{-\alpha}_{\infty,\infty}}\,.
\end{gathered}$$

Then

$$\begin{aligned}
\|(-\Delta)^{\frac{1-\alpha}{2}}(w\cdot\nabla u)\|_{L^2}&\leq C\|w\cdot\nabla u\|_{\dot H^{1-\alpha}}\\
&\leq C\big(\|w\|_{\dot B^{-\alpha}_{\infty,\infty}}\|\nabla u\|_{\dot H^1}+\|\nabla u\|_{\dot B^{-\alpha}_{\infty,\infty}}\|w\|_{\dot H^1}\big)\\
&\leq C\big(\|w\|_{\dot B^{-\alpha}_{\infty,\infty}}\|\Delta u\|_{L^2}+\|w\|_{\dot B^{-\alpha}_{\infty,\infty}}\|\nabla w\|_{L^2}\big)\\
&\leq C\|w\|_{\dot B^{-\alpha}_{\infty,\infty}}\|\nabla w\|_{L^2}\,.
\end{aligned}$$

Then we deduce, from (3.4) and the interpolation inequality, that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla w(t)\|_{L^2}^2+\|\Delta w(t)\|_{L^2}^2 &\le C\|w(t)\|_{\dot B^{-\alpha}_{\infty,\infty}}\|\nabla w(t)\|_{L^2}\|(-\Delta)^{\frac{1+\alpha}{2}}(w)\|_{L^2}\\ &\le C\|w(t)\|_{\dot B^{-\alpha}_{\infty,\infty}}\|\nabla w(t)\|_{L^2}\|\nabla w(t)\|_{L^2}^{1-\alpha}\|\Delta w(t)\|_{L^2}^{\alpha}\\ &\le C\|w(t)\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}}\|\nabla w(t)\|_{L^2}^2+\frac{1}{2}\|\Delta w(t)\|_{L^2}^2\,.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{dt}\|\nabla w(t)\|_{L^2}^2 &\le C\|w(t)\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}}\|\nabla w(t)\|_{L^2}^2\\ &= C\frac{\|w(t)\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}}}{1+\log^+\|w(t)\|_{\dot H^{s_0}}}\left(1+\log^+\|w(t)\|_{\dot H^{s_0}}\right)\|\nabla w(t)\|_{L^2}^2\,.\end{aligned}$$

By Gronwall's inequality, we obtain

$$\begin{aligned}\|\nabla w(t_2)\|_{L^2}^2 \le \|\nabla w(t_1)\|_{L^2}^2\exp\Big(C\log\Big(e+\sup_{t\in[t_1,t_2]}\|w(t)\|_{\dot H^{s_0}}\Big)\\ \times\int_{t_1}^{t_2}\frac{\|w(\tau)\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}}}{1+\log^+\|w(\tau)\|_{\dot H^{s_0}}}d\tau\Big)\end{aligned}\tag{3.6}$$

for all $0<t_1<t_2<T$. Moreover, it follows from (1.10) that for every $\varepsilon>0$, there exists $0<T^*<T$ such that

$$\int_{T^*}^{T}\frac{\|w(t)\|_{\dot B^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}}}{1+\log^+\|w(t)\|_{\dot H^{s_0}}}\,dt<\varepsilon\,.$$

This and (3.6) imply that there exists a constant $C_0>0$ (independent of $w(t)$) such that

$$\begin{aligned}\|\nabla w(t)\|_{L^2}^2 &\le \|\nabla w(T^*)\|_{L^2}^2\Big(e+\sup_{t\in[T^*,t]}\|w(t)\|_{\dot H^{s_0}}\Big)^{C_0\varepsilon}\\ &\le C\Big(e+\sup_{t\in[T^*,t]}\|w(t)\|_{\dot H^{s_0}}\Big)^{C_0\varepsilon},\end{aligned}\tag{3.7}$$

for all $T^*<t<T$.
To obtain the conclusion, it suffices to prove that $\|w(t)\|_{\dot H^{s_0}}$ is bounded on $[T^*,T]$.

