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MONOTONE SOLUTIONS OF FIRST ORDER NONLINEAR DIFFERENTIAL SYSTEMS

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Abstract. This article concerns the classification, continuablity, boundedness, and existence of solutions for a system of first order nonlinear differential equations. First, we prove that all solutions of the system are eventually monotonic and can be separated into two classes. Then we discuss the continuability of solutions. After that we establish necessary and sufficient conditions for the boundedness of all solutions. Also, we study the existence of monotone solutions in certain classes.

1. Introduction

We study the system of first order nonlinear differential equations

$$\begin{aligned} x'(t) &= p(t)f(y(t)), \\ y'(t) &= q(t)g(x(t)), \end{aligned} \tag{1.1}$$

where $p(t)$, $q(t)$, $f(r)$, and $g(r)$ are functions satisfying the conditions:

(H0) $p(t), q(t) : [a, \infty) \to (0, \infty)$ are continuous; $f(r) : \mathbb{R} \to \mathbb{R}$ is continuous and $rf(r) > 0$ or $r \neq 0$; $g(r) : \mathbb{R} \to \mathbb{R}$ is continuous and $rg(r) > 0$ for $r \neq 0$.

A pair of functions (x, y) is called a solution of (1.1) with maximal existence interval $[a, \alpha)$, $a < \alpha \le \infty$, if both $x(t)$ and $y(t)$ are differentiable and satisfy system (1.1) on $[a, \alpha)$. A solution (x, y) is said to be eventually monotone if there exists a $t_1 \ge a$ such that both $x(t)$ and $y(t)$ are monotone on $[t_1, \alpha)$. A solution (x, y) is said to be eventually trivial if there exists a $t_1 \ge a$ such that $(x(t), y(t)) \equiv (0, 0)$ on $[t_1, \alpha)$. We only consider eventually nontrivial solutions.

The following lemma shows that all solutions of (1.1) are eventually monotone and can be separated into two classes.

Lemma 1.1. *If (x, y) is a solution of (1.1) with maximal existence interval $[a, \alpha)$, $a < \alpha \le \infty$, then (x, y) is eventually monotone and belongs to one of the two classes:*

$$\begin{gathered} A = \big\{(x, y) : \exists\, b \ge a \text{ such that } x(t)y(t) > 0, \forall t \in [b, \alpha)\big\}, \\ B = \big\{(x, y) : x(t)y(t) < 0, \forall t \in [a, \alpha)\big\}. \end{gathered}$$

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Proof. The proof is similar to that of [3, Lemma 1] with minor modifications. Let $F(t) = x(t)y(t)$. In view of condition (H0) we have

$$F'(t) = x'(t)y(t) + x(t)y'(t) = y(t)p(t)f(y(t)) + x(t)q(t)g(x(t)) \geq 0.$$

Then F is increasing on $[a, \alpha)$ and there are three possible cases:

(1) $F(t) < 0$ for all $t \in [a, \alpha)$;
(2) There exists $b \geq a$ such that $F(t) > 0$ for all $t \in [b, \alpha)$;
(3) There exists $b \geq a$ such that $F(t) \equiv 0$ for all $t \in [b, \alpha)$.

Clearly, $(x, y) \in B$ in the first case and $(x, y) \in A$ in the second case. We claim that (x, y) is eventually trivial in the third case. Otherwise, if there exists $t_1 \in [b, \alpha)$ such that $x(t_1) \neq 0$, then $y(t_1) = 0$ and there exists an open interval U_1 containing t_1 such that $x(t) \neq 0$ for any $t \in U_1$. This implies that $y(t) \equiv 0$ on U_1. This contradicts the fact that $y'(t) = q(t)g(x(t)) \neq 0$ for any $t \in U_1$. Hence, $x(t) \equiv 0$ on $[b, \alpha)$. Similarly, we can prove that $y(t) \equiv 0$ on $[b, \alpha)$. □

Remark 1.2. From (H0) and Lemma 1.1, each class A solution (x, y) of (1.1) belongs to one of the two subclasses, one subclass is comprise of positive and increasing functions x and y, and the other subclass is comprise of negative and decreasing functions x and y. The former is called the subclass of positive class A solutions and the latter is called the subclass of negative class A solutions.

The classification, boundedness, existence, asymptotic behavior, and other properties of solutions of special cases of system (1.1) (second order nonlinear differential equations) have been extensively studied; see [1, 3, 5, 6, 7, 8, 12, 13, 14, 15, 16, 17] and other publications, but there is less discussion for differential system. A special nonlinear differential system

$$\begin{aligned} x'(t) &= p(t)y^{\lambda_1}(t), \\ y'(t) &= q(t)x^{\lambda_2}(t) \end{aligned} \tag{1.2}$$

is considered in [2, 11]. For system (1.1), classification and existence of positive class A solutions are investigated in [10]. Nonoscillatory and existence of solutions are explored in [4, 9] with the assumption that $q(t) < 0$.

The following assumptions are imposed for later discussions.