We can proceed as the proof of Theorem 1.1 of [5] and obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta w(t)\|_{L^2}^2\le C\left(1+\|\nabla w(t)\|_{L^2}\right)^6\,.\tag{3.8}$$

We now divide the rest of our proof into the two following cases:

(i) If $\alpha\in(0,\frac{1}{2})$, then $s_0\in(1,\frac{3}{2})$, and it follows from the inequality of Gagliardo-Nirenberg type that

$$\|w(t)\|_{\dot H^{s_0}}\lesssim\|w(t)\|_{\dot H^1}^{2-s_0}\|w(t)\|_{\dot H^2}^{s_0-1}\lesssim\|\nabla w(t)\|_{L^2}^{2-s_0}\|\Delta w(t)\|_{L^2}^{s_0-1}\,.\tag{3.9}$$

Combining (3.7), (3.8), and (3.9) yields that there exists a constant $C_1>0$ (independent of $w(t)$) such that

$$\|\nabla w(t)\|_{L^2}^2\le C+C\sup_{\tau\in[T^*,t]}\|\nabla w(\tau)\|_{L^2}^{C_1\varepsilon}\,,\tag{3.10}$$

for all $t \in [T^*, T)$. This implies that $\|\nabla w(t)\|_{L^2}^2$ is uniformly bounded in (T^*, T) if $\varepsilon > 0$ is chosen such that $C_1\varepsilon < 2$.
Therefore,

$$w \in L^\infty(\tau, T; L^6(\mathbb{R}^3)), \tag{3.11}$$

for any $\tau \in (0, T)$.
Then, by the result by Beirão da Veiga [2], u is regular in $(0, T]$.

(ii) If $\alpha \in (\frac{1}{2}, 1]$ then $s_0 \in (\frac{1}{2}, 1]$. Applying the inequality of Gagliardo–Nirenberg type we obtain

$$\begin{aligned}
\|w(t)\|_{\dot{H}^{s_0}} &\lesssim \|w(t)\|_{L^2}^{1-\frac{s_0}{2}} \|\Delta w(t)\|_{L^2}^{\frac{s_0}{2}} \\
&\lesssim \|\nabla u(t)\|_{L^2}^{1-\frac{s_0}{2}} \|\Delta w(t)\|_{L^2}^{\frac{s_0}{2}} \\
&\lesssim \Big(\|u(t)\|_{L^2}^{1/2} \|\Delta u(t)\|_{L^2}^{1/2}\Big)^{1-\frac{s_0}{2}} \|\Delta w(t)\|_{L^2}^{\frac{s_0}{2}} \\
&\le \Big(\|u_0\|_{L^2}^{1/2} \|\Delta u(t)\|_{L^2}^{1/2}\Big)^{1-\frac{s_0}{2}} \|\Delta w(t)\|_{L^2}^{\frac{s_0}{2}} \\
&\lesssim \Big(\|u_0\|_{L^2}^{1/2} \|\nabla w(t)\|_{L^2}^{1/2}\Big)^{1-\frac{s_0}{2}} \|\Delta w(t)\|_{L^2}^{\frac{s_0}{2}} .
\end{aligned} \tag{3.12}$$

Note that the last inequality was obtained by using the Biot-Savart law

$$u(x,t) = C \int_{\mathbb{R}^3} K(x-y) w(y,t)\, dy\,,$$

where $K(x)$ is homogeneous of degree -2. As a result, $\nabla K(x)$ is a singular kernel of Calderón-Zygmund type.