(H1) There exists a real number $M > 0$ such that
$$|f(uv)| \leq M|f(u)||f(v)|, \quad |g(uv)| \leq M|g(u)||g(v)|, \quad \forall u, v \in \mathbb{R}.$$
(H2) There exists a real number $m > 0$ such that $f(r)$ and $g(r)$ are increasing for $|r| \geq m$.
(H3A) There exists a real number $r_0 > 0$ such that
$$\int_{\pm r_0}^{\pm\infty} \frac{dr}{f(g(r))} = \infty.$$
(H3B) There exists a real number $r_0 > 0$ such that
$$\int_{\pm r_0}^{\pm\infty} \frac{dr}{g(f(r))} = \infty.$$
(H4A) There exists a real number $r_1 > 0$ such that
$$\int_{0}^{\pm r_1} \frac{dr}{f(g(r))} = \infty.$$

(H4B) There exists a real number $r_1 > 0$ such that
$$\int_0^{\pm r_1} \frac{dr}{g(f(r))} = \infty.$$

Remark 1.3. (H1) holds for all homogeneous functions such as the so-called p-Laplacian operator
$$f(r) = \Phi_p(r) = |r|^{p-2}r, p > 1.$$
(H1) is also satisfied for some non-homogeneous functions as well, for example,
$$f(r) = \begin{cases} \frac{1+r^2}{r}, & |r| > 1, \\ 2r, & |r| \le 1. \end{cases}$$
It is easy to check that f is non-homogeneous, continuous, and increasing on $(-\infty, \infty)$, that $rf(r) > 0$ for all $r \neq 0$, and that
$$|f(uv)| \le |f(u)||f(v)|, \quad \forall u, v \in \mathbb{R}.$$

Our goal is to investigate the classification, continuability, boundedness, and existence of solutions of (1.1). This article is organized in the follows: Section 1 is for the introduction. The background, motivation, classification, and assumptions are addressed in this section. Continuability of solutions is discussed in Section 2. After that, necessary and sufficient conditions for boundedness of all solutions are established in Section 3. Moreover, examples are provided to illustrate that the boundedness conditions are optimal in some sense. Finally, the existence of monotone solutions in certain classes is obtained in Section 4.

2. Continuability of solutions

The maximal existence interval of a solution of (1.1) might be finite or infinite. Since many applications require the infinite existence interval of solutions, the next theorem provides conditions for the continuability of solutions.

Theorem 2.1. *Suppose that conditions* (H1), (H2), (H3A), (H3B) *hold. Then all solutions of* (1.1) *can be extended to* $[a, \infty)$.

Proof. We focus on class A solutions since all class B solutions can be extended to infinity. Without loss of generality we consider a positive class A solution (x, y), in other words, $x(t) > 0$ and $y(t) > 0$ for $b \le t < \alpha$ and both are increasing. If $\alpha < \infty$, we claim $\lim_{t\to\alpha^-} x(t) = \infty$ and $\lim_{t\to\alpha^-} y(t) = \infty$. Indeed, if $\lim_{t\to\alpha^-} x(t) < \infty$, it follows from
$$y(t) = y(a) + \int_a^t q(s)g(x(s))ds$$
that
$$\lim_{t\to\alpha^-} y(t) = y(a) + \int_a^\alpha q(t)g(x(t))dt < \infty.$$
So (x, y) can be extended to $[a, \alpha]$ and further to a small neighborhood at the right of α. This contradicts the assumption that $[a, \alpha)$ is the maximal existence interval of (x, y). Therefore, $\lim_{t\to\alpha^-} x(t) = \infty$. Similarly, we can show that $\lim_{t\to\alpha^-} y(t) = \infty$. Thus, there exists a real number $c > a$ such that $x(t) \ge m$ and

$y(t) \geq m$ for all $c \leq t < \alpha$, where the number m is defined in (H2). By (H2) both $f(x(t))$ and $g(y(t))$ are increasing on $[c, \alpha)$, then

$$\begin{aligned} y(t) &= y(c) + \int_c^t q(s)g(x(s))ds \\ &\leq y(c) + g(x(t)) \int_c^t q(s)ds \\ &= g(x(t))\Big(\frac{y(c)}{g(x(t))} + \int_c^t q(s)ds\Big) \\ &\leq g(x(t))\Big(\frac{y(c)}{g(x(c))} + \int_c^t q(s)ds\Big). \end{aligned}$$

Choosing $k > 1$ and $t_1 \geq c$ such that for $t \geq t_1$,

$$\frac{y(c)}{g(x(c))} + \int_c^t q(s)ds \leq k \int_c^t q(s)ds,$$

we have

$$y(t) \leq kg(x(t)) \int_c^t q(s)ds,$$

$$p(t)f(y(t)) \leq p(t)f\Big(kg(x(t)) \int_c^t q(s)ds\Big).$$

By (H1)

$$x'(t) = p(t)f(y(t)) \leq M^2 f(k)p(t)f\big(g(x(t))f\Big(\int_c^t q(s)ds\Big).$$

Then

$$\frac{x'(t)}{f\big(g(x(t)\big)} \leq M^2 f(k)p(t)f\Big(\int_c^t q(s)ds\Big).$$

Integrating from t_1 to t we have

$$\int_{x(t_1)}^{x(t)} \frac{dr}{f(g(r))} = \int_{t_1}^t \frac{x'(s)ds}{f(g(x(s))} \leq M^2 f(k) \int_{t_1}^t p(s)f\Big(\int_c^s q(\sigma)d\sigma\Big)ds.$$

Letting $t \to \alpha^-$ yields

$$\int_{x(t_1)}^{\infty} \frac{dr}{f(g(r))} \leq M^2 f(k) \int_{t_1}^{\alpha} p(s)f\Big(\int_c^s q(\sigma)d\sigma\Big)ds,$$

which contradicts (H3A), and hence, $x(t)$ can be extended to infinity.

Following the similar procedure, we have

$$\int_{y(t_1)}^{\infty} \frac{dr}{g(f(r))} \leq M^2 g(k) \int_{t_1}^{\alpha} q(s)g\Big(\int_c^s p(\sigma)d\sigma\Big)ds.$$

This contracts (H3B) and implies that $y(t)$ can be extended to infinity. □

Theorem 2.1 generalizes part of [14, Theorem 1] for system (1.1).

3. Boundedness of solutions

In this section we consider the boundedness of all solutions of (1.1) and assume that all solutions can be extended to infinity. Notice that it is possible only one component of a solution is bounded. For example, consider a system defined on $[1,\infty)$

$$\begin{aligned} x'(t) &= \frac{1}{(t^3+t)}y(t), \\ y'(t) &= \frac{1}{\arctan t}x(t). \end{aligned} \tag{3.1}$$

All the conditions of Theorem 2.1 are satisfied, so all solutions can be extended to infinity. It is easy to verify that $(x(t),y(t)) = (\arctan t, t)$ is a class A solution of (3.1) with bounded x component and unbounded y component.

Now we provide conditions for the boundedness of the first component of all solutions of (1.1).

Theorem 3.1. *Suppose that conditions* (H1), (H2), (H3A) *hold. Then the x components of all solutions of* (1.1) *are bounded if and only if*

$$\int_a^\infty p(t)f\Big(\int_a^t q(s)ds\Big)dt < \infty.$$

Proof. We consider class A solutions since all class B solutions are bounded. Without loss of generality we assume that (x,y) is a positive class A solution.

If x is bounded, then $\lim_{t\to\infty} x(t) = L_1 < \infty$. Define

$$K_1 = \min_{x(b)\le r\le L_1} g(r) > 0.$$

Then

$$y(t) = y(b) + \int_b^t q(s)g(x(s))ds \ge K_1 \int_b^t q(s)ds.$$

In view of (H1), we have

$$f\Big(\int_b^t q(s)ds\Big) \le f\Big(\frac{y(t)}{K_1}\Big) \le Mf\Big(\frac{1}{K_1}\Big)f(y(t)).$$

Thus

$$p(t)f\Big(\int_b^t q(s)ds\Big) \le Mf\Big(\frac{1}{K_1}\Big)x'(t).$$

Integrating from b to infinity implies

$$\int_b^\infty p(t)f\Big(\int_b^t q(s)ds\Big)ds \le Mf\Big(\frac{1}{K_1}\Big)(L_1 - x(b)) < \infty.$$

Therefore,

$$\int_a^\infty p(t)f\Big(\int_a^t q(s)ds\Big)dt < \infty.$$

For sufficiency, using arguments similar to the proof of Theorem 2.1, we have

$$\int_{x(t_1)}^{x(t)} \frac{dr}{f(g(r))} \le M^2 f(k)\int_{t_1}^t p(s)f\Big(\int_c^s q(\sigma)d\sigma\Big)ds. \tag{3.2}$$

If x is unbounded, then $\lim_{t\to\infty} x(t) = \infty$. Letting $t\to\infty$ to (3.2) we have

$$\int_{x(t_1)}^\infty \frac{dr}{f(g(r))} \le M^2 f(k)\int_{t_1}^\infty p(t)f\Big(\int_c^t q(s)ds\Big)ds < \infty.$$

This contradicts to (H3A) and hence, x is bounded. □

The following corollary is directly derived from the proof of Theorem 3.1.

Corollary 3.2. *Let* (H1), (H2), (H3A) *hold. If the x component of one class A solution of* (1.1) *is bounded, then the x components of all solutions are bounded. On the other hand, if the x component of one class A solution is unbounded, then the x components of all class A solutions are unbounded.*

Remark 3.3. Condition (H3A) in Theorem 3.1 is sharp. For example consider the differential system, defined for $t \geq 1$,

$$\begin{aligned} x'(t) &= \frac{1}{t^{4/3}} y^{1/3}(t), \\ y'(t) &= \frac{4}{t^2} x^5(t). \end{aligned} \tag{3.3}$$

Here, $p(t) = 1/t^{4/3}$, $q(t) = 4/t^2$, $f(r) = r^{1/3}$, and $g(r) = r^5$. Clearly, (H1), (H1), and (H2) are satisfied, but (H3A) does not hold because

$$\int_{\pm 1}^{\pm\infty} \frac{dr}{f(g(r))} = \int_{\pm 1}^{\pm\infty} \frac{dr}{r^{\frac{5}{3}}} < \infty.$$

Note that

$$\int_1^\infty p(t) f\Big(\int_1^t q(s)ds\Big)dt = \sqrt[3]{4} \int_1^\infty \frac{1}{t^{4/3}} \Big(1 - \frac{1}{t}\Big)^{1/3} < \infty.$$

However, $(x, y) = (t, t^4)$ is an unbounded solution of (3.3).

Remark 3.4. Theorem 3.1 can be applied to some differential equations but [3, Theorem 8] is not applicable. For example, let $p(t) = \Phi_{p^*}(1/a(t))$, $f(r) = \Phi_{p^*}(r)$, $q(t) = b(t)$, and $g(r) = f(r)$ (the function f is defined in [3]), where $\Phi_p(r)$ is the p-Laplacian operator and $p^* = \frac{p}{p-1}$. Then system (1.1) becomes equation (1) in [3],

$$(a(t)\Phi_p(x'))' = b(t) f(x). \tag{3.4}$$

Define

$$f(r) = \begin{cases} \Phi_p(r \ln |r|), & |r| > e, \\ e^{p-2} r, & |r| \leq e. \end{cases}$$

It is easy to check that f is continuous and increasing on $(-\infty, \infty)$ and that $rf(r) > 0$ for all $r \neq 0$. The major condition (22) in [3] does not hold because

$$\lim_{|r|\to\infty} \frac{f(r)}{\Phi_p(r)} = \infty.$$

Thus, [3, Theorem 8] cannot be applied to equation (3.4). However, (H3A) is satisfied because

$$\int_{\pm e}^{\pm\infty} \frac{1}{f(g(r))} dr = \int_e^\infty \frac{dr}{r \ln r} = \infty.$$

Note that the x component of a solution of (1.1) is a solution of equation (3.4). Theorem 3.1 implies that all solutions of equation (3.4) are bounded.