Note that

$$\Delta u(x,t) = C \int_{\mathbb{R}^3} \nabla K(x-y) \cdot \nabla w(y,t)\, dy\,.$$

It follows from the standard Calderón-Zygmund theory that

$$\|\Delta u(t)\|_{L^2} \lesssim \|\nabla w(t)\|_{L^2}\,.$$

A combination of (3.7), (3.8), and (3.12) implies that there exists a constant $C_2 > 0$ such that

$$\|\nabla w(t)\|_{L^2}^2 \le C + C \sup_{\tau \in [T^*, t]} \|\nabla w(\tau)\|_{L^2}^{C_2 \varepsilon}, \text{ for } t \in (T^*, T)\,.$$

Therefore, we also obtain the conclusion as in (3.11).
This completes the proof of Theorem 1.6. □

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References

[1] J. T. Beale, T. Kato, A. Majda; *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Commun. Math. Phys., **94** (1984), 61–66.
[2] H. Beirão da Veiga; *A new regularity class for the Navier-Stokes equations in* $\mathbb{R}^n$, Chin. Ann. Math., **16** (1995), 407–412.
[3] N. A. Dao, J. I. Díaz, Q. H. Nguyen; *Generalized Gagliardo-Nirenberg inequalities using Lorentz spaces andz BMO*, Nonlinear Analysis: Theory, Methods, Applications, **173** (2018), 146–153.

[4] L. Ecauriaza, G. A. Seregin, V. Sverák; *$L^{3,\infty}$-solutions of the Navier-Stokes equations and backward uniqueness*, Russ. Math. Surv., **58** (2003), 211–248.

[5] J. Fan, S. Jiang, G. Nakamura; *On logarithmically improved regularity criteria for the Navier-Stokes equations in $\mathbb{R}^n$*, IMA Journal of Applied Mathematics, **76** (2011), 298–311.

[6] J. Fan, S. Jiang, G. Nakamura, Y. Zhou; *Logarithmically improved regularity criteria for the Navier-Stokes and MHD equations*, J. Math. Fluid Mech., **13** (2011), 557-571.

[7] Y. Giga; *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Diff. Equ., **62** (1986), 186–212.

[8] E. Hopf; *Über die Anfangswertaufabe für die hydrodynamischen Grundgleichungen,* Math. Nachr., **4** (1951), 213–23l.

[9] B. Jawerth; *Some observations on Besov and Lizorkin-Triebel spaces*, Math. Scand., **40** (1977), 94–104.

[10] T. Kato, Strong; *L^p-solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions*, Math. Z., **187** (1984), 471–480.

[11] T. Kato, G. Ponce; *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math., **41** (1988), 891–907.

[12] H. Kozono, T. Ogawa, Y. Taniuchi; *The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations*, Math. Z., **242** (2002), 251–278.

[13] H. Kozono, Y. Shimada; *Bilinear estimates in homogeneous Triebel–Lizorkin spaces and the Navier-Stokes equations*, Math. Nachr., **276** (2004), 63–74.

[14] H. Kozono, Y. Taniuchi; *Bilinear estimates in BMO and the Navier-Stokes equations*, Math. Z., **235** (2000), 173–194.

[15] H. Kozono, Y. Taniuchi; *Limitting case of the Sobolev inequality in BMO with application to the Euler equations*, Comm. Math. Phys., **214** (2000), 191–200.

[16] P.G. Lemarié-Rieusset; *Recent Developments in the Navier-Stokes Problem*, Chapman and Hall/CRC, Boca Raton, 2002.

[17] J. Leray; *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193-248.

[18] S. Montgomery-Smith; *Conditions implying regularity of the three dimensional Navier-Stokes equation*, Appl. Math. **50** (2005), 451–464.

[19] K. Nakao, Y. Taniuchi; *An alternative proof of logarithmically improved Beale-Kato-Majda type extension criteria for smooth solutions to the Navier-Stokes equations*, Nonl. Anal., **176** (2018), 48–55.

[20] J. Necas, M. Ruzicka, V. Sverák; *On Leray's self-similar solutions of the Navier-Stokes equations*, Acta Math., **176** (1996), 283–294.

[21] T. Ohyama; *Interior regularity of weak solutions to the Navier-Stokes equation.* Proceedings of the Japan Academy, **36** (1960), 273–277.

[22] G. Prodi; *Un teorema di unicità per le equazioni di Navier-Stokes*, Ann. Mat. Pura Appl., **48** (1959), 173–182.

[23] J. Serrin; *On the interior regularity of weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal., **9** (1962), 187–195.

[24] H. Triebel; *Theory of function spaces II*. Birkhäuser, 1992.

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