Theorem 3.1 generalizes part of [14, Theorem 1] for the differential system (1.1). Because of the symmetric manner of (1.1), we have the conditions for the boundedness of the y components of all solutions of (1.1).

Theorem 3.5. *Suppose that conditions* (H1), (H2), (H3B) *hold. Then the y components of all solutions of* (1.1) *are bounded if and only if*

$$\int_a^\infty q(t)g\Big(\int_a^t p(s)ds\Big)dt < \infty.$$

Corollary 3.6. *Let* (H1), (H2), (H3B) *hold. If the y component of one class A solution of* (1.1) *is bounded, then the y components of all solutions are bounded. On the other hand, if the y component of one class A solution is unbounded, then the y components of all class A solutions are unbounded.*

Remark 3.7. Condition (H3B) in Theorem 3.5 is sharp. We already know that $(x, y) = (t, t^4)$ is an unbounded solution of (3.3), but

$$\int_1^\infty q(t)g\Big(\int_a^t p(s)ds\Big)dt = 972\int_1^\infty \frac{1}{t^2}\Big(1-\frac{1}{t^{1/3}}\Big)^5 < \infty,$$

and (H3B) does not hold because

$$\int_{\pm 1}^{\pm\infty} \frac{dr}{g(f(r))} = \int_{\pm 1}^{\pm\infty} \frac{dr}{r^{\frac{5}{3}}} < \infty.$$

Combining Theorem 3.1 and Theorem 3.5, we have the following theorem.

Theorem 3.8. *Suppose that conditions* (H1), (H2), (H3A), (H3B) *hold. Then all solutions of* (1.1) *are bounded if and only if*

$$\int_a^\infty p(t)f\Big(\int_a^t q(s)ds\Big)dt < \infty,$$

and

$$\int_a^\infty q(t)g\Big(\int_a^t p(s)ds\Big)dt < \infty.$$

Corollary 3.9. *Let* (H1), (H2), (H3A), (H3B) *hold. If system* (1.1) *has one bounded class A solution, then all solutions are bounded. On the other hand, if system* (1.1) *has one unbounded class A solution, then all class A solutions are unbounded.*

Clearly, (H1), (H2), (H3A), and (H3B) are satisfied for system (1.2) when $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_1\lambda_2 \le 1$ because

$$\int_{\pm r_0}^{\pm\infty} \frac{dr}{f(g(r))} = \int_{\pm r_0}^{\pm\infty} \frac{dr}{g(f(r))} = \int_{\pm r_0}^{\pm\infty} \frac{dr}{r^{\lambda_1\lambda_2}} = \infty$$

for any $r_0 > 0$. We have the following result for system (1.2).

Corollary 3.10. *Suppose that $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_1\lambda_2 \le 1$. Then all solutions of* (1.2) *can be extended to infinity. Moreover, the x components of all solutions are bounded if and only if*

$$\int_a^\infty p(t)\Big(\int_a^t q(s)ds\Big)^{\lambda_1} dt < \infty,$$

the y components of all solutions are bounded if and only if

$$\int_a^\infty q(t)\Big(\int_a^t p(s)ds\Big)^{\lambda_2} dt < \infty,$$

and all solutions are bounded if and only if

$$\int_a^\infty p(t)\Big(\int_a^t q(s)ds\Big)^{\lambda_1} dt < \infty,$$

and

$$\int_a^\infty q(t)\Big(\int_a^t p(s)ds\Big)^{\lambda_2} dt < \infty.$$

If assumptions (H2), (H3A), (H3B) are dropped but the boundedness of functions f and g is required, we have the following results.

Theorem 3.11. *Let* (H1) *hold. Assume that there exists $K > 0$ such that $|g(r)| \le K$ for all $r \in \mathbb{R}$. Then the x components of all solutions of* (1.1) *are bounded if and only if*

$$\int_a^\infty p(t)f\Big(\int_a^t q(s)ds\Big)dt < \infty.$$

Proof. The proof of necessity is similar to the proof of Theorem 3.1. For sufficiency, without loss of generality we consider positive class A solution (x, y). By the boundedness assumption of g we have

$$y(t) = y(b) + \int_b^t q(s)g(x(s))ds \le y(b) + K\int_b^t q(s)ds.$$

Choosing $k > 1$ and $t_1 \ge b$ such that for $t \ge t_1$

$$y(b) + K\int_b^t q(s)ds \le k\int_b^t q(s)ds.$$

Then

$$p(t)f(y(t)) \le p(t)f\Big(k\int_b^t q(s)ds\Big).$$

Applying (H1) we have

$$x'(t) = p(t)f(y(t)) \le Mf(k)p(t)f\Big(\int_b^t q(s)ds\Big).$$

Then

$$x(t) - x(t_1) \le Mf(k)\int_{t_1}^t p(s)f\Big(\int_b^s q(\sigma)d\sigma\Big)ds,$$

and x is bounded. $\square$

Theorem 3.11 generalizes part of [14, Theorem 2] for the differential system (1.1).

Corollary 3.12. *Let* (H1) *hold. Assume that there exists $K > 0$ such that $|g(r)| \le K$ for all $r \in \mathbb{R}$. If the x component of one class A solution of* (1.1) *is bounded, then the x components of all solutions are bounded. On the other hand, if the x component of one class A solution is unbounded, then the x components of all class A solutions are unbounded.*

Similar to Theorem 3.11 we have the following theorem.

Theorem 3.13. *Let* (H1) *hold. Assume that there exists $K > 0$ such that $|f(r)| \le K$ for all $r \in \mathbb{R}$. Then the y components of all solutions of* (1.1) *are bounded if and only if*

$$\int_a^\infty q(t)g\Big(\int_a^t p(s)ds\Big)dt < \infty.$$

Corollary 3.14. *Let* (H1) *hold. Assume that there exists $K > 0$ such that $|f(r)| \leq K$ for all $r \in \mathbb{R}$. If the y component of one class A solution of* (1.1) *is bounded, then the y components of all solutions are bounded. On the other hand, if the y component of one class A solution is unbounded, then the y components of all class A solutions are unbounded.*

Combining Theorem 3.11 and Theorem 3.13 we have the following theorem.

Theorem 3.15. *Let* (H1) *hold. Assume that there exists $K > 0$ such that $|f(r)| \leq K$ and $|g(r)| \leq K$ for all $r \in \mathbb{R}$. Then all solutions of* (1.1) *are bounded if and only if*
$$\int_a^\infty p(t)f\Big(\int_a^t q(s)ds\Big)dt < \infty,$$
and
$$\int_a^\infty q(t)g\Big(\int_a^t p(s)ds\Big)dt < \infty.$$

Corollary 3.16. *Let* (H1) *hold. Assume that there exists $K > 0$ such that $|f(r)| \leq K$ and $|g(r)| \leq K$ for all $r \in \mathbb{R}$. If system* (1.1) *has one bounded class A solution, then all solutions are bounded. On the other hand, if system* (1.1) *has one unbounded class A solution, then all class A solutions are unbounded.*

4. Existence of solutions

In this section we discuss the existence of solutions for each class and require the existence and uniqueness of all initial value problems of system (1.1), in other words, the IVP problem
$$\begin{aligned} x'(t) &= p(t)f(y(t)), \quad x(a) = x_0,\\ y'(t) &= q(t)g(x(t)), \quad y(a) = y_0 \end{aligned} \tag{4.1}$$
has a unique solution for any pair (x_0, y_0). It is well-known that the existence and uniqueness of IVP problems of system (1.1) holds under general conditions.

Theorem 4.1. *System* (1.1) *has both positive and negative class A solutions.*

Proof. Let (x, y) be the solution of (4.1) with initial condition $(x(a), y(a)) = (x_0, y_0)$, where $x_0 > 0$ and $y_0 > 0$. We claim that (x, y) is a positive class A solution of system (1.1). Indeed, from the proof of Lemma 1.1 we have $F(t) > 0$ for $t \in [a, \alpha)$, so (x, y) is a positive class A solution. Similarly, if $x_0 < 0$ and $y_0 < 0$, (x, y) is a negative class A solution of system (1.1). $\square$

Now, we focus on the existence of class B solutions.

Theorem 4.2. *Let* (H1), (H2), (H4A) *hold. Assume that*
$$\lim_{r\to\pm\infty} f(r) = \pm\infty. \tag{4.2}$$
Then system (1.1) *has class B solutions.*

Proof. Let $x_0 > 0$ and c be a real number. Denote the solution of (1.1) with the initial condition $(x(a), y(a)) = (x_0, c)$ by $(x(t, c), y(t, c))$. We define two sets
$$U = \{c \in \mathbb{R} : \text{there exists } t_1 \geq a \text{ such that } x(t_1, c) < 0\},$$
$$V = \{c \in \mathbb{R} : \text{there exists } t_2 \geq a \text{ such that } y(t_2, c) > 0\}.$$

Clearly, both U and V are open sets and V is nonempty. We claim that U is also nonempty. Indeed, by (4.2) we can select $c \in \mathbb{R}$ such that

$$c < -g(x_0) \int_a^{a+1} q(s) ds$$

and

$$x_0 + f\Big(c + g(x_0) \int_a^{a+1} q(s) ds\Big)\Big(\int_a^{a+1} p(s) ds\Big) < 0. \tag{4.3}$$

Obviously, U is nonempty if there exists $\bar{t} \in (a, a+1]$ such that $x(\bar{t}, c) < 0$, so we assume $x(t, c) \geq 0$ on $[a, a+1]$ in the following discussion.

It is easy to check that $y(t, c) < 0$ for $a \leq t \leq a+1$. If not, there exists $t_1 \in (a, a+1]$ such that $y(t_1, c) = 0$ and $y(t, c) < 0$ for $t \in [a, t_1)$. Consequently,

$$0 = y(t_1, c) = c + \int_a^{t_1} q(s) g(x(s, c)) ds \leq c + g(x_0) \int_a^{a+1} q(s) ds < 0.$$

This is a contradiction. Therefore, $y(t, c) < 0$ on $a \leq t \leq a+1$ and $x(t, c)$ is decreasing on $[a, a+1]$. Moreover, for $t \in [a, a+1]$ we have

$$y(t, c) = c + \int_a^t q(s) g(x(s, c)) ds \leq c + g(x_0) \int_a^{a+1} q(s) ds. \tag{4.4}$$

In account of (4.3) and (4.4) we have

$$\begin{aligned} x(a+1, c) &= x_0 + \int_a^{a+1} p(t) f(y(t, c)) dt \\ &\leq x_0 + \int_a^{a+1} p(t) f\Big(c + g(x_0) \int_a^{a+1} q(s) ds\Big) dt \\ &\leq x_0 + f\Big(c + g(x_0) \int_a^{a+1} q(s) ds\Big) dt \int_a^{a+1} p(s) ds < 0, \end{aligned}$$

which contradicts the fact $x(t, c) \geq 0$ on $[a, a+1]$ and hence U is nonempty.

Clearly, $U \cap V = \emptyset$. Hence, $\mathbb{R} - (U \cup V) \neq \emptyset$. Let $c \in \mathbb{R} - (U \cup L)$. Then $x(t, c)$ is a nonincreasing nonnegative function and $y(t, c)$ is a nondecreasing and nonpositive function on $[a, \infty)$. We will show that $x(t, c) > 0$ and $y(t, c) < 0$ on $[a, \infty)$. If not, there exists $t^* > a$ such that $x(t, c) > 0$, $y(t, c) < 0$ for $t \in [a, t^*)$, and $x(t, c) = 0$, $y(t, c) = 0$ for $t \geq t^*$. Note that for $t \in [a, t^*]$

$$\begin{aligned} y(t, c) &= y(t^*) - \int_t^{t^*} q(s) g(x(s, c)) ds \\ &\geq - \int_t^{t^*} q(s) g(x(s, c)) ds \\ &\geq -g(x(t)) \int_t^{t^*} q(s) ds. \end{aligned}$$

Then

$$\begin{aligned} x'(t) = p(t) f(y(t)) &\geq p(t) f\Big(- g(x(t)) \int_t^{t^*} q(s) ds\Big) \\ &\geq M f(g(x(t))) p(t) f\Big(- \int_t^{t^*} q(s) ds\Big). \end{aligned}$$

Dividing both sides by $f(g(x(t)))$ and integrating from a to t^*, we have

$$\int_0^{x_0} \frac{dr}{f(g(r))} \le -M \int_a^{t^*} p(t) f\Big(-\int_t^{t^*} q(s)ds\Big) dt < \infty,$$

which contradicts (H4A). Therefore, $(x(t,c), y(t,c))$ is a class B solution. $\square$

Theorem 4.2 can be applied to some differential equations but [3, Theorem 6] is not applicable. Consider equation (3.4) again and define

$$f(r) = \begin{cases} \Phi_p(r), & |r| > 1/e, \\ \Phi_p(-r\ln|r|), & |r| \le 1/e, \ r \ne 0, \\ 0, & r = 0, \end{cases}$$

where $p > 1$. It is easy to check that f is continuous and nondecreasing on $(-\infty, \infty)$, that $rf(r) > 0$ for all $r \ne 0$, and that $\lim_{r\to\pm\infty} f(r) = \pm\infty$. Note that the major condition (18) in [3] does not hold because

$$\lim_{r\to 0} \frac{f(r)}{\Phi_p(r)} = \infty,$$

So, [3, Theorem 6] cannot be applied to equation (3.4). However, (H4A) is satisfied because

$$\int_0^{\pm 1/e} \frac{1}{f(g(r))} dr = -\int_0^{\pm 1/e} \frac{dr}{r\ln|r|} = \infty.$$

By Theorem 4.2 equation (3.4) has class B solutions.

Theorem 4.3. *Let* (H1), (H2), (H4B) *hold. Assume that*

$$\lim_{r\to\pm\infty} g(r) = \pm\infty. \tag{4.5}$$

Then system (1.1) *has class B solutions.*

The proof of the above theorem is similar to Theorem 4.2 by switching the role of x and y. We omit it here. We already know that class A solutions could be bounded or unbounded. The next results provide the existence of certain bounded class A solutions.

Theorem 4.4. *Let* (H1), (H2) *hold. Then system* (1.1) *has a class A solution with bounded x component if and only if*

$$\int_a^\infty p(t) f\Big(\int_a^t q(s)ds\Big) dt < \infty.$$

Proof. The proof of necessity is similar to the proof of Theorem 3.1. For the sufficiency, let

$$M_2 = \max_{1\le r\le 2} g(r) > 0.$$

From the assumption we can choose $d \ge b$ such that

$$\int_d^\infty p(s) f\Big(\frac{1}{M_2} + \int_d^s q(\sigma)d\sigma\Big) ds \le \frac{1}{Mf(M_2)}.$$

Let $CB[d,\infty)$ be the Banach space of all bounded continuous functions defined on $[d,\infty)$ with the supremum norm. Define a nonempty subset of $CB[d,\infty)$ as

$$X = \{x \in CB[d,\infty) : 1 \le x(t) \le 2, \ \forall t \ge d\}.$$

Clearly, X is a bounded, closed, and convex set. Define a mapping $F : X \to CB[d,\infty)$ by

$$(Fx)(t) = 1 + \int_d^t p(s) f\Big(1 + \int_d^s q(\sigma) g(x(\sigma)) d\sigma\Big) ds.$$

It is a routine practice to show that F maps X into X, F is continuous in X, and $F(X)$ is pre-compact in $CB[d,\infty)$.

F maps X into X because for any $x \in X$ we have

$$\begin{aligned} 1 \le (Fx)(t) &= 1 + \int_d^t p(s) f\Big(1 + \int_d^s q(\sigma) g(x(\sigma)) d\sigma\Big) ds \\ &\le 1 + \int_d^t p(s) f\Big(1 + M_2 \int_d^s q(\sigma) d\sigma\Big) ds \\ &\le 1 + M f(M_2) \int_d^\infty p(s) f\Big(\frac{1}{M_2} + \int_d^s q(\sigma) d\sigma\Big) ds \le 2. \end{aligned}$$

To show the continuity of F in X, we need to prove that $\|Fx_n - Fx^*\| \to 0$ if $\{x_n\}$, $x^* \in X$ such that $\|x_n - x^*\| \to 0$ as $n \to \infty$. Indeed, for any $s \in [d,\infty)$, since $x_n(s) \to x^*(s)$ as $n \to \infty$, we have

$$\Big| p(s) f\Big(1 + \int_d^s q(\sigma) g(x_n(\sigma)) d\sigma\Big) - p(s) f\Big(1 + \int_d^s q(\sigma) g(x^*(\sigma)) d\sigma\Big)\Big| \to 0.$$

Moreover, for any $s \in [d,\infty)$

$$\begin{aligned} &\Big| p(s) f\Big(1 + \int_d^s q(\sigma) g(x_n(\sigma)) d\sigma\Big) - p(s) f\Big(1 + \int_d^s q(\sigma) g(x^*(\sigma)) d\sigma\Big)\Big| \\ &\le 2M f(M_2) p(s) f\Big(\frac{1}{M_2} + \int_d^s q(\sigma) d\sigma\Big) := J(s), \end{aligned} \tag{4.6}$$

and

$$\int_d^\infty J(s) ds = 2M f(M_2) \int_d^\infty p(s) f\Big(\frac{1}{M_2} + \int_d^s q(\sigma) d\sigma\Big) ds < \infty. \tag{4.7}$$

It follows from (4.6), (4.7), and the Dominated Convergence Theorem that

$$\begin{aligned} \|Fx_n - Fx^*\| &= \sup_{b \le t < \infty} |(Fx_n)(t) - (Fx^*)(t)| \\ &= \sup_{b \le t < \infty} \Big| \int_d^t p(s) f\Big(1 + \int_d^s q(\sigma) g(x_n(\sigma)) d\sigma\Big) ds \\ &\quad - \int_d^t p(s) f\Big(1 + \int_d^s q(\sigma) g(x^*(\sigma)) d\sigma\Big) ds \Big| \\ &\le \sup_{b \le t < \infty} \int_d^t \Big| p(s) f\Big(1 + \int_d^s q(\sigma) g(x_n(\sigma)) d\sigma\Big) \\ &\quad - p(s) f\Big(1 + \int_d^s q(\sigma) g(x^*(\sigma)) d\sigma\Big)\Big| ds \\ &\le \int_d^\infty \Big| p(s) f\Big(1 + \int_d^s q(\sigma) g(x_n(\sigma)) d\sigma\Big) \\ &\quad - p(s) f\Big(1 + \int_d^s q(\sigma) g(x^*(\sigma)) d\sigma\Big)\Big| ds \to 0. \end{aligned}$$

To prove that $F(X)$ is pre-compact in $CB[d,\infty)$, we need to show that for any sequence $\{x_n\} \in X$, $\{Fx_n\}$ has a convergent subsequence in $CB[d,\infty)$. If we can prove that $\{Fx_n\}$ has a convergent subsequence in $C[b_1,b_2]$ for any closed and bounded interval $[b_1,b_2]$ of $[d,\infty)$, then $\{Fx_n\}$ has a convergent subsequence in $CB[d,\infty)$ by the diagonal rule. By Arzelà–Ascoli Theorem, if we can show that $\{Fx_n\}$ is uniformly bounded and equicontinuous in $C[b_1,b_2]$, then $\{Fx_n\}$ has a convergent subsequence in $C[b_1,b_2]$.

Obviously, $\{Fx_n\}$ is uniformly bounded because $1 \le (Fx_n)(t) \le 2$ for all $t \in [d,\infty)$ and $n \in \mathbb{N}$. Notice that

$$
\begin{aligned}
(Fx_n)'(t) &= p(t)f\Big(1+\int_d^t q(s)g(x_n(s))ds\Big)\\
&\le Mf(M_2)p(t)f\Big(\frac{1}{M_2}+\int_d^t q(s)ds\Big).
\end{aligned}
$$

By the Mean Value Theorem, there exists a $\xi \in [t_1,t_2]$ such that

$$
\begin{aligned}
&|(Fx_n)(t_1)-(Fx_n)(t_2)|\\
&=|(Fx_n)'(\xi)(t_1-t_2)|\\
&\le Mf(M_2)\max_{b_1\le t\le b_2}\Big(p(t)f\Big(\frac{1}{M_2}+\int_d^t q(s)ds\Big)\Big)|t_1-t_2|.
\end{aligned}
$$

Hence, $\{Fx_n\}$ is equicontinuous on $[b_1,b_2]$ and therefore $F(X)$ is pre-compact in $CB[d,\infty)$. Applying Schauder Fixed-Point Theorem we claim that F has a fixed point $\bar{x}$ in X, that is

$$
\bar{x}(t)=1+\int_d^t p(s)f\Big(1+\int_d^s q(\sigma)g(\bar{x}(\sigma))d\sigma\Big)ds.
$$

We define

$$
\bar{y}(t)=1+\int_d^t q(s)g(\bar{x}(s))ds.
$$

It is easy to verify that $(\bar{x}(t),\bar{y}(t))$ is a positive class A solution of system (1.1) with bounded x component. $\square$

Theorem 4.4 generalizes [3, Theorem 3] and [14, Theorem 3] for system (1.1). Following the similar arguments we can prove the following Theorem.

Theorem 4.5. *Suppose that (H1) and (H2) hold. Then system* (1.1) *has a class A solution with bounded y component if and only if*

$$
\int_a^\infty q(t)g\Big(\int_a^t p(s)ds\Big)dt<\infty.
$$

Combining Theorems 4.4 and 4.5 we have the following result.

Theorem 4.6. *Suppose that* (H1), (H2) *hold. Then system* (1.1) *has a bounded class A solution if and only if*

$$
\int_a^\infty p(t)f\Big(\int_a^t q(s)ds\Big)dt<\infty,
$$

and

$$
\int_a^\infty q(t)g\Big(\int_a^t p(s)ds\Big)dt<\infty.
$$

If (x,y) is a class B solution, then both $\lim_{t\to\infty} x(t) = u_x$ and $\lim_{t\to\infty} y(t) = u_y$ are finite. We will discuss the results of nonzero u_x or u_y.

Theorem 4.7. *Suppose that* (H1), (H2) *hold. Then system* (1.1) *has a class B solution* (x,y) *with* $\lim_{t\to\infty} x(t) = u_x \neq 0$ *if and only if*

$$\int_a^\infty p(t)f\Big(\int_t^\infty q(s)ds\Big)dt < \infty.$$

Proof. Let (x,y) be a class B solution. Without loss of generality we assume $x(t) > 0$, $y(t) < 0$, then $x'(t) < 0$, $y'(t) > 0$ for $t \geq a$. If $\lim_{t\to\infty} x(t) = u_x > 0$, then $\lim_{t\to\infty} y(t) = u_y \leq 0$ and $K_1 := \min_{x(b)\leq r\leq u_x} g(r) > 0$. Notice that

$$-y(t) \geq u_y - y(t) = \int_t^\infty q(s)g(x(s))ds \geq K_1\int_t^\infty q(s)ds.$$

In view of (H1), we have

$$f\Big(\int_t^\infty q(s)ds\Big) \leq f\Big(-\frac{y(t)}{K_1}\Big) \leq -Mf\Big(-\frac{1}{K_1}\Big)f(y(t)).$$

Thus

$$p(t)f\Big(\int_t^\infty q(s)ds\Big) \leq -Mf\Big(\frac{1}{K_1}\Big)x'(t).$$

Integrating from a to infinity,

$$\int_a^\infty p(t)f\Big(\int_t^\infty q(s)ds\Big)ds \leq -Mf\Big(\frac{1}{K_1}\Big)(u_x - x(a)) < \infty.$$

For the sufficiency, let

$$M_2 = \max_{1\leq r\leq 2} g(r) > 0.$$

From the assumption we can choose $d \geq a$ such that

$$\int_d^\infty p(s)f\Big(\int_s^\infty q(\sigma)d\sigma\Big)ds \leq \frac{1}{Mf(M_2)}.$$

Define

$$X = \{x \in CB[d,\infty) : 1 \leq x(t) \leq 2,\ \forall t \geq d\}$$

and $F : X \to CB[d,\infty)$:

$$(Fx)(t) = 1 - \int_t^\infty p(s)f\Big(-\int_s^\infty q(\sigma)g(x(\sigma))d\sigma\Big)ds.$$

Then F maps X into X as we have

$$\begin{aligned}1 \leq (Fx)(t) &\leq 1 - \int_d^\infty p(t)f\Big(-\int_t^\infty q(s)g(x(s))ds\Big)ds\\ &\leq 1 + Mf(M_2)\int_d^\infty p(t)f\Big(\int_t^\infty q(\sigma)d\sigma\Big)ds \leq 2.\end{aligned}$$

As in the proof of Theorem 4.4, we can show that F is continuous in X and $F(X)$ is pre-compact in $CB[d,\infty)$. By Schauder Fixed-Point Theorem F has a fixed point $\bar{x}$ in X, that is

$$\bar{x}(t) = 1 - \int_t^\infty p(s)f\Big(-\int_s^\infty q(\sigma)g(\bar{x}(\sigma))d\sigma\Big)ds.$$

Define

$$\bar{y}(t) = -\int_t^\infty q(s)g(\bar{x}(s))ds.$$

It is easy to verify that $(\bar{x}(t), \bar{y}(t))$ is a class B solution of (1.1) with $\lim_{t\to\infty} x(t) = 1$. $\square$

Theorem 4.7 generalizes [3, Theorem 1] and [14, Theorem 5] for system (1.1). Using the similar arguments we can prove the following Theorem.

Theorem 4.8. *Suppose that* (H1), (H2) *hold. Then system* (1.1) *has a class B solution* (x, y) *with* $\lim_{t\to\infty} y(t) = u_y \neq 0$ *if and only if*

$$\int_a^\infty q(t)g\Big(\int_t^\infty p(s)ds\Big)dt < \infty.$$

Combining Theorem 4.7 and Theorem 4.8 we have the following theorem.

Theorem 4.9. *Suppose that* (H1), (H2) *hold. Then system* (1.1) *has a class B solution* (x, y) *with* $\lim_{t\to\infty} x(t) = u_x \neq 0$ *and* $\lim_{t\to\infty} y(t) = u_y \neq 0$ *if and only if*

$$\int_a^\infty p(t)f\Big(\int_t^\infty q(s)ds\Big)dt < \infty,$$

and

$$\int_a^\infty q(t)g\Big(\int_t^\infty p(s)ds\Big)dt < \infty.$$

